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THE IDEAL OF WEAKLY p-COMPACT OPERATORS AND ITS APPROXIMATION PROPERTY FOR BANACH SPACES

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Abstract. We investigate the ideal $\mathcal{W}_p$ of weakly p-compact operators and its approximation property ($\mathcal{W}_p$-AP). We prove that

$$\mathcal{W}_p = \mathcal{W}_p \circ \mathcal{W}_p \quad \text{and} \quad \mathcal{V}_p = \mathcal{K}_{up} \circ \mathcal{W}_p^{-1}$$

and that for $1 < p \le \infty$, a Banach space X has the $\mathcal{W}_p$-AP if and only if the identity map on X is approximated by finite rank operators on X in the topology of uniform convergence on weakly p-compact sets. Also, we study the $\mathcal{W}_p$-AP for classical sequence spaces and dual spaces.

1. Introduction

The main subject of this paper originates from the classical *approximation property* (AP) and an operator ideal introduced by Sinha and Karn [SK1]. A Banach space X is said to have the AP if

$$\mathrm{id}_X \in \overline{\mathcal{F}(X,X)}^{\tau_c},$$

where id_X is the identity map on X, $\mathcal{F}$ is the ideal of finite rank operators between Banach spaces and τ_c is the topology of uniformly compact convergence on the ideal $\mathcal{L}$ of all operators between Banach spaces. Grothendieck [G] systematically investigated the AP and one of the basic tools in [G] was a criterion of classical compactness such as the following.

A subset K of a Banach space X is relatively compact if and only if for every $\varepsilon > 0$, there exists $(x_n)_n \in c_0(X)$, the space of all null sequences in X, with $\|(x_n)_n\|_\infty := \sup_n \|x_n\| \le \sup_{x\in K} \|x\| + \varepsilon$ such that

$$K \subset \left\{ \sum_{n=1}^\infty \alpha_n x_n \colon (\alpha_n)_n \in B_{\ell_1} \right\}, \tag{†}$$

where we denote by B_Z the unit ball of a Banach space Z. It follows from this result that for every $T \in \mathcal{K}(Y,X)$, where $\mathcal{K}$ is the ideal of compact operators between Banach spaces,

$$\|T\| = \inf \left\{ \|(x_n)_n\|_\infty \colon (x_n)_n \in c_0(X),\ T(B_Y) \subset \left\{ \sum_{n=1}^\infty \alpha_n x_n \colon (\alpha_n)_n \in B_{\ell_1} \right\} \right\}. \tag{††}$$

Sinha and Karn [SK1] was motivated by (†) to introduce a new compactness. Let $1 \le p \le \infty$. A subset K of X is said to be *p-compact* (respectively, *weakly*

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p-compact) if there exists $(x_n)_n \in \ell_p(X)$ (respectively, $\ell_p^w(X)$) such that

$$K \subset p\text{-}co(x_n)_n := \left\{ \sum_{n=1}^{\infty} \alpha_n x_n \colon (\alpha_n) \in B_{\ell_{p^*}} \right\},$$

where $1/p + 1/p^* = 1$ and $\ell_p(X)$ is the Banach space with the norm $\|\cdot\|_p$ of all X-valued absolutely p-summable sequences (respectively, $\ell_p^w(X)$ is the Banach space with the norm $\|\cdot\|_p^w$ of all X-valued weakly p-summable sequences). When $p=\infty$, $\ell_p(X)$ (respectively, $\ell_p^w(X)$) is replaced by $c_0(X)$ (respectively, the space $c_0^w(X)$ of all weakly null sequences in X). Also, when $p=1$, the unit ball $B_{\ell_{p^*}}$ is replaced by B_{c_0}. Note that every p-compact set is relatively compact and every weakly p-compact set $(1 < p \leq \infty)$ is relatively weakly compact.

A linear map $T\colon Y \to X$ is *p-compact* (respectively, *weakly p-compact*) if $T(B_Y)$ is a p-compact (respectively, weakly p-compact) subset of X. The collection of all p-compact (respectively, weakly p-compact) operators from Y to X is denoted by $\mathcal{K}_p(Y,X)$ (respectively, $\mathcal{W}_p(Y,X)$). We remark that the notion of weakly p-compact set (the ideal of weakly p-compact operators) was already introduced and studied by Castillo and Sanchez as an another concept (see [CS, Definition 1.3]).

In view of (††), it is natural to consider the same way to measure p-compact operators. Delgado, Piñeiro and Serrano [DPS1] introduced an operator ideal norm on $\mathcal{K}_p$ in that way. For $T \in \mathcal{K}_p(Y,X)$, let

$$\|T\|_{\mathcal{K}_p} := \inf\left\{\|(x_n)_n\|_p \colon (x_n)_n \in \ell_p(X), T(B_Y) \subset p\text{-}co(x_n)_n\right\}.$$

Then $[\mathcal{K}_p, \|\cdot\|_{\mathcal{K}_p}]$ is a Banach operator ideal [DPS1]. Ain, Lillemets and Oja [ALO] introduced and studied a more general form of the ideal $[\mathcal{K}_p, \|\cdot\|_{\mathcal{K}_p}]$. We define a norm on $\mathcal{W}_p$ as in $\mathcal{K}_p$. For $T \in \mathcal{W}_p(Y,X)$, let

$$\|T\|_{\mathcal{W}_p} := \inf\left\{\|(x_n)_n\|_p^w \colon (x_n)_n \in \ell_p^w(X), T(B_Y) \subset p\text{-}co(x_n)_n\right\}.$$

Then $[\mathcal{W}_p, \|\cdot\|_{\mathcal{W}_p}]$ is a Banach operator ideal (see Theorem 2.1).

Grothendieck [G] proved that a Banach space X has the AP if and only if

$$\mathcal{K}(Y,X) = \overline{\mathcal{F}(Y,X)}^{\|\cdot\|}$$

for every Banach space Y, where the closure is in the operator norm topology. A more general notion extending this criterion was introduced by Lassalle and Turco [LT1], and Oja [O2]. For a Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$, a Banach space X is said to have the *$\mathcal{A}$-approximation property* ($\mathcal{A}$-AP) if $\mathcal{A}(Y,X) = \overline{\mathcal{F}(Y,X)}^{\|\cdot\|_{\mathcal{A}}}$ for every Banach space Y. Therefore a Banach space X is said to have the $\mathcal{K}_p$-AP (respectively, $\mathcal{W}_p$-AP) if

$$\mathcal{K}_p(Y,X) = \overline{\mathcal{F}(Y,X)}^{\|\cdot\|_{\mathcal{K}_p}} \quad (\text{respectively}, \mathcal{W}_p(Y,X) = \overline{\mathcal{F}(Y,X)}^{\|\cdot\|_{\mathcal{W}_p}})$$

for every Banach space Y. The ideal $\mathcal{K}_p$, the $\mathcal{K}_p$-AP and their related subjects were investigated in [AO, ALO, CK, DP, DPS1, DPS2, DOPS, GLT, K1, K2, K3, K4, K6, K7, LT1, LT2, LT3, O2, P, PD, SK1, SK2] and so on. In this paper, we investigate the ideal $\mathcal{W}_p$ and the $\mathcal{W}_p$-AP as the following organization.

