Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 45, 2020, 1103–1109

A NOTE ON LUSIN'S CONDITION (N) FOR $W^{1,n}_{\mathrm{loc}}$-MAPPINGS WITH CONVEX POTENTIALS

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Abstract. Given an open convex set $\Omega \subset \mathbf{R}^n$ and a convex function $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$, a short proof of the fact that $|\nabla u(E)| = 0$ for every subset $E \subset \Omega$ with $|E| = 0$ is presented.

1. Introduction and main result

Let $|A|$ denote the Lebesgue measure of $A \subset \mathbf{R}^n$. Given an open set $\Omega \subset \mathbf{R}^n$ we will write $C(\Omega)$ for $C(\Omega; \mathbf{R}^n)$, $W^{k,p}(\Omega)$ for $W^{k,p}(\Omega; \mathbf{R}^n)$, etc. to indicate regularity of mappings defined in Ω.

A mapping $F \colon \Omega \to \mathbf{R}^n$ with $F \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is said to satisfy Lusin's condition (N), which will be denoted as $F \in N(\Omega)$, if $|F(E)| = 0$ for every set $E \subset \Omega$ with $|E| = 0$. The literature on Lusin's condition (N) is vast and we will only mention a few essential results to establish some context. For instance, in [10, Corollary B], Malý and Martio proved that $C(\Omega) \cap W^{1,n}_{\mathrm{loc}}(\Omega) \cap \{F \colon \Omega \to \mathbf{R}^n \colon F \text{ open}\} \subset N(\Omega)$ and, in [10, Theorem C], that $C^{\alpha}_{\mathrm{loc}}(\Omega) \cap W^{1,n}_{\mathrm{loc}}(\Omega) \subset N(\Omega)$ for every $\alpha \in (0,1)$ (see also Malý's Theorem 1.3 in [9]). However, $C(\Omega) \cap W^{1,n}_{\mathrm{loc}}(\Omega) \not\subset N(\Omega)$ (see [10, Section 1] and references therein). In [11], Martio and Ziemer introduced and studied analytic and topological conditions on mappings $F \in W^{1,n}_{\mathrm{loc}}(\Omega)$ with a.e. nonnegative Jacobian determinant (that is, $\det DF \geq 0$ a.e. Ω) that guarantee $F \in N(\Omega)$, for example in [11, Corollary 3.13] they proved that $W^{1,n}_{\mathrm{loc}}(\Omega) \cap \{F \colon \Omega \to \mathbf{R}^n \colon \det DF > 0 \text{ a.e. } \Omega\} \subset N(\Omega)$.

When $\Omega \subset \mathbf{R}^n$ is open and convex, the class of mappings $F \in W^{1,n}_{\mathrm{loc}}(\Omega)$ with a.e. nonnegative Jacobian determinant includes those with convex potentials, that is, $F = \nabla u$ for a convex function $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$. In the case of mappings with convex potentials, the inclusion $\{\nabla u \colon u \in W^{2,n}_{\mathrm{loc}}(\Omega), u \text{ convex}\} \subset N(\Omega)$ can be deduced from Theorem 5.11 and Remark 5.15 in the work of Alberti and Ambrosio [1], in the context of maximal monotone operators in $W^{1,n}_{\mathrm{loc}}(\Omega)$. The exponent n in the inclusion $\{\nabla u \colon u \in W^{2,n}_{\mathrm{loc}}(\Omega),\ u \text{ convex}\} \subset N(\Omega)$ is sharp in the sense that a construction from [1, Section 8] yields a differentiable convex function $u \colon \mathbf{R}^n \to \mathbf{R}$ such that $u \in W^{2,p}_{\mathrm{loc}}(\mathbf{R}^n)$ for every $p \in (1,n)$, $\nabla u \in C^{\alpha}_{\mathrm{loc}}(\mathbf{R}^n)$ for every $\alpha \in (0,1)$, and $\nabla u \notin N(\mathbf{R}^n)$. Moreover, in [8] Liu and Malý constructed a *strictly* convex function $u \colon (0,1)^n \to \mathbf{R}$ such that $u \in W^{2,p}_{\mathrm{loc}}((0,1)^n)$ for every $p \in (1,n)$, $\nabla u \in C^{\alpha}_{\mathrm{loc}}((0,1)^n)$ for every $\alpha \in (0,1)$, and $\nabla u \notin N((0,1)^n)$. Both constructions satisfy $\det D^2 u = 0$ a.e. in Ω.

