

RIGIDITY OF WEIGHTED COMPOSITION OPERATORS ON H^p

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Abstract. We show that every non-compact weighted composition operator $f \mapsto u \cdot (f \circ \phi)$ acting on a Hardy space H^p for $1 \leq p < \infty$ fixes an isomorphic copy of the sequence space ℓ^p and therefore fails to be strictly singular. We also characterize those weighted composition operators on H^p which fix a copy of the Hilbert space ℓ^2 . These results extend earlier ones obtained for unweighted composition operators.

1. Introduction and main results

Let \mathbf{D} be the open unit disc in the complex plane \mathbf{C} and fix analytic maps $u: \mathbf{D} \rightarrow \mathbf{C}$ and $\phi: \mathbf{D} \rightarrow \mathbf{D}$. The weighted composition operator uC_ϕ is defined by

$$(uC_\phi)f = u \cdot (f \circ \phi)$$

for $f: \mathbf{D} \rightarrow \mathbf{C}$ analytic. Boundedness and compactness properties of such operators acting on the classical Hardy spaces H^p were characterized in terms of Carleson measures in [1, 2] (see also [3]). An obvious necessary condition for the boundedness of $uC_\phi: H^p \rightarrow H^q$ is that $u = (uC_\phi)1 \in H^q$.

The purpose of this work is to study the qualitative properties of non-compact weighted composition operators on the Hardy spaces H^p , extending the results obtained in [6] for unweighted composition operators. It turns out that the weighted composition operators exhibit the exact same rigidity phenomena as the unweighted ones. We also refer the reader to the recent parallel work [7] in the context of Volterra-type integral operators, where some of our ideas originate from.

Recall that if X is a Banach space and $T: X \rightarrow X$ is a linear operator, then T is called *strictly singular* if the restriction of T to any infinite-dimensional subspace of X is not an isomorphism (equivalently, it is not bounded below). For a detailed discussion of strict singularity and related notions in the context of Hardy spaces and composition operators, see [6] and the references therein.

Our first result is a generalization of [6, Thm 1.2] and shows, in particular, that the notions of compactness and strict singularity coincide for weighted composition operators on H^p . Here we employ the usual test functions

$$g_a(z) = \frac{(1 - |a|^2)^{1/p}}{(1 - \bar{a}z)^{2/p}}, \quad z \in \mathbf{D},$$

where $a \in \mathbf{D}$. They always satisfy $\|g_a\|_{H^p} = 1$.

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Theorem 1. *Let $1 \leq p < \infty$ and suppose that uC_ϕ is bounded and non-compact $H^p \rightarrow H^p$. Then uC_ϕ fixes an isomorphic copy of ℓ^p in H^p . More precisely, there exists a sequence (a_n) in \mathbf{D} such that (g_{a_n}) is equivalent to the natural basis of ℓ^p and uC_ϕ is bounded below on the closed linear span of (g_{a_n}) .*

We next determine under which conditions a weighted composition operator on H^p with $p \neq 2$ fixes a copy of the Hilbert space ℓ^2 . In the unweighted case (see [6, Thm 1.4]) this is the case precisely when the boundary contact set

$$E_\phi = \{\zeta \in \mathbf{T} : |\phi(\zeta)| = 1\}$$

has positive measure. It turns out that this result holds in the weighted case as well. The first half is established in the following theorem, where we also allow for the possibility that the target space of the operator is a larger Hardy space than the domain.

Theorem 2. *Let $1 \leq q \leq p < \infty$ and suppose that uC_ϕ is bounded $H^p \rightarrow H^q$. If $u \neq 0$ and $m(E_\phi) > 0$, then uC_ϕ fixes an isomorphic copy of ℓ^2 in H^p . Here m is the normalised Lebesgue measure on \mathbf{T} .*

In the converse direction we have the following result.

Theorem 3. *Let $1 \leq p < \infty$ and suppose that uC_ϕ is bounded $H^p \rightarrow H^p$ with $m(E_\phi) = 0$. If uC_ϕ is bounded below on an infinite-dimensional subspace $M \subset H^p$, then M contains an isomorphic copy of ℓ^p . In particular, if $p \neq 2$, then uC_ϕ does not fix any isomorphic copies of ℓ^2 in H^p .*

The last statement of the theorem is due to the fact that ℓ^p and ℓ^2 are totally incomparable spaces for $p \neq 2$.

2. Proofs

Towards the proof of Theorem 1 we first state the following lemma.

