Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 24, 1999, 231–252

# NORMAL FAMILIES, MULTIPLICITY AND THE BRANCH SET OF QUASIREGULAR MAPS

**O. Martio, U. Srebro, and J. Väisälä**

University of Helsinki, Department of Mathematics
P.O. Box 4, FIN-00014 Helsinki, Finland; olli.martio@helsinki.fi
Technion, Department of Mathematics
Haifa 32000, Israel; srebro@math.technion.ac.il
University of Helsinki, Department of Mathematics
P.O. Box 4, FIN-00014 Helsinki, Finland; jussi.vaisala@helsinki.fi

**Abstract.** A criterion for normality and compactness of families of $K$-quasiregular mappings of bounded multiplicity is established and then applied to the study of the branch set and its image.

## 1. Introduction

Let $D$ be a domain in $\mathbf{R}^n$, $n \geq 2$, and let $f\colon D \to \mathbf{R}^n$ be a discrete and open map. By a theorem of Chernavskii [C1]–[C2], see also [V1], both the branch set $B_f$ of $f$, i.e. the set where $f$ fails to define a local homeomorphism, and $fB_f$ have topological dimension $\leq n-2$. For $n = 2$, $B_f$ consists of isolated points, the local behavior of $f$ at a point $x \in D$ is quite simple, and it is classified by its local topological index $i(x, f)$. Contrary to the planar case, little is known of the structure of $B_f$ for $n \geq 3$, and maps with the same index $i(x, f)$ at $x$ may have different topological behavior in any neighborhood of $x$. Even for $n = 3$ and for small values of $i(x, f)$, the local behavior of a discrete open map can be complicated unless the image of the branch set is relatively simple near the point $f(x)$; see [MRV3, 3.20] and [MSr, 3.8].

Suppose that $f\colon D \to \mathbf{R}^n$, $n \geq 2$, is quasiregular. This means that $f$ is continuous, locally in the Sobolev space $W^{1,n}$, and for some $K \geq 1$

$$|f'(x)|^n \leq KJ(x, f) \tag{1.1}$$

a.e. in $D$. Here $f'(x)$ is the formal derivative of $f$ at $x$, $|f'(x)| = \sup_{|h|=1} |f'(x)h|$ and $J(x, f) = \det f'(x)$ is the Jacobian determinant of $f$ at $x$. By a theorem of

---

1991 Mathematics Subject Classification: Primary 30C65.

The research of the second author was partially supported by grants from the Finnish Academy, the Fund of Promotion of Research at the Technion and Japan Technion Society Research Fund.

Reshetnyak [Re, pp. 183–184], $f$ is either constant or a discrete, open and sense-preserving map. We shall only consider the latter case. Since quasiregular maps form a natural generalization of plane analytic functions to higher dimensional euclidean spaces, rather many studies have been devoted to the metric structure of their branch sets.

In [MRV1] it was shown that $m(B_f) = 0 = m(fB_f)$, where $m$ refers to the Lebesgue measure in $\mathbf{R}^n$; see also [Re, p. 224]. Moreover, on each $(n-1)$-hyperplane $T$, $m_{n-1}(T \cap B_f) = 0 = m_{n-1}(T \cap fB_f)$; see [Re, p. 221] and [MR, 3.1] for these results. Sarvas [S2, 4.10] showed that for any compact set $C \subset D$, $\dim_{\mathscr{H}}(B_f \cap C) < n$ where $\dim_{\mathscr{H}}$ refers to the Hausdorff dimension. In [MRV3, 4.4] it was proved that if $n \geq 3$ and if $B_f$ omits an open cone $C_x(\alpha)$ with vertex at $x$ and opening angle $\alpha > 0$, then $i(x,f) \leq N(\alpha, K, n)$. We replace the cone $C_x(\alpha)$ by a curvilinear cone and show in Section 5 that this result is quantitatively the best possible.

In this paper we study metric properties of the domain $D \setminus B_f$, assuming that $f\colon D \to \mathbf{R}^n$ is a $K$-quasiregular map of finite multiplicity

$$N(f) = \sup\{\#f^{-1}(y) : y \in \mathbf{R}^n\} < \infty.$$

For example, we show in Section 3 that if $D = \mathbf{R}^n$, then $\mathbf{R}^n \setminus B_f$ and $\mathbf{R}^n \setminus fB_f$ are uniform domains, and hence contain arbitrarily large balls. In fact, there are arbitrarily large balls in which $f$ is injective. The proofs are based on normal family properties of quasiregular maps of finite multiplicity. These are studied in Section 2, and they differ considerably from the quasiconformal case. In Section 4 we show that the class of nullsets for uniform domains and the class of porous sets are invariant under quasiregular maps $f\colon \mathbf{R}^n \to \mathbf{R}^n$ of finite multiplicity. The results hold in all dimensions $n \geq 2$.

Iwaniec [Iw] has studied normal families and injectivity of quasiregular mappings. His studies were mainly devoted to the stability problem, i.e. to quasiregular mappings in $\mathbf{R}^n$, $n \geq 3$, whose dilatation coefficient $K$ is close to $1$. He also uses a different type of normalization.

Our notation is standard. In particular, $B^n(x,r)$ or $B(x,r)$ denotes the open ball centered at $x \in \mathbf{R}^n$ with radius $r > 0$, $B^n(r) = B^n(0,r)$ and $B^n = B^n(1)$. Also $S^{n-1}(x,r) = \partial B^n(x,r)$, $S^{n-1}(r) = S^{n-1}(0,r)$ and $S^{n-1} = S^{n-1}(1)$. For $A \subset \mathbf{R}^n$ and $r > 0$, we let $B(A,r) = \{x : \operatorname{dist}(x,A) < r\}$ denote the $r$-neighborhood of $A$ with $B(A,\infty) = \mathbf{R}^n$. The one-point extension of $\mathbf{R}^n$ is $\dot{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$. For real numbers $r$, $s$ we write $r \wedge s = \min(r,s)$.

## 2. Normalization and normality

Let $D$ be a domain in $\mathbf{R}^n$, $n \geq 2$, let $1 \leq N < \infty$, and let $\mathscr{F}$ denote the family of all $K$-quasiregular maps $f\colon D \to \mathbf{R}^n$ with $N(f) \leq N$. Clearly, $\mathscr{F}$ is invariant under the action of sense preserving similarities of $\mathbf{R}^n$, i.e. $A \circ f \in \mathscr{F}$

if $f \in \mathscr{F}$ and $A$ is a sense preserving similarity. Next, let $\varphi\colon \mathscr{F} \to \mathbf{R}$ be a functional, and suppose that $\varphi$ is invariant under sense preserving similarities, i.e. $\varphi(A \circ f) = \varphi(f)$, where $A$ and $f$ are as above.

In studying the infimum of $\varphi$ on $\mathscr{F}$ one often considers a sequence $(f_k)$ of elements of $\mathscr{F}$ such that

$$\lim_{k\to\infty} \varphi(f_k) = \inf_{f\in\mathscr{F}} \varphi(f).$$

In view of the similarity invariance of $\mathscr{F}$ and $\varphi$, one may replace each $f_k$ by another element $g_k \in \mathscr{F}$ which satisfies certain normalization conditions, such as

$$g_k(a) = a \qquad \text{and} \qquad g_k(b) = b \tag{2.1}$$

for two fixed points $a$ and $b$ in $D$.

In the case where $n = 2$, $K = 1$ and $N = 1$, the maps are complex analytic univalent functions, and one can normalize the maps $g_k$ also by the condition

$$g_k(a) = a \qquad \text{and} \qquad g_k'(a) = 1. \tag{2.2}$$

In this case, each of the conditions (2.1) and (2.2) implies normality, and this fact is widely used in the theory of analytic univalent functions, cf. [Po2] and [Sc]. This, however, is not the case as soon as $N > 1$, as noted already in [Po2] and can be seen from the following two examples. The functions

$$g_k(z) = (k+1)z - kz^2, \qquad z \in \mathbf{C},$$

$k = 1, 2, \ldots$, are analytic and 2-valent in $B^2(r)$ for any $r > 1$. They satisfy (2.1) with $a = 0$ and $b = 1$, but $(g_k)$ is not normal in any neighborhood of $0$ since $g_k(1/k) = 1$, $k = 1, 2, \ldots$. The functions

$$g_k(z) = z - kz^2, \qquad z \in \mathbf{C},$$

$k = 1, 2, \ldots$, are analytic and 2-valent in $B^2(r)$ for any $r > \frac{1}{2}$. They satisfy (2.2) with $a = 0$, but $(g_k)$ is not normal in any neighborhood of $0$ because $g_k(k^{-1/3}) \to \infty$ as $k \to \infty$. These two examples can be generalized to quasiregular maps in all dimensions $n \geq 2$ showing that another normalization is needed for noninjective maps.

Let $\mathscr{F}$ and $\varphi$ be as above. Choose a point $a \in D$ and a number $R > 0$ such that $\overline{B}(a, R) \subset D$. Then, by the similarity invariance of $\mathscr{F}$ and $\varphi$, for each $f \in \mathscr{F}$ there exists $g \in \mathscr{F}$ with $\varphi(g) = \varphi(f)$ such that

$$g(a) = 0 \qquad \text{and} \qquad \max_{|x-a|=R} |g(x)| = 1.$$

We show in 2.5 that this normalization yields a normal, and even compact, family of elements $g$ of $\mathscr{F}$ in $B(a, R)$. This will follow from a more general result 2.4, which will be needed in Section 3. We recall that a family $\mathscr{F}$ of maps $f\colon D \to \mathbf{R}^n$ is normal if from each sequence of functions $f_k \in \mathscr{F}$ it is possible to extract a subsequence $(f_{k_i})$ which converges locally uniformly in $D$ to a function $f\colon D \to \dot{\mathbf{R}}^n$.

We first prove a distortion lemma.

