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MULTILINEAR FRACTIONAL INTEGRAL OPERATORS: A COUNTER-EXAMPLE

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Abstract. By means of a counter-example we show that the multilinear fractional operator $\mathcal{I}_\gamma$ $(1 < \gamma < 2)$ is not bounded from $H^1(\mathbf{R}) \times H^p(\mathbf{R})$ into $H^q(\mathbf{R})$, for $0 < p \leq \gamma^{-1}$ and $\frac{1}{q} = 1 + \frac{1}{p} - \gamma$.

1. Introduction

Given positive integers m, n and a real number $0 < \gamma < mn$, it is define the multilinear fractional operator $\mathcal{I}_\gamma$ by

$$\mathcal{I}_\gamma(f_1, \ldots, f_m)(x) = \int_{(\mathbf{R}^n)^m} \frac{f_1(y_1) \cdots f_m(y_m)}{(|x - y_1| + \cdots + |x - y_m|)^{nm-\gamma}} \, dy_1 \cdots dy_m, \quad x \in \mathbf{R}^n.$$

Lin and Lu in [3] proved Hardy space estimates for the multilinear fractional operator $\mathcal{I}_\gamma$. More precisely, they proved that if $0 < \gamma < n$, $0 < p_1, \ldots, p_m, q \leq 1$, and q such that $\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \frac{\gamma}{n} > 0$, then

$$\|\mathcal{I}_\gamma(f_1, \ldots, f_m)\|_{L^q} \leq C \|f_1\|_{H^{p_1}} \cdots \|f_m\|_{H^{p_m}}.$$

Recently, Cruz-Uribe, Moen and Nguyen in [1] generalized the result of Lin and Lu to weighted Hardy spaces on the full range $0 < \gamma < nm$.

The purpose of this note is to give a counter-example to show that the multilinear fractional operator $\mathcal{I}_\gamma$ is not bounded from a product of Hardy spaces into a Hardy space. For them, we consider $n = 1$, $m = 2$, $\gamma = \alpha + 1$ with $0 < \alpha < 1$, so $1 < \gamma < 2$, $2 - \gamma = 1 - \alpha$ and the multilinear fractional operator $\mathcal{I}_{\alpha+1}$ in this case is given by

$$\mathcal{I}_{\alpha+1}(f_1, f_2)(x) = \iint_{\mathbf{R}^2} \frac{f_1(s) f_2(t)}{(|x - s| + |x - t|)^{1-\alpha}} \, ds \, dt, \quad x \in \mathbf{R}.$$

We will prove that the operator $\mathcal{I}_{\alpha+1}$ is not bounded from $H^1(\mathbf{R}) \times H^p(\mathbf{R})$ into $H^q(\mathbf{R})$, for $0 < p \leq (\alpha + 1)^{-1}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$.

We briefly recall the definition of Hardy space on $\mathbf{R}^n$. The Hardy space $H^p(\mathbf{R}^n)$ (for $0 < p < \infty$) consists of tempered distributions $f \in \mathcal{S}'(\mathbf{R}^n)$ such that for some Schwartz function φ with $\int \varphi = 1$, the maximal operator

$$(\mathcal{M}_\varphi f)(x) = \sup_{t>0} |(\varphi_t * f)(x)|$$

is in $L^p(\mathbf{R}^n)$, where $\varphi_t(x) := \frac{1}{t^n} \varphi(\frac{x}{t})$. In this case we define $\|f\|_{H^p} := \|\mathcal{M}_\varphi f\|_p$ as the H^p "norm". It can be shown that this definition does not depend on the choice of the function φ. For $1 < p < \infty$, it is well known that $H^p(\mathbf{R}^n) \cong L^p(\mathbf{R}^n)$, $H^1(\mathbf{R}^n) \subset L^1(\mathbf{R}^n)$ strictly, and for $0 < p < 1$ the spaces $H^p(\mathbf{R}^n)$ and $L^p(\mathbf{R}^n)$ are not comparable.

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The following sentences hold in Hardy spaces $H^p(\mathbf{R}^n)$ for $0 < p \leq 1$ (see pp. 128–129 in [4]):

(S1) A bounded compactly supported function f belongs to $H^p(\mathbf{R}^n)$ if and only if it satisfies the moment conditions $\int x^\beta f(x)\, dx = 0$ for all $|\beta| \leq n(p^{-1} - 1)$.

(S2) If $f \in L^1(\mathbf{R}^n) \cap H^p(\mathbf{R}^n)$, then $\int x^\beta f(x)\, dx = 0$, whenever $|\beta| \leq n(p^{-1} - 1)$ and the function $x^\beta f(x)$ is in $L^1(\mathbf{R}^n)$.