Lemma 4. *Let $u \in H^p$ and $\phi: \mathbf{D} \rightarrow \mathbf{D}$ be analytic. For $\epsilon > 0$, define*

$$F_\epsilon = \{\zeta \in \mathbf{T} : |\phi(\zeta) - 1| < \epsilon\}.$$

Then

$$\begin{aligned} \lim_{a \rightarrow 1} \int_{\mathbf{T} \setminus F_\epsilon} |(uC_\phi)g_a|^p dm &= 0 \quad \text{for each } \epsilon > 0, \\ \lim_{\epsilon \rightarrow 0} \int_{F_\epsilon} |(uC_\phi)g_a|^p dm &= 0 \quad \text{for each } a \in \mathbf{D}. \end{aligned}$$

Proof. Let $\epsilon > 0$ be fixed and consider $\zeta \in \mathbf{T} \setminus F_\epsilon$ for which the radial limit $\phi(\zeta)$ exists. Then, if $|a - 1| < \epsilon/2$, we have

$$|1 - \bar{a}\phi(\zeta)| \geq |1 - \phi(\zeta)| - |1 - a| > \epsilon/2,$$

and so

$$\int_{\mathbf{T} \setminus F_\epsilon} |(uC_\phi)g_a|^p dm \leq (1 - |a|^2) \int_{\mathbf{T} \setminus F_\epsilon} \frac{|u|^p}{|1 - \bar{a}\phi|^2} dm \leq \frac{4(1 - |a|^2)}{\epsilon^2} \|u\|_{H^p}^p.$$

Since this tends to 0 as $a \rightarrow 1$, we obtain the first part of the lemma.

The second part follows from the absolute continuity of the measure $F \mapsto \int_F |(uC_\phi)g_a|^p dm$ and the fact that $m(F_\epsilon) \rightarrow m(\{\zeta \in \mathbf{T} : \phi(\zeta) = 1\}) = 0$ as $\epsilon \rightarrow 0$. Note that $g_a \in H^\infty$ and hence $(uC_\phi)g_a \in L^p(\mathbf{T}, m)$. \square

Proof of Theorem 1. Since uC_ϕ is non-compact, we may find a sequence (a_n) in \mathbf{D} such that $|a_n| \rightarrow 1$ and $\|(uC_\phi)g_{a_n}\|_{H^p} \geq c > 0$ for all n . This is a consequence of the compactness characterization of uC_ϕ in terms of vanishing Carleson measures; see [1, Theorem 3.5]. By passing to a convergent subsequence of (a_n) and utilizing a suitable rotation, we may assume that $a_n \rightarrow 1$.

We now proceed exactly as in the unweighted case (see the proof of Theorem 1.2 in [6] for the details of the following argument). First, by invoking Lemma 4 repeatedly, we may extract a subsequence of (a_n) , still denoted by (a_n) , such that the image sequence $((uC_\phi)g_{a_n})$ in H^p is equivalent to the standard basis of ℓ^p , that is,

$$\left\| \sum_{n=1}^{\infty} \alpha_n (uC_\phi)g_{a_n} \right\|_{H^p} \sim \|(\alpha_n)\|_p \quad \text{for } (\alpha_n) \in \ell^p,$$

where $A \sim B$ for quantities A and B means that $cA \leq B \leq CA$ for some positive numbers c and C .

Then a second application of Lemma 4 to the functions g_{a_n} (taking $u = 1$ and $\phi(z) = z$) produces a further subsequence of (a_n) , which we continue to denote by (a_n) , such that also

$$\left\| \sum_{n=1}^{\infty} \alpha_n g_{a_n} \right\|_{H^p} \sim \|(\alpha_n)\|_p \quad \text{for } (\alpha_n) \in \ell^p.$$

By combining the preceding two norm estimates we see that uC_ϕ restricts to a linear isomorphism on the closed linear span of (g_{a_n}) . \square

Proof of Theorem 2. Since $m(E_\phi) > 0$, [6, Prop. 3.2] shows that there exists a sequence of integers (n_k) satisfying $\inf_k (n_{k+1}/n_k) > 1$ and a constant $c_1 > 0$ such that $\|\sum_k \alpha_k \phi^{n_k}\|_{H^1} \geq c_1 \|(\alpha_k)\|_2$ for all $(\alpha_k) \in \ell^2$. Our goal is to prove a weighted version of this estimate, that is, for some constant $c > 0$,

$$(1) \quad \left\| u \sum_k \alpha_k \phi^{n_k} \right\|_{H^1} \geq c \|(\alpha_k)\|_2.$$

Since Paley’s theorem (see e.g. [4, p. 104]) implies that the closed linear span $M = [z^{n_k} : k \geq 1]$ in H^p is isomorphic to ℓ^2 , inequality (1) implies that uC_ϕ is an isomorphism from M into H^1 . This yields the theorem because $\|f\|_{H^q} \geq \|f\|_{H^1}$ for all $f \in H^q$.