2.3. **Lemma.** *Let* $0 < r < s < R \le \infty$, $1 \le K < \infty$, $N \ge 1$ *and* $n \ge 2$. *Then there is* $c = c(r,s,R,K,N,n)$ *with the following property: If* $f\colon B^n(R) \to \mathbf{R}^n$ *is a* $K$*-quasiregular map with* $f(0) = 0$ *and* $N(f) \le N$, *and if* $A \subset \overline{B}^n(r)$ *is a continuum joining* $0$ *and* $S^{n-1}(r)$, *then*

$$\max\{|f(x)| : |x| \le s\} \le c \max\{|f(x)| \in A\}.$$

*Proof.* Let $m_0$ and $m_1$ be the maximum of $|f(x)|$ over $x \in A$ and $|x| \le s$, respectively. Choose $x \in S^{n-1}(s)$ with $|f(x)| = m_1$, and define a path $\alpha\colon [1,\infty) \to \mathbf{R}^n$ by $\alpha(t) = tf(x)$. Let $\alpha^*\colon [1,t_0) \to B^n(R)$ be a maximal lift of $\alpha$ starting at $x$; see [MRV3, 3.11]. Then $|\alpha^*(t)| \ge s$ for all $t$, and $|\alpha^*(t)| \to R$ as $t \to t_0$.

Let $\Gamma$ be the family of all paths joining $A$ and the locus of $\alpha^*$ in $B^n(R)$. For the modulus $\mathsf{M}(\Gamma)$, a standard estimate gives a lower bound

$$\mathsf{M}(\Gamma) \ge q(r,s,R,n) > 0;$$

see [GM, 2.6 and 2.12]. We may assume that $m_0 < m_1$. Since each member of $f\Gamma$ meets the spheres $S^{n-1}(m_0)$ and $S^{n-1}(m_1)$, we have

$$\mathsf{M}(f\Gamma) \le \omega_{n-1}\left(\log\frac{m_1}{m_0}\right)^{1-n}.$$

Since $f$ is $K$-quasiregular with $N(f) \le N$, the $K_O(f)$-modulus inequality [MRV1, 3.2] yields $\mathsf{M}(\Gamma) \le KN\mathsf{M}(f\Gamma)$. Combining these inequalities we obtain the lemma.

2.4. **Theorem.** *Suppose that* $0 < r < R \le \infty$, $0 < r' < \infty$, $1 \le K < \infty$, $N \ge 1$, *and that* $\mathscr{F}$ *is a family of* $K$*-quasiregular maps* $f\colon B^n(R) \to \mathbf{R}^n$ *such that* $N(f) \le N$, $f(0) = 0$, *and such that for each* $f \in \mathscr{F}$ *there is a continuum* $A(f)$ *with the properties*

$$0 \in A(f), \qquad \max\{|x| : x \in A(f)\} = r, \qquad \max\{|f(x)| : x \in A(f)\} = r'.$$

*Then* $\mathscr{F}$ *is a normal family. If* $f_k \in \mathscr{F}$ *and if* $f_k \to f$ *locally uniformly in* $B^n(R)$, *then* $f$ *is a* $K$*-quasiregular map with* $N(f) \le N$.

*Proof.* For $r < s < R$, Lemma 2.3 implies that $|f(x)| \le c(r,s,R,K,N,n)r'$ for all $|x| \le s$ and $f \in \mathscr{F}$. Thus $\mathscr{F}$ is uniformly bounded in $B^n(s)$, and the normality of $\mathscr{F}$ follows from [MRV2, 3.17] or from [Re, p. 220].

Next let $(f_k)$ be a sequence in $\mathscr{F}$ converging to a map $f$ locally uniformly in $B^n(R)$. By a theorem of Reshetnyak [Re, p. 218], $f$ is $K$-quasiregular. For each $k$ there is a point $x_k \in A(f_k) \cap S^{n-1}(R)$ with $|f_k(x_k)| = r'$. Hence $|f(x)| = r'$ for some $x \in \overline{B}^n(r)$. Since $f(0) = 0$, $f$ is nonconstant. The inequality $N(f) \le N$ follows by an easy degree argument.

2.5. **Corollary.** *Let $1 < R \le \infty$, $1 \le K < \infty$ and $1 \le N < \infty$, and let $\mathscr{F}$ be a family of $K$-quasiregular maps $f\colon B^n(R) \to \mathbf{R}^n$ with $N(f) \le N$ satisfying*

$$f(0) = 0 \qquad \text{and} \qquad \max_{|x|=1} |f(x)| = 1.$$

*Then $\mathscr{F}$ is a normal family. Moreover, if $f_k \in \mathscr{F}$ and $f_k \to f$ locally uniformly in $B^n(R)$, then $f$ is $K$-quasiregular and $N(f) \le N$.*

2.6. **Remark.** The assumption in 2.5 that $N(f) \le N$ for all $f \in \mathscr{F}$ is indispensable as can be seen by considering $z^k$, $k = 1, 2, \dots$, for $n = 2$, and a sequence of polynomial-like $K$-quasiregular maps $f\colon \mathbf{R}^n \to \mathbf{R}^n$ in the sense of [Ri, I.3.2] or [Mr, Th. 2] for $n > 2$.

## 3. The branch set and multiplicity

3.1. *Terminology.* Let $c \ge 1$. A set $A \subset \mathbf{R}^n$ is *$c$-plump* if for each $x \in \bar{A}$ and for each $r > 0$ with $A \setminus B(x,r) \ne \varnothing$ there is $z \in \overline{B}(x,r)$ such that $B(z, r/c) \subset A$. A set $F \subset \mathbf{R}^n$ is *$c$-porous* if $\operatorname{int} F = \varnothing$ and if $\mathbf{R}^n \setminus F$ is $c$-plump.

Let $D$ be a proper subdomain of $\mathbf{R}^n$. For each $x \in D$ we write

$$\delta(x) = \delta_D(x) = \operatorname{dist}(x, \partial D).$$

A domain $D$ is *$c$-uniform* if $D = \mathbf{R}^n$ or if each pair of points $a$, $b$ in $D$ can be joined by a rectifiable path $\gamma\colon [0, l(\alpha)] \to D$, parametrized by arc length, such that $l(\gamma) \le c|a-b|$ and such that

$$t \wedge \bigl(l(\alpha) - t\bigr) \le c\delta\bigl(\gamma(t)\bigr) \tag{3.2}$$

for all $t \in \bigl(0, l(\alpha)\bigr)$; see [MS] and [V4]. Recall that $t \wedge s$ denotes $\min(t,s)$.

We recall from [MRV1] some basic properties of a discrete open map $f\colon D \to \mathbf{R}^n$. A domain $U$ is a *normal domain* of $f$ if $\overline{U}$ is compact in $D$ and if $f\partial U = \partial fU$. For $x \in D$, the $x$-component $U(x,f,r)$ of $f^{-1}B\bigl(f(x), r\bigr)$ is a normal domain of $f$ whenever its closure is compact in $D$. Then $fU(x,f,r) = B\bigl(f(x), r\bigr)$. If, in addition, $U(x,f,r)$ meets $f^{-1}\bigl(f(x)\bigr)$ only at $x$, it is called a *normal neighborhood* of $x$. If $U$ is a normal domain of $f$, then $f$ defines a proper map $U \to fU$, that is, the preimage of every compact set is compact.

For each $x \in D$ there is $r_0 > 0$ such that $U(x,f,r_0)$ is a normal neighborhood of $x$ for each $r \le r_0$, and $\operatorname{diam} U(x,f,r) \to 0$ as $r \to 0$. Moreover, the topological degree $\mu\bigl(f(x), f, U(x,f,r)\bigr)$ is independent of $r \in (0, r_0]$, and it is the local index $i(x,f)$ of $f$ at $x$. We also have $|i(x,f)| = N(f \,|\, V)$ for every neighborhood $V \subset U(x,f,r_0)$ of $x$. A point $x \in D$ is in $B_f$ if and only if $|i(x,f)| \ge 2$. Nonconstant quasiregular maps are sense-preserving, that is, $i(x,f) > 0$ for all $x \in D$.

If $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is quasiregular with $N(f) = N < \infty$, then $f$ extends to a continuous map $f\colon \dot{\mathbf{R}}^n \to \dot{\mathbf{R}}^n$ by $f(\infty) = \infty$. This follows, for example, from [Ri, III.2.11]. Consequently, $f$ is a proper map and also a closed map onto $\mathbf{R}^n$.

As main results of this section we shall show that if $f\colon D \to \mathbf{R}^n$ is $K$-quasiregular with $N(f) \le N < \infty$ and if $D$ is $c$-plump or $c$-uniform, then $D \setminus B_f$ has the same properties with a constant $c' = c'(c, N, K, n)$.

For plumpness this follows from the following stronger result.

3.3. **Theorem.** *For each $n \ge 2$, $K \ge 1$ and $N \ge 1$ there exists $q = q(N, K, n) > 0$ such that $f$ is injective in some ball $B(x, q) \subset B^n$ whenever $f\colon B^n \to \mathbf{R}^n$ is $K$-quasiregular with $N(f) \le N$.*

*Proof.* Assume that the theorem is false for some triple $(N, K, n)$. Then there is a sequence of $K$-quasiregular maps $f_k\colon B^n \to \mathbf{R}^n$ with $N(f_k) \le N$ such that $f_k$ is not injective in any ball $B(x, 1/k) \subset B^n$. By auxiliary similarities we can normalize the maps $f_k$ so that

$$f_k(0) = 0, \qquad \max\bigl\{|f_k(x)| : |x| \le \tfrac{1}{2}\bigr\} = 1.$$

Applying 2.5 and passing to a subsequence we may assume that the sequence $(f_k)$ converges locally uniformly to a nonconstant $K$-quasiregular map $f\colon B^n \to \mathbf{R}^n$. Choose a ball $\overline{B}(a, r) \subset B^n$ in which $f$ is injective. Then the topological degree $\mu\bigl(f(x), f, B(a, r)\bigr)$ is 1 for all $x \in B(a, r)$. Since $f_k \to f$ uniformly in $\overline{B}(a, r)$, there is $k_0$ such that $\mu\bigl(f_k(x), f_k, B(a, r)\bigr) = 1$ for all $k \ge k_0$ and $x \in B(a, r/2)$. Hence $f_k \mid B(a, r/2)$ is injective for all $k \ge k_0$, which gives a contradiction.

3.4. **Theorem.** *Suppose that $D \subset \mathbf{R}^n$ is a $c$-plump domain and that $f\colon D \to \mathbf{R}^n$ is a $K$-quasiregular map with $N(f) \le N < \infty$. Then $D \setminus B_f$ is $c'$-plump with $c' = c'(c, N, K, n)$.*

*Proof.* Assume that $x \in \overline{D}$, $r > 0$, and $D \setminus B(x, r) \ne \varnothing$. Since $D$ is $c$-plump, there is $z \in \overline{B}(x, r)$ with $B(z, r/c) \subset D$. By 3.3, $f$ is injective in some ball $B(y, qr/c) \subset B(z, r/c)$ where $q = q(N, K, n)$. Hence $D \setminus B_f$ is $(c/q)$-plump.

