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WANDERING DOMAINS IN THE ITERATION OF COMPOSITIONS OF ENTIRE FUNCTIONS

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Abstract. If p is entire, $g(z) = a + b\exp(2\pi i/c)$, where a, b, c are non-zero constants and the normal set of $g(p)$ has no wandering components, then the same is true for the normal set of $p(g)$.

Let f be a rational function of degree at least 2 or a nonlinear entire function. Let f^n, for $n \in \mathbf{N}$ denote the nth iterate of f. Denote the set of normality by $N(f)$ and the Julia set by $J(f)$. Thus

$$N(f) = \{z : (f^n) \text{ is normal in some neighbourhood of } z\},$$
$$J(f) = \mathbf{C} - N(f).$$

By definition $N(f)$ is open (and possibly empty) and it is well known (see for example [8], [9]) that $J(f)$ is nonempty and perfect and $J(f)$ is completely invariant under f, that is, $z \in J(f)$ implies $f(z) \in J(f)$ and $z_0 \in J(f)$ for any z_0 such that $f(z_0) = z$. Consequently $N(f)$ is completely invariant.

If U is a component of $N(f)$ then $f(U)$ lies in some component V of $N(f)$. In fact $V\setminus f(U)$ is either $\emptyset$ or a single point, by an unpublished result of M. Herring. By a slight abuse of language we write $V = f(U)$ even when $V\setminus f(U)$ is a singleton. If all $f^n(U)$ with $n \in \mathbf{N}$ are different components of $N(f)$ then U is called a wandering domain.

D. Sullivan [13] proved that the set of normality of a rational function has no wandering domain, thus solving a problem open since the papers of Fatou and Julia. On the other hand this is not so for transcendental entire functions. In [1] the first author constructed an entire function f such that $N(f)$ has wandering domains. Since then several entire functions which have wandering domains with various different properties have been constructed, see for instance [3], [7]. Also at the same time there has been a move to classify those entire functions which do not have wandering domains [2], [6], [11]. In particular this is the case for functions which have only a finite number of asymptotic or critical values. Such functions are denoted as having finite type. In this paper we shall identify a class of composite entire functions which have no wandering domains and a class of composite entire functions which have wandering domains. We shall prove

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Theorem 1. *Let $p(z)$ be a nonconstant entire function and let $g(z) = a + be^{2\pi iz/c}$ where a, b, c are nonzero constants. If $h = g(p)$ has no wandering domains then neither does $p(g)$.*

In particular for a polynomial $p(z)$ it is known [2] $e^{p(z)}$ has no wandering domain and consequently it follows immediately that $p(e^z)$ has no wandering domain (also proved in [2]). As another application of the above theorem we shall show that $e^{e^z} - e^z$ has no wandering domain. Also $e^{e^z} - e^z$ is not of finite type, and so provides an example of an entire function which is of not finite type and having no wandering domain.

Proof of Theorem 1. Suppose $f = p(g)$ has a wandering domain say U_1. Then $U_n = f^{n-1}(U_1)$ are distinct for all $n = 1, 2, \dots$. Since $g(z + c) = g(z)$ we have $N(f) = N(f) + c$. Now suppose $U_j \cap (U_k + cl) \neq \emptyset$ for some j, k, l then $U_{j+1} \cap U_{k+1} \neq \emptyset$ and so $j = k$. Thus for $j \neq k$, $U_j \cap (U_k + cl) = \emptyset$ and consequently $g(U_j) \cap g(U_k) = \emptyset$.

For each k, let $V_k = g(U_k)$. Then for $j \neq k$, $V_j \cap V_k = \emptyset$ and $h(V_k) = h\big(g(U_k)\big) = g\big(f(U_k)\big) \subseteq V_{k+1}$. Thus $h^n(V_k) \subseteq V_{k+n}$ and so does not meet V_k, $n > 1$. Thus (h^n) is normal in each V_k and so V_k belongs to a component of $N(h)$.