In Section 2, we prove that

$$\mathcal{W}_p = \mathcal{W}_p \circ \mathcal{W}_p \quad \text{and} \quad \mathcal{V}_p = \mathcal{K}_{up} \circ \mathcal{W}_p^{-1}.$$

In Section 3, we establish some characterizations of the $\mathcal{W}_p$-AP. Among them, for $1 < p \leq \infty$, a Banach space X has the $\mathcal{W}_p$-AP if and only if

$$\mathrm{id}_X \in \overline{\mathcal{F}(X,X)}^{\tau_{wp}},$$

where τ_{wp} is the *topology of uniform convergence on weakly p-compact sets.* In Section 4, we check whether the classical sequence spaces have the $\mathcal{W}_p$-AP. As a consequence, it is shown that the AP does not imply the $\mathcal{W}_p$-AP and the $\mathcal{W}_p$-AP $(1 < p < 2)$ does not imply the AP in general. Also, we study the $\mathcal{W}_p$-AP for dual spaces. As a consequence, it is shown that for $1 < p < \infty$, the dual space of a Banach space X has the $\mathcal{W}_p$-AP if and only if for every Banach space Y, $\mathcal{F}(X,Y)$ is dense in the space of quasi weakly p-nuclear operators from X to Y.

2. The ideal of weakly p-compact operators

First, we need to show the following for the sake of the completeness of presentation.

Theorem 2.1. *For every $1 \leq p \leq \infty$, $[\mathcal{W}_p, \|\cdot\|_{\mathcal{W}_p}]$ is a Banach operator ideal.*

Lemma 2.2. [K5, Corollary 3.6] *Let $1 \leq p < \infty$ and let $T\colon X \to Y$ be a linear map.*

(a) *If $(y_n)_n \in \ell_p^w(Y)$, then $T(B_X) \subset p\text{-}co(y_n)_n$ if and only if $\|T^*y^*\| \leq \|(y^*(y_n))_n\|_p$ for every $y^* \in Y^*$.*
(b) *If $(y_n)_n \in c_0^w(Y)$, then $T(B_X) \subset \infty\text{-}co(y_n)_n$ if and only if $\|T^*y^*\| \leq \|(y^*(y_n))_n\|_\infty$ for every $y^* \in Y^*$.*

Proof of Theorem 2.1. Let X and Y be Banach spaces. We only show the linearity of $\mathcal{W}_p(X,Y)$, the triangle inequality and completeness of $\|\cdot\|_{\mathcal{W}_p}$. The other conditions for an operator ideal are clear. Let $(T_k)_k$ be a sequence in $\mathcal{W}_p(X,Y)$ with $\sum_k \|T_k\|_{\mathcal{W}_p} < \infty$. Then $\sum_{k=1}^\infty \|T_k\| < \infty$ and so there exists a $T \in \mathcal{L}(X,Y)$ such that $\|\sum_{k=1}^l T_k - T\| \longrightarrow 0$ as $l \to \infty$.

Let $\varepsilon > 0$ be given. For each $k \in \mathbf{N}$, let $(y_n^k)_n \in \ell_p^w(Y)$ be such that

$$T_k(B_X) \subset p\text{-}co(y_n^k)_n \quad \text{and} \quad \|(y_n^k)_n\|_p^w \leq \|T_k\|_{\mathcal{W}_p} + \frac{\varepsilon}{2^k}.$$

In the case $p = \infty$, let $(y_n^k)_n \in c_0^w(Y)$ and $\|(y_n^k)_n\|_\infty \leq \|T_k\|_{\mathcal{W}_\infty} + \varepsilon/2^k$.

For each $k, n \in \mathbf{N}$, let

$$z_n^k := \frac{y_n^k}{(\|T_k\|_{\mathcal{W}_p} + \varepsilon/2^k)^{1/p^*}} \in Y.$$

In the case $p = \infty$, let

$$z_n^k := \frac{y_n^k}{\beta_k(\|T_k\|_{\mathcal{W}_\infty} + \varepsilon/2^k)},$$

where $\beta_k > 1, \lim_{k\to\infty} \beta_k = \infty$ and

$$\sum_{k=1}^\infty \beta_k(\|T_k\|_{\mathcal{W}_\infty} + \varepsilon/2^k) \leq (1+\varepsilon)\sum_{k=1}^\infty (\|T_k\|_{\mathcal{W}_\infty} + \varepsilon/2^k).$$

The sequence $(z_m)_m$ in Y is defined as the following array:

$$\begin{array}{cccccccc}
z_1^1 & \rightarrow & z_2^1 & & z_3^1 & \cdots & z_n^1 & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & \\
z_1^2 & \leftarrow & z_2^2 & & z_3^2 & \cdots & z_n^2 & \cdots \\
 & & & & \downarrow & & \downarrow & \\
z_1^3 & \leftarrow & z_2^3 & \leftarrow & z_3^3 & \cdots & z_n^3 & \cdots \\
 & & & & & & \downarrow & \\
 & & & \vdots & & \vdots & & \cdots \\
 & & & & & & & \\
z_1^n & \leftarrow & z_2^n & \leftarrow & \cdots & \leftarrow & z_n^n & \cdots \\
 & & & \vdots & & \vdots & & \vdots
\end{array}$$

Then

$$\sup_{y^* \in B_{Y^*}} \sum_{m=1}^{\infty} |y^*(z_m)|^p \leq \sum_{k=1}^{\infty} \frac{(\|(y_n^k)_n\|_p^w)^p}{(\|T_k\|_{\mathcal{W}_p} + \varepsilon/2^k)^{p/p^*}} \leq \sum_{k=1}^{\infty} \Big(\|T_k\|_{\mathcal{W}_p} + \frac{\varepsilon}{2^k}\Big).$$

Thus

$$\|(z_m)_m\|_p^w \leq \left(\sum_{k=1}^{\infty} \|T_k\|_{\mathcal{W}_p} + \varepsilon\right)^{1/p}.$$

In the case $p = \infty$, we see that $(z_m)_m \in c_0^w(Y)$ and $\|(z_m)_m\|_\infty \leq 1$.