3.5. **Corollary.** *If $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is $K$-quasiregular with $N(f) \le N < \infty$, then $B_f$ is $c$-porous with $c = c(N, K, n)$. In particular, $\mathbf{R}^n \setminus B_f$ contains arbitrarily large balls.*

3.6. **Remark.** Every $K$-quasiconformal map $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is $\eta$-quasisymmetric with $\eta = \eta_{K,n}$; see [V3, 2.5]. Hence $f$ maps each $c$-porous set in $\mathbf{R}^n$ onto a $c'$-porous set, $c' = c'(c, K, n)$. In particular, the image of each $(n-2)$-dimensional plane $T \subset \mathbf{R}^n$ is $c'$-porous with $c' = c'(K, n)$. We remark that this result can also be obtained from 3.5. Indeed, there is a 2-valent quasiregular winding map $w\colon \mathbf{R}^n \to \mathbf{R}^n$ with $B_w = T$, and then $fT$ is the branch set of the quasiregular map $w \circ f^{-1}$.

To obtain the uniform version of 3.4 we need some auxiliary results.

3.7. *Terminology.* The *relative distance* between points $a$, $b$ in a domain $D \neq \mathbf{R}^n$ is the number

$$r_D(a,b) = \frac{|a-b|}{\delta(a) \wedge \delta(b)},$$

where $\delta(x) = \operatorname{dist}(x, \partial D)$ as before. For $c \geq 1$, we say that a pair $(a,b)$ of points in $D$ is a *$c$-pair* in $D$ if $1 \leq r_D(a,b) \leq c$. This is a simplified version of the notion considered in [V4, 2.13].

3.8. **Lemma.** *Suppose that $D \neq \mathbf{R}^n$ is a domain and that*
(1) *$D$ is $c$-plump.*
(2) *For each $8c$-pair $(a,b)$ in $D$ there is an arc $\gamma$ joining $a$ and $b$ such that*

$$\delta(a) \wedge \delta(b) \leq c_0 \operatorname{dist}(\gamma, \partial D), \qquad \operatorname{diam}\gamma \leq c_0|a-b|.$$

*Then $D$ is a $c'$-uniform domain with $c' = c'(c, c_0, n)$.*

*Proof.* Suppose that $B(a,r)$ and $B(b,s)$ are balls in $D$ such that $r/s \in [1/2, 2]$ and $|a-b| \leq 4c\max(r,s)$. By [V4, 2.15 and 2.10], it suffices to show that $a$ and $b$ can be joined by an arc $\gamma$ such that $\operatorname{diam}\gamma \leq c_1|a-b|$ and such that

$$|x-a| \wedge |x-b| \leq c_1\delta(x)$$

for all $x \in \gamma$ with $c_1 = c_1(c, c_0)$.

If $r_D(a,b) \leq 1$, we can choose $\gamma$ to be the line segment $[a,b]$. If $r_D(a,b) \geq 1$, then $(a,b)$ is obviously an $8c$-pair in $D$. Hence there is $\gamma$ satisfying (2). For each $x \in \gamma$ we have

$$\begin{aligned}|x-a| \wedge |x-b| &\leq \operatorname{diam}\gamma \leq c_0|a-b| \leq 8cc_0\big(\delta(a) \wedge \delta(b)\big) \\ &\leq 8cc_0^2 \operatorname{dist}(\gamma, \partial D) \leq 8cc_0^2\delta(x),\end{aligned}$$

and the lemma is proved.

3.9. **Lemma.** *Suppose that $a$ and $b$ are points in a $c$-uniform domain $D \subset \mathbf{R}^n$ such that $0 < |a-b| \leq c'\big(\delta(a) \wedge \delta(b)\big)$. Then there is $L = L(c, c') \geq 1$ and an $L$-bilipschitz map $F\colon B^n(|a-b|) \to D$ such that $F(0) = a$ and $F(|a-b|e_1/2) = b$.*

*Proof.* Set $r = \delta(a) \wedge \delta(b)$ and $t = |a-b|$. Then $0 < t \leq c'r$. The assertion is clear if $r < t$, since then $B\big((a+b)/2, t\sqrt{3}/2\big) \subset D$. By a result of G. Martin [Ma, 5.1], there is $L = L(c)$ and an $L$-bilipschitz map $f\colon \overline{B}^n(t) \to D$ such that $\{a,b\} \subset f\overline{B}^n(t)$. Set $U = fB^n(t)$. Since $f$ is $L$-bilipschitz, $U$ is easily seen to be $2L^2$-plump. Hence there is a ball $B(z,s) \subset U \cap B(a, r/4)$ with $s = r/16L^2$. It follows that there is $L_1 = L_1(c)$ and an $L_1$-bilipschitz homeomorphism $g\colon \overline{B}(a, r/2) \to \overline{B}(a, r/2)$ such that $g \mid \partial B(a, r/2) = \mathrm{id}$ and such that $gB(z,s) = B(a,s)$. Since $r \leq t$, the balls $B(a, r/2)$ and $B(b, r/2)$ are disjoint. Hence we can use the same construction in $B(b,r/2)$ to extend $g$ to an $L_1$-bilipschitz homeomorphism $g\colon \mathbf{R}^n \to \mathbf{R}^n$ such that

(1) $B(a,s) \cup B(b,s) \subset gU$,

(2) $g = \mathrm{id}$ outside $B(a,r/2) \cup B(b,r/2)$.

Setting $h(x) = g\big(f(x)\big)$ we obtain an $LL_1$-bilipschitz homeomorphism $h\colon B^n(t) \to gU$. Moreover,

$$\big(t - |h^{-1}(a)|\big) \wedge \big(t - |h^{-1}(b)|\big) \geq s/LL_1 \geq t/c_1$$

with $c_1 = c_1(c,c') = 16c'L^3L_1$. Hence there is $L_2 = L_2(c,c')$ and an $L_2$-bilipschitz homeomorphism $u\colon B^n(t) \to B^n(t)$ with $u\big(h^{-1}(a)\big) = 0$, $u\big(h^{-1}(b)\big) = te_1/2$. The desired map is then $F = h \circ u^{-1}\colon B^n(t) \to gU \subset D$.

3.10. **Lemma.** *Suppose that $f_k\colon D \to \mathbf{R}^n$ is a sequence of discrete open maps converging locally uniformly to a discrete open map $f\colon D \to \mathbf{R}^n$. Then a point $a \in D$ is in $B_f$ if and only if there are points $x_k \in B_{f_k}$ such that $x_k \to a$.*

*Proof.* The 'if' part is given in [MR, 3.2]. To prove the converse, it suffices to show that if each $f_k$ is a local homeomorphism, then $|i(a,f)| = 1$.

Choose $r > 0$ such that $U(a,f,3r)$ is a normal neighborhood of $a$; see 3.1. Next choose $k \in \mathbf{N}$ such that $|f_k(x) - f(x)| < r/2$ for all $x \in \overline{U}(a,f,r)$. Then $f_k(a) \in B\big(f(a),r/2\big)$. Let $V_k$ be the $a$-component of $f_k^{-1}B\big(f(a),2r\big)$. Then $V_k$ does not meet $\partial U(a,f,3r)$, and hence $V_k \subset U(a,f,3r)$. Moreover, $V_k$ is a normal domain of $f_k$, and hence $f_k$ defines a covering map of $V_k$ onto $B\big(f(a),2r\big)$. Since $B\big(f(a),2r\big)$ is simply connected, this map is a homeomorphism. Since $U(a,f,r) \subset V_k$, we have

$$i(a,f) = \mu\big(f(a),f,U(a,f,r)\big) = \mu\big(f_k(a),f_k,U(a,f,r)\big) = \pm 1;$$

see [RR, Th. 6, p. 131].

3.11. **Theorem.** *Suppose that $D \subset \mathbf{R}^n$ is a $c$-uniform domain and that $f\colon D \to \mathbf{R}^n$ is a $K$-quasiregular map with $N(f) \leq N < \infty$. Then the domain $D \setminus B_f$ is $c'$-uniform with $c' = c'(c,N,K,n)$.*

*Proof.* We show that the domain $G = D \setminus B_f$ satisfies the conditions of 3.8. From the definitions it easily follows that a $c$-uniform domain is $2c$-plump. Hence $G$ is $c'$-plump with $c' = c'(c,N,K,n)$ by 3.4.

It suffices to show that there is $c_0 = c_0(c,N,K,n)$ such that each $8c'$-pair $(a,b)$ in $G$ can be joined by an arc $\gamma$ such that

$$\delta_G(a) \wedge \delta_G(b) \leq c_0 \operatorname{dist}(\gamma, \partial G), \qquad \operatorname{diam}\gamma \leq c_0|a-b|.$$

Assume that this is false for some $(c,N,K,n)$. Then there is a sequence of $K$-quasiregular maps $f_k\colon D_k \to \mathbf{R}^n$ such that $N(f_k) \leq N$, the domains $D_k \subset \mathbf{R}^n$

are $c$-uniform, and there are $8c'$-pairs $(a_k, b_k)$ in $G_k = D_k \setminus B_{f_k}$ such that for any arc $\gamma$ joining $a_k$ and $b_k$ in $G_k$ we have

$$\text{(3.12)} \qquad \delta_k(a_k) \wedge \delta_k(b_k) > k\,\mathrm{dist}\,(\gamma, \partial G_k) \qquad \text{or} \qquad \mathrm{diam}\,\gamma > k|a_k - b_k|;$$

here $\delta_k(x) = \mathrm{dist}\,(x, \partial G_k)$.

Setting $r_k = \delta_k(a_k) \wedge \delta_k(b_k)$ and $t_k = |a_k - b_k|$ we have

$$r_k \le t_k \le 8c' r_k \le 8c'\big(\mathrm{dist}\,(a_k, \partial D_k) \wedge \mathrm{dist}\,(b_k, \partial D_k)\big).$$

By 3.9 there is $L = L(c, c')$ and an $L$-bilipschitz map $F_k\colon B^n(t_k) \to D_k$ such that $F_k(0) = a_k$, $F_k(t_k e_1/2) = b_k$. Define $g_k\colon B^n \to \mathbf{R}^n$ by $g_k(x) = f_k\big(F_k(t_k x)\big)$. Then $g_k$ is $K_1$-quasiregular with $K_1 = K_1(K, c, c', n)$, and $N(g_k) \le N$. By auxiliary similarities we can normalize the situation so that

$$g_k(0) = 0, \qquad \max\{|g_k(x)| : |x| \le 1/2\} = 1.$$

Applying 2.5 and passing to a subsequence we may assume that the sequence $(g_k)$ converges locally uniformly in $B^n$ to a nonconstant $K_1$-quasiregular map $g\colon B^n \to \mathbf{R}^n$.