To establish (1), we first note that since $u \neq 0$, we have $|u| > 0$ a.e. on \mathbf{T} . Thus, for a given $\epsilon > 0$ there exists a set $F \subset \mathbf{T}$ with $m(\mathbf{T} \setminus F) < \epsilon$ such that $|u| > c_2$ on F for some constant $c_2 = c_2(\epsilon) > 0$. Then, using Hölder’s inequality and the boundedness of C_ϕ on H^2 , we get

$$\int_{\mathbf{T} \setminus F} \left| \sum_k \alpha_k \phi^{n_k} \right| dm \leq \sqrt{m(\mathbf{T} \setminus F)} \left\| \sum_k \alpha_k \phi^{n_k} \right\|_2 \leq \sqrt{\epsilon} \cdot c_3 \|(\alpha_k)\|_2$$

for some constant $c_3 > 0$. On combining these estimates we obtain

$$\left\| u \sum_k \alpha_k \phi^{n_k} \right\|_{H^1} \geq c_2 \int_F \left| \sum_k \alpha_k \phi^{n_k} \right| dm \geq c_2 (c_1 - \sqrt{\epsilon} \cdot c_3) \|(\alpha_k)\|_2.$$

In particular, choosing $\epsilon = (c_1/2c_3)^2$ here proves (1) with $c = \frac{1}{2}c_2c_1$. This completes the proof of the theorem. \square

Proof of Theorem 3. Since M is infinite-dimensional, there exists a sequence (f_n) in M such that $\|f_n\|_{H^p} = 1$ and $f_n \rightarrow 0$ uniformly on compact subsets of \mathbf{D} ; for instance, we can choose $f_n(0) = f'_n(0) = \dots = f_n^{(n)}(0) = 0$ for all n .

For each $k \geq 1$, define $E_k = \{\zeta \in \mathbf{T} : |\phi(\zeta)| > 1 - \frac{1}{k}\}$. We have $m(E_k) \rightarrow m(E_\phi) = 0$ as $k \rightarrow \infty$ and therefore

$$(2) \quad \lim_{k \rightarrow \infty} \int_{E_k} |(uC_\phi)f_n|^p dm = 0.$$

On the other hand, since $f_n \circ \phi$ converges to zero uniformly on $\mathbf{T} \setminus E_k$ as $n \rightarrow \infty$ and $u \in H^p$,

$$(3) \quad \lim_{n \rightarrow \infty} \int_{\mathbf{T} \setminus E_k} |(uC_\phi)f_n|^p dm = 0.$$

Since uC_ϕ is bounded below on M , we also have $\|(uC_\phi)f_n\|_{H^p} \geq c$ for all n and some constant $c > 0$. Using a gliding hump argument based on a repeated application of (2) and (3) (akin to the proof of [6, Prop. 3.3] in the unweighted case), we may extract a subsequence (f_{n_j}) such that the sequence $((uC_\phi)f_{n_j})$ is equivalent to the standard basis of ℓ^p . Since uC_ϕ is bounded below on the closed linear span $[f_{n_j} : j \geq 1] \subset M$, we conclude that uC_ϕ fixes a copy of ℓ^p in M . \square

To conclude, we record a couple of questions on the space $BMOA$, which consists of those functions in H^2 that have boundary values of bounded mean oscillation.

Q1. Does every bounded and non-compact $uC_\phi : BMOA \rightarrow BMOA$ fix an isomorphic copy of c_0 (the space of null sequences)? This is known to be true for (unweighted) composition operators; see [5, Section 3] and the discussion in [6, p. 625].

Q2. Do Theorems 2 and 3 have versions for uC_ϕ on $BMOA$? More specifically, is it true that $uC_\phi : BMOA \rightarrow BMOA$ fixes an isomorphic copy of ℓ^2 if and only if $u \neq 0$ and $m(E_\phi) > 0$? In the unweighted case, a partial result of this kind is obtained in [6, Prop. 4.1].

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