To obtain our result we will compute explicitly in Section 2 the Fourier transform of the kernel $(|x-s| + |x-t|)^{\alpha-1}$ in the x variable, this allows us to get the following identity

$$\int_{\mathbf{R}} \mathcal{I}_{\alpha+1}(a_1, a_2)(x)\, dx = \frac{\alpha - 1}{\alpha} \iint_{\mathbf{R}^2} a_1(s)\, a_2(t)\, |t-s|^\alpha\, ds\, dt$$

valid for bounded functions a_1 and a_2 having compact support with $\int a_1 = 0$ or $\int a_2 = 0$. Then, from (S2), the counter-example will follow to consider $a_1 \in H^1(\mathbf{R})$ and $a_2 \in H^p(\mathbf{R})$ such that $\iint_{\mathbf{R}^2} a_1(s)\, a_2(t)\, |t-s|^\alpha\, ds\, dt \neq 0$.

Notation. We use the following convention for the Fourier transform in $\mathbf{R}$ $\widehat{f}(\xi) = \int f(x) e^{-ix\xi}\, dx$. As usual we denote with $\mathcal{S}(\mathbf{R})$ the Schwartz space on $\mathbf{R}$.

2. Preliminaries

We start with the following lemma.

Lemma 1. *For $0 < \alpha < 1$ and $s \neq t \in \mathbf{R}$ fixed, let $K^\alpha_{s,t}$ be the function defined in $\mathbf{R}$ by*

$$K^\alpha_{s,t}(x) = (|x-s| + |x-t|)^{\alpha-1}, \quad x \in \mathbf{R}.$$

Then

$$\widehat{K^\alpha_{s,t}}(\xi) = -2^\alpha\, \Gamma(\alpha) \sin\left(\frac{(\alpha-1)\pi}{2}\right) e^{-i\xi(\frac{s+t}{2})} |\xi|^{-\alpha} + |t-s|^{\alpha-1} \operatorname{sgn}(t-s) \int_s^t e^{-ix\xi}\, dx$$
$$- \frac{|t-s|^\alpha}{\alpha} e^{-i\frac{(s+t)}{2}\xi} \cos\left(\frac{|t-s|\xi}{2}\right) + \frac{i\, 2^\alpha\, \xi e^{-i\xi(\frac{s+t}{2})}}{\alpha} \int_0^{\frac{|t-s|}{2}} x^\alpha \sin(x\xi)\, dx,$$

in the distributional sense.

Proof. First we assume that $s < t$. Then for each $\phi \in \mathcal{S}(\mathbf{R})$ fixed, we have

$$\begin{aligned}\left(\widehat{K^\alpha_{s,t}}, \phi\right) &= \left(K^\alpha_{s,t}, \widehat{\phi}\right) = \int_{\mathbf{R}} K^\alpha_{s,t}(x) \widehat{\phi}(x)\, dx \\ &= \int_t^{+\infty} K^\alpha_{s,t}(x)\widehat{\phi}(x)\, dx + \int_s^t K^\alpha_{s,t}(x)\widehat{\phi}(x)\, dx + \int_{-\infty}^s K^\alpha_{s,t}(x)\widehat{\phi}(x)\, dx \\ &= I + II + III.\end{aligned}$$

Let us now proceed to compute each one of these integrals,

$$\begin{aligned} I &= \int_{\mathbf{R}} \chi_{(t,+\infty)}(x)\, (2x - (s+t))^{\alpha-1}\, \widehat{\phi}(x)\, dx \\ &= 2^{\alpha-1} \int_{\mathbf{R}} x^{\alpha-1} \chi_{(\frac{t-s}{2}, +\infty)}(x)\, (e^{-i(\cdot)\frac{(s+t)}{2}} \phi)\widehat{\ }(x)\, dx \\ &= 2^{\alpha-1} \int_{\mathbf{R}} x_+^{\alpha-1} (e^{-i(\cdot)\frac{(s+t)}{2}} \phi)\widehat{\ }(x)\, dx - 2^{\alpha-1} \int_{\mathbf{R}} x^{\alpha-1} \chi_{(0, \frac{t-s}{2})}(x) (e^{-i(\cdot)\frac{(s+t)}{2}} \phi)\widehat{\ }(x)\, dx \\ &= 2^{\alpha-1} \int_{\mathbf{R}} x_+^{\alpha-1} (e^{-i(\cdot)\frac{(s+t)}{2}} \phi)\widehat{\ }(x)\, dx \end{aligned}$$