Since $f_k$ is locally injective in $B(a_k, r_k)$, and since $t_k \le 8c' r_k$, the map $g_k$ is locally injective in $B^n(1/8c'L)$ for each $k$. Hence also $g \mid B^n(1/8c'L)$ is locally injective, which implies that $0 \notin B_g$ by 3.10. Similarly we obtain $e_1/2 \notin B_g$. Since $B^n \setminus B_g$ is connected, we can join $0$ and $e_1/2$ by an arc $\beta$ in $B^n \setminus B_g$. Setting $\lambda = \mathrm{dist}\,(\beta, S^{n-1} \cup B_g)$ we have $\mathrm{dist}\,(\beta, B_{g_k}) \ge \lambda/2$ for large $k$ by [MR, 3.2]. The arc $\gamma_k = F_k[t_k\beta]$ joins $a_k$ and $b_k$ in $D_k$. Since

$$\mathrm{dist}\,(\gamma_k, \partial G_k) \ge \frac{t_k\lambda}{2L} \ge \frac{r_k\lambda}{2L},$$

for large $k$, the first inequality of (3.12) fails for large $k$. Since $\mathrm{diam}\,\gamma_k \le L t_k \mathrm{diam}\,\beta \le 2L t_k$, the second inequality of (3.12) is not true for large $k$, and we have reached a contradiction.

3.13. **Remark.** Theorem 3.11 was proved in [MV, 4.25–4.26] for maps of bounded length distortion. These maps form a proper subclass of the maps considered in 3.11. The case of quasiregular maps is more complicated, since a sequence of maps of $L$-bounded distortion never converges to a constant.

3.14. *The set* $fB_f$. Suppose that $f\colon D \to \mathbf{R}^n$ is a $K$-quasiregular map with $N(f) \le N < \infty$. Without further restrictions, very little can be said about the set $fD \setminus fB_f$. It need not be open, and if it is open, in need not be plump, even if $D$ and $fD$ are plump.

However, if $D = \mathbf{R}^n$, then $f$ is a closed map onto $\mathbf{R}^n$, and we prove in 3.16 that $\mathbf{R}^n \setminus fB_f$ is a uniform domain. For maps of bounded distortion, this was proved in [MV, 4.25]. A local version is given in 3.17. Both results are corollaries of the more general Theorem 3.15.

3.15. **Theorem.** *Suppose that* $f\colon D \to \mathbf{R}^n$ *is a* $K$*-quasiregular map and that* $U \subset D$ *is a domain such that*

(1) $B(U, 2\operatorname{diam} U) \subset D$,

(2) $fU$ *is a ball or* $fU = \mathbf{R}^n$,

(3) $f$ *defines a proper map* $U \to fU$ *with* $N(f \mid U) \le N < \infty$.

*Then* $fU \setminus f[B_f \cap U]$ *is a* $c$*-uniform domain with* $c = c(N, K, n)$.

Recall that $B(A, r)$ denotes the $r$-neighborhood $\{x \in \mathbf{R}^n : \operatorname{dist}(x, A) < r\}$ of $A$. Before giving the proof we remark that the conditions of 3.15 hold if $U = D = \mathbf{R}^n$ and $N(f) \le N$. If $D \ne \mathbf{R}^n$, then (1) implies that $U$ is bounded with $\overline{U} \subset D$. Then (3) means that $U$ is a normal domain of $f$. In view of the discussion in 3.1, we obtain the following two corollaries of 3.15.

3.16. **Theorem.** *If* $f\colon \mathbf{R}^n \to \mathbf{R}^n$ *is a* $K$*-quasiregular map with* $N(f) \le N < \infty$*, then* $\mathbf{R}^n \setminus fB_f$ *is a* $c$*-uniform domain with* $c = c(N, K, n)$.

3.17. **Theorem.** *Suppose that* $f\colon D \to \mathbf{R}^n$ *is* $K$*-quasiregular and non-constant. Then for each* $x \in D$ *with* $i(x, f) = N \ge 2$ *there is* $r_0 > 0$ *such that for* $0 < r \le r_0$, $U = U(x, f, r)$ *is a normal neighborhood of* $x$*, and* $fU \setminus fB_{f|U} = B\bigl(f(x), r\bigr) \setminus f[U \cap B_f]$ *is a* $c$*-uniform domain with* $c = c(N, K, n)$.

3.18. *Proof of Theorem* 3.15. *Part* 1. We show that the domain $G = fU \setminus f[B_f \cap U]$ is $c$-plump with $c = c(N, K, n)$. Assume that this is false for some $(N, K, n)$. Then, for each $k \in \mathbf{N}$ we can find a $K$-quasiregular map $f_k\colon D_k \to \mathbf{R}^n$ and a domain $U_k \subset D_k$ such that:

(i) Conditions (1)–(3) hold with $D = D_k$, $f = f_k$, $U = U_k$.

(ii) The domain $G_k = fU_k \setminus f_k[B_{f_k} \cap U_k]$ is not $k$-plump.

By (ii), for each $k \in \mathbf{N}$ there are $y'_k \in \overline{G}_k$ and $s_k > 0$ such that $G_k \setminus B(y'_k, s_k) \ne \varnothing$ and such that

$$B(z, s_k/k) \not\subset G_k \tag{3.19}$$

for every $z \in \overline{B}(y'_k, s_k)$. By (2), there is a ball $V'_k = B(y_k, s_k/3)$ with $\overline{V}'_k \subset B(y'_k, s_k) \cap fU_k$. The set $U_k \cap f_k^{-1}(y_k)$ is nonempty and contains at most $N$ points, which we number as $a_{1k}, \dots, a_{Nk}$, using repetition if necessary. Let $V_{jk}$ be the $a_{jk}$-component of $f_k^{-1}V'_k$. Then $V_{jk}$ is a normal domain of $f_k$, and

$$f_k V_{jk} = V'_k, \qquad U_k \cap f_k^{-1} V'_k = V_{1k} \cup \dots \cup V_{Nk}; \tag{3.20}$$

see 3.1.

Set $t_{jk} = \max\{|x - a_{jk}| : x \in \overline{V}_{jk}\}$, and define similarities $S_{jk}$ and $T_k$ of $\mathbf{R}^n$ by

$$S_{jk}(x) = (x - a_{jk})/t_{jk}, \qquad T_k(x) = 3(x - y_k)/s_k.$$

From (3) it follows that $B(a_{jk},2t_{jk})\subset D_k$. Hence we can define $K$-quasiregular maps $g_{jk}\colon B^n(2)\to\mathbf{R}^n$ by $g_{jk}=T_k\circ f_k\circ S_{jk}^{-1}\mid B^n(2)$. Setting $W_{jk}=S_{jk}V_{jk}$ we have $g_{jk}W_{jk}=B^n$.

Applying 2.4 with $r=1$, $R=2$, $r'=1$, $A(g_{jk})=\overline{W}_{jk}$, and passing successively $N$ times to subsequences, we may assume that for each $j=1,\dots,N$, the sequence $(g_{jk})_{k\in\mathbf{N}}$ converges locally uniformly in $B^n(2)$ to a nonconstant $K$-quasiregular map $g_j\colon B^n(2)\to\mathbf{R}^n$ with $g_j(0)=0$. Set

$$F_j=g_j[B_{g_j}\cap\overline{B}^n],\qquad F=F_1\cup\cdots\cup F_N. \tag{3.21}$$

Since $F$ is a compact set with empty interior, we can find a ball $B(w,\lambda)\subset B^n\setminus F$. From 3.10 it follows that there is $k_0\in\mathbf{N}$ such that $B(w,\lambda/2)$ does not meet $g_{jk}[B_{g_{jk}}\cap\overline{B}^n]$ whenever $k\ge k_0$ and $1\le j\le N$. For $z=T_k^{-1}(w)$ we then have $B(z,\lambda s_k/6)\subset G_k$ for $k\ge k_0$. By (3.19), this gives a contradiction for large $k$.

*Part* 2. Let $c=c(N,K,n)$ be the number given by Part 1. By 3.8, it suffices to find a number $c_0=c_0(N,K,n)$ such that each $8c$-pair $(y,z)$ in $G$ can be joined by an arc $\gamma$ with the properties

$$\delta_G(y)\wedge\delta_G(z)\le c_0\operatorname{dist}(\gamma,\partial G),\qquad \operatorname{diam}\gamma\le c_0|y-z|.$$

We shall modify the proof of Part 1. Assume that $c_0$ does not exist for some $(N,K,n)$. Then, for each $k\in\mathbf{N}$ we can find a $K$-quasiregular map $f_k\colon D_k\to\mathbf{R}^n$, a domain $U_k\subset D_k$, and an $8c$-pair $(y_k,z_k)$ in $G_k=fU_k\setminus f_k[B_{f_k}\cap U_k]$ such that:

(i) Conditions (1)–(3) hold with $D=D_k$, $f=f_k$, $U=U_k$.

(ii) If $\gamma$ is an arc joining $y_k$ and $z_k$ in $G_k$, then

$$\delta_{G_k}(y_k)\wedge\delta_{G_k}(z_k)\ge k\operatorname{dist}(\gamma,\partial G_k)\quad\text{or}\quad\operatorname{diam}\gamma\ge k|y_k-z_k|.$$

Set

$$q=1/9c,\qquad V_k'=B([y_k,z_k],q|y_k-z_k|).$$

Since $(y_k,z_k)$ is an $8c$-pair in $G_k$, we have $\overline{V}_k'\subset fU_k$, and the set $U_k\cap f_k^{-1}(y_k)$ contains precisely $N$ points $a_{1k},\dots,a_{Nk}$. For each $j=1,\dots,N$, we let $V_{jk}$ denote the $a_{jk}$-component of $f_k^{-1}V_k'$. Then $V_{jk}$ is a normal domain of $f_k$, and (3.20) holds. It is possible that $V_{ik}=V_{jk}$ for some $i\ne j$.

