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REALITY OF ZEROS OF DERIVATIVES OF MEROMORPHIC FUNCTIONS

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Abstract. Let f be a meromorphic non-entire function in the plane, and suppose that for every $n \geq 0$, the derivative $f^{(n)}$ has only real zeros. We have proved that then there are real numbers a and b where $a \neq 0$, such that f is of the form $f(az+b) = P(z)/Q(z)$ where $Q(z) = z^n$ or $Q(z) = (z^2+1)^n$ for some positive integer n, and P is a polynomial with only real zeros such that $\deg P \leq \deg Q + 1$; or $f(az+b) = C(z-i)^{-n}$ or $f(az+b) = C(z-\alpha)/(z-i)$ where α is real and C is a non-zero complex constant. In this paper we explain the structure of the proof (which is divided into several cases), and give the proof in those cases that can be dealt with by reasonably elementary methods.

1. Introduction and results

1.1. Let f be a function meromorphic in the complex plane $\mathbf{C}$. We consider the question of under what circumstances all the derivatives of f, including f itself, can have only real zeros. We may and will assume that f is not a polynomial so that none of the derivatives $f^{(n)}$ vanishes identically. We shall show that if f has the above property and if f is not entire, then f is a rational function of a suitable type, and we determine all cases that can occur. The complete proof is long, and is divided into three papers (this paper and [8], [9]). Such a division is natural as there are different cases to be considered that require quite different methods of proof. To discuss the problem further, we need to recall a number of definitions.

We say that f is *real* if $f(z)$ is real or $f(z) = \infty$ whenever z is real. If f is not a constant multiple of a real function, then f is called *strictly non-real*. We now define some classes of functions. We say that $f \in V_{2p}$ where p is an integer with $p \geq 0$ if f is of the form

$$f(z) = g(z) \exp\{-az^{2p+2}\}$$

where $a \geq 0$ and g is a constant multiple of a real entire function with genus not exceeding $2p+1$ and with only real zeros. We set $U_0 = V_0$ and $U_{2p} = V_{2p} \setminus V_{2p-2}$

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for $p \geq 1$. The class U_0 is the so-called Laguerre–Pólya class. We have $f \in U_0$ if, and only if, there are real polynomials P_n with only real zeros such that $P_n \to f$ locally uniformly in $\mathbf{C}$. Also, $f \in U_0$ if, and only if, we may write

$$f(z) = cz^m e^{-az^2+bz} \prod_{n=1}^{N} \Big(1 - \frac{z}{z_n}\Big) e^{z/z_n}$$

where $0 \leq N \leq \infty$, c is a non-zero complex constant, m is a non-negative integer, $a \geq 0$, b is a real number, $z_n \in \mathbf{R} \setminus \{0\}$ for all $n \geq 1$, and $\sum_{n=1}^{N} z_n^{-2} < \infty$. Here $\mathbf{R}$ denotes the real axis, and an empty product (with $N = 0$) is equal to 1. If $f \in U_0$ then $f^{(n)} \in U_0$ and so $f^{(n)}$ has only real zeros for all $n \geq 0$.

In 1914, Pólya posed the problem of determining all entire functions f such that $f^{(n)}$ has only real zeros for all $n \geq 0$. (Of course, $f^{(0)} = f$.) Hellerstein and Williamson [2], [3] proved that if f is real entire and f, f' and f'' have only real zeros then f is in the Laguerre–Pólya class. If, instead, f is strictly non-real and entire, then by a result of Hellerstein, Shen and Williamson [5] we have $f(z) = Ae^{Bz}$ or $f(z) = A(e^{icz} - e^{id})$, so that $f^{(n)}$ has only real zeros for all $n \geq 0$, or

$$\begin{aligned} f(z) &= A\exp\big\{e^{i(cz+d)}\big\} \qquad \text{or} \\ f(z) &= A\exp\big\{K\big[i(cz+d) - e^{i(cz+d)}\big]\big\}, \end{aligned}$$

in which case f''' has some non-real zeros. Here A and B are non-zero complex constants while c, d and K are real constants with $c \neq 0$ and $K \leq -1/4$.

These results answer Pólya's question completely, for entire functions. In fact, in this case, no knowledge is required of the location of the zeros of $f^{(n)}$ for $n \geq 3$. Recently, there have been various improvements in the direction that even the assumption on f' can sometimes be dropped, notably by Sheil–Small [16], [17] (see also [1], [6], [7]).

1.2. One can ask what can be said if f is a non-entire meromorphic function. Here two remarks are in order. Recall the notion of the *final set* of f introduced by Pólya ([12, p. 37], [13, p. 205]). We say that $z \in \mathbf{C}$ lies in the final set of f if for every neighbourhood U of z there are infinitely many values of n such that some zero of $f^{(n)}$ lies in U. In particular, this is the case if $f^{(n)}(z) = 0$ for infinitely many n. Pólya [12], [13] proved that if f has at least two distinct poles then the final set is non-empty, and a point z belongs to the final set if, and only if, $f(z) \neq \infty$ and the circle centred at z with radius $r(z) > 0$ contains at least two distinct poles of f, where $r(z)$ is the Euclidean distance from z to the nearest pole of f. The multiplicities of the poles are not of importance here. Pólya [12], [13] also proved that if f has exactly one pole (of any multiplicity) then the final set of f is empty. Now if all the $f^{(n)}$ have only real zeros then the final set of f must be contained in the real axis $\mathbf{R}$. On the other hand, from Pólya's description of the final set it is easily seen that if f has at least two distinct poles then the final set of f can be contained in $\mathbf{R}$ only if there are exactly two distinct poles (ignoring multiplicities) which furthermore are complex conjugates of each other. (More precisely, if f has at least three distinct poles, then elementary geometric considerations show that the final set cannot be contained in a single straight line.) So in this case f must be of the form $f(z) = g(z)(z-w)^{-m}(z-\overline{w})^{-n}$ where w is a non-real complex number, g is entire with $g(w)g(\overline{w}) \neq 0$, and m and n are positive integers. This restriction on the pole distribution of f, while still leaving many functions to be studied, considerably simplifies the problem of determining all such f.