$$- 2^{\alpha-1}\int_{\mathbf{R}}\left(\frac{(t-s)^\alpha e^{-i\xi\frac{(t-s)}{2}}}{2^\alpha\alpha}+\frac{i\xi}{\alpha}\int_0^{\frac{t-s}{2}} x^\alpha e^{-ix\xi}\,dx\right)e^{-i\xi\frac{(s+t)}{2}}\phi(\xi)\,d\xi,$$

to compute III we proceed as in I, thus

$$\begin{aligned}III &= 2^{\alpha-1}\int_{\mathbf{R}} x_-^{\alpha-1}(e^{-i(\cdot)\frac{(s+t)}{2}}\phi)\widehat{\ }(x)\,dx\\ &\quad - 2^{\alpha-1}\int_{\mathbf{R}}\left(\frac{(t-s)^\alpha}{2^\alpha\alpha}e^{i\xi(\frac{t-s}{2})}-\frac{i\xi}{\alpha}\int_0^{\frac{t-s}{2}} x^\alpha e^{ix\xi}\,dx\right)e^{-i\xi\frac{(s+t)}{2}}\phi(\xi)\,d\xi,\end{aligned}$$

so

$$\begin{aligned}I+III &= 2^{\alpha-1}\int_{\mathbf{R}}|x|^{\alpha-1}(e^{-i(\cdot)\frac{(s+t)}{2}}\phi)\widehat{\ }(x)\,dx\\ &\quad -\int_{\mathbf{R}}\frac{(t-s)^\alpha}{\alpha}e^{-i\frac{(s+t)}{2}\xi}\cos\left(\frac{(t-s)\xi}{2}\right)\phi(\xi)\,d\xi\\ &\quad +\int_{\mathbf{R}}\left(\frac{i\,2^\alpha\,\xi e^{-i\xi(\frac{s+t}{2})}}{\alpha}\int_0^{\frac{(t-s)}{2}} x^\alpha\sin(x\xi)\,dx\right)\phi(\xi)\,d\xi.\end{aligned}$$

Now II is easy, indeed

$$II=\int_{\mathbf{R}}\chi_{(s,t)}(x)(t-s)^{\alpha-1}\,\widehat{\phi}(x)\,dx=\int_{\mathbf{R}}\left((t-s)^{\alpha-1}\int_s^t e^{-ix\xi}\,dx\right)\phi(\xi)\,d\xi.$$

Since

$$\widehat{|x|^{\alpha-1}}(\xi)=-2\Gamma(\alpha)\sin\left(\frac{(\alpha-1)\pi}{2}\right)|\xi|^{-\alpha}$$

(see equation (12), p. 173, in [2]), the lemma follows for the case $s<t$. Finally, exchanging the roles of s and t we obtain the statement of the lemma. □

Corollary 2. *If a_1 and a_2 are two bounded functions on $\mathbf{R}$ with compact support and such that $\int a_1=0$ or $\int a_2=0$, then*

$$\int_{\mathbf{R}}\mathcal{I}_{\alpha+1}(a_1,a_2)(x)\,dx=\frac{\alpha-1}{\alpha}\iint_{\mathbf{R}^2}a_1(s)\,a_2(t)\,|t-s|^\alpha\,ds\,dt.$$

Proof. It is easy to check that $\mathcal{I}_{\alpha+1}(a_1,a_2)(\cdot)\in L^1(\mathbf{R})$. Let $\varphi\in\mathcal{S}(\mathbf{R})$ be an **even** function such that $\varphi(0)=1$ and for $\epsilon>0$ let $\varphi_\epsilon(x)=\varphi(\epsilon x)$. Since

$$\int_{\mathbf{R}}\mathcal{I}_{\alpha+1}(a_1,a_2)(x)\,dx=\lim_{\epsilon\to0^+}\int_{\mathbf{R}}\mathcal{I}_{\alpha+1}(a_1,a_2)(x)\varphi_\epsilon(x)\,dx,$$

we will proceed to compute this limit.