The proof of the aforementioned Theorem 5.11 in [1] relies on methods from geometric measure theory involving n-currents associated to graphs, the area formula on Lipschitz manifolds, and degree theory. The purpose of this note is to provide a short, simple proof of the inclusion $\{\nabla u \colon u \in W^{2,n}_{\mathrm{loc}}(\Omega),\ u \text{ convex}\} \subset N(\Omega)$ based on

https://doi.org/10.5186/aasfm.2020.4555
2010 Mathematics Subject Classification: Primary 26B25, 26B10; Secondary 46E35, 28A75.
Key words: Sobolev mappings with convex potentials, Monge–Ampère measure, area formulas.

the notion of Monge–Ampère measure. Also, we will point out how one of the main theorems from the work of Braga, Figalli, and Moreira in [2] implies that convex functions in $W^{2,n}_{\mathrm{loc}}(\Omega)$ are continuously differentiable in Ω. Thus, our main result is

Theorem 1. *Let $\Omega \subset \mathbf{R}^n$ be an open convex set and let $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ be a convex function. Then $\nabla u \in C(\Omega) \cap N(\Omega)$.*

Some consequences of Theorem 1, related to the change-of-variable formulas and to the notions of weak and strong solutions of the Monge–Ampère equation, will be included in Section 3.

2. Proof of Theorem 1

Given an open convex set $\Omega \subset \mathbf{R}^n$ and a convex function $u\colon \Omega \to \mathbf{R}$, the normal mapping or subdifferential of u is the set-valued function defined for $x_0 \in \Omega$ as

$$\partial u(x_0) := \{v \in \mathbf{R}^n \colon u(x) \geq u(x_0) + v \cdot (x - x_0) \ \text{ for all } x \in \Omega\}, \tag{2.1}$$

and, given $E \subset \Omega$, $\partial u(E) := \bigcup_{x \in E} \partial u(x)$. If u is differentiable at x_0 we identify $\partial u(x_0)$ with $\nabla u(x_0)$. The Monge–Ampère measure associated to u, denoted by μ_u, is the nonnegative locally finite measure

$$\mu_u(E) := |\partial u(E)| \tag{2.2}$$

defined on the Borel σ-algebra $\{E \subset \Omega \colon \partial u(E) \text{ is Lebesgue measurable}\}$, see [6, Section 2.1] or [7, Section 1.1] for further details.

Our proof of Theorem 1 will be based on the following compactness result for Monge–Ampère measures (see Proposition 2.6 from Figalli's book [6, p. 12] or Lemmas 1.2.2 and 1.2.3 from Gutiérrez's book [7]): let $\{u_\varepsilon\}_{\varepsilon > 0}$ and u be convex functions defined in Ω, let $U \subset \Omega$ be an open set and suppose that u_ε converges uniformly to u on compact subsets of U, then μ_{u_ε} converges weakly* to μ_u, that is,

$$\int_U g \, d\mu_{u_\varepsilon} \to \int_U g \, d\mu_u \quad \forall g \in C_c(U). \tag{2.3}$$

Let us start with a lemma comparing the weight $\det D^2 u$ and the measure μ_u for convex functions $u \in W^{2,1}_{\mathrm{loc}}(\Omega)$. Notice that, due to the convexity of u, $\det D^2 u(x)$ exists and is nonnegative for a.e. $x \in \Omega$.

Lemma 2. *Let $\Omega \subset \mathbf{R}^n$ be an open convex set and let $u \in W^{2,1}_{\mathrm{loc}}(\Omega)$ be a convex function. Then, the inequality*

$$\int_U \det D^2 u(x) \, dx \leq |\partial u(U)| \tag{2.4}$$

holds true for every open set $U \subset\subset \Omega$. In particular, $\det D^2 u \in L^1_{\mathrm{loc}}(\Omega)$.