The second remark is that various results have been obtained by making the extra assumption that all the *poles* of f (and hence of all the $f^{(n)}$) are *real*, and that then it has been sufficient to assume that only the first few derivatives (usually f, f' and f'') of f have only real zeros. Of course, Pólya's result on the final set is not available to limit the pole distribution of f when we do not have the reality assumption on the zeros of *all* the $f^{(n)}$. Indeed, Hellerstein, Shen and Williamson [5] proved that if f is a strictly non-real non-entire meromorphic function such that f, f' and f'' have only real zeros and poles then

$$f(z) = Ae^{-i(cz+d)}/\sin(cz+d)$$

or

$$f(z) = A\,\frac{\exp\bigl[-2i(cz+d) - 2\exp\{2i(cz+d)\}\bigr]}{\sin^2(cz+d)},$$

where A is a non-zero complex constant, c and d are real constants, and $c \neq 0$. In both cases, f''' has infinitely many non-real zeros. Analogous results for reciprocals of entire functions have been established by Hellerstein and Williamson [4] and by Rossi [15]. It is an open question to determine all real non-entire meromorphic functions f such that f, f' and f'' have only real zeros and poles. There are rather large families of transcendental functions f with these properties, as is shown by an example due to Lounesto and Toppila [11] and by an extension of this example due to the author [6, Theorem 5, p. 633].

1.3. We have proved the following result. The proof is given partly in this paper and partly in the two companion papers [8], [9].

Theorem 1. *Let f be a non-entire meromorphic function in the complex plane, and suppose that for every integer $k \geq 0$, the derivative $f^{(k)}$ has only real zeros. Then there are real numbers a and b where $a \neq 0$, and a polynomial P with only real zeros (possibly a non-zero constant), such that*

(i) $f(az+b) = P(z)/Q(z)$, *where* $Q(z) = z^n$ *or* $Q(z) = (z^2+1)^n$, n *is a positive integer, and* $\deg P \leq \deg Q + 1$; *or*

(ii) $f(az+b) = C(z-i)^{-n}$ *where* C *is a non-zero complex constant*; *or*

(iii) $f(az+b) = C(z-\alpha)/(z-i)$, *where* α *is a real number and* C *is a non-zero complex constant.*

Conversely, if f *is as in* (i) *with* $\deg P \le \deg Q$, *or if* f *is as in* (ii) *or* (iii), *then* $f^{(k)}$ *has only real zeros for all* $k \ge 0$. *If* f *is as in* (i) *with* $\deg P = \deg Q + 1$ *then* $f^{(k)}$ *has only real zeros for all* $k \ge 0$ *if, and only if,* f' *(or, equivalently,* $zP'(z) - nP(z)$ *or* $(z^2+1)P'(z) - 2nzP(z)$*) has only real zeros.*

Of course, $\deg P$ denotes the degree of the polynomial P.

Note that if f is as in (i) of Theorem 1 and $\deg P = \deg Q + 1$ then there are polynomials P for which f' has only real zeros, and other polynomials P for which f' has at least two non-real zeros. If $f(z) = P(z)/z^n$ where $\deg P = n+1$ and all the zeros of P (are real and non-zero and) have the same sign then Rolle's theorem together with simple arguments shows that $zP'(z) - nP(z)$ has only real zeros. If $n \ge 2$ and $P(z) = (z-1)^n\bigl(z + (n-1)^2/(4n)\bigr)$ then $zP'(z) - nP(z) = (z-1)^{n-1}\bigl(z + \frac{1}{2}(n-1)\bigr)^2$, which has only real zeros, and hence for $n \ge 2$, the function $zP'(z) - nP(z)$ may have only real zeros even if the zeros of P are not all of the same sign. If $P(z) = (z-A)^{2n+1}$ where A is a real number and $f(z) = P(z)/(z^2+1)^{2n}$ then f' (or $(z^2+1)P'(z) - 2nzP(z)$) has only real zeros if, and only if, $n^2A^2 \ge 2n+1$.

1.4. As the preceding discussion clearly indicates, to prove Theorem 1, we may assume that f has at most two distinct poles. If there is only one pole and if the pole is real, we may replace $f(z)$ by $f(z+b)$ for a suitable real number b and assume that the pole is at the origin. In this case

$$f(z) = \frac{g(z)}{z^n} \tag{1.1}$$

where g is an entire function with only real zeros with $g(0) \ne 0$ and n is a positive integer. If there is only one pole z_0, which is non-real, we may replace $f(z)$ by $f(az+b)$ where $a = \operatorname{Im} z_0 \ne 0$ and $b = \operatorname{Re} z_0$, and assume that

$$f(z) = \frac{g(z)}{(z-i)^n} \tag{1.2}$$

where g is an entire function with only real zeros with $g(i) \ne 0$ and n is a positive integer. Finally if f has exactly two poles then they are at the points z_0 and $\overline{z_0}$ where $\operatorname{Im} z_0 > 0$. Now replacing $f(z)$ by $f(az+b)$ where $a = \operatorname{Im} z_0 \ne 0$ and $b = \operatorname{Re} z_0$, we may assume that

$$f(z) = \frac{g(z)}{(z+i)^m(z-i)^n} \tag{1.3}$$

where g is an entire function with only real zeros with $g(i)g(-i) \ne 0$ and m and n are positive integers, not necessarily equal to each other.