$$\begin{aligned}\lim_{\epsilon\to0^+}\int_{\mathbf{R}}\mathcal{I}_{\alpha+1}(a_1,a_2)(x)\varphi_\epsilon(x)\,dx &= \lim_{\epsilon\to0^+}\iint_{\mathbf{R}^2}a_1(s)a_2(t)\left(\int_{\mathbf{R}}K^\alpha_{s,t}(x)\varphi_\epsilon(x)\,dx\right)ds\,dt\\ &= \lim_{\epsilon\to0^+}\iint_{\mathbf{R}^2}a_1(s)a_2(t)\left(\int_{\mathbf{R}}\widehat{K^\alpha_{s,t}}(\xi)\widehat{\varphi_\epsilon}(\xi)\,d\xi\right)ds\,dt\\ &= \iint_{\mathbf{R}^2}a_1(s)a_2(t)\lim_{\epsilon\to0^+}\left(\int_{\mathbf{R}}\widehat{K^\alpha_{s,t}}(\epsilon\,\xi)\widehat{\varphi}(\xi)\,d\xi\right)ds\,dt\\ &= \frac{\alpha-1}{\alpha}\iint_{\mathbf{R}^2}a_1(s)\,a_2(t)\,|t-s|^\alpha\,ds\,dt,\end{aligned}$$

where the last equality follows from Lemma 1, the moment condition of a_1 (or a_2) and that $\varphi(0) = 1$. □

3. A counter-example

We take $a_1(s) = \chi_{(-1,0)}(s) - \chi_{(0,1)}(s)$ and $a_2(t) = a_1(t-2)$. From (S1) it follows that $a_1 \in H^1(\mathbf{R})$ and $a_2 \in H^{(\alpha+1)^{-1}}(\mathbf{R})$ for each $0 < \alpha < 1$. A computation gives

$$\iint a_1(s) a_2(t) |t-s|^\alpha \, ds \, dt = \frac{4 \cdot 3^{\alpha+2} - 4^{\alpha+2} - 6 \cdot 2^{\alpha+2} + 4}{(\alpha+1)(\alpha+2)} \neq 0.$$

From (S2) and corollary 2 it obtains that $\mathcal{I}_{\alpha+1}(a_1, a_2)(\cdot) \notin H^1(\mathbf{R})$, for each $0 < \alpha < 1$. For $0 < p < (\alpha+1)^{-1}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$, we take N as any fixed integer with $N > p^{-1} - 1$, then the set of all bounded, compactly supported functions for which $\int x^\beta f(x)\, dx = 0$, for all $|\beta| \leq N$ is dense in $H^r(\mathbf{R})$ for each $p \leq r \leq 1$ (see 5.2 b), pp. 128, in [4]). In particular, there exists $b \in H^p(\mathbf{R})$ such that $\|a_1\|_{H^1} \|a_2 - b\|_{H^{(\alpha+1)^{-1}}} < \left| \int_{\mathbf{R}} \mathcal{I}_{\alpha+1}(a_1, a_2)(x)\, dx \right| / 2C$. Then

$$\begin{aligned} \left| \int_{\mathbf{R}} \mathcal{I}_{\alpha+1}(a_1, b)(x)\, dx \right| &\geq \left| \int_{\mathbf{R}} \mathcal{I}_{\alpha+1}(a_1, a_2)(x)\, dx \right| - \int_{\mathbf{R}} |\mathcal{I}_{\alpha+1}(a_1, a_2 - b)(x)|\, dx \\ &\geq \left| \int_{\mathbf{R}} \mathcal{I}_{\alpha+1}(a_1, a_2)(x)\, dx \right| - C \|a_1\|_{H^1} \|a_2 - b\|_{H^{(\alpha+1)^{-1}}} \\ &> \frac{1}{2} \left| \int_{\mathbf{R}} \mathcal{I}_{\alpha+1}(a_1, a_2)(x)\, dx \right| > 0, \end{aligned}$$

where the second inequality follows from Theorem 1.1 in [1] with $p_1 = 1$, $p_2 = (\alpha+1)^{-1}$ and $q = 1$. But then the operator $\mathcal{I}_{\alpha+1}$ is not bounded from $H^1(\mathbf{R}) \times H^p(\mathbf{R})$ into $H^q(\mathbf{R})$ for $0 < p \leq (\alpha+1)^{-1}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$, since $\int_{\mathbf{R}} \mathcal{I}_{\alpha+1}(a_1, b)(x)\, dx \neq 0$.

We conclude this note by summarizing our main result in the following theorem.

Theorem 3. *For $1 < \gamma < 2$, let $\mathcal{I}_\gamma$ be the multilinear fractional integral operator given by*

$$\mathcal{I}_\gamma(f_1, f_2)(x) = \iint_{\mathbf{R}^2} \frac{f_1(s) f_2(t)}{(|x-s| + |x-t|)^{2-\gamma}}\, ds\, dt, \quad x \in \mathbf{R}.$$

Then the operator $\mathcal{I}_\gamma$ is not bounded from $H^1(\mathbf{R}) \times H^p(\mathbf{R})$ into $H^q(\mathbf{R})$ for $0 < p \leq \gamma^{-1}$ and $\frac{1}{q} = 1 + \frac{1}{p} - \gamma$.

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