Proof. Given $U \subset\subset \Omega$, let $\varepsilon_0 := \operatorname{dist}(U, \partial\Omega)$ and for $\varepsilon \in (0, \varepsilon_0)$ and $x \in U$ define

$$u_\varepsilon(x) := u * \eta_\varepsilon(x) = \int_{\mathbf{R}^n} u(x - y) \eta_\varepsilon(y) \, dy, \tag{2.5}$$

where $\eta \in C_c^\infty(\mathbf{R}^n)$ is supported in the unit Euclidean ball $\mathbf{B}(0,1)$ with $\int_{\mathbf{R}^n} \eta(y) \, dy = 1$ and $\eta_\varepsilon(y) := \varepsilon^{-n} \eta(\varepsilon^{-1} y)$. Then, u_ε converges uniformly to u on compact subsets of U and (2.3) holds. Now, given $\delta > 0$, set

$$U_\delta := \{x \in U \colon \operatorname{dist}(x, \partial U) > \delta\} \tag{2.6}$$

and

$$V_\delta := \{x \in \mathbf{R}^n \colon \operatorname{dist}(x, \mathbf{R}^n \setminus U) < \delta/2\}.$$

It then follows that $\overline{U_\delta} \cap \overline{V_\delta} = \emptyset$, since otherwise there would be an $x \in U$ such that

$$\frac{\delta}{2} \geq \operatorname{dist}(x, \mathbf{R}^n \setminus U) = \operatorname{dist}(x, \partial(\mathbf{R}^n \setminus U)) = \operatorname{dist}(x, \partial U) \geq \delta,$$

a contradiction. Hence, there exists a continuous function $g\colon \mathbf{R}^n \to [0,1]$ such that $g \equiv 1$ on $\overline{U_\delta}$ and $g \equiv 0$ on $\overline{V_\delta}$; in particular, $\operatorname{supp}(g) \subset \overline{U \setminus V_\delta} = \overline{U_{\delta/2}} \subset U$. By using that u_ε is a smooth function in U, we get (see for instance [6, Example 2.2] or [7, Example 1.1.4])

$$\int_{U_\delta} \det D^2 u_\varepsilon(x)\, dx = |\nabla u_\varepsilon(U_\delta)| = \int_{U_\delta} d\mu_{u_\varepsilon} \leq \int_U g\, d\mu_{u_\varepsilon}. \tag{2.7}$$

On the other hand, since $D^2 u \in L^1_{\mathrm{loc}}(\Omega)$, we have that $D^2 u_\varepsilon(x)$ (or a subsequence) converges to $D^2 u(x)$ as $\varepsilon \to 0^+$ for (Lebesgue) a.e. $x \in U$ and consequently $\det D^2 u_\varepsilon(x)$ converges to $\det D^2 u(x)$ for a.e. $x \in U$. Thus, by combining (2.7) and (2.3) with Fatou's lemma, we get

$$\begin{aligned} \int_{U_\delta} \det D^2 u(x)\, dx &\leq \liminf_{\varepsilon \to 0^+} \int_{U_\delta} \det D^2 u_\varepsilon(x)\, dx \leq \liminf_{\varepsilon \to 0^+} \int_U g\, d\mu_{u_\varepsilon} \\ &= \int_U g\, d\mu_u \leq \int_U d\mu_u = |\partial u(U)| \end{aligned}$$

and (2.4) follows from the monotone convergence theorem by letting $\delta \searrow 0^+$. □

The next result is based on Theorem 2.9 from Braga–Figalli–Moreira [2] and will allow us to write ∇u, instead of ∂u, for convex functions $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$.