We finally show that V_k is a component of $N(h)$. We first show that if $\beta \in \partial V_k$ has the form $\beta = g(\alpha)$, $\alpha \in \partial U_k$ then $\beta \in J(h)$. Indeed since U_k is a component of f, $\partial U_k \subseteq J(f)$, and so α is a limit point of repelling periodic points z_n $(\neq \alpha)$ of f say $f^{\nu_n}(z_n) = z_n$. Since $g(f^n) = h^n(g)$ for all n, one obtains $h^{\nu_n}\big(g(z_n)\big) = g(z_n)$. Thus $g(z_n)$ are periodic points of h (of arbitrarily large order). Also $g(z_n) \to g(\alpha) = \beta$ (but $g(z_n) \neq g(\alpha)$ for large n). Thus $\beta \in J(h)$.

To complete the proof we assume that there exists $\beta \in \partial V_k$ with $\beta \notin J(h)$. Then β is (by the above) not a limit of points of $J(h)$, hence not a limit of points in $g(\partial U_k)$. Thus there is a disc $D = D(\beta, r)$, $r > 0$ which contains no points of $g(\partial U_k \setminus \{\infty\})$. Since $\beta \in \partial g(U_k) = \partial V_k$, there exists $w' \in D(\beta, r)$ with $w' = g(z')$, $z' \in U_k$ and without any loss of generality we assume $w' \neq a$. We can continue $g^{-1}(w) = c/(2\pi i)\log\big((w - a)/b\big)$ from w' to β along a path γ in D which avoids a and the values of g^{-1} lie in U_k since they can never meet ∂U_k. Since $\beta \notin g(U_k)$ the only possibility is that $g^{-1}(\gamma) \to \infty$ in U_k and hence β is an asymptotic value of g on this path, i.e. $\beta = a$.

Summing up, $\partial V_k \subseteq J(h)$ except perhaps for a single isolated point. If there is such an isolated point we add it to V_k and then have, since the V_k are distinct, that the V_k are wandering components of h with $h(V_k) = V_{k+1}$. This contradicts the hypothesis and the proof is complete.

As an application of Theorem 1 we shall show that $e^{e^z} - e^z$ has no wandering domain. For its proof we shall need the following lemma.

Lemma 1. *Let f and g be entire functions having a finite number of asymptotic values. Then so does $f(g)$.*

Proof. Let c be an asymptotic value for $f(g)$. Thus there exists a curve $\Gamma \to \infty$ on which $f(g) \to c$. Associated with this curve Γ we have either $g(\Gamma) \to \infty$ or $g(\Gamma) \not\to \infty$. If $g(\Gamma) \to \infty$ then c is an asymptotic value for f. Since f has only finitely many asymptotic values, such c's must be finite in number.

We next consider the case when $g(\Gamma) \not\to \infty$. Since Γ is a curve tending to ∞, we can find a sequence $z_n \to \infty$ on this curve for which $\lim_{n\to\infty} g(z_n) = w_0$ for some finite w_0. Thus $f(w_0) = \lim_{n\to\infty} f\bigl(g(z_n)\bigr) = c$. Consider $\varrho > 0$ fixed, but arbitrarily small. Then $|f(w)-c| > \varepsilon > 0$ for $w \in (|w-w_0| = \varrho)$. Next, as c is an asymptotic value for $f(g)$, $\bigl|f\bigl(g(z)\bigr)-c\bigr| < \varepsilon$ for all $|z| > A$ on Γ where A is some constant. In particular if $|z_n|$ are sufficiently large on Γ then $\bigl|f\bigl(g(z)\bigr) - c\bigr| < \varepsilon$ for all z beyond z_n on Γ and $|g(z_n) - w_0| < \varrho$. Thus $|g(z) - w_0| < \varrho$ for all sufficiently large z on Γ. Thus w_0 is an asymptotic value for $g(z)$ on Γ, where $f(w_0) = c$. But the number of asymptotic values of g is finite. This completes the proof.

Lemma 2. *If f and g are entire functions of finite order then $f(g)$ has at most a finite number of asymptotic values.*

The proof is immediate from Lemma 1 and the fact that an entire function of order k has at most $2k$ different asymptotic values [12, p. 307].