Set $t_{jk}=\max\{|x-a_{jk}|:x\in\overline{V}_{jk}\}$, and choose similarities $S_{jk}$ and $T_k$ of $\mathbf{R}^n$ such that

$$S_{jk}(x)=(x-a_{jk})/t_{jk},\qquad T_k(y_k)=0,\qquad T_k(z_k)=e_1.$$

By (1) we again have $B(a_{jk},2t_{jk})\subset D_k$, and we can define $K$-quasiregular maps $g_{jk}\colon B^n(2)\to\mathbf{R}^n$ by $g_{jk}=T_k\circ f_k\circ S_{jk}^{-1}\mid B^n(2)$. Setting $W_{jk}=S_{jk}V_{jk}$ and $W'=B([0,e_1],q)$ we have

$$\max\{|x|:x\in\overline{W}_{jk}\}=1,\qquad g_{jk}W_{jk}=W',\qquad g_{jk}(0)=0.$$

Applying 2.4 with $r = 1$, $R = 2$, $r' = 1 + q$, $A(g_{jk}) = \overline{W}_{jk}$, and passing successively to subsequences, we may assume that for each $j = 1, \dots, N$, the sequence $(g_{jk})_{k\in\mathbf{N}}$ converges locally uniformly in $B^n(2)$ to a nonconstant $K$-quasiregular map $g_j\colon B^n(2) \to \mathbf{R}^n$ with $g_j(0) = 0$.

Define $F_j$ and $F$ as in (3.21). We show that $\operatorname{dist}(\{0, e_1\}, F) \ge q$. Assume, for example, that there is $u \in B_{g_j} \cap B^n$ with $|g_j(u) - e_1| < q$. By 3.10 we can find a sequence of points $u_k \in B_{g_{jk}}$ converging to $u$. For large $k$ we have $|g_{jk}(u) - e_1| < q$. Since $T^{-1}\bigl(g_{jk}(u_k)\bigr) = f_k\bigl(S_{jk}^{-1}(u_k)\bigr) \in f_k[B_{f_k} \cap U_k]$, this gives the contradiction $\delta_{G_k}(z_k) < |y_k - z_k|/8c$.

Since $F$ is a compact set with $\dim F \le n - 2$, we can join 0 and $e_1$ by an arc $\alpha \subset W' \setminus F$. Set $\lambda = \operatorname{dist}(\alpha, F \cup \partial W') > 0$. By 3.10 we can find $k_0 \ge 2$ such that $\operatorname{dist}(\alpha, g_{jk}[B_{g_{jk}} \cap \overline{B}^n]) \ge \lambda/2$ whenever $k \ge k_0$ and $1 \le j \le N$. Let $k \ge k_0$. The arc $\gamma = T_k^{-1}\alpha$ joins $y_k$ and $z_k$ in $V_k'$, and

$$\operatorname{dist}(\gamma, f_k[B_{f_k} \cap V_{jk}] \cup \partial V_k') \ge \lambda|y_k - z_k|/2$$

for all $1 \le j \le N$. Since

$$V_k' \cap f_k[B_{f_k} \cap U_k] = \bigcup_{j=1}^{N} f_k[B_{f_k} \cap V_{jk}],$$

we obtain $\operatorname{dist}(\gamma, \partial G_k) \ge \lambda|y_k - z_k|/2$. Since

$$\operatorname{diam} \gamma \le \operatorname{diam} V_k' = (1 + 2q)|y_k - z_k| < 2|y_k - z_k|,$$

it follows from (ii) that

$$k \operatorname{dist}(\gamma, \partial G_k) \le \delta_{G_k}(y_k) \wedge \delta_{G_k}(z_k) \le |y_k - z_k|,$$

where the second inequality follows from the property $r_{G_k}(y_k, z_k) \ge 1$ of a $c$-pair; see 3.7. Hence

$$k\lambda|y_k - z_k|/2 \le |y_k - z_k|,$$

which gives a contradiction for large $k$.

## 4. Invariance of NUD and porous sets

4.1. *Terminology.* A closed set $F \subset \mathbf{R}^n$ is a *$c$-nullset for uniform domains* or briefly $c$-NUD if $\operatorname{int} F = \varnothing$ and if $\mathbf{R}^n \setminus F$ is a $c$-uniform domain.

Let $D \subset \mathbf{R}^n$ be a domain, let $a, b \in \overline{D}$ and let $c \ge 1$. We say that a continuum $\alpha \subset \overline{D}$ containing $a$ and $b$ satisfies the *$c$-uniformity conditions* in $D$ if

$$\operatorname{diam} \alpha \le c|a - b|, \qquad |x - a| \wedge |x - b| \le c \operatorname{dist}(x, \partial D)$$

for all $x \in \alpha$. This implies that $\alpha \cap \partial D \subset \{a, b\}$.

If $D$ is $c$-uniform, it follows from the definition in 3.1 that each pair of distinct points $a, b \in D$ can be joined by an arc satisfying the $c$-uniformity conditions in $D$. A simple limiting process involving Ascoli's theorem shows that this is true for all $a, b \in \overline{D}$.

Conversely, if each pair of points in a domain $D$ can be joined by a continuum with the $c$-uniformity properties in $D$, then $D$ is $c'$-uniform with $c' = c'(c, n)$; see [V4, 2.11].

Suppose that $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is $K$-quasiregular with $N(f) \le N < \infty$. Then $B_f$ and $fB_f$ are $c_0$-NUD with $c_0 = c_0(N, K, n)$ by 3.11 and 3.16. In this section we show that if $F \subset \mathbf{R}^n$ is $c$-NUD, then $fF$ and $f^{-1}F$ are $c'$-NUD with $c' = c'(c, N, K, n)$. Similar results hold for porosity.

4.2. **Lemma.** *Suppose that $f\colon B^n \to \mathbf{R}^n$ is a $K$-quasiregular local homeomorphism with $N(f) \le N < \infty$. Then $f$ is injective in a ball $B^n(\psi)$ with $\psi = \psi(N, K, n) > 0$.*

*Proof.* For $n \ge 3$, [MRV3, 2.3] gives the stronger result where $\psi = \psi(K, n)$. For $n = 2$, one can make use of factorization and basic properties of quasiconformal maps to reduce the question to the case where $f$ is complex analytic. This case follows from the results of C. Pommerenke [Po1, Satz 1.3 and Lemma 1.3].

We give an alternative proof, based on Theorem 2.4, which is valid for all dimensions $n \ge 2$. We may assume that $f(0) = 0$. With the notation of 3.1, we let $r_0(f)$ be the supremum of all $r > 0$ such that $U(0, f, r) \subset B^n(1/2)$. Clearly $0 < r_0(f) < \infty$. Replacing $f$ by $f/r_0(f)$ we may assume that $r_0(f) = 1$. For $0 < r < 1$, $f$ maps $U(0, f, r)$ homeomorphically onto $B^n(r)$ by [MRV3, 2.2]. It follows that $f$ maps $V(f) = \bigcup\{U(0, f, r) : r < 1\}$ onto $B^n$ and that $V(f) = U(0, f, 1)$. Hence it suffices to find $\psi = \psi(N, K, n) > 0$ such that $B^n(\psi) \subset V(f)$.

Let $\mathscr{F} = \mathscr{F}(N, K, n)$ be the family of all $K$-quasiregular local homeomorphisms $g\colon B^n \to \mathbf{R}^n$ such that $N(g) \le N$, $g(0) = 0$, and $r_0(g) = 1$. Then $\mathscr{F}$ satisfies the conditions of Theorem 2.4 with $r = 1/2$, $R = 1$, $r' = 1$, $A(g) = \overline{V(g)}$. Indeed, $g$ is injective in $A(g)$ by [MRV3, 2.2], and hence in a neighborhood of $A(g)$ by [Zo, p. 422]. By the definition of $r_0(g)$, the continuum $A(g)$ meets $S^{n-1}(1/2)$.

By Theorem 2.4, $\mathscr{F}$ is a normal family and hence equicontinuous. Consequently, there is $\psi > 0$ such that $gB^n(\psi) \subset B^n(1/2)$ for all $g \in \mathscr{F}$. This implies that $B^n(\psi) \subset V(g)$, and the lemma is proved.

4.4. **Theorem.** *Suppose that $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is $K$-quasiregular with $N(f) \le N < \infty$, and that $F \subset \mathbf{R}^n$ is $c$-NUD. Then $f^{-1}F$ is $c'$-NUD with $c' = c'(c, N, K, n)$.*

*Proof.* Let $a, b \in \mathbf{R}^n$, $a \ne b$. Write $\lambda(x) = |x-a| \wedge |x-b|$. By 4.1, it suffices to find a continuum $\beta$ containing $a$ and $b$ such that

$$\operatorname{diam} \beta \le c'|a-b|, \qquad \lambda(x) \le c' \operatorname{dist}(x, f^{-1}F) \tag{4.5}$$

for all $x \in \beta$.

The set $B_f$ is $c_0$-NUD with $c_0 = c_0(N,K,n)$ by 3.11. By 4.1, we can join $a$ and $b$ by an arc $\alpha$ such that

$$\operatorname{diam} \alpha \le c_0|a-b|, \qquad \lambda(x) \le c_0 \operatorname{dist}(x, B_f) \tag{4.6}$$

for all $x \in \alpha$.

Let $\psi = \psi(N,K,n)$ be the number given by 4.2, and set $q = q(N,K,n) = \psi/6c_0$. Then $q \le 1/6$. Orient $\alpha$ from $a$ to $b$. Pick $x_0 \in \alpha$ with $|x_0 - a| = |x_0 - b|$. Define a sequence of successive points $x_0, x_1, \ldots$ of $\alpha$ such that $x_{j+1}$ is the last point of $\alpha$ with $|x_{j+1} - x_j| = q\lambda(x_j)$. Similarly, define $x_{-1}, x_{-2}, \ldots$ such that $x_{-j-1}$ is the first point of $\alpha$ with $|x_{-j-1} - x_{-j}| = q\lambda(x_{-j})$, $j \ge 0$. The sequence $x_1, x_2, \ldots$ converges to a point $b' \in \alpha$. Since $q\lambda(x_j) = |x_j - x_{j+1}| \to 0$, we have $b' = b$. Similarly $x_j \to a$ as $j \to -\infty$. Since $q \le 1/6$, we easily see that

$$\tfrac{5}{6}\lambda(x_{j-1}) \le \lambda(x_j) \le \tfrac{6}{5}\lambda(x_{j-1}), \qquad \{x_{j-1}, x_{j+1}\} \subset \overline{B}\big(x_j, \tfrac{6}{5}q\lambda(x_j)\big) \tag{4.7}$$

for all $j \in \mathbf{Z}$.