Before getting into the specific results whose complete proofs will be given in this paper, we explain the results to be proved in the companion papers [8], [9] that will provide part of the proof of Theorem 1. Suppose that f is as in (1.3). The case when $m \neq n$ and $mn \neq 0$ cannot occur, as is shown by the following theorem, which is proved in [8].

Theorem 2. *Let f be given by* (1.3) *where g is an entire function with only real zeros with $g(i)g(-i) \neq 0$ and m and n are integers, not necessarily equal, with $m \geq 0$ and $n \geq 1$. Suppose that f' and f'' have only real zeros.*

If $m = 0$ then there is a non-zero complex constant C such that

(i) $g(z) \equiv C$; *or*

(ii) $g(z) \equiv C(z-\alpha)$ *for some real α, and then $n = 1$; or*

(iii) $g(z) \equiv Ce^{inz}$.

If g is as in (i) *or* (ii) *then $f^{(k)}$ has only real zeros for all $k \geq 0$. If g is as in* (iii) *then f''' has at least one non-real zero.*

If $m \geq 1$ then $m = n$ and g is a constant multiple of a real entire function.

In this paper we shall show, in Theorems 4 and 5 below, that if f is as in (1.3) and $m = n$ then g is a constant multiple of a real entire function of finite order, that is, we prove that the case of g being of infinite order cannot occur. For this, it suffices to assume that f, f', and f'' have only real zeros. The case when g is of finite order is dealt with by the following result, proved in [9].

Theorem 3. *Let f be given by*

$$f(z) = \frac{g(z)}{(z^2+1)^n}$$

where g is a real entire function of finite order with $g(i)g(-i) \neq 0$ and n is a positive integer. If f, f', f'', and f''' have only real zeros then g is a polynomial of degree at most $2n+1$.

Conversely, if f is of this form where g is a polynomial of degree at most $2n$ then $f^{(k)}$ has only real zeros for all $k \geq 0$. If the degree of g is $2n+1$ then $f^{(k)}$ has only real zeros for all $k \geq 0$ if, and only if, f and f' have only real zeros.

In fact, as will be explained in [9], if g is a real entire function of finite order with only real zeros, and if g is not in the Laguerre–Pólya class, then it suffices to assume in Theorem 3 that merely f, f', and f'' have only real zeros. Clearly Theorems 2 and 3 together with Theorems 4 and 5 to be stated and proved below complete the proof of Theorem 1 when f is as in (1.3).

Suppose that f is as in (1.2). This case corresponds to taking $m = 0$ in (1.3), and is therefore settled by Theorem 2.

1.5. Suppose then that f is as in (1.1). We shall prove Theorem 1 for such functions f in this paper. We write $M(r,g) = \max\{|g(z)| : |z| = r\}$ for the maximum modulus of g.

Theorem 4. *Let g be a real transcendental entire function, let Q be a real non-constant polynomial, and set*

$$f(z) = \frac{g(z)}{Q(z)}.$$

If f has only real zeros, then f'' has infinitely many non-real zeros or g satisfies

$$\log\log M(r,g) = O(r\log r), \qquad \text{as } r \to \infty. \tag{1.4}$$

Theorem 5. *Suppose that $f(z) = g(z)/Q(z)$, where Q is a real polynomial and g is a real entire function of infinite order satisfying* (1.4)*, and that f and f' have only real zeros. Then f'' has infinitely many non-real zeros.*

As was pointed out to the author by John Rossi, Theorem 4 is a straightforward consequence of earlier results, essentially due to Levin and Ostrovskiĭ [10], and is the same as [10, Theorem 2], except that f may have finitely many poles instead of being entire, and the proof of Theorem 5 can be based on methods developed by Hellerstein and Williamson [3]. The necessary modifications in the proofs will be given in §§ 7–8.

If f is as in (1.1) and if g is strictly non-real, then also f is strictly non-real. Since we are assuming, in particular, that f, f', and f'' have only real zeros, and since, in this case, f has only real poles, it follows from the result of Hellerstein, Shen and Williamson [5, Theorem 1, p. 320] that there is no function f satisfying all these conditions. So we may assume that g, and hence f, is real. In view of Theorems 4 and 5, we may now further assume that g, and hence f, is of finite order. The following results address this situation.

Theorem 6. *Suppose that*

$$f(z) = g(z)z^{-n},$$

where n is an integer, $n \ge 1$, $g(0) \ne 0$, and g is a real entire function. If g is transcendental of order < 2, then $f^{(k)}$ has some non-real zeros for all $k \ge 2$, and if g is a polynomial with $\deg g = d \ge n+2$, then $f^{(k)}$ has some non-real zeros for all k with $2 \le k \le d-n$.

Suppose that g is a polynomial with only real zeros and with $\deg g \le n+1$. Then $f^{(k)}$ has only real zeros for all $k \ge 1$, if, and only if, f' has only real zeros. If, furthermore, $\deg g \le n$, then f' has only real zeros. If $\deg g = n+1$ and the zeros of g have the same sign, then f' has only real zeros. This last condition is necessary for $n = 1$, but not for any $n \ge 2$.

Note that if $\deg g \ge n+2$, no assumption on the reality of the zeros of f or f' is needed.

Next we show that if g in (1.1) is transcendental and of finite order, then $f^{(k)}$ will eventually have non-real zeros.

Theorem 7. *Suppose that f is given by* (1.1), *where* $n \geq 1$, $g(0) \neq 0$, *and g is a real transcendental entire function of order $< q$, where q is an integer with $q \geq 1$. Then $f^{(k)}$ has non-real zeros for all $k \geq qn+1$.*

Clearly Theorems 6 and 7 together with the preceding remarks complete the proof of Theorem 1.