Now let $y^* \in Y^*$. Then, for each $k \in \mathbf{N}$, since $T_k(B_X) \subset p\text{-}co(y_n^k)_n$, by Lemma 2.2 $\|T_k^* y^*\| \leq \|(y^*(y_n^k))_n\|_p$. Then we have

$$\begin{aligned}
\|T^* y^*\| &\leq \sum_{k=1}^{\infty} \|T_k^* y^*\| \leq \sum_{k=1}^{\infty} \|(y^*(y_n^k))_n\|_p = \sum_{k=1}^{\infty} (\|T_k\|_{\mathcal{W}_p} + \varepsilon/2^k)^{1/p^*} \|(y^*(z_n^k))_n\|_p \\
&\leq \left(\sum_{k=1}^{\infty} \|T_k\|_{\mathcal{W}_p} + \varepsilon\right)^{1/p^*} \|(y^*(z_m))_m\|_p.
\end{aligned}$$

In the case $p = \infty$, we see that

$$\|T^* y^*\| \leq (1 + \varepsilon) \left(\sum_{k=1}^{\infty} \|T_k\|_{\mathcal{W}_\infty} + \varepsilon\right) \|(y^*(z_m))_m\|_\infty.$$

For each $m \in \mathbf{N}$, let

$$w_m := \left(\sum_{k=1}^{\infty} \|T_k\|_{\mathcal{W}_p} + \varepsilon\right)^{1/p^*} z_m.$$

In the case $p = \infty$, let

$$w_m := (1 + \varepsilon) \left(\sum_{k=1}^{\infty} \|T_k\|_{\mathcal{W}_\infty} + \varepsilon\right) z_m.$$

Then since $(w_m)_m \in \ell_p^w(Y)$ or $(w_m)_m \in c_0^w(Y)$ for the case $p = \infty$, by Lemma 2.2,

$$T(B_X) \subset p\text{-}co(w_m)_m.$$

Therefore $T \in \mathcal{W}_p(X, Y)$ and

$$\|T\|_{\mathcal{W}_p} \leq \|(w_m)_m\|_p^w \leq (1 + \varepsilon) \left(\sum_{k=1}^{\infty} \|T_k\|_{\mathcal{W}_p} + \varepsilon\right).$$

Since $\varepsilon > 0$ was arbitrary, $\|T\|_{\mathcal{W}_p} \leq \sum_{k=1}^{\infty} \|T_k\|_{\mathcal{W}_p}$.

The above proof can be applied to show that for every $l \in \mathbf{N}$, $\sum_{k>l} T_k \in \mathcal{W}_p(X,Y)$ and $\|\sum_{k>l} T_k\|_{\mathcal{W}_p} \leq \sum_{k>l} \|T_k\|_{\mathcal{W}_p}$. Hence

$$\Big\| \sum_{k=1}^{l} T_k - T \Big\|_{\mathcal{W}_p} \leq \sum_{k>l} \|T_k\|_{\mathcal{W}_p} \longrightarrow 0$$

as $l \to \infty$. □

Remark 2.3. Sinha and Karn [SK1, SK2] studied factorizations of p-compact and weakly p-compact operators, and defined the Banach operator ideal norms $\kappa_p(\cdot)$ and $\omega_p(\cdot)$, respectively, on $\mathcal{K}_p$ and $\mathcal{W}_p$, using those factorizations. Delgado, Piñeiro and Serrano [DPS1, Proposition 3.15] showed that $\kappa_p(\cdot) = \|\cdot\|_{\mathcal{K}_p}$. We can also show that $\omega_p(\cdot) = \|\cdot\|_{\mathcal{W}_p}$ using their proof.

Let $1 \leq p \leq \infty$ and let X and Y be Banach spaces. For $\hat{y} := (y_n)_n \in \ell_p^w(Y)$ ($(y_n)_n \in c_0^w(Y)$ when $p = \infty$), define the map $E_{\hat{y}} \colon \ell_{p^*} \to Y$ by

$$E_{\hat{y}}(\alpha_n)_n = \sum_{n=1}^{\infty} \alpha_n y_n.$$

Here ℓ_{p^*} is replaced by c_0 when $p = 1$. For an operator $T \colon X \to Y$, the injective operator $T_{inj} \colon X/\ker(T) \to Y$ is defined by

$$T_{inj}[x] = Tx.$$

The following result is essentially due to [SK1, Theorem 3.1].

Proposition 2.4. *Let $1 \leq p < \infty$ and let X and Y be Banach spaces. Let $T \colon X \to Y$ be a linear map. Then $T \in \mathcal{W}_p(X,Y)$ if and only if there exist a quotient space Z of ℓ_{p^*} (Z is a quotient subspace of c_0 when $p = 1$), $R \in \mathcal{W}_p(X,Z)$ and injective $S \in \mathcal{W}_p(Z,Y)$ such that $T = SR$. In this case, $\|T\|_{\mathcal{W}_p} = \inf \|S\|_{\mathcal{W}_p} \|R\|_{\mathcal{W}_p}$, where the infimum is taken over all such factorizations.*

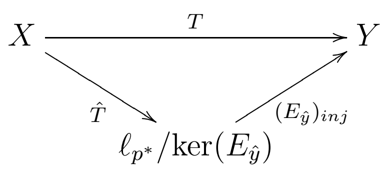
Proof. The "if" part is clear and, in this case, $\|T\|_{\mathcal{W}_p} \leq \inf \|\cdot\|_{\mathcal{W}_p} \|\cdot\|_{\mathcal{W}_p}$.

Let $T \in \mathcal{W}_p(X,Y)$ and let $\varepsilon > 0$ be given. Choose $(y_n)_n \in \ell_p^w(Y)$ such that $T(B_X) \subset p\text{-}co(y_n)_n$ and $\|(y_n)_n\|_p^w \leq (1+\varepsilon)\|T\|_{\mathcal{W}_p}$. Then we see that the maps $E_{\hat{y}} \colon \ell_{p^*} \to Y$ and $(E_{\hat{y}})_{inj} \colon \ell_{p^*}/\ker(E_{\hat{y}}) \to Y$ are weakly p-compact and $\|(E_{\hat{y}})_{inj}\|_{\mathcal{W}_p} \leq \|(y_n)_n\|_p^w$.

Now, for each $x \in X$, there exists $(\alpha_n)_n \in \ell_{p^*}$ such that $Tx = \sum_{n=1}^{\infty} \alpha_n y_n$. Define the map

$$\hat{T} \colon X \to \ell_{p^*}/\ker(E_{\hat{y}}) \quad \text{by} \quad \hat{T}x = [(\alpha_n)_n].$$

Then $\hat{T}$ is well defined and linear, and we have the following commutative diagram.



