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THE CHARACTERIZATIONS OF HARDY–SOBOLEV SPACES BY FRACTIONAL SQUARE FUNCTIONS RELATED TO SCHRÖDINGER OPERATORS

Jizheng Huang, Pengtao Li, Yu Liu and Jie Xin

Beijing University of Posts and Telecommunications, School of Science
Beijing 100876, P. R. China; hjzheng@163.com

Qingdao University, College of Mathematics
Qingdao, Shandong 266071, P. R. China; ptli@qdu.edu.cn

University of Science and Technology Beijing, School of Mathematics and Physics
Beijing 100083, P. R. China; liuyu75@pku.org.cn

Qufu Normal University, School of Mathematical Sciences
Qufu 273165, P. R. China; fdxinjie@sina.com

Abstract. Let $L = -\Delta + V$ be a Schrödinger operator, where the potential V satisfies the reverse Hölder condition. In this paper, via the heat semigroup e^{-tL} and the Poisson semigroup $e^{-t\sqrt{L}}$, we introduce several classes of fractional square functions associated with L including the Litttlewood–Paley g-function, the area integral and the g^*_λ-function, respectively. By the regularities of semigroup, we establish several square function characterizations of the Hardy space and the Hardy–Sobolev space related to the Schrödinger operator.

1. Introduction

It is a well-known fact that, as a class of important Calderón–Zygmund singular integrals, the square functions are usually applied to characterize function spaces. In 1972, Fefferman and Stein [18] proved that the square functions can be used to characterize Hardy spaces $H^p(\mathbf{R}^d)$ for $0 < p \le 1$. From then on, such characterizations were extended to other settings. We refer the reader to [13, 11, 21] and the references therein. In the last decades, many researchers pay their attentions to the equivalent characterizations of Hardy spaces associated with operators, such as [16, 13, 21, 6, 26, 25, 29, 38, 7, 39, 40, 41] and the references therein. In particular, Dziubański and Zienkiewicz [16] dealt with the Hardy space associated with the Schrödinger operator. In this paper, based on the fractional square functions, we will establish several equivalent characterizations of the Hardy space and the Hardy–Sobolev space associated with the Schrödinger operator.

In the following, we recall some backgrounds of the Hardy space and the Hardy–Sobolev space associated with the Schrödinger operator, respectively. Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbf{R}^d$, $d \ge 3$, where $V \not\equiv 0$ is a nonnegative potential. Assume that V belongs to the reverse Hölder class $B_q, q \ge d/2$, i.e.,

$$\left(\frac{1}{|B|}\int_B V^q(x)\,dx\right)^{1/q} \le C\left(\frac{1}{|B|}\int_B V(x)\,dx\right) \quad \text{for every ball } B.$$

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Let $\{T_t^L\}_{t>0}$ be the semigroup of linear operators generated by $-L$ and let $K_t^L(\cdot,\cdot)$ be their kernels. Since V is nonnegative, the Feynman–Kac formula implies that

$$0 \le K_t^L(x,y) \le \widetilde{T}_t(x,y) := (4\pi t)^{-d/2} \exp\left(-|x-y|^2/4t\right).$$

Dziubański and Zienkiewicz [16] defined the Hardy space $H_L^1(\mathbf{R}^d)$ as

$$H_L^1(\mathbf{R}^d) = \left\{ f \in L^1(\mathbf{R}^d) \colon Mf \in L^1(\mathbf{R}^d) \right\},$$

where

$$Mf(x) = \sup_{t>0} \left| T_t^L f(x) \right|.$$

The Hardy space $H_L^1(\mathbf{R}^d)$ is a Banach space with the norm $\|f\|_{H_L^1} = \|Mf\|_{L^1}$. In order to give the atomic decomposition of the Hardy spaces associated with L, we introduce the function

$$\rho(x) := \sup\left\{ r > 0 \colon \frac{1}{r^{d-2}} \int_{B(x,r)} V(y)\,dy \le 1 \right\}.$$

Definition 1.1. For $1 \le q \le \infty$, a function a is an H_L^q-atom associated with a ball $B(x_0,r)$ if

(i) $\operatorname{supp} a \subset B(x_0,r)$;
(ii) $\|a\|_{L^q} \le |B(x_0,r)|^{1/q-1}$;
(iii) if $r < \rho(x_0)$, then $\int a(x)\,dx = 0$.