Proposition 3. *Let $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ be a convex function. Then $u \in C^1(\Omega)$.*

Proof. Given a convex function $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ set $f := \Delta u \in L^n_{\mathrm{loc}}(\Omega)$. Fix an arbitrary $x_0 \in \Omega$, let $R > 0$ such that $B_R(x_0) \subset\subset \Omega$. Then, $\Delta u(x) = f(x)$ for a.e. $x \in B_R(x_0)$. In the terminology of Caffarelli–Crandall–Kocan–Święch [4, p. 366], this means that u is an L^n-strong solution of $\Delta u = f$, which, due to the fact that $u \in W^{2,n}(B_R(x_0))$, is equivalent to u being an L^n-viscosity solution of $\Delta u = f$ in $B_R(x_0)$ (see [4, Lemma 2.5 and Corollary 3.7]). Now, by [2, Theorem 2.9] on the $C^{1,\alpha}$-regularity of convex L^n-viscosity supersolutions of fully nonlinear equations used with $\lambda = \Lambda = 1$ and $\gamma \equiv 0$ (so that, in the notation from [2, Section 2.2], we get $\mathcal{P}^-_{\lambda,\Lambda,\gamma} = \Delta$ applied to $\varphi = u$ with $\omega \equiv 0$) and $q = n$, it follows that $u \in C^1(B_{R/64}(x_0))$ and then, since $x_0 \in \Omega$ and $R > 0$ were arbitrary with $B_R(x_0) \subset\subset \Omega$, we obtain $u \in C^1(\Omega)$. □

Remark 4. As mentioned, the only role of Proposition 3 is to allow us to write ∇u, instead of ∂u, for convex functions $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$. All of the results in this note are true, with ∂u instead of ∇u, without assuming $u \in C^1(\Omega)$.

The next lemma provides the reverse inequality to the one from Lemma 2 for convex functions $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$.

Lemma 5. *Let $\Omega \subset \mathbf{R}^n$ be a convex set and let $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ be a convex function. Then, the inequality*

$$|\nabla u(U)| \leq \int_U \det D^2 u(x)\, dx \tag{2.8}$$

holds true for every open set $U \subset\subset \Omega$.

Proof. Let $U_\delta \subset U$, g, and u_ε be as in the proof of Lemma 2. Then, we have

$$|\nabla u(U_\delta)| = \int_{U_\delta} d\mu_u \le \int_U g\, d\mu_u = \lim_{\varepsilon\to 0^+}\int_U g\, d\mu_{u_\varepsilon} \le \lim_{\varepsilon\to 0^+}\int_U d\mu_{u_\varepsilon}$$

$$= \lim_{\varepsilon\to 0^+}|\nabla u_\varepsilon(U)| = \lim_{\varepsilon\to 0^+}\int_U \det D^2 u_\varepsilon(x)\, dx. \tag{2.9}$$

Next, let $\Omega_U \subset \Omega$ denote a set such that $U \subset\subset \Omega_U \subset\subset \Omega$ and define $H := (\Delta u)\,\chi_{\Omega_U}$ so that for $0 < \varepsilon < \operatorname{dist}(U, \partial\Omega_U)$ and for $x \in U$ we have $\Delta u * \eta_\varepsilon(x) = (H * \eta_\varepsilon)(x)$ and then, always for $x \in U$,

$$\det D^2 u_\varepsilon(x) \le \Delta u_\varepsilon(x)^n = (\Delta u * \eta_\varepsilon)(x)^n = (H * \eta_\varepsilon)(x)^n \le \mathcal{M}(H)(x)^n,$$

where $\mathcal{M}$ denotes the Hardy–Littlewood maximal function. If $n > 1$, the (n,n)-strong type of $\mathcal{M}$ and the hypothesis $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ give

$$\int_U \mathcal{M}(H)(x)^n\, dx \le \|\mathcal{M}(H)\|^n_{L^n(\mathbf{R}^n)} \le C_n \|H\|^n_{L^n(\mathbf{R}^n)} = C_n \int_{\Omega_U} \Delta u(x)^n\, dx < \infty,$$

and then the Lebesgue dominated convergence theorem implies that

$$\lim_{\varepsilon\to 0^+}\int_U \det D^2 u_\varepsilon(x)\, dx = \int_U \det D^2 u(x)\, dx$$

which combined with (2.9), and after letting $\delta \searrow 0^+$, proves (2.8). If $n = 1$ then we just use that $\det D^2 u_\varepsilon = u''_\varepsilon = u'' * \eta_\varepsilon$ converges in $L^1_{\mathrm{loc}}(U)$ to $u'' = \det D^2 u$. □

Theorem 6. *Let $S \subset \mathbf{R}^n$ be an open convex set and let $u \in W^{2,n}(S)$ be a convex function. Then,*

$$\int_E \det D^2 u(x)\, dx = |\nabla u(E)| \tag{2.10}$$

for every Borel set $E \subset S$.