Again, in Theorem 7 it is not necessary to assume that g or f or certain first few derivatives of f have only real zeros. This gain is obtained at the expense of getting the conclusion only for $f^{(k)}$ when $k \geq qn+1$, where the lower bound $qn+1$ could be arbitrarily large.

It may be of some interest to note the following consequence of Theorem 1 (or, already, of Theorems 6 and 7).

Corollary 1. *The largest class of meromorphic non-entire functions with only real zeros and poles, closed under differentiation, consists of functions f of the form*

$$f(z) = AP(z)(z-a)^{-n},$$

where A is a non-zero complex number, a is a real number, n is an integer with $n \geq 1$, and P is a real polynomial with only real zeros and with $P(a) \neq 0$, such that $\deg P \leq n$, or such that $\deg P = n+1$ and $P'(z)(z-a) - nP(z)$ has only real zeros.

1.6. When f is a rational function (that is, g in (1.1)–(1.3) is a polynomial), there are some slightly more accurate results than those provided by the preceding theorems, using only the first few derivatives of f. Theorems 8 and 9 below provide some such results as well as counterexamples that indicate the sharpness of the converse to part (i) of Theorem 1.

Theorem 8. (i) *Suppose that f is given by* (1.2), *where g is a polynomial with only real zeros. If f' has only real zeros, then g is constant, or $g(z) = B(z-A)^n$, where $B \neq 0$ and A is real. In the latter case, f'' has non-real zeros if $n \geq 2$.*

(ii) *Suppose that f is given by* (1.3), *where g is a real polynomial. If f' has only real zeros, then $m = n$. Suppose that $m = n$ and that g has only real zeros. If $\deg g \leq 2n+1$, then $f^{(k)}$ has only real zeros for all $k \geq 1$ if, and only if, f' has only real zeros. If $\deg g \leq 2n$, then f' has only real zeros. If $\deg g = 2n+1$, then f' may or may not have only real zeros. For example, if $g(z) = (z-A)^{2n+1}$, then f' has only real zeros if, and only if, $n^2A^2 \geq 2n+1$.*

Theorem 8 shows that there is a class of rational functions of the form

$$f(z) = g(z)(z^2+1)^{-n},$$

with only real zeros, closed under differentiation (of course, the same conclusion is a very special case of the more elaborate Theorem 1). The following theorem is related to the situation when f is given by (1.1) and considers the sharpness of the conclusion of part (i) of Theorem 1.

Theorem 9. *Let P be a real polynomial with only real zeros and with $\deg P = d \geq 1$ and $P(0) \neq 0$. If $d \leq n$, or if $d = n+1$ and the zeros of P have the same sign, then $h(z) \equiv z^{n+1}(P/z^n)' = P'(z)z - nP(z)$ has only real zeros.*

If $n \geq 2$ and $P(z) = (z-1)^n(z+\alpha)$, then $h(z) = (z-1)^{n-1}\bigl(z^2+(n-1)z+n\alpha\bigr)$, and h has only real zeros if, and only if, $\alpha \leq (n-1)^2/(4n)$. Thus when $\deg P = d = n+1 \geq 3$, there are polynomials P whose zeros do not all have the same sign, for which h has only real zeros, and other such polynomials P for which h has some non-real zeros.

If $m \geq 2$, $n+1-m \geq 2$, and $P(z) = (z-1)^m(z+\alpha)^{n+1-m}$, then there exist α_1 and α_2 with $0 < \alpha_1 < \alpha_2$, such that if $\alpha > 0$, then h has only real zeros if, and only if, $0 < \alpha \leq \alpha_1$ or $\alpha \geq \alpha_2$.

For $n = 1 = d-1$, the function h has only real zeros if, and only if, the two zeros of P are of the same sign.

1.7. Finally, we mention a result whose content and proof, albeit an application of Rolle's theorem, seems rather novel. If P is a real polynomial (possibly with some non-real zeros) with exactly N real zeros with due count of multiplicity, where $N \geq 1$, then P' has certain $N-1$ real zeros, which we describe as "zeros arising from Rolle's theorem". More precisely, any real zero of P of order $m \geq 2$ gives a zero of P' of order $m-1$. Between two successive zeros of P there is at least one zero of P', and we pick one of these. Of course, P' might have altogether more than $N-1$ real zeros. Similarly, we say that for $1 \leq k \leq N$, the polynomial $P^{(k)}$ has exactly $N-k$ (real) "zeros arising from Rolle's theorem", these points being zeros that arise from repeated application of Rolle's theorem in the above fashion. The following result, which will be used in the proof of Theorem 7, is stated here separately as it may be of independent interest.

Theorem 10. *Let P be a real polynomial with M distinct non-real zeros of any multiplicity. If $l \geq 1$, then $P^{(l)}$ has at most Ml zeros z, with due count of multiplicity, such that*

(i) *$P(z) \neq 0$; and such that*
(ii) *z does not arise from Rolle's theorem if z is real.*

So this includes all non-real zeros z of $P^{(l)}$ such that $P(z) \neq 0$.

In this paper we prove Theorems 4–10. Theorems 4 and 5 are independent of the others, and are proved by simple modifications of the proofs of certain known results. Hence they will be proved last, in Sections 7 and 8.

We start by proving Theorem 10 in §2 and its consequence Theorem 7 in §3. These results are the most original contributions of this paper. Theorem 6 will be proved in §5, and Theorems 8 and 9 that elaborate on the situation, will be proved in Sections 6 and 4, respectively. To a small extent, the proofs of Theorems 6 and 8 rely on the proof of Theorem 9, so that the latter proof is presented first.