The atomic quasi-norm of $H_L^1(\mathbf{R}^d)$ is defined by $\|f\|_{L\text{-atom},q} = \inf\left\{ \sum |c_j| \right\}$, where the infimum is taken over all decompositions $f = \sum c_j a_j$ and $\{a_j\}$ are H_L^q-atoms.

We have the following H_L^q-atomic decomposition of $H_L^1(\mathbf{R}^d)$.

Proposition 1.2. *Let $1 \le q \le \infty$. The norms $\|f\|_{H_L^1}$ and $\|f\|_{L\text{-atom},q}$ are equivalent, i.e., there exists a constant $C > 0$ such that*

$$C^{-1}\|f\|_{H_L^1} \le \|f\|_{L\text{-atom},q} \le C\|f\|_{H_L^1}.$$

Proof. The case $q = \infty$ is obtained by Dziubański and Zienkiewicz [16]. For $1 \le q < \infty$, it is easy to see that any H_L^∞-atom is also an H_L^q-atom. On the other hand, by Proposition 2.1, we can use a direct computation to deduce that there exists a constant C such that for any H_L^q-atom a, $\|M(a)\|_1 \le C$. We omit the details. □

Now, we can define the square function. For $k \in \mathbf{N}$, let

$$Q_t^k f(x) = t^{2k}\left(\partial_s^k T_s^L|_{s=t^2} f\right)(x),$$

then square function associated with $\{Q_t^k\}$ is defined as

$$(S_k^L(f))(x) := \left(\int_0^{+\infty} \int_{|x-y|<t} \left|Q_t^k(f)(y)\right|^2 \frac{dy\,dt}{t^{d+1}} \right)^{1/2}.$$

The following square function characterization was obtained by Hoffmann et al. (cf. [21]).

Proposition 1.3. *Let $k \in \mathbf{N}$. A function $f \in H_L^1(\mathbf{R}^d)$ if and only if $f \in L^1(\mathbf{R}^d)$ and the square function $S_L^k(f) \in L^1(\mathbf{R}^d)$. Moreover,*

$$\|f\|_{H_L^1} \sim \|S_L^k(f)\|_{L^1} + \|f\|_{L^1}.$$

In order to define the fractional square function, we first introduce the Poisson semigroup. Let $\{P_t^L\}_{t>0}$ be the semigroup of linear operators generated by $-\sqrt{L}$. Denote by $P_t^L(\cdot,\cdot)$ the integral kernel of P_t^L. Since V is nonnegative, it can be seen from the Feynman–Kac formula that $P_t^L(\cdot,\cdot)$ is dominated by the classical Poisson kernel $\widetilde{P}_t(\cdot,\cdot)$:

$$0 \le P_t^L(x,y) \le \widetilde{P}_t(x,y) := \frac{C_d\, t}{(t^2+|x-y|^2)^{(d+1)/2}}.$$

Segovia and Wheeden [34] studied several types of fractional square and Lusin functions associated with $e^{-t(-\Delta)^{1/2}}$. Betancor et al. in [5] introduced the fractional square functions associated with P_t^L. The fractional derivatives $\partial_t^\alpha P_t^L, \alpha > 0$, are defined as follows (cf. [34]):

$$\partial_t^\alpha P_t^L(x,y) := \frac{e^{-i\pi(m-\alpha)}}{\Gamma(m-\alpha)} \int_0^\infty \partial_t^m P_{t+s}^L(x,y) s^{m-\alpha} \frac{ds}{s}, \quad m = [\alpha]+1. \tag{1.1}$$

Using the Poisson semigroup $\{P_t^L\}_{t>0}$, we define the fractional square functions as follows:

$$\begin{cases} g_\alpha(f)(x) := \Big(\int_0^\infty \Big| t^\alpha \partial_t^\alpha P_t^L f(x) \Big|^2 \frac{dt}{t} \Big)^{1/2}, \\ G_\alpha(f)(x) := \Big(\int_0^\infty \int_{|x-y|<t} \Big| t^\alpha \partial_t^\alpha P_t^L f(x) \Big|^2 \frac{dy\,dt}{t^{d+1}} \Big)^{1/2}, \\ g_{\alpha;\lambda}^*(f)(x) := \Big(\int_0^\infty \int_{\mathbf{R}^d} \Big(\frac{t}{|x-y|+t} \Big)^{2\lambda} \Big| t^\alpha \partial_t^\alpha P_t^L f(x) \Big|^2 \frac{dy\,dt}{t^{d+1}} \Big)^{1/2}. \end{cases}$$

The first main result of this paper is the following characterizations of the Hardy space $H_L^1(\mathbf{R}^d)$.