From (4.6) it follows that $\operatorname{dist}(x_j, B_f) \ge \lambda(x_j)/c_0$ for all $j \in \mathbf{Z}$. By 4.2, $f \mid B(x_j, r)$ is injective, where $r = \psi\lambda(x_j)/c_0 = 6q\lambda(x_j)$. By [V3, 2.4], $f$ is $\eta$-quasisymmetric in $B\big(x_j, 3q\lambda(x_j)\big)$ with $\eta = \eta_{K,n}$. We let $c_1, c_2, \ldots$ denote constants $c_j \ge 1$ depending only on $(c, K, n)$.

*Fact* 1. If $0 < t \le 3q\lambda(x_j)$, then $B(x_j, t) \setminus f^{-1}F$ is a $c_1$-uniform domain.

Since uniformity is quantitatively preserved by quasisymmetric maps, the domain $fB(x_j, t)$ is $c_2$-uniform. Since $F$ is $c$-NUD, the domain $fB(x_j, t) \setminus F$ is $c_3$-uniform by [V4, 5.4], and Fact 1 follows by quasisymmetry.

By (4.7) and Fact 1, we can join $x_{j-1}$ and $x_j$ by an arc $\alpha_j$ satisfying the $c_1$-uniformity conditions in $B\big(x_{j-1}, \frac{6}{5}q\lambda(x_{j-1})\big) \setminus f^{-1}F$. Then $\alpha_j \subset B(x_j, r)$ with

$$r = |x_{j-1} - x_j| + \tfrac{6}{5}q\lambda(x_{j-1}) \le \tfrac{6}{5}q\lambda(x_j) + (\tfrac{6}{5})^2 q\lambda(x_j) < 3q\lambda(x_j).$$

Pick $y_j \in \alpha_j$ with $|y_j - x_{j-1}| = |y_j - x_j|$. Then $\{y_j, y_{j+1}\} \subset B\big(x_j, 3q\lambda(x_j)\big)$. By Fact 1, we can thus join $y_j$ and $y_{j+1}$ by an arc $\beta_j$ satisfying the $c_1$-uniformity conditions in the domain $G_j = B\big(x_j, 3q\lambda(x_j)\big) \setminus f^{-1}F$. Let $\beta$ be the union of $\{a, b\}$ and all continua $\beta_j$, $j \in \mathbf{Z}$. We show that $\beta$ is the desired continuum.

Since $\operatorname{diam} \beta_j \le 6q\lambda(x_j) \to 0$ as $j \to \pm\infty$, $\beta$ is indeed a continuum. If $x \in \beta_j$, then

$$\operatorname{dist}(x, \alpha) \le |x - x_j| < 3q\lambda(x_j) \le \tfrac{1}{2}\operatorname{diam} \alpha.$$

This and (4.6) yield the first inequality of (4.5) with $c' = 2c_0$.

To prove the second inequality, assume that $x \in \beta_j$. Then

$$\lambda(x) \le \lambda(x_j) + |x - x_j| \le \lambda(x_j) + 3q\lambda(x_j) \le \tfrac{3}{2}\lambda(x_j). \tag{4.8}$$

If $j \geq 1$, then $G_j$ contains the ball $B(y_j, t)$ with

$$t = |y_j - x_j|/c_1 \geq |x_{j-1} - x_j|/2c_1 = q\lambda(x_{j-1})/2c_1 > q\lambda(x_j)/3c_1$$

by (4.7). Moreover, $G_j$ contains $B(y_{j+1}, t')$ with

$$t' = |y_{j+1} - x_j|/c_1 \geq |x_j - x_{j+1}|/2c_1 = q\lambda(x_j)/2c_1 > q\lambda(x_j)/3c_1.$$

Similar arguments show that these estimates hold also for $j \leq 0$. By the choice of $\beta_j$ we thus obtain $\operatorname{dist}(x, f^{-1}F) \geq q\lambda(x_j)/6c_1^2$. By (4.8), this gives the second inequality of (4.5) with $c' = 9c_1^2/q$.

4.9. **Lemma.** *Suppose that $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is $K$-quasiregular with $N(f) = N < \infty$, and that $V \subset \mathbf{R}^n \setminus fB_f$ is a simply connected $c$-uniform domain. Then $f^{-1}V$ has precisely $N$ components $V_1, \ldots, V_N$, and $f$ defines $\eta$-quasisymmetric homeomorphisms $f_j\colon V_j \to V$, $1 \leq j \leq N$, with $\eta$ depending only on $c$, $N$, $K$, $n$. Moreover, the domains $V_j$ are $c'$-uniform with $c' = c'(c, N, K, n)$.*

*Proof.* The map $f$ is closed and proper by 3.1. Arguing as in [MRV3, 2.2] we see that for each component $V_j$ of $f^{-1}V$, $f$ defines a covering map $f_j\colon V_j \to V$. Since $V$ is simply connected, $f_j$ is a homeomorphism. To prove the quasisymmetry of $f_j$, we consider a triple $(x, y, z)$ of distinct points in $V_j$ with $|x - y| \leq |x - z|$. By [V5, 2.9], it suffices to show that

$$|f(x) - f(y)| \leq H|f(x) - f(z)| \tag{4.10}$$

with $H = H(c, N, K, n)$.

Since $V$ is $c$-uniform, there is an arc $\alpha \subset V$ joining $f(x)$ and $f(z)$ such that $\operatorname{diam}\alpha \leq c|f(x) - f(z)|$. We may assume that $|f(x) - f(y)| > c|f(x) - f(z)|$. Define $\beta\colon [1, \infty) \to R^n$ by $\beta(t) = f(x) + t\big(f(y) - f(x)\big)$. Let $\beta^*$ be a maximal lift of $\beta$, starting at $y$. Then $\beta^*$ is unbounded. Let $\Gamma$ be the family of all paths joining $\alpha^* = f_j^{-1}\alpha$ and $|\beta^*|$. Since $|x - y| \leq |x - z|$, a standard estimate gives the lower bound $\mathsf{M}(\Gamma) \geq b_n > 0$. Since $\alpha \subset \overline{B}\big(f(x), c|f(x) - f(z)|\big)$, we have

$$\mathsf{M}(f\Gamma) \leq \omega_{n-1}\left(\log\frac{|f(x) - f(y)|}{c|f(x) - f(z)|}\right)^{1-n}.$$

Since $\mathsf{M}(\Gamma) \leq KN\mathsf{M}(f\Gamma)$, these estimates yield (4.10). Hence $f\,|\,V_j$ is $\eta$-quasisymmetric, and the rest of the lemma follows from the quasisymmetric invariance of uniform domains.

4.11. **Lemma.** *Suppose that $D \subset G \subset \mathbf{R}^n$ are domains, that $F$ is closed in $G$ with $\operatorname{int} F = \varnothing$, and that $D$ and $G \setminus F$ are $c$-uniform domains. Then $D \setminus F$ is a $c_1$-uniform domain with $c_1 = c_1(c, n)$.*

*Proof.* The case $G = \mathbf{R}^n$ was proved in [V4, 5.4], but the same proof is valid in the general case.

4.12. **Corollary.** *Suppose that $D \subset \mathbf{R}^n$ is a domain, that $F_1, \dots, F_N$ are closed in $D$ with $\operatorname{int} F_j = \varnothing$, and that each $D \setminus F_j$ is a $c$-uniform domain. Then $D \setminus \bigcup_j F_j$ is a $c_1$-uniform domain with $c_1 = c_1(c, N, n)$.*

4.13. **Theorem.** *Suppose that $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is $K$-quasiregular with $N(f) = N < \infty$, and that $F \subset \mathbf{R}^n$ is $c$-NUD. Then $fF$ is $c'$-NUD with $c' = c'(c, N, K, n)$.*

*Proof.* The basic idea is the same as in 4.4 and in [V4, 5.4]. Let $a, b \in \mathbf{R}^n$, $a \neq b$. Since $fB_f$ is $c_0$-NUD with $c_0 = c_0(N, K, n)$ by 3.16, there is an arc $\alpha$ joining $a$ and $b$ such that

$$\operatorname{diam} \alpha \le c_0 |a - b|, \qquad \lambda(x) \le c_0 \operatorname{dist}(x, fB_f)$$

for all $x \in \alpha$, where $\lambda(x) = |x - a| \wedge |x - b|$ as in 4.4. Let $c_1, c_2, \dots$ denote constants $c_j \ge 1$ depending only on $(c, N, K, n)$, and set $q = 1/6c_0$. Choose $x_0 \in \alpha$ with $|x_0 - a| = |x_0 - b|$, and define the points $x_j \in \alpha$, $j \in \mathbf{Z}$, as in the proof of 4.4. Then (4.7) is again valid for all $j \in \mathbf{Z}$.

*Fact* 1. If $0 < t \le 6q\lambda(x_j)$, then $B(x_j, t) \setminus fF$ is a $c_1$-uniform domain.

By 4.9, the set $f^{-1}B(x_j, t)$ has $N$ components $V_{j1}, \dots, V_{jN}$, and each $V_{jk}$ is a $c_2$-uniform domain. Moreover, $f \mid V_{jk}$ is an $\eta$-quasisymmetric homeomorphism onto $B(x_j, t)$ with $\eta = \eta_{N,K,n}$. Since $F$ is $c$-NUD, each $V_{jk} \setminus F$ is $c_3$-uniform. Hence $f[V_{jk} \setminus F] = B(x_j, t) \setminus f[F \cap V_{jk}]$ is $c_4$-uniform by quasisymmetry. Fact 1 follows now from 4.12.

The proof can now be completed as in 4.4. The points $x_{j-1}$ and $x_j$ are joined by an arc $\alpha_j$ satisfying the $c_1$-uniformity conditions in $B\big(x_{j-1}, \frac{6}{5}\lambda(x_{j-1})\big) \setminus fF$. Then choose $y_j \in \alpha_j$ with $|y_j - x_{j-1}| = |y_j - x_j|$ and join $y_j$ to $y_{j+1}$ by an arc $\beta_j$ satisfying the $c_1$-uniformity conditions in $B\big(x_j, 3\lambda(x_j)\big) \setminus fF$. The desired continuum $\beta$ from $a$ to $b$ is then obtained as the union of all $\beta_j$ and of $\{a, b\}$.

4.14. **Corollary.** *If $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is $K$-quasiregular with $N(f) \le N < \infty$, then $f^{-1}fB_f$ is $c$-NUD with $c = c(N, K, n)$.*

*Proof.* This follows from 3.16 and 4.13.