Proof. From lemmas 2 and 5 we have

$$\int_U \det D^2 u(x)\, dx = |\nabla u(U)| \tag{2.11}$$

for every open set $U \subset\subset S$ and by considering increasing sequences of sets, the equality (2.11) can be extended to every open set $U \subset S$. In addition, the hypothesis $u \in W^{2,n}(S)$ implies

$$\mu_u(S) = |\nabla u(S)| = \int_S \det D^2 u(x)\, dx \le \int_S \Delta u(x)^n\, dx < \infty.$$

Thus, μ_u and $\det D^2 u$ are two finite Borel measures on S that coincide on the open subsets of S. Therefore, the equality (2.11) can be extended to every Borel set $E \subset S$ by means of the π-λ-theorem (see for instance [5, Theorem 1.5]). □

Proof of Theorem 1. First, let us just observe that Theorem 6 can be extended to $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$. Indeed, for $\delta > 0$ define

$$\Omega_\delta := \{x \in U \colon \operatorname{dist}(x, \partial\Omega) > \delta\} \tag{2.12}$$

so that $u \in W^{2,n}(\Omega_\delta)$ for every $\delta > 0$. Given a Borel set $E \subset \Omega$ consider $E_\delta := E \cap \Omega_\delta$ so that Theorem 6 with $S := \Omega_\delta$ gives

$$\int_{E_\delta} \det D^2 u(x)\, dx = |\nabla u(E_\delta)| \tag{2.13}$$

and (2.10) follows by taking limits as $\delta \searrow 0^+$. Finally, given $E \subset \Omega$ with $|E| = 0$, (2.10) yields $|\nabla u(E)| = 0$, which means $\nabla u \in N(\Omega)$. □

3. Some consequences of Theorems 1 and 6

Let us begin with a change-of-variable formula for $W^{2,n}_{\mathrm{loc}}$-mappings with strictly convex potentials.

Corollary 7. *Let $\Omega \subset \mathbf{R}^n$ be an open convex set and let $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ be a strictly convex function. Then, for every measurable set $S \subset \Omega$ and every nonnegative Borel measurable function W defined on $\mathbf{R}^n$ the change-of-variable formula holds true*

$$\int_S W(\nabla u(x)) \det D^2u(x)\, dx = \int_{\nabla u(S)} W(y)\, dy. \tag{3.14}$$

Proof. Given $F \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and a measurable set $S \subset \Omega$, the following are equivalent (see, for instance, [9, Proposition 1.1]):

(a) $|F(E)| = 0$ for every set $E \subset S$ with $|E| = 0$ (i.e., Lusin's condition (N) on S).

(b) For every measurable $S' \subset S$ the area formula

$$\int_{S'} |\det \nabla F(x)|\, dx = \int_{\mathbf{R}^n} \mathcal{N}(y, F, S')\, dy \tag{3.15}$$

holds true, where $\mathcal{N}(y, F, S') := \#\{x \in S' \colon F(x) = y\}$.

(c) The change-of-variable formula holds for F on S, that is,

$$\int_S W(F(x))|\det \nabla F(x)|\, dx = \int_{\mathbf{R}^n} W(y)\mathcal{N}(y, F, S)\, dy, \tag{3.16}$$

for every nonnegative Borel measurable function W defined on $\mathbf{R}^n$.