Acknowledgement. The author formulated and proved Theorems 4–10 above during the academic year 1982–83. At that time he was a Postdoctoral Fellow at Imperial College, University of London, London, England, funded by the Osk. Huttunen Foundation, Helsinki, Finland, whose support is hereby gratefully acknowledged. Furthermore, at that time, the author wrote a handwritten manuscript containing these results and their proofs and circulated it to a few people. However, he has not published the results before, due to the lack of a more complete result such as Theorem 1 above until now. After receiving the author's manuscript at that time, Professor Hellerstein pointed out to the author that Li-Chien Shen had independently formulated and proved a result corresponding to Theorem 6 above in 1982. It would appear that so far Professor Shen has not published his result, either. Shen's theorem states that if f is of the form (1.1) and if g is a polynomial of degree $d \geq n+2$, or if g is transcendental with $g \in U_0$, then f'' has some non-real zeros (and, in fact, exactly two non-real zeros if all the zeros of g are positive and the order of g is < 2). Also Shen notes that there are polynomials g of degree $d \leq n+1$ such that then f'' and indeed $f^{(k)}$ for every $k \geq 0$ has only real zeros, for example, those g that have only positive zeros. Shen's proof is the same as ours; the only difference in the proof, which also explains the difference between the statement of his result and of our Theorem 6, is that he noted that if $g \in U_0$ and $z^{n+2} f'' \notin U_0$ then f'' must have some non-real zeros (this is not totally obvious when g has order 2).

2. Proof of Theorem 10

Let the assumptions of Theorem 10 be satisfied. We can write $P^{(l)} = P_1 P_2$, where P_2 vanishes exactly at the zeros arising from Rolle's theorem. Let N be the number of real zeros of P, and let the non-real zeros of P be z_i with multiplicities m_i for $1 \leq i \leq M$. By the assumptions, the number of those zeros of $P^{(l)}$ in question does not exceed, even if $m_i - l < 0$ for some i, the number

$$\begin{aligned}
\deg P_1 - \sum_{i=1}^{M} (m_i - l) &= \deg P^{(l)} - \deg P_2 - \sum_{i=1}^{M} m_i + Ml \\
&= \deg P - \deg P_2 - \sum_{i=1}^{M} m_i + (M-1)l \\
&= N - \deg P_2 + (M-1)l \leq Ml,
\end{aligned}$$

since $\deg P_2 \geq N - l$, with equality if $l \leq N$. This proves Theorem 10.

3. Proof of Theorem 7

Suppose that the assumptions of Theorem 7 are satisfied. Let us write

$$g(z)=\sum_{m=0}^{\infty}a_m z^m. \tag{3.1}$$

Then

$$f'(z)z^{n+1}=g'(z)z-ng(z)=\sum_{m=0}^{\infty}(m-n)a_m z^m.$$

By induction, for $k\ge 1$,

$$f^{(k)}(z)z^{n+k}\equiv h_k(z)=\sum_{m=0}^{\infty}C(m,n,k)a_m z^m, \tag{3.2}$$

where $C(m,n,k)=0$ if, and only if, $n\le m\le n+k-1$. In particular, $C(0,n,k)\neq 0$ so that $h_k(0)\neq 0$, since $a_0=g(0)\neq 0$. Now consider a fixed $k\ge qn+1$. Then $h=h_k$ is a real transcendental entire function of order $<q$, and $h(0)\neq 0$. Suppose that $f^{(k)}$ has only real zeros, so that the same applies to h. We will derive a contradiction from this.

We can write

$$h(z)=e^{P(z)}\prod_{m=1}^{\infty}\Bigl(1-\frac{z}{z_m}\Bigr)E(z,z_m)$$

where P is a real polynomial of degree at most $q-1$, all the z_m are real and non-zero,

$$E(z,z_m)=\exp\biggl(\frac{z}{z_m}+\frac{1}{2}\Bigl(\frac{z}{z_m}\Bigr)^2+\cdots+\frac{1}{q-1}\Bigl(\frac{z}{z_m}\Bigr)^{q-1}\biggr),$$

and $\sum|z_m|^{-q}<\infty$.

For $N\ge 1$, we define the real polynomial P_N by

$$P_N(z)=\Bigl(1+\frac{Q_N(z)}{r_N}\Bigr)^{r_N}\prod_{|z_m|\le N}\Bigl(1-\frac{z}{z_m}\Bigr),$$

where

$$Q_N(z)=P(z)+\sum_{|z_m|\le N}\sum_{r=1}^{q-1}\frac{1}{r}\Bigl(\frac{z}{z_m}\Bigr)^r,$$

and r_N is a positive integer and so large that

$$|Q_N(z)|\le \tfrac{1}{2}r_N \qquad \text{for } |z|\le 1, \tag{3.3}$$

and

$$\left|\left(1+\frac{Q_N(z)}{r_N}\right)^{r_N}-\exp\{Q_N(z)\}\right|\cdot\prod_{|z_m|\le N}\left|1-\frac{z}{z_m}\right|\le\frac{1}{N}$$

for $|z|\le N$.

Next we show that $P_N\to h$ locally uniformly as $N\to\infty$. We set

$$H_N(z)=e^{P(z)}\prod_{|z_m|\le N}\left(1-\frac{z}{z_m}\right)E(z,z_m).$$

Then $H_N\to h$ locally uniformly as $N\to\infty$. Thus if $R\ge 2$ and $\varepsilon>0$ are given, there is N_0 such that

$$|H_N(z)-h(z)|<\tfrac{1}{2}\varepsilon$$

for $N\ge N_0$ and $|z|\le R$. Further, if also $N\ge R$ and $N\ge 2/\varepsilon$, we have for $|z|\le R$,

$$|H_N(z)-P_N(z)|\le\left|\exp\{Q_N(z)\}-\left(1+\frac{Q_N(z)}{r_N}\right)^{r_N}\right|\cdot\prod_{|z_m|\le N}\left|1-\frac{z}{z_m}\right|\le\frac{1}{N}\le\frac{\varepsilon}{2},$$

so that $|h(z)-P_N(z)|<\varepsilon$ for $|z|\le R$ whenever $N\ge\max\{R,2/\varepsilon,N_0\}$. Thus $P_N\to h$ locally uniformly as $N\to\infty$.