4.15. **Theorem.** *Suppose that $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is a $K$-quasiregular map with $N(f) \le N < \infty$, and that $F \subset \mathbf{R}^n$ is $c$-porous. Then $fF$ and $f^{-1}F$ are $c_1$-porous with $c_1 = c_1(c, N, K, n)$.*

*Proof.* The idea of the proof is somewhat similar to that in 4.4 and in 4.13, but the present case is much easier. The proofs make use of the porosity of $B_f$ and $fB_f$, the quasisymmetric invariance of plumpness, and the fact that the union of two porous sets is porous. The details are omitted.

## 5. The branch set and the local index

Let $f\colon D \to \mathbf{R}^n$ be a $K$-quasiregular map, $n \geq 3$, and let $a \in B_f$. In [MRV3, 4.4] it was proved that if $D \backslash B_f$ contains an open cone with vertex at $a$ and opening angle $\alpha$, then the local index $i(a,f)$ has an upper bound $i(a,f) \leq N(\alpha, K, n)$. See also [S1, 3.4 and 4.3].

The direct converse of this result is false. In 5.4 below we give an example of a quasiregular map $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ with $N(f) = 2$ such that $B_f$ meets every half open line segment $(0, y]$, $y \neq 0$.

However, we show that replacing the cone by a curvilinear cone we obtain a condition that is both necessary and sufficient for the existence of such a bound for the local index. This means that we can estimate $i(a,f)$ from above and from below purely in terms of $K$, $n$ and $B_f$; see 5.2.

5.1. **Theorem.** *Suppose that $f\colon D \to \mathbf{R}^n$ is $K$-quasiregular and nonconstant with $n \geq 3$. Let $a \in D$, $N \geq 2$, and $c \geq 1$. Then the following conditions are $(K,n)$-quantitatively equivalent*:

(1) $i(a,f) \leq N$,

(2) *there is an arc $\gamma \subset D$ with $a$ as an endpoint such that $|x - a| \leq c \operatorname{dist}(x, B_f)$ for all $x \in \gamma$.*

*More precisely,* (1) *implies* (2) *with $c = c(N,K,n)$, and* (2) *implies* (1) *with $N = N(c,K,n)$.*

*Proof.* (1) $\Rightarrow$ (2): Choose a ball $B = B(a,r) \subset D$ with $N(f \mid B) \leq N$. The domain $G = B \setminus B_f$ is $c_0$-uniform with $c_0 = c_0(N,K,n)$ by 3.11. Fix a point $b \in G$. By 4.1, we can join $a$ and $b$ by an arc $\alpha$ satisfying the $c$-uniformity conditions in $B \setminus B_f$. Let $\gamma$ be a subarc of $\alpha$ with endpoint $a$ and contained in $B(a, |a-b|/2)$. Then $\gamma$ satisfies (2).

(2) $\Rightarrow$ (1): By [MRV3, 5.2], there is a number $M = M(K,n) > 1$ such that for all $x \in D$ we have $\limsup_{r \to 0} H^*(x,f,r) < M$, where $H^*(x,f,r) = L^*(x,f,r)/l^*(x,f,r)$ and $L^*$ and $l^*$ are defined as in the proof of 4.2. Assume that (2) does not imply (1) for some $(c,N,K,n)$. Then there is a sequence of $K$-quasiregular maps $f_k\colon B^n(2M) \to \mathbf{R}^n$ with the following properties:

(i) $f_k^{-1}\{0\} = \{0\}$,

(ii) $i(0, f_k) \geq k$,

(iii) $U_k = U(0, f_k, 1)$ is a normal neighborhood of $0$,

(iv) $H^*(0, f_k, 1) < M$,

(v) $e_1 \in \partial U_k$, and there is an arc $\gamma_k \subset \overline{U}_k$ joining $0$ and $e_1$ such that $\operatorname{dist}(x, B_{f_k}) \geq |x|/c$ for all $x \in \gamma_k$,

(vi) $f_k(e_1) = e_1$.

From (iv) and (vi) it follows that $B^n(1/M) \subset U_k \subset B^n(M)$. Passing to a subsequence we may assume that the sequence $(\gamma_k)$ converges to a continuum $F$ in the Hausdorff metric of all nonempty compact subsets of $\overline{B}^n(M)$. Then $\{0, e_1\} \subset F \subset \overline{B}^n(M)$. Let $\psi = \psi(K,n)$ be the local injectivity number given by

[MRV3, 2.3], and let $D$ be the union of $B^n(1/M)$ and the balls $B(x,\psi|x|/2c)$ over $x \in F\setminus B^n(1/M)$. Since $F$ is connected, $D$ is a domain. Moreover, $D \subset B^n(2M)$, and we can define the maps $g_k = f_k \mid D$.

We show that the sequence $(g_k)$ is equicontinuous. Since $B^n(1/M) \subset U_k$, we have $|f_k(x)| < 1$ for all $|x| < 1/M$ and for all $k$. Hence $(g_k)$ is equicontinuous in $B^n(1/M)$ by [MRV2, 3.17] or by [Re, p. 220]. Suppose that $x \in F \setminus B^n(1/M)$. For each $y \in \gamma_k \setminus \{0\}$, $f_k$ is injective in $B(y,\psi|y|/c)$ by (v) and by the choice of $\psi$. Since $\gamma_k \to F$, the maps $f_k \mid B(x,\psi|x|/2c)$ are injective for large $k$. Since they omit 0 and $\infty$ by (i), it follows from [V2, 19.3] that $(g_k)$ is equicontinuous in $B(x,\psi|x|/2c)$, and hence in $D$.

Passing to a subsequence we may assume that $(g_k)$ converges locally uniformly to a $K$-quasiregular map $g\colon D \to \mathbf{R}^n$. Since $g(0) = 0$ and $g(e_1) = e_1$, $g$ is nonconstant. By [MRV3, 4.5], $i(0,g) \ge \limsup_{k\to\infty} i(0,g_k)$. This contradicts (ii) and completes the proof.

We give a slightly different formulation of Theorem 5.1.

5.2. **Theorem.** *Suppose that $f\colon D \to \mathbf{R}^n$ is $K$-quasiregular and nonconstant with $n \ge 3$. For $a \in D$, let $u = u(a, B_f)$ denote the infimum of all $c \ge 1$ satisfying condition (2) of Theorem 5.1. Then*

$$N_1(u,K,n) \le i(a,f) \le N_2(u,K,n) < \infty,$$

*where $N_1(u,K,n) \to \infty$ as $u \to \infty$.*

*Proof.* Let $c(N,K,n)$ and $N(c,K,n)$ be the functions given by 5.1. The second inequality of 5.2 holds, for example, with $N_2(u,K,n) = N(u+1,K,n)$.

If $(K,n)$ is a pair such that $i(a,f)$ is bounded by a number $M(K,n)$ for all $f$ and $a$, then $u \le c\big(M(K,n),K,n\big)$, and the first inequality of 5.2 is an empty condition. Assume that $(K,n)$ is a pair such that $i(a,f)$ may have arbitrarily large values. Such pairs exist for all $n \ge 3$ by [MRV3, 4.9]. By 5.1, $c(N,K,n) \to \infty$ as $N \to \infty$. For $t \ge 1$, let $N_1(t,K,n)$ be the maximum of all integers $m$ such that $c(m,K,n) < t$, with $N_1(t,K,n) = 1$ if there are no such integers. From 5.1 it follows that the theorem holds with this function $N_1$.

5.3. *Open problem.* Is it possible to replace the arc in 5.1 by a sequence of points converging to $a$? More precisely, suppose that $(x_j)$ is a sequence of points in $D \setminus B_f$ converging to $a$ such that $|x_j - a| \le c\,\mathrm{dist}\,(x_j, B_f)$ for all $j$. Is $i(a,f)$ bounded by a constant $N(c,K,n)$?

5.4. **Theorem.** *There is a quasiregular map $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ with $N(f) = 2$ such that $B_f$ meets $(0,y]$ for each $y \in \mathbf{R}^3 \setminus \{0\}$.*

*Proof.* The theorem follows from Lemmas 5.5 and 5.11 below.

5.5. **Lemma.** *Let $Z \subset \mathbf{R}^3$ be the line* span$(e_3)$, *and let $g\colon \mathbf{R}^3 \to \mathbf{R}^3$ be a $K$-quasiconformal map. Then there is a $4K$-quasiregular map $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ with $N(f) = 2$ such that $B_f = gZ$.*

*Proof.* Let $w\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the 4-quasiregular winding map, defined by $w(r, \varphi, z) = (r, 2\varphi, z)$ in the cylindrical coordinates. Then $f = w \circ g^{-1}$ is the desired map.

5.6. *Remark.* There is also a map $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ such that $B_f = fB_f = gZ$ and such that $f$ is of $L$-bounded length distortion with $L = L(K)$. See [MV, 4.27].

If $Q \subset \mathbf{R}^3$ is a closed ball of radius $r$ and if $t > 0$, we let $Q(t)$ denote the concentric ball with radius $tr$.

5.7. **Lemma.** *There is $t \in (1/2, 1)$ and a finite family $\mathscr{B}$ of disjoint closed balls in $B^3 \setminus \overline{B}^3(t)$ such that if $R \subset \mathbf{R}^3$ is a ray from a point in $\overline{B}^3(t)$, then $R$ meets $Q(t)$ for some $Q \in \mathscr{B}$.*

*Proof.* A construction for the corresponding result in $\mathbf{R}^2$ is given in Figure 1 with $t = 9/10$. The construction in $\mathbf{R}^3$ is rather similar but somewhat more complicated. We omit the details.