Consider the sequence $([e_n])_n$ in $\ell_{p^*}/\ker(E_{\hat{y}})$, where each e_n is the n-th standard unit vector in ℓ_{p^*}. Then a simple verification shows that $([e_n])_n \in \ell_p^w(\ell_{p^*}/\ker(E_{\hat{y}}))$ and

$$\hat{T}(B_X) \subset p\text{-}co([e_n])_n.$$

Thus $\hat{T}$ is weakly p-compact and $\|\hat{T}\|_{\mathcal{W}_p} \leq 1$. Consequently,

$$\inf \|\cdot\|_{\mathcal{W}_p}\|\cdot\|_{\mathcal{W}_p} \leq \|(y_n)_n\|_p^w \leq (1+\varepsilon)\|T\|_{\mathcal{W}_p}.$$

Since $\varepsilon > 0$ was arbitrary, we complete the proof. □

Recall the composition operator ideal $[\mathcal{A}\circ\mathcal{B}, \|\cdot\|_{\mathcal{A}\circ\mathcal{B}}]$ of operator ideals $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ and $[\mathcal{B}, \|\cdot\|_{\mathcal{B}}]$ (cf. [DF, Section 9.10]). Then by Proposition 2.4, we have:

Corollary 2.5. *Let $1 \leq p < \infty$. Then $[\mathcal{W}_p, \|\cdot\|_{\mathcal{W}_p}] = [\mathcal{W}_p \circ \mathcal{W}_p, \|\cdot\|_{\mathcal{W}_p\circ\mathcal{W}_p}]$.*

Remark 2.6. For the case $p = \infty$, the proof of Propsition 2.4 is invalid. Indeed, if the map

$$\hat{T}\colon X \to \ell_1/\ker(E_{\hat{y}})$$

in the proof of Proposition 2.4 is a weakly ∞-compact operator, then it is a weakly compact operator. Hence the map $\hat{T}$ is a compact operator because $\ell_1/\ker(E_{\hat{y}})$ has the Schur property (see, e.g., the proof of [JLO, Theorem 1.1]). Consequently, every weakly ∞-compact operator would be a compact operator. This is a contradiction.

We need a space of other vector valued sequences to introduce a stronger notion of the weakly p-compact operator. For $1 \leq p \leq \infty$, the closed subspace $\ell_p^u(X)$ of $\ell_p^w(X)$ consists of all sequences $(x_n)_n$ in a Banach space X satisfying that

$$\lim_{m\to\infty} \sup_{x^*\in B_{X^*}} \sum_{n\geq m} |x^*(x_n)|^p = 0$$

(cf. [DF, Section 8.2]). The sequence was called the *unconditionally p-summable sequence* in [K1]. Note that $\ell_\infty^u(X) = c_0(X)$. It is well known that for a sequence $(x_n)_n$ in X, $(x_n)_n$ is unconditionally 1-summable if and only if $(x_n)_n$ is unconditionally summable (cf. [DJT, Theorem 1.9]). The ideal $[\mathcal{K}_{up}, \|\cdot\|_{\mathcal{K}_{up}}]$ of *unconditionally p-compact operators* was defined in [K1] by replacing $\ell_p(X)$ with $\ell_p^u(X)$ in the definition of the ideal of p-compact operators.

Let $\mathcal{V}$ be the ideal of completely continuous operators which take weakly null sequences to null sequences. For $1 \leq p < \infty$, let $\mathcal{V}_p$ be the ideal of operators which take weakly p-summable sequences to unconditionally p-summable sequences.

Recall that the *right-hand quotient* $\mathcal{A}\circ\mathcal{B}^{-1}$ of operator ideals $\mathcal{A}$ and $\mathcal{B}$ is the operator ideal that consists of all $T \in \mathcal{L}(X,Y)$ such that $TS \in \mathcal{A}(Z,Y)$ for every Banach space Z and every $S \in \mathcal{B}(Z,X)$. It was shown in [JLO, Theorem 1.1] that

$$\mathcal{V} = \mathcal{W}_\infty \circ \mathcal{W}^{-1},$$

where $\mathcal{W}$ is the ideal of weakly compact operators.

Lemma 2.7. *Let $1 \leq p \leq \infty$ and let $(x_n)_n \in \ell_p^w(X)$ ($(x_n)_n \in c_0^w(X)$ when $p = \infty$). The operator $E_{\hat{x}}\colon \ell_{p^*} \to X$ is compact if and only if $(x_n)_n \in \ell_p^u(X)$.*

Proof. We see that the adjoint operator $E_{\hat{x}}^*\colon X^* \to \ell_p$ (ℓ_p is replaced by c_0 when $p = \infty$) is defined by

$$E_{\hat{x}}^* x^* = (x^*(x_n))_n.$$

It is well known that the subset $\{(x^*(x_n))_n\colon x^* \in B_{X^*}\}$ of ℓ_p is relatively compact if and only if $(x_n)_n \in \ell_p^u(X)$ (cf. [D, Exercises I.6 and II.6(i)]). Hence the conclusion follows. □

Theorem 2.8. *For $1 \leq p < \infty$,*

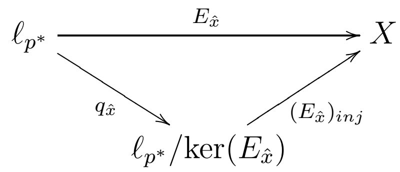
$$\mathcal{V}_p = \mathcal{K}_{up} \circ \mathcal{W}_p^{-1}.$$

Proof. Let X and Y be Banach spaces. By definitions, it is clear that

$$\mathcal{V}_p(X,Y) \subset \mathcal{K}_{up} \circ \mathcal{W}_p^{-1}(X,Y).$$

In order to show the other part, let $T \in \mathcal{K}_{up} \circ \mathcal{W}_p^{-1}(X,Y)$. Suppose that $T \notin \mathcal{V}_p(X,Y)$. Then there exists $(x_n)_n \in \ell_p^w(X)$ such that $(Tx_n)_n \notin \ell_p^u(Y)$.

Consider the following commutative diagram, where $q_{\hat{x}} \colon \ell_{p^*} \to \ell_{p^*}/\ker(E_{\hat{x}})$ is the quotient operator.



Note that $(E_{\hat{x}})_{inj}$ is a weakly p-compact operator. Since $T \in \mathcal{K}_{up} \circ \mathcal{W}_p^{-1}(X,Y)$, $T(E_{\hat{x}})_{inj} \in \mathcal{K}_{up}(\ell_{p^*}/\ker(E_{\hat{x}}), Y)$. Thus there exists $(y_n)_n \in \ell_p^u(Y)$ such that

$$T(E_{\hat{x}})_{inj}(B_{\ell_{p^*}/\ker(E_{\hat{x}})}) \subset p\text{-}co(y_n)_n.$$

Then we have

$$E_{\widehat{Tx}}(B_{\ell_{p^*}}) = TE_{\hat{x}}(B_{\ell_{p^*}}) = T(E_{\hat{x}})_{inj} q_{\hat{x}}(B_{\ell_{p^*}}) \subset p\text{-}co(y_n)_n.$$

Hence by Lemma 2.7, $(Tx_n)_n \in \ell_p^u(Y)$. This is a contradiction. □

Remark 2.9. The proof of Theorem 2.8 can be also applied to show that

$$\mathcal{V} = \mathcal{K} \circ \mathcal{W}_\infty^{-1}.$$

3. Characterizations of the $\mathcal{W}_p$-approximation property

Let $1 \le p \le \infty$. For Banach spaces X and Y, let τ_{wp} be the locally convex topology on $\mathcal{L}(X,Y)$ of *uniform convergence on weakly p-compact sets*, which is given by the seminorms

$$p_K(T) = \sup_{x \in K} \|Tx\|,$$

where K ranges over all weakly p-compact subsets of X. By definition of the weakly p-compact set, we see that the topology τ_{wp} is given by the seminorms

$$p_{\hat{x}}(T) = \|(Tx_n)_n\|_p^w,$$

where $(x_n)_n \in \ell_p^w(X)$. Then for a net $(T_\alpha)_\alpha$ in $\mathcal{L}(X,Y)$, $\lim_\alpha T_\alpha = 0$ in $(\mathcal{L}(X,Y), \tau_{wp})$ if and only if

$$\lim_\alpha \|(T_\alpha x_n)_n\|_p^w = 0$$

for every $(x_n)_n \in \ell_p^w(X)$.