Now we note that $\deg Q_N\le q-1$ for all N, and apply Theorem 10 to P_N with $M\le q-1$ and $l=n$. It follows that in addition to the zeros of $1+Q_N(z)/r_N$, which have modulus ≥ 1 by (3.3), the function $P_N^{(n)}$ has at most $n(q-1)$ zeros not arising from Rolle's theorem. Further, if z_1 and z_2, where $z_1<0<z_2$, are the zeros of h closest to 0, then for all large N, the function $P_N^{(n)}$ has at most n zeros arising from Rolle's theorem between $\frac{1}{2}z_1$ and $\frac{1}{2}z_2$, hence altogether at most qn zeros in $\{z:|z|\le\varepsilon\}$, where $0<\varepsilon<1$ and $2\varepsilon<\max(-z_1,z_2)$. By the argument principle, $h^{(n)}$ has at most qn zeros in $\{z:|z|\le\varepsilon\}$. But by (3.2), $h^{(n)}$ has a zero of order at least k at the origin, since $C(m,n,k)=0$ for $n\le m\le n+k-1$, and we have $k\ge qn+1$. This is a contradiction. Theorem 7 is proved.

4. Proof of Theorem 9

Let the assumptions in the first paragraph of Theorem 9 be satisfied. We may assume that the leading coefficients of P and h are 1 and $d-n$, respectively (for h only if $d\ne n$). If x_0 is a zero of P of order m, then x_0 is a zero of h of order $m-1$. Let x_1 and x_2 be successive zeros of P of the same sign, of order μ and ν, with $x_1<x_2$.

We have

$$P(x) = A(x - x_1)^{\mu} + O\big((x - x_1)^{\mu+1}\big), \qquad \text{as } x \to x_1, \tag{4.1}$$

$$P(x) = B(x - x_2)^{\nu} + O\big((x - x_2)^{\nu+1}\big), \qquad \text{as } x \to x_2, \tag{4.2}$$

for some real A, B with $AB(-1)^{\nu} > 0$. Thus

$$h(x) = P'(x)(x - x_1) + P'(x)x_1 - nP(x) = \mu A x_1 (x - x_1)^{\mu-1} + O\big((x - x_1)^{\mu}\big)$$

as $x \to x_1$, and

$$h(x) = \nu B x_2 (x - x_2)^{\nu-1} + O\big((x - x_2)^{\nu}\big)$$

as $x \to x_2$. Hence $h(x_3)h(x_4) < 0$ for some x_3 and x_4 with $x_1 \le x_3 < x_4 \le x_2$, so that h has a zero on (x_1, x_2).

Suppose next that $d < n$ and that P has at least one positive zero, x_2 being the largest of them. By the assumption on the leading coefficients of P and h, we have $h(x) \to -\infty$ as $x \to \infty$, and we can show as above that $h(x) > 0$ for some $x > x_2$ since we must have $B > 0$ in (4.2). Thus h has a zero on (x_2, ∞).

Similarly, if $d < n$ and if P has at least one negative zero, x_1 being the smallest of them, then $(-1)^d h(x) \to -\infty$ as $x \to \infty$, while $A(-1)^{d+\mu} > 0$ in (4.1), so that $(-1)^d h(x) > 0$ for some $x < x_1$. Thus h has a zero on $(-\infty, x_1)$.

Suppose now that $d \le n + 1$ and that P has p positive and $d - p$ negative zeros. It follows that h has at least $p - 1$ positive and $d - p - 1$ negative zeros, hence at least $d - 2$ real zeros, and furthermore at least $d - 1$ real zeros if $p = 0$ or $p = d$. Also, if $d < n$ and $p \ne 0$, then h has at least $d - 1$ real zeros. We note that $\deg h \le d$ and that $\deg h \le d - 1$ if $d = n$.

Thus in all cases h has at least $(\deg h) - 1$ real zeros. Since h is a real polynomial, all the zeros of h are real. This proves the statements made in the first paragraph on Theorem 9. The other statements of Theorem 9 are easily verified, and we omit the details. Theorem 9 is proved.

5. Proof of Theorem 6

Let f be as in (1.1) and let us use the notation in (3.1) and (3.2). Recall that $C(m, n, k) = 0$ if, and only if, $n \le m \le n + k - 1$. In particular, $C(0, n, k) \ne 0$ so that $h_k(0) \ne 0$, since $a_0 = g(0) \ne 0$.

Suppose that g is real and transcendental of order < 2 (in the Laguerre–Pólya class) or a real polynomial with $\deg g = d \ge n + 2$. Suppose that $k \ge 2$ if g is transcendental, and that $2 \le k \le d - n$ if g is a polynomial. If $f^{(k)}$ has only real zeros, then $h_k \in U_0$ since h_k is real with only real zeros and of order < 2. Hence by a result of Pólya and Schur ([14], [13, pp. 104, 121]), h_k cannot have two or more successive vanishing coefficients between non-vanishing ones. This together with the above gives a contradiction, since $a_m \ne 0$ for some $m \ge n + k$. Hence $h_k \notin U_0$, and $f^{(k)}$ has some non-real zeros.

Suppose that g is a polynomial with only real zeros and with $\deg g = d \leq n+1$. Since

$$f'(z)z^{n+1} = g'(z)z - ng(z),$$

it follows from Theorem 9 that f' has only real zeros of $d \leq n$, or if $d = n+1$ and the zeros of g have the same sign. It is easy to check that the condition of the zeros of g having the same sign for $d = n+1$ is necessary for $n = 1$. It follows from the example given in the statement of Theorem 9 that this condition is not necessary for any $n \geq 2$. We leave it to the reader to verify that this example has the properties stated in Theorem 9.