Lemma 3 [6]. *Let $\mathscr{I}$ denote the collection of transcendental entire functions of finite type. Functions in $\mathscr{I}$ have no wandering domains.*

Theorem 2. *The function $e^{e^z} - e^z$ has no wandering domain.*

Proof. Set $g(z) = e^z$ and $p(z) = e^z - z$, then by Lemma 2 $f(z) = g\bigl(p(z)\bigr)$ has a finite number of asymptotic values. Also clearly $f(z)$ has e as the only critical value. Thus e^{e^z-z} is of finite type and so by Lemma 3, e^{e^z-z} has no wandering domain. We now apply Theorem 1 to conclude $p\bigl(g(z)\bigr) = e^{e^z} - e^z$ has no wandering domain.

We next prove the following theorem.

Theorem 3. *Let g be a transcendental entire function having at least one fixed point. Then there exists an entire function f such that $g(f)$ has a wandering domain.*

The proof of this theorem is based on the proof of theorems in [4], [5] and so on the method of construction of wandering domain first introduced by A. Eremenko and M. Lyubich [7]. We first recall the following facts: If F denotes a closed subset of $\mathbf{C}$ and $C_a(F)$ the functions which are continuous on F and analytic in F^0 then F is called a Carleman set (for $\mathbf{C}$) if, for any g in $C_a(F)$ and for any positive continuous function ε on F, there is an entire function f such that $|g(z) - f(z)| < \varepsilon(z)$, $z \in F$. By Arakelyan's theorem (e.g. [10, p. 137]) we have (i) $\widehat{\mathbf{C}} \setminus F$ must be connected and also locally connected at ∞. If in addition to

(i) we have (ii) for each compact K the union $W(K)$ of those components of F^0 which meet K is relatively compact in $\mathbf{C}$, then F is indeed a Carleman set [10, p. 157].

Proof of Theorem 3. Without any loss of generality let the fixed point z_0 of g satisfy $\mathrm{Re}(z_0) < -2$ and $\mathrm{Im}(z_0) = 0$. Let

$$\begin{aligned} B &= \{z : |z - z_0| \le 1\}, \\ L_m &= \{z : \mathrm{Re}(z) = 4m\}, \qquad m \ge 10, \\ A_m &= \{z : |z - (4m+2)| \le 1\}, \qquad m \ge 10. \end{aligned}$$

Then clearly $F = B \bigcup \bigl\{ \bigcup_{m=10}^{\infty} \{A_m \cup B_m\} \bigr\}$ is a Carleman set. As g is continuous at the fixed point z_0, we can choose $\delta > 0$ so small that $|g(z) - z_0| < \frac{1}{2}$ whenever $|z - z_0| < \delta$. Now consider a branch value $g^{-1}(4m+2)$ where $m \ge m_0$. Then again by the continuity of g, there exist $\delta_m > 0$ such that $|g(z) - (4m+2)| < \frac{1}{2}$ for all $|z - g^{-1}(4m+2)| < \delta_m$. By the above remark it follows that there exists an entire function f such that

$$\begin{aligned} |f(z) - z_0| &< \delta, \qquad z \in B, \\ |f(z) - z_0| &< \delta, \qquad z \in L_m,\ m \ge 10, \\ |f(z) - g^{-1}(4m+6)| &< \delta_{m+1}, \qquad z \in A_m. \end{aligned}$$

And clearly $g(f) = h$ is an entire function with $h(A_m) \subseteq A_{m+1}$. Also $h^n(z) \to \infty$ in each A_m and so $A_m \in N(h)$. On the other hand h maps B into a smaller disc $|z - z_0| < \frac{1}{2}$ and so h contains an attractive fixed point ξ such that $h^n \to \xi$ in B. Finally h maps L_m $(m \ge 10)$ into B similarly. Further L_m belongs to a component of $N(h)$ different from a component of G_m to which A_m belongs. Thus each G_m is a wandering domain mapping to G_{m+1} under $z \to h(z)$.

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