First, we apply Proposition 2.4 to a characterization of the $\mathcal{W}_p$-AP.

Proposition 3.1. *Let $1 \le p < \infty$. A Banach space X has the $\mathcal{W}_p$-AP if and only if for every quotient space Z of ℓ_{p^*} (Z is a quotient space of c_0 when $p = 1$) and every injective $R \in \mathcal{W}_p(Z,X)$,*

$$R \in \overline{\mathcal{F}(Z,X)}^{\tau_{wp}}.$$

Proof. We only need to show the "if" part. Let Y be a Banach space and let $T \in \mathcal{W}_p(Y,X)$. Let $\varepsilon > 0$ be given. By Proposition 2.4, there exist a Banach space W, a quotient space Z of ℓ_{p^*}, $R_1 \in \mathcal{W}_p(Y,W)$, $R_2 \in \mathcal{W}_p(W,Z)$ and injective $R \in \mathcal{W}_p(Z,X)$ such that the following diagram is commutative.

$$\begin{array}{ccc} Y & \xrightarrow{T} & X \\ {\scriptstyle R_1}\downarrow & & \uparrow{\scriptstyle R} \\ W & \xrightarrow[R_2]{} & Z \end{array}$$

Then by our assumption, there exists an $S \in \mathcal{F}(Z,X)$ such that

$$\varepsilon \geq \|R_1\|_{\mathcal{W}_p} \sup_{z \in R_2(B_W)} \|Rz - Sz\| = \|R_1\|_{\mathcal{W}_p} \|RR_2 - SR_2\| \geq \|T - SR_2R_1\|_{\mathcal{W}_p}.$$

Since $SR_2R_1 \in \mathcal{F}(Y,X)$, we complete the proof. $\square$

We now obtain a similar characterization with the AP for the $\mathcal{W}_p$-AP.

Theorem 3.2. *Let $1 < p < \infty$. The following statements are equivalent for a Banach space X.*

(a) *X has the $\mathcal{W}_p$-AP.*

(b) *For every quotient space Z of ℓ_{p^*} and every injective $R \in \mathcal{W}_p(Z,X)$, $R \in \overline{\mathcal{F}(Z,X)}^{\|\cdot\|}$.*

(c) $\mathrm{id}_X \in \overline{\mathcal{F}(X,X)}^{\tau_{wp}}$.

(d) *For every $(x_n)_n \in \ell_p^w(X)$, $E_{\hat{x}} \in \overline{\{SE_{\hat{x}} \colon S \in \mathcal{F}(X,X)\}}^{\|\cdot\|_{\mathcal{W}_p}}$.*

(e) *For every Banach space Y and every $R \in \mathcal{W}_p(Y,X)$, $R \in \overline{\{SR \colon S \in \mathcal{F}(X,X)\}}^{\|\cdot\|_{\mathcal{W}_p}}$.*

Proof. We show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) and (e)$\Rightarrow$(a) are trivial.

(b) $\Rightarrow$ (c): Let $(x_n)_n \in \ell_p^w(X)$ and let $\varepsilon > 0$ be given. Consider the maps

$$E_{\hat{x}} \colon \ell_{p^*} \to X \quad \text{and} \quad (E_{\hat{x}})_{inj} \colon \ell_{p^*}/\ker(E_{\hat{x}}) \to X.$$

Then by (b), there exists an $S \in \mathcal{F}(\ell_{p^*}/\ker(E_{\hat{x}}), X)$ such that

$$\|S - (E_{\hat{x}})_{inj}\| \leq \frac{\varepsilon}{2}.$$

We may write $S = \sum_{k=1}^m y_k^* \otimes x_k$, where $y_k^* \in (\ell_{p^*}/\ker(E_{\hat{x}}))^*$, $x_k \in X$ for each $k = 1, \ldots, m$ and $\sum_{k=1}^m \|x_k\| = 1$. Since $(E_{\hat{x}})_{inj}$ is injective and $\ell_{p^*}/\ker(E_{\hat{x}})$ is reflexive, ${(E_{\hat{x}})_{inj}}^{**}$ is injective. Thus $(\ell_{p^*}/\ker(E_{\hat{x}}))^* = \overline{(E_{\hat{x}})_{inj}^*(X^*)}$. Thus for each $k = 1, \ldots, m$, there exists an $x_k^* \in X^*$ such that

$$\|y_k^* - (E_{\hat{x}})_{inj}^*(x_k^*)\| \leq \frac{\varepsilon}{2}.$$

Consider the operator $\sum_{k=1}^m x_k^* \otimes x_k \in \mathcal{F}(X,X)$. Then for every $(\alpha_n) \in B_{\ell_{p^*}}$, we have

$$\begin{aligned} \Big\| \sum_{k=1}^m x_k^* \Big(\sum_{n=1}^\infty \alpha_n x_n \Big) x_k - \sum_{n=1}^\infty \alpha_n x_n \Big\| &= \Big\| \sum_{k=1}^m x_k^*(E_{\hat{x}}(\alpha_n)_n) x_k - E_{\hat{x}}(\alpha_n)_n \Big\| \\ &= \Big\| \sum_{k=1}^m x_k^*((E_{\hat{x}})_{inj}[(\alpha_n)_n]) x_k - (E_{\hat{x}})_{inj}[(\alpha_n)_n]) \Big\| \end{aligned}$$

$$
\begin{aligned}
&= \Big\| \sum_{k=1}^{m} ((E_{\hat{x}})^*_{inj} x_k^*)([(\alpha_n)_n]) x_k - (E_{\hat{x}})_{inj}[(\alpha_n)_n]) \Big\| \\
&\le \Big\| \sum_{k=1}^{m} ((E_{\hat{x}})^*_{inj} x_k^*)([(\alpha_n)_n]) x_k - \sum_{k=1}^{m} y_k^*([(\alpha_n)_n]) x_k \Big\| \\
&\quad + \Big\| \sum_{k=1}^{m} y_k^*([(\alpha_n)_n]) x_k - (E_{\hat{x}})_{inj}[(\alpha_n)_n]) \Big\| \le \varepsilon.
\end{aligned}
$$

Hence $\mathrm{id}_X \in \overline{\mathcal{F}(X,X)}^{\tau_{wp}}$.