Finally, if $\deg g \leq n+1$ and if f and f' have only real zeros, then for $k \geq 1$, we have $f^{(k)}(z) = h_k(z)z^{-(n+k)}$, where $\deg h_k \leq n+1 \leq n+k$. Now induction on k shows that $f^{(k)}$ has only real zeros for all $k \geq 1$. Theorem 6 is proved.

6. Proof of Theorem 8

Suppose that f is given by (1.2), where g is a polynomial with $\deg g \geq 2$ and with only real zeros. We may assume that g is real. Then if

$$f'(z)(z-i)^{n+1} \equiv h(z) = g'(z)(z-i) - ng(z),$$

has only real zeros, h must be a constant multiple of a real polynomial. But in fact, h must be real, unless g is of the form $B(z-A)^d$, since then there is a real number x_0 with $g'(x_0) = 0 \neq g(x_0)$, so that $h(x_0)$ is real and non-zero. But if

$$g(z) = \sum_{m=0}^{d} a_m z^m, \qquad a_d \neq 0,\ a_m \in \mathbf{R},\ a_{d+1} = 0, \tag{6.1}$$

then

$$h(z) = \sum_{m=0}^{d} b_m z^m = \sum_{m=0}^{d} \{(m-n)a_m - i(m+1)a_{m+1}\} z^m,$$

and if b_{d-1} is real, then $a_d = 0$, a contradiction.

If $g(z) = B(z-A)^d$, where $d \geq 2$, $B \neq 0$, and A is real, then h has only real zeros if, and only if, $d = n$. If $d = n$, then $f'(z)(z-i)^{n+1} = C(z-A)^{n-1}$ for some non-zero complex constant C, so that f'' has non-real zeros if $n \geq 2$. The case when $\deg g \leq 1$ is easy to check. This proves part (i) of Theorem 8.

Suppose that f is given by (1.3), where g is a real polynomial, given by (6.1). Then

$$f'(z)(z+i)^{m+1}(z-i)^{n+1} \equiv h(z) = g'(z)(z^2+1) - g(z)\{(m+n)z - (m-n)i\},$$

so that $\deg h \leq d+1$. Suppose that $m \neq n$. If $d \neq m+n$, then the leading coefficient of h is real, but not all coefficients of h are real, so that h is not a constant multiple of a real polynomial and has therefore non-real zeros. If $d = m + n$ and if h is a multiple of a real polynomial, then, since

$$h(z) = \sum_{k=0}^{d} \{(k-1-d)a_{k-1} + (k+1)a_{k+1} + i(m-n)a_k\} z^k,$$

where $a_{-1} = a_{d+1} = 0$, it follows that for $0 \le k \le d$,

$$(k+1)a_{k+1} + (k-1-d)a_{k-1} = Aa_k,$$

where A is a real constant with $A \neq 0$. (The polynomial h cannot have only purely imaginary coefficients, since that would imply that $g'/g = (m+n)z/(z^2+1)$, or $g = C(z^2+1)^{(m+n)/2}$.) Then by induction, we have

$$a_k = a_0 A^k C_k \qquad \text{for } 1 \le k \le d, \text{ where } C_k = C_k(k, d, A) > 0,$$

which gives a contradiction since $a_{d-1} = -Aa_d \neq 0$. Hence f' has non-real zeros if $m \neq n$.

Suppose that g has only real zeros, and that $m = n$ and $\deg g = d \le 2n$. Then

$$h(z) = g'(z)(z^2+1) - 2nzg(z).$$

As in the proof of Theorem 9 we see that h has at least $d-1$ real zeros on $[x_1, x_2]$, where x_1 and x_2 are the smallest and largest zeros of g, respectively, namely at the multiple zeros of g, and between any two successive zeros of g, since $z^2 + 1 > 0$ for all real z and since g' changes sign between such zeros. If $d = 2n$, then $\deg h \le d$, so that h, being real, has only real zeros. If $d < 2n$, we may assume without loss of generality that $a_d = 1$, so that the leading coefficient of h is $d - 2n < 0$, and $h(x) \to -\infty$ as $x \to \infty$. But as in the proof of Theorem 9 we see that $h(x) > 0$ for some $x > x_2$, so that h has a zero on (x_2, ∞), and hence at least d real zeros. Since $\deg h \le d+1$, the function h has only real zeros. One can show that the remaining zero of h is on $(-\infty, x_1)$.

Suppose finally that $\deg g \le 2n+1$ and that f' has only real zeros. Then for $k \ge 1$,

$$f^{(k)}(z)(z-i)^{n+k} = g_k(z),$$

where $\deg g_k \le 2n + 1 + k \le 2(n+k)$. Hence it follows by induction on k that $f^{(k)}$ has only real zeros. The last statement of Theorem 8 is trivial to verify, and so Theorem 8 is proved.

7. Proof of Theorem 4

Theorem 4 is a straightforward consequence of earlier results, essentially due to Levin and Ostrovskiĭ [10], and is the same as [10, Theorem 2], except that f may have finitely many poles instead of being entire. As remarked in [10, p. 334], [10, Theorem 1] still applies, and all growth estimates in the proof of [10, Theorem 2] remain valid in spite of the presence of Q. The only thing to be modified is the definition of

$$\pi(z) = \prod_k \frac{1 - \dfrac{z}{b_k}}{1 - \dfrac{z}{a_k}}, \tag{7.1}$$

where the a_k are the distinct zeros of f, and $b_k \in (a_k, a_{k+1})$ is a zero of f' whose existence is ensured by Rolle's theorem. For each interval (a_k, a_{k+1}) containing at least one zero of Q, we drop the term $(1 - z/b_k)/(1 - z/a_k)$ from (7.1). In this case no b_k might exist. Otherwise the definition of $\pi(z)$ is unaltered, and we write

$$\frac{f'}{f}(z) = \frac{\phi(z)\pi(z)}{S(z)},$$

where ϕ is entire and S is a polynomial. Now the proof can proceed essentially unaltered. Theorem 4 is proved.