Theorem 1.4. *Let $\alpha \ge \frac{1}{2}$ and $\lambda > d/2$. The following assertions are equivalent:*
(i) $f \in H_L^1(\mathbf{R}^d)$;
(ii) $f \in L^1(\mathbf{R}^d)$ *and* $g_\alpha(f) \in L^1(\mathbf{R}^d)$;
(iii) $f \in L^1(\mathbf{R}^d)$ *and* $G_\alpha(f) \in L^1(\mathbf{R}^d)$;
(iv) $f \in L^1(\mathbf{R}^d)$ *and* $g_{\alpha;\lambda}^*(f) \in L^1(\mathbf{R}^d)$.

Moreover, for every $f \in H_L^1(\mathbf{R}^d)$,

$$\|f\|_{H_L^1} \sim \|f\|_1 + \|g_\alpha(f)\|_1 \sim \|f\|_1 + \|G_\alpha(f)\|_1 \sim \|f\|_1 + \|g_{\alpha;\lambda}^*(f)\|_1.$$

In what follows, we consider the Hardy–Sobolev space associated with the Schrödinger operator, which are also important in harmonic analysis due to the fact that they can be used to give the strong boundedness of some linear operators instead of the weak boundeness. Denote by $\mathcal{S}'$ the space of tempered distributions and denote by $\mathcal{P}$ the space of polynomials, respectively. Let $I_\alpha\colon \mathcal{S}'/\mathcal{P} \to \mathcal{S}'/\mathcal{P}$ be the Bessel potential operator. The function space $I_\alpha(H^p)$ which is called Hardy–Sobolev spaces are natural generalizations of the homogeneous Sobolev space $I_\alpha(L^p)$ to the range $0 < p \le 1$. Hardy–Sobolev spaces associated with the Laplacian were studied by many authors. In [35], Strichartz investigated the space $I_\alpha(H^p), 0 < p \le 1$, and proved that $I_{n/p}(H^p)$ was an algebra and found equivalent norms for the Hardy–Sobolev space or, more generally, for the corresponding space with fractional smoothness and Lebesgue exponents in the range $p > n/(n+1)$. The trace properties of the space $I_\alpha(H^p)$ were discussed by Torchinsky [36]. Miyachi [32] characterized the Hardy–Sobolev spaces in terms of maximal functions related to mean oscillation of the function in cubes. In particular, he obtained a counterpart of previous results of Calderón and of the general theory of De Vore and Sharpley [12].

Recently, many scholars pay their attentions to Hardy–Sobolev spaces and their variants on $\mathbf{R}^d$, or on subdomains. In [8], Chang, Dafni, and Stein studied Hardy–Sobolev spaces in connection with estimates for elliptic operators. Furthermore, Aucher, Emmanuel, and Tchamitchian [1] studied these spaces with applications to square roots of elliptic operators. A simple strictly pointwise characterization of the Hardy–Sobolev space in terms of first difference was obtained by Koskela and Saksman in [28]. Lou and Yang [30] gave the atomic decomposition of the Hardy–Sobolev space and proved the endpoint case of the div-curl theorem in [10]. More about Hardy–Sobolev spaces can be found in [20, 9, 24, 33]. In [3] and [4], the authors considered Hardy–Sobolev spaces on the manifold. In [22] and [23], Huang and his coauthors studied the Hardy–Sobolev spaces associated with Hermite and special Hermite operators.

Now, we give the definitions of Hardy–Sobolev spaces associated with the Schrödinger operator L.