(c) $\Rightarrow$ (d): Let $(x_n) \in \ell_p^w(X)$ and let $\varepsilon > 0$ be given. Then by (c), there exists an $S \in \mathcal{F}(X,X)$ such that

$$\|((S - \mathrm{id}_X)x_n)_n\|_p^w \le \varepsilon.$$

Since $(SE_{\hat{x}} - E_{\hat{x}})(B_{\ell_{p^*}}) = p\text{-}co((S - \mathrm{id}_X)x_n)_n$ and $((S - \mathrm{id}_X)x_n)_n \in \ell_p^w(X)$, we have

$$\|SE_{\hat{x}} - E_{\hat{x}}\|_{\mathcal{W}_p} \le \|((S - \mathrm{id}_X)x_n)_n\|_p^w \le \varepsilon.$$

(d) $\Rightarrow$ (e): Let Y be a Banach space and let $R \in \mathcal{W}_p(Y,X)$. Let $\varepsilon > 0$ be given. Then there exists $(x_n)_n \in \ell_p^w(X)$ such that $R(B_Y) \subset p\text{-}co(x_n)_n$. By (d), there exists an $S \in \mathcal{F}(X,X)$ such that

$$\|SE_{\hat{x}} - E_{\hat{x}}\|_{\mathcal{W}_p} \le \varepsilon/2.$$

Now, let $(z_n)_n \in \ell_p^w(X)$ be such that $(SE_{\hat{x}} - E_{\hat{x}})(B_{\ell_{p^*}}) \subset p\text{-}co(z_n)_n$ and $\|(z_n)_n\|_p^w \le \|SE_{\hat{x}} - E_{\hat{x}}\|_{\mathcal{W}_p} + \varepsilon/2$. Since $(SR - R)(B_Y) \subset p\text{-}co((S - \mathrm{id}_X)x_n)_n = (SE_{\hat{x}} - E_{\hat{x}})(B_{\ell_{p^*}})$, we have

$$\|SR - R\|_{\mathcal{W}_p} \le \|(z_n)_n\|_p^w \le \|SE_{\hat{x}} - E_{\hat{x}}\|_{\mathcal{W}_p} + \varepsilon/2 \le \varepsilon.$$

Hence $R \in \overline{\{SR \colon S \in \mathcal{F}(X,X)\}}^{\|\cdot\|_{\mathcal{W}_p}}$. □

Remark 3.3. In Theorem 3.2, (c), (d) and (e) are also equivalent for the case $p = 1$ and $p = \infty$.

Lima, Nygaard, and Oja [LNO] proved that if K is a balanced convex and weakly compact set in the unit ball B_X of a Banach space X, then there exists a linear subspace Z of X, equipped with a different norm which makes it a reflexive Banach space, such that the formal identity map $J_Z \colon Z \longrightarrow X$ is a weakly compact operator and $K \subset B_Z \subset B_X$. Moreover, Oja showed

Lemma 3.4. [O1, Corollary 4.3]

$$\mathcal{F}(Z,X) \subset \overline{\{SJ_Z \colon S \in \mathcal{F}(X,X)\}}^{\|\cdot\|}.$$

We denote by τ_{wc} the *topology of uniform convergence on weakly compact sets* on $\mathcal{L}$.

Theorem 3.5. *The following statements are equivalent for a Banach space X.*

(a) *X has the $\mathcal{W}_\infty$-AP.*
(b) *X has the AP and Schur's property.*
(c) *X has the $\mathcal{W}$-AP.*
(d) $\mathrm{id}_X \in \overline{\mathcal{F}(X,X)}^{\tau_{wc}}$.
(e) $\mathrm{id}_X \in \overline{\mathcal{F}(X,X)}^{\tau_{w\infty}}$.

Proof. We show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a).

(d) $\Rightarrow$ (e) is trivial and (e) $\Rightarrow$ (a) follows from Remark 3.3.

(a) $\Rightarrow$ (b): If X has the $\mathcal{W}_\infty$-AP, then clearly it has the AP.

A Banach space Z has the Schur property if (and only if)

$$\mathcal{W}_\infty(Y,Z) \subset \mathcal{K}(Y,Z)$$

for every Banach space Y. Indeed, if $(z_n)_n \in c_0^w(Z)$, then the operator

$$E_{\hat{z}} \in \mathcal{W}_\infty(\ell_1, Z).$$

If $\mathcal{W}_\infty(\ell_1,Z) \subset \mathcal{K}(\ell_1,Z)$, then we see that $\{z_n\}_{n=1}^\infty$ is a relatively compact subset of Z. Therefore $(z_n)_n \in c_0(Z)$.

By (a), X has the Schur property.

(b) $\Rightarrow$ (c): By (b), for every Banach space Y,

$$\mathcal{W}(Y,X) = \mathcal{K}(Y,X) = \overline{\mathcal{F}(Y,X)}^{\|\cdot\|}.$$

(c) $\Rightarrow$ (d): Let K be a weakly compact subset of X and let $\varepsilon > 0$ be given. We may assume that K is a balanced convex and weakly compact subset of B_X. Then by (c) and Lemma 3.4,

$$J_Z \in \overline{\mathcal{F}(Z,X)}^{\|\cdot\|} = \overline{\{SJ_Z \colon S \in \mathcal{F}(X,X)\}}^{\|\cdot\|}.$$

Hence there exists an $S \in \mathcal{F}(X,X)$ such that

$$\varepsilon \geq \|J_Z - SJ_Z\| \geq \sup_{x\in K} \|J_Z x - SJ_Z x\| = \sup_{x\in K} \|x - Sx\|. \qquad \square$$

In the present paper, we do not know whether the $\mathcal{W}_1$-AP for a Banach space X is equivalent to that $\mathrm{id}_X \in \overline{\mathcal{F}(X,X)}^{\tau_{w1}}$.

4. The $\mathcal{W}_p$-approximation property for the spaces ℓ_p, c_0, ℓ_∞, and dual spaces

In [K2], the $\mathcal{K}_{up}$-AP was investigated and it was shown that if a Banach space X has the AP, then X has the $\mathcal{K}_{up}$-AP for every $1 \leq p < \infty$.

Example 4.1. (The $\mathcal{W}_1$-AP) It follows from a result of Bessaga and Pelczyński that a Banach space X does not contain an isomorphic copy of c_0 if and only if $\ell_1^w(X) = \ell_1^u(X)$ (cf. [M, Theorem 4.3.12]). Then for those Banach spaces X, $\mathcal{W}_1(Y,X)$ is isometrically equal to $\mathcal{K}_{u1}(Y,X)$ for every Banach space Y. Consequently, for $1 \leq p < \infty$, since ℓ_p has the AP, it has the $\mathcal{W}_1$-AP.