8. Proof of Theorem 5

To prove Theorem 5, we modify the proof of [3, Theorem, p. 497], due to Hellerstein and Williamson. Their $\psi(z)$ [3, p. 498] corresponds to $\pi(z)$ in the proof of Theorem 4 above, and is modified in the same way. So we can write

$$\frac{f'}{f}(z) = \frac{\phi(z)\psi(z)}{S(z)}, \tag{8.1}$$

where ϕ is entire and S is a polynomial. Both ϕ and S are real, and ϕ has only real zeros. Similarly,

$$\frac{f''}{f'}(z) = \frac{\phi_1(z)\psi_1(z)}{S_1(z)}. \tag{8.2}$$

Now the proof can proceed unaltered apart from some natural changes due to the presence of the polynomials S and S_1 in (8.1) and (8.2). As remarked in [15, p. 667], the statement in [3, Lemma 1] that $\phi_1 \in U_0$ is false, but this is never used. In [3, (1.21)], the polynomial Q is replaced by Q/S, but since $S(a_k) \neq 0$ for all k and the real numbers a_k considered can cluster at $\pm\infty$ only, one can find $M > 0$ and $d \geq 1$ such that $|Q(a_k)| \leq M|a_k|^d$ for all k, as required. We conclude that

ϕ_1 has infinitely many zeros, and we want to show that ϕ_1 has infinitely many non-real zeros. We now assume that ϕ_1 has only finitely many non-real zeros and derive a contradiction. In fact, under this assumption we shall prove that apart from finitely many exceptions, the zeros of ϕ_1 are not real. Since ϕ_1 has infinitely many zeros, this clearly gives a contradiction, as desired.

What we want to prove is an analogue of [3, Lemma 4], except that we have made the additional a priori assumption that ϕ_1 has only finitely many non-real zeros. To prove our statement, following the method of proof of [2, Lemma 7] (which, apart from minor modifications, is also the method of proof of [3, Lemma 4]), we note that the proof of [2, Lemma 7], to which we now refer, has to be modified as follows. Instead of (3.1) and (3.2) in [2], we have (8.1) and (8.2) above. Let the notation in the proof of [2, Lemma 7] be changed accordingly. Moreover, the proof of [3, Lemma 2] can be inverted to show that since ϕ_1 has infinitely many zeros but only finitely many non-real zeros, it follows that ϕ has infinitely many zeros. (More precisely, if ϕ has only finitely many zeros, so that $\phi(z) = S_2(z)e^{\alpha_2 z}$ for some real α_2 and some polynomial S_2, then we have, with $z = iy$ where $y \to \infty$,

$$|\phi_1(z)| = \left|\frac{S_1(z)}{\psi_1(z)}\left\{\frac{\phi(z)\psi(z)}{S(z)} + \frac{\phi'(z)}{\phi(z)} + \frac{\psi'(z)}{\psi(z)} - \frac{S'(z)}{S(z)}\right\}\right| = O(y^k)$$

for some positive number k, which is impossible if ϕ_1 is the product of a polynomial and a function that belongs to U_0 and has infinitely many zeros, on the same basis as explained in [3, p. 500].)

Now the proof of [2, Lemma 7] remains valid unaltered on intervals (γ_n, γ_{n+1}) such that (γ_n, γ_{n+1}) contains no poles of f and $|\gamma_n|$, $|\gamma_{n+1}|$ are large enough. In this way only finitely many real zeros of ϕ_1 are omitted. We only have to show that in case (I), the function

$$H_k(x) = (x - a_k)\frac{f'(x)}{f(x)}$$

satisfies

$$\left(\frac{H_k'(x)}{H_k(x)}\right)' < 0, \qquad \text{for } \gamma_n < x < \gamma_{n+1}, \tag{8.3}$$

if $|x|$ is large enough, and that in case II the function $H(x) = f'(x)/f(x)$ satisfies

$$\left(\frac{H'(x)}{H(x)}\right)' < 0, \qquad \text{for } \gamma_n < x < \gamma_{n+1}, \tag{8.4}$$

if $|x|$ is large enough. The remaining arguments can then be copied word by word.

We prove (8.3). The proof of (8.4) is similar. We have

$$\left(\frac{H_k'(x)}{H_k(x)}\right)' = \left(\frac{\phi'(x)}{\phi(x)}\right)' - \left(\frac{S'(x)}{S(x)}\right)' + \left(\frac{\psi'(x)}{\psi(x)}\right)',$$

where

$$\left(\frac{\psi'(x)}{\psi(x)}\right)' = \sum_{\substack{j=\alpha \\ j\neq k}}^{\omega} \left[(x-a_j)^{-2} - (x-b_j)^{-2}\right] - (x-b_k)^{-2} < 0$$

as shown in [2]. Further, ϕ has infinitely many zeros α_k, say, all of them real, and S has finitely many zeros, β_j, say, some of which may be non-real. However, if $S(\beta_j) = 0$, then $S(\overline{\beta_j}) = 0$. Hence an estimate which takes into account one more a_k than there are points β_j, shows that

$$\left(\frac{\phi'(x)}{\phi(x)}\right)' - \left(\frac{S'(x)}{S(x)}\right)' = -\sum_k (x-\alpha_k)^{-2} + \sum_j (x-\beta_j)^{-2} < 0$$

if $|x|$ is large enough. This completes the proof of Theorem 5.

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