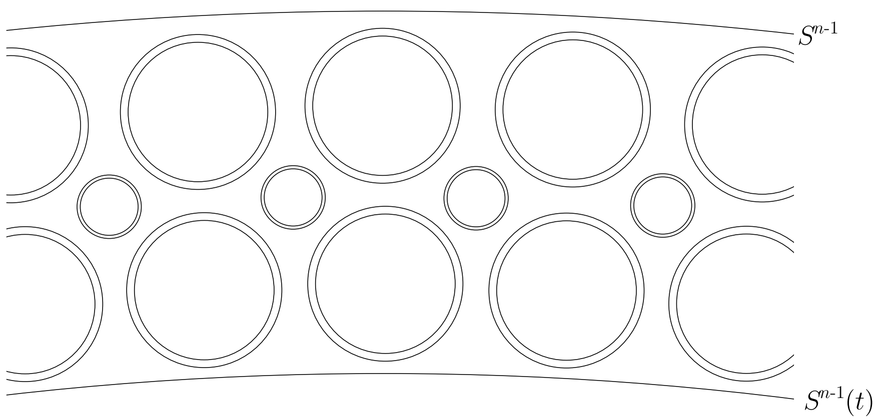


Figure 1

5.8. *The Cantor set $C$.* By a *parallel similarity* we mean a map $f\colon \mathbf{R}^3 \to \mathbf{R}^3$ of the form $f(x) = \lambda x + b$, $\lambda > 0$, $b \in \mathbf{R}^3$. Let $t > 0$ and $\mathscr{B} = \{Q_1, \dots, Q_m\}$ be the number and the family of balls given by 5.7. For $1 \le j \le m$, let $\beta_j$ be the parallel similarity with $\beta_j \overline{B}^3 = Q_j$. These maps define in the familiar way a self similar Cantor set $C$. More precisely, $C$ is the intersection of the descending sequence of compact sets $C_k$, where $C_1 = Q_1 \cup \dots \cup Q_m$, $C_2$ is the union of the balls $\beta_i Q_j$, etc.

5.9. **Lemma.** *Let $R \subset \mathbf{R}^3$ be a ray from a point in $\overline{B}^3(t)$. Then $R$ meets $C$.*

*Proof.* Let $k \in \mathbf{N}$. It suffices to show that $R$ meets $C_k$. For $k=1$ this follows from 5.7. In fact, $R$ meets $Q_j(t)$ for some $j$. Hence $\beta_j^{-1}R$ meets $\overline{B}^3(t)$. By 5.7, there is $i$ such that $\beta_j^{-1}R$ meets $Q_i(t)$. It follows that $R$ meets $\beta_j Q_i(t) \subset C_2$. Proceeding inductively in this manner we obtain the lemma.

5.10. **Lemma.** *There is a quasiconformal map $h\colon \mathbf{R}^3 \to \mathbf{R}^3$ such that $h(x) = x$ for $|x| \le t$ and for $|x| \ge 1$, and such that $h[te_3, e_3]$ contains $C$.*

*Proof.* Choose disjoint closed balls $A_1, \ldots, A_m$ in $B^3 \setminus \overline{B}^3(t)$ with centers on the line segment $[te_3, e_3]$. For each $j = 1, \ldots, m$, let $\alpha_j$ be the parallel similarity with $\alpha_j \overline{B}^3 = A_j$. Choose a homeomorphism $h_1\colon \mathbf{R}^3 \to \mathbf{R}^3$ such that

$h_1(x) = x$ for $|x| \le t$ and for $|x| \ge 1$,

$h_1(x) = \beta_j \alpha_j^{-1}(x)$ for $x \in A_j$, $1 \le j \le m$,

and such that $h_1$ is $K$-quasiconformal with some $K$. Next define $h_2\colon \mathbf{R}^3 \to \mathbf{R}^3$ by $h_2 = h_1$ outside the balls $A_j$, and by $h_2 = \beta_j h_1 \alpha_j^{-1}$ in $A_j$. Iterating the construction in the natural way we obtain a sequence $(h_k)$ of $K$-quasiconformal maps converging to a $K$-quasiconformal map $h\colon \mathbf{R}^3 \to \mathbf{R}^3$ with the desired properties.

5.11. **Lemma.** *There is a quasiconformal map $g\colon \mathbf{R}^3 \to \mathbf{R}^3$ such that $gZ$ meets every line segment $(0, y]$, $y \ne 0$.*

*Proof.* We set $g(x) = x$ if $x = 0$ or if $|x| \ge 1$. Let $0 < |x| < 1$. Then $t^{k+1} \le |x| < t^k$ for a unique integer $k \ge 0$, and we set $g(x) = t^k h(x/t^k)$, where $h$ is given by 5.10. Then $g$ is clearly a $K$-quasiconformal homeomorphism. From 5.9 and 5.10 it follows that $g[te_3, e_3]$ meets $[te, e]$ for every $e \in S^2$. By construction, $g[0, e_3]$ meets every $(0, y]$.

5.12. *Remarks.* 1. If $g\colon \mathbf{R}^3 \to \mathbf{R}^3$ is bilipschitz, then the Hausdorff dimension of $gZ$ is 1, and hence the set of all $e \in S^2$ with $(0, e] \cap gZ \ne \varnothing$ is of area zero. However, there is a bilipschitz map $g$ such that for every $r > 0$, $gZ$ meets $B^3(r) \cap V$ for each open cone $V$ with vertex at 0; see [LV, 4.11].

2. On the other hand, the map $f$ of 5.4 can be chosen to be of bounded length distortion in view of Remark 5.6.

3. Similar examples exist in $\mathbf{R}^n$ for all $n \ge 3$.

We finally give a result in a direction converse to the cone theorem [MRV3, 4.4]. For $y \in S^{n-1}$ and $0 \le \alpha \le \pi/2$, let $C(y, \alpha)$ denote the open cone $\{x \in \mathbf{R}^n : x \cdot y > |x| \cos \alpha\}$.

5.13. **Theorem.** *Suppose that $n \ge 3$, that $f\colon B^n \to \mathbf{R}^n$ is a nonconstant $K$-quasiregular map with $f(0) = 0$, and that $0 < \alpha < 1/2$. If for some $r_0 > 0$,*

$$B_f \cap C(y, \alpha) \cap B^n(r) \setminus B^n\big((1-\alpha)r\big) \ne \varnothing$$

*for all $y \in S^{n-1}$ and for all $r \in (0, r_0]$, then $i(0, f) \ge N_1(\alpha, K, n)$, where $N_1(\alpha, K, n) \to \infty$ as $\alpha \to 0$.*

*Proof.* This follows easily either from 3.4 or from 5.2.

**References**

[C1] Chernavskii, A.V.: Discrete and open mappings on manifolds. - Mat. Sb. 65, 1964, 357–369 (Russian).

[C2] Chernavskii, A.V.: Continuation to "Discrete and open mappings on manifolds". - Mat. Sb. 66, 1965, 471–472 (Russian).

[GM] Gehring, F.W., and O. Martio: Quasiextremal distance domains and extension of quasiconformal mappings. - J. Analyse Math. 45, 1985, 181–206.

[GO] Gehring, F.W., and B.G. Osgood: Uniform domains and the quasi-hyperbolic metric. - J. Analyse Math. 36, 1979, 50–74.

[Iw] Iwaniec, T.: Stability property of Möbius mappings. - Proc. Amer. Math. Soc. 100, 1987, 61–69.

[LV] Luukkainen, J., and J. Väisälä: Elements of Lipschitz topology. - Ann. Acad. Sci. Fenn. Math. 3, 1977, 85–122.

[Ma] Martin, G.J.: Quasiconformal and bi-Lipschitz homeomorphisms, uniform domains and the quasihyperbolic metric. - Trans. Amer. Math. Soc. 292, 1985, 169–191.

[MR] Martio, O., and S. Rickman: Measure properties of the branch set and its image of quasiregular mappings. - Ann. Acad. Sci. Fenn. Math. 541, 1973, 1–16.

[MRV1] Martio, O., S. Rickman, and J. Väisälä: Definitions for quasiregular mappings. - Ann. Acad. Sci. Fenn. Math. 448, 1969, 1–40.

[MRV2] Martio, O., S. Rickman, and J. Väisälä: Distortion and singularities of quasiregular mappings. - Ann. Acad. Sci. Fenn. Math. 465, 1970, 1–13.

[MRV3] Martio, O., S. Rickman, and J. Väisälä: Topological and metric properties of quasiregular mappings. - Ann. Acad. Sci. Fenn. Math. 488, 1971, 1–31.

[MS] Martio, O., and J. Sarvas: Injectivity theorems in plane and space. - Ann. Acad. Sci. Fenn. Math. 4, 1981, 383–401.

[MSr] Martio, O., and U. Srebro: On the local behavior of quasiregular maps and branched covering maps. - J. Analyse Math. 36, 1979, 198–212.

[MV] Martio, O., and J. Väisälä: Elliptic equations and maps of bounded length distortion. - Math. Ann. 282, 1988, 423–443.

[Mr] Mayer, V.: Uniformly quasiregular mappings of Lattés type. - Conf. Geom. Dynam. 1, 1997, 104–111.

[Po1] Pommerenke, C.: Linear-invariante Familien analytischer Funktionen I. - Mat. Ann. 155, 1964, 108–154.

[Po2] Pommerenke, C.: Univalent functions. - Vandenhoeck & Ruprecht, Göttingen, 1973.

[RR] Rado, T., and P.V. Reichelderfer: Continuous transformations in analysis. - Springer-Verlag, 1955.

[Re] Reshetnyak, Yu.G.: Space mappings with bounded distortion. - Translations of Mathematical Monographs 73. American Mathematical Society, Providence, RI, 1989.

[Ri] Rickman, S.: Quasiregular mappings. - Ergebnisse der Mathematik 26, Springer-Verlag, 1993.

[S1] Sarvas, J.: On the local behavior of quasiregular mappings. - Ann. Acad. Sci. Fenn. Math. 1, 1975, 221–226.

[S2] Sarvas, J.: The Hausdorff dimension of the branch set of a quasiregular mapping. - Ann. Acad. Sci. Fenn. Math. 1, 1975, 297–307.

[Sc] Schober, G.: Univalent functions – Selected topics. - Lecture Notes in Math. 478, Springer-Verlag, 1975.

[V1] Väisälä, J.: Discrete open mappings on manifolds. - Ann. Acad. Sci. Fenn. Math. 392, 1966, 1–10.

[V2] Väisälä, J.: Lectures on $n$-dimensional quasiconformal mappings. - Lecture Notes in Math. 229, Springer-Verlag, 1971.

[V3] Väisälä, J.: Quasi-symmetric embeddings in euclidean spaces. - Trans. Amer. Math. Soc. 264, 1981, 191–204.

[V4] Väisälä, J.: Uniform domains. - Tôhoku Math. J. 40, 1988, 101–118.

[V5] Väisälä, J.: Quasiconformal maps of cylindrical domains. - Acta Math. 162, 1989, 201–225.

[Zo] Zorich, V.A.: The theorem of M.A. Lavrent'ev on quasiconformal mappings in space. - Mat. Sb. 74, 1967, 417–433 (Russian).

Received 23 September 1997
Received in revised form 5 June 1998