On the other hand, $\mathcal{W}_1(c_0,X) = \mathcal{L}(c_0,X)$ for every Banach space X. Indeed, if $T \in \mathcal{L}(c_0,X)$, then

$$T = \sum_{n=1}^\infty e_n^* \otimes Te_n,$$

where each e_n and e_n^*, respectively, are the standard unit vectors in c_0 and ℓ_1, respectively. Since $(Te_n)_n \in \ell_1^w(X)$, we see that $T \in \mathcal{W}_1(c_0,X)$. Then c_0 and ℓ_∞ do not have the $\mathcal{W}_1$-AP because the inclusion map from c_0 to c_0 (respectively, ℓ_∞) is not compact.

Example 4.2. (The $\mathcal{W}_p$-AP $(1 < p < \infty)$) Let $1 < p < \infty$ be fixed. It is known that $1 \leq q < p^*$ if and only if $\ell_p^w(\ell_q) = \ell_p^u(\ell_q)$ (cf. [DF, Ex. 8.4(b)]). Thus for $1 \leq q < p^*$, $\mathcal{W}_p(Y,\ell_q)$ is isometrically equal to $\mathcal{K}_{up}(Y,\ell_q)$ for every Banach space Y. It follows that ℓ_q has the $\mathcal{W}_p$-AP for $1 \leq q < p^*$.

On the other hand, as in Example 4.1, we see that $\mathcal{W}_p(\ell_{p^*}, X) = \mathcal{L}(\ell_{p^*}, X)$ for every Banach space X. Then ℓ_q $(q \geq p^*)$, c_0 and ℓ_∞ do not have the $\mathcal{W}_p$-AP because the inclusion map from ℓ_{p^*} to ℓ_q (respectively, c_0 and ℓ_∞) is not compact.

Example 4.3. (The $\mathcal{W}_\infty$-AP) From Theorem 3.5(a) $\Leftrightarrow$ (b), ℓ_1 has the $\mathcal{W}_\infty$-AP but ℓ_p $(1 < p < \infty)$, c_0 and ℓ_∞ do not have the $\mathcal{W}_\infty$-AP.

Example 4.4. In view of the above examples, the AP does not imply the $\mathcal{W}_p$-AP in general. Now, let $1 < p < 2$ be fixed and let S_p be Szankowski's subspace [S] of ℓ_p, which fails to have the AP. Since $1 < p < 2 < p^*$, $\ell_p^w(\ell_p) = \ell_p^u(\ell_p)$. In particular, $\ell_p^w(S_p) = \ell_p^u(S_p)$. Thus $\mathcal{W}_p(Y, S_p)$ is isometrically equal to $\mathcal{K}_{up}(Y, S_p)$ for every Banach space Y. In [K4, Section 5], it was observed that S_p has the $\mathcal{K}_{up}$-AP, hence it has the $\mathcal{W}_p$-AP. Also, the $\mathcal{W}_\infty$-AP implies the AP. In the present paper, for $p = 1$ or $2 \leq p < \infty$, we do not know whether the $\mathcal{W}_p$-AP implies the AP.

We now consider the $\mathcal{W}_p$-AP for dual spaces. In [K5], a weaker notion of the p-nuclear operator was introduced (see, e.g., [DJT, p. 111] for the p-nuclear operator). For $1 \leq p \leq \infty$, we say that an operator $T\colon X \to Y$ is *weakly p-nuclear* if it is represented as

$$T = \sum_{n=1}^{\infty} x_n^* \otimes y_n,$$

where $(x_n^*)_n \in \ell_p^w(X^*)$ $((x_n^*)_n \in c_0^{w^*}(X^*)$ when $p = \infty)$ and $(y_n)_n \in \ell_{p^*}^w(Y)$ $((y_n)_n \in c_0^w(Y)$ when $p = 1)$. Here $c_0^{w^*}(X^*)$ is the space of all *weak** null sequences in X^*. We denote the space of all weakly p-nuclear operators from X to Y by $\mathcal{N}_{wp}(X, Y)$ and define a norm on $\mathcal{N}_{wp}(X, Y)$ by

$$\|T\|_{\mathcal{N}_{wp}} := \inf \|(x_n^*)_n\|_p^w \|(y_n)_n\|_{p^*}^w,$$

where the infimum is taken over all such weakly p-nuclear representations of T. Then $[\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]$ is a Banach operator ideal [K5, Theorem 2.1]. It was shown in [K5, Theorem 3.2] that $[\mathcal{W}_p, \|\cdot\|_{\mathcal{W}_p}]$ is equal to the surjective hull of $[\mathcal{N}_{wp^*}, \|\cdot\|_{\mathcal{N}_{wp^*}}]$.

In [K5], a weaker notion of the quasi p-nuclear operator of Persson and Pietsch [PP] was introduced. For $1 \leq p \leq \infty$, a linear map $T\colon X \to Y$ is called *quasi weakly p-nuclear* if there exists $(x_n^*)_n \in \ell_p^w(X^*)$ $((x_n^*)_n \in c_0^{w^*}(X^*)$ when $p = \infty)$ such that

$$\|Tx\| \leq \|(x_n^*(x))_n\|_p$$

for every $x \in X$. We denote the space of all quasi weakly p-nuclear operators from X to Y by $\mathcal{N}_{wp}^Q(X, Y)$. For $T \in \mathcal{N}_{wp}^Q(X, Y)$, let $\|T\|_{\mathcal{N}_{wp}^Q} := \inf \|(x_n^*)_n\|_p^w$, where the infimum is taken over all such inequalities. It was shown in [K5, Theorem 3.3] that $[\mathcal{N}_{wp}^Q, \|\cdot\|_{\mathcal{N}_{wp}^Q}]$ is equal to the injective hull of $[\mathcal{N}_{wp}, \|\cdot\|_{\mathcal{N}_{wp}}]$.

Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]^{dual}$ be the *dual ideal* of a Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ (cf. [DF, Section 9.9]).