Definition 1.5. For $\alpha > 0$, the inhomogeneous Hardy–Sobolev space $H_L^{1,\alpha}(\mathbf{R}^d)$ is defined as the set of all functions $f \in H_L^1(\mathbf{R}^d)$ such that $(I+L)^{\alpha/2} f \in H_L^1(\mathbf{R}^d)$ with the norm

$$\|f\|_{H_L^{1,\alpha}} := \|(I+L)^{\alpha/2} f\|_{H_L^1} + \|f\|_{H_L^1}.$$

In 1969, Segovia and Wheeden [34] obtained the characterization of the Sobolev space $L^{p,\alpha}$ via the fractional square function defined as

$$g_{k,\alpha}(f) = \left(\int_0^\infty \left| t^{k-\alpha} \frac{\partial^k \widetilde{P}_t f}{\partial t^k} \right|^2 \frac{dt}{t} \right)^{1/2}, \quad k \geq \alpha > 0.$$

In [5], the authors introduced the fractional square functions in the case of the Schrödinger operator L and characterized the potential spaces associated with L. The fractional Littlewood–Paley g-function associated with $\{P_t^L\}_{t>0}$ is defined as

$$g_{k,\alpha}(f) = \left(\int_0^\infty \left| t^{k-\alpha} \frac{\partial^k P_t^L f}{\partial t^k} \right|^2 \frac{dt}{t} \right)^{1/2}, \quad k \geq \alpha > 0.$$

The fractional square function and the fractional g_λ^*-function associated to $\{P_t^L\}_{t>0}$ are defined, respectively, by

$$G_{k,\alpha}(f) = \left(\int_0^{+\infty} \int_{|x-y|<t} \left| t^{k-\alpha} \frac{\partial^k P_t^L f}{\partial t^k} \right|^2 \frac{dy\,dt}{t^{d+1}} \right)^{1/2}, \quad k \geq \alpha > 0$$

and

$$g_{k,\alpha;\lambda}^*(f) = \left[\int_0^{+\infty} \int_{\mathbf{R}^d} \left(\frac{t}{|x-y|+t} \right)^{2\lambda} \left| t^{k-\alpha} \frac{\partial^k P_t^L f}{\partial t^k} \right|^2 \frac{dy\,dt}{t^{d+1}} \right]^{1/2}, \quad k \geq \alpha > 0.$$

Define $L_V^{p,\alpha}(\mathbf{R}^d)$ as the collection of all functions f such that

$$f = (I+L)^{-\alpha/2} g, \quad g \in L^p(\mathbf{R}^d),$$

where $1 < p < \infty$. The main result of [5] is the following proposition.

Proposition 1.6. *Let $\alpha > 0$, $k \in \mathbf{N} \setminus \{0\}$, $1 < p < \infty$ and $\lambda > \max\{d/p, d/2\}$. Then the following assertions are equivalent:*

(i) $f \in L_V^{p,\alpha}(\mathbf{R}^d)$;

(ii) $f \in L^p(\mathbf{R}^d)$ *and* $g_{k,\alpha}(f) \in L^p(\mathbf{R}^d)$ *for* $k > \alpha$;

(iii) $f \in L^p(\mathbf{R}^d)$ *and* $G_{k,\alpha}(f) \in L^p(\mathbf{R}^d)$ *for* $\alpha < k - (d+1)/2$;

(iv) $f \in L^p(\mathbf{R}^d)$ *and* $g^*_{k,\alpha;\lambda}(f) \in L^p(\mathbf{R}^d)$ *for* $\alpha < k - (d+1)/2$.

Moreover, for every $f \in L^{p,\alpha}_V$,

$$\|f\|_{L^{p,\alpha}_V} \sim \|f\|_{L^p} + \|g_{k,\alpha}(f)\|_{L^p} \sim \|f\|_{L^p} + \|G_{k,\alpha}(f)\|_{L^p} \sim \|f\|_{L^p} + \|g^*_{k,\alpha;\lambda}(f)\|_{L^p}.$$

In this paper, we will give the characterizations of the Hardy–Sobolev space $H^{1,\alpha}_L(\mathbf{R}^d)$ as follows.

Theorem 1.7. *Let* $\alpha \geq \frac{1}{2}$, $k \in \mathbf{N} \setminus \{0\}$ *and* $\lambda > d$. *Then the following assertions are equivalent:*

(i) $f \in H^{1,\alpha}_L(\mathbf{R}^d)$;
(ii) $f \in H^1_L(\mathbf{R}^d)$ *and* $g_{k,\alpha}(f) \in L^1(\mathbf{R}^d)$ *for* $k > \alpha$;
(iii) $f \in H^1_L(\mathbf{R}^d)$ *and* $G_{k,\alpha}(f) \in L^1(\mathbf{R}^d)$ *for* $\alpha < k - (d+1)/2$;
(iv) $f \in H^1_L(\mathbf{R}^d