The fact that u is strictly convex is equivalent to ∇u being 1-1, thus $\mathcal{N}(y, \nabla u, S) = \#\{x \in S \colon \nabla u(x) = y\} = \chi_{\nabla u(S)}(y)$ and (3.14) follows from Theorem 1 and (3.16). □

Next, let us relate the notions of weak and strong solutions of the Monge–Ampère equation $\det D^2u = f$ in Ω. Let us fix an open convex set $\Omega \subset \mathbf{R}^n$ and an a.e. nonnegative $f \in L^1_{\mathrm{loc}}(\Omega)$. Recall that a convex function $u \in C(\Omega)$ is said to be a weak (i.e. Aleksandrov) solution of the Monge–Ampère equation $\det D^2u = f$ in Ω if

$$\mu_u(E) = \int_E f(x)\, dx \tag{3.17}$$

for every Borel set $E \subset \Omega$. By definition, a strong solution satisfies $\det D^2u = f$ in a.e. in Ω.

Corollary 8. *Fix an a.e. nonnegative $f \in L^1_{\mathrm{loc}}(\Omega)$. A convex function $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ is a weak (Aleksandrov) solution of the Monge–Ampère equation $\det D^2u = f$ in Ω if and only if it is a strong solution.*

Proof. Let us suppose first that u is a weak solution. From (3.17) and (2.10) it follows that

$$\int_E \det D^2u(x)\, dx = \int_E f(x)\, dx$$

for every Borel set $E \subset \Omega$. Then, from Lebesgue's differentiation theorem we obtain $\det D^2u(x) = f(x)$ for a.e. $x \in \Omega$, which means that u is a strong solution of $\det D^2u = f$ in Ω. Conversely, if u is a strong solution then $\int_E \det D^2u(x)\, dx = \int_E f(x)\, dx$ for every Borel set $E \subset \Omega$ which, along with (2.10), yields (3.17). □

Remark 9. Let us recall a correspondence between weak and viscosity solutions. If $f \in C(\Omega)$ and $f \geq 0$ in Ω, by [3, Lemma 3(a)] (see also [7, Proposition 1.3.4]), every weak solution of $\det D^2u = f$ in Ω is also a viscosity solution. On the other hand, if u is a viscosity solution of $\det D^2u = f$ in Ω with $f \in C(\overline{\Omega})$ and $f > 0$ in $\overline{\Omega}$, then u is a weak solution (see [7, Proposition 1.7.1]).

Corollary 10. *Fix an open convex set $\Omega \subset \mathbf{R}^n$ and $f \in C(\Omega)$ with $f > 0$ in Ω. If $u \in C(\Omega)$ is a strictly convex weak (Aleksandrov) solution of the Monge–Ampère equation $\det D^2u = f$ in Ω, then it is also a strong solution.*

Proof of Corollary 10. For $x_0 \in \Omega$, $q \in \partial u(x_0)$, and $t > 0$ set

$$S(x_0, q, t) := \{x \in \Omega\colon u(x) - u(x_0) - q \cdot (x - x_0) < t\}.$$

Since u is strictly convex in Ω, there exists t_0 such that $S := S(x_0, q, t_0) \subset\subset \Omega$. Introduce $v(x) := u(x) - u(x_0) - q \cdot (x - x_0) - t_0$ so that v is a weak solution of

$$\begin{cases} \det D^2v = f & \text{in } S, \\ v = 0 & \text{on } \partial S. \end{cases} \tag{3.18}$$

Since f is continuous and positive in S by Remark 9 we have that v is also a viscosity solution of (3.18). Thus, given $1 < p < \infty$, Caffarelli's $W^{2,p}$-estimate for viscosity solutions (see [3, Theorem 1(b)] or [6, Corollary 4.38]) apply, so that we have $v \in W^{2,p}(\frac{1}{2}S)$ (where $\frac{1}{2}S$ denotes the $\frac{1}{2}$-contraction of S with respect to its center of mass), since $D^2u = D^2v$ we get $u \in W^{2,p}(\frac{1}{2}S)$ and then, after a covering argument, $u \in W^{2,p}_{\text{loc}}(\Omega)$. By taking $p = n$, it follows that $u \in W^{2,n}_{\text{loc}}(\Omega)$ and then Corollary 8 guarantees that u is also a strong solution. □

Acknowledgements. The author is grateful to Professors Alessio Figalli and Jan Malý for useful conversations as well as to the referees for thoughtful comments that improved the presentation.

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Received 16 May 2019 • Accepted 22 January 2020