Lemma 4.5 (K6, Proposition 4.9)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach operator ideals. If $\mathcal{A} \subset \mathcal{B}^{dual}$ and $\mathcal{B} \subset \mathcal{A}^{dual}$, then the dual space of a Banach space X has the $\mathcal{A}$-AP if and only if for every Banach space Y, $\mathcal{B}(X, Y^*) = \overline{\mathcal{F}(X, Y^*)}^{\|\cdot\|_{\mathcal{B}}}$.*

In [K5, Theorem 3.7], it was shown that for $1 \leq p < \infty$, $\mathcal{W}_p \subset (\mathcal{N}_{wp}^Q)^{dual}$ and $\mathcal{N}_{wp}^Q \subset \mathcal{W}_p^{dual}$. From Lemma 4.5, we have:

Corollary 4.6. *Let $1 \le p < \infty$. The dual space of a Banach space X has the $\mathcal{W}_p$-AP (respectively, $\mathcal{N}^Q_{wp}$-AP) if and only if for every Banach space Y, $\mathcal{N}^Q_{wp}(X, Y^*) = \overline{\mathcal{F}(X, Y^*)}^{\|\cdot\|_{\mathcal{N}^Q_{wp}}}$ (respectively, $\mathcal{W}_p(X, Y^*) = \overline{\mathcal{F}(X, Y^*)}^{\|\cdot\|_{\mathcal{W}_p}}$).*

Proposition 4.7. *Let $1 \le p \le \infty$ and let X and Y be Banach spaces. Let $T \colon X \to Y$ be a linear map. Then $T \in \mathcal{N}^Q_{wp}(X, Y)$ if and only if there exist a closed subspace Z of ℓ_p (Z is a closed subspace of c_0 when $p = \infty$), $R \in \mathcal{N}^Q_{wp}(X, Z)$ and $S \in \mathcal{N}^Q_{wp}(Z, Y)$ such that $T = SR$. In this case, $\|T\|_{\mathcal{N}^Q_{wp}} = \inf \|S\|_{\mathcal{N}^Q_{wp}} \|R\|_{\mathcal{N}^Q_{wp}}$, where the infimum is taken over all such factorizations.*

Proof. The "if" part is clear and, in this case, $\|T\|_{\mathcal{N}^Q_{wp}} \le \inf \|\cdot\|_{\mathcal{N}^Q_{wp}} \|\cdot\|_{\mathcal{N}^Q_{wp}}$.

Let $T \in \mathcal{N}^Q_{wp}(X, Y)$. Let $\varepsilon > 0$ be given. Then there exists $(x^*_n)_n \in \ell^w_p(X^*)$ such that

$$\|Tx\| \le \|(x^*_n(x))_n\|_p$$

for every $x \in X$ and $\|(x^*_n)_n\|^w_p \le \|T\|_{\mathcal{N}^Q_{wp}} + \varepsilon$. Consider the linear subspace

$$Z_0 = \{(x^*_n(x))_n \colon x \in X\}$$

of ℓ_p (ℓ_p is replaced by c_0 when $p = \infty$) and the map

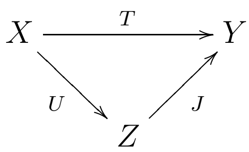
$$J_0 \colon Z_0 \to Y, \quad J_0(x^*_n(x))_n \mapsto Tx.$$

Then it follows that J_0 is well defined and linear, and $\|J_0\| \le 1$. Let $J \colon Z := \overline{Z_0} \to Y$ be the linear continuous extension of J_0. Define the operator $U \colon X \to Z$ by

$$Ux = (x^*_n(x))_n.$$

Then U is quasi weakly p-nuclear and $\|U\|_{\mathcal{N}^Q_{wp}} \le \|T\|_{\mathcal{N}^Q_{wp}} + \varepsilon$.

Now, we obtain the following commutative diagram:



Consider the sequence $(e^*_n)_n$ of standard unit vectors in ℓ_{p^*}. Then $(e^*_n|_Z)_n \in \ell^w_p(Z^*)$ and $\|(e^*_n|_Z)_n\|^w_p \le 1$. Since for every $x \in X$,

$$\|J(x^*_k(x))_k\|_Y = \|Tx\|_Y \le \|(x^*_k(x))_k\|_p = \|(\langle e^*_n|_Z, (x^*_k(x))_k\rangle)_n\|_p,$$

we can check that $\|Jz\|_Y \le \|(\langle e^*_n|_Z, z\rangle)_n\|_p$ for every $z \in Z$. Hence J is quasi weakly p-nuclear and $\|J\|_{\mathcal{N}^Q_{wp}} \le 1$, and $\inf \|\cdot\|_{\mathcal{N}^Q_{wp}} \|\cdot\|_{\mathcal{N}^Q_{wp}} \le \|T\|_{\mathcal{N}^Q_{wp}} + \varepsilon$. □

Corollary 4.8. *Let $1 \le p \le \infty$. Then $[\mathcal{N}^Q_{wp}, \|\cdot\|_{\mathcal{N}^Q_{wp}}] = [\mathcal{N}^Q_{wp} \circ \mathcal{N}^Q_{wp}, \|\cdot\|_{\mathcal{N}^Q_{wp} \circ \mathcal{N}^Q_{wp}}]$.*

Proposition 4.9. *Let $1 \le p \le \infty$. A Banach space X has the $\mathcal{N}^Q_{wp}$-AP if (and only if) for every closed subspace Y of ℓ_p (ℓ_p is replaced by c_0 when $p = \infty$), $\mathcal{N}^Q_{wp}(Y, X) \subset \overline{\mathcal{F}(Y, X)}^{\|\cdot\|}$.*

Proof. Let Y be a Banach space and let $T \in \mathcal{N}^Q_{wp}(Y, X)$. Let $\varepsilon > 0$ be given. By Proposition 4.7, there exists a closed subspace Z of ℓ_p, $R \in \mathcal{N}^Q_{wp}(Y, Z)$ and $S \in \mathcal{N}^Q_{wp}(Z, X)$ such that $T = SR$. Then by assumption, there exists an $S_0 \in \mathcal{F}(Z, X)$ such that

$$\varepsilon \ge \|S - S_0\| \|R\|_{\mathcal{N}^Q_{wp}} \ge \|T - S_0 R\|_{\mathcal{N}^Q_{wp}}.$$

Since $S_0 R \in \mathcal{F}(Y, X)$, we complete the proof. □

Proposition 4.10. *Let $1 < p < \infty$. The following statements are equivalent for a Banach space X.*

(a) *X^* has the $\mathcal{W}_p$-AP.*

(b) *For every closed subspace Y of ℓ_p, $\mathcal{N}^Q_{wp}(X,Y) \subset \overline{\mathcal{F}(X,Y)}^{\|\cdot\|}$.*

(c) *For every Banach space Y, $\mathcal{N}^Q_{wp}(X,Y) = \overline{\mathcal{F}(X,Y)}^{\|\cdot\|_{\mathcal{N}^Q_{wp}}}$.*

Proof. (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) follow from Corollary 4.6.
(b) $\Rightarrow$ (c): Adapt the proof of Proposition 4.9 using Proposition 4.7. □

Proposition 4.11. *Let $1 < p < \infty$. The following statements are equivalent for a Banach space X.*

(a) *X^* has the $\mathcal{N}^Q_{wp}$-AP.*

(b) *For every quotient space Y of ℓ_{p^*}, $\mathcal{W}_p(X,Y) \subset \overline{\mathcal{F}(X,Y)}^{\|\cdot\|}$.*

(c) *For every Banach space Y, $\mathcal{W}_p(X,Y) = \overline{\mathcal{F}(X,Y)}^{\|\cdot\|_{\mathcal{W}_p}}$.*

Proof. (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) follow from Corollary 4.6.
(b) $\Rightarrow$ (c): Adapt the proof of Proposition 4.9 using Proposition 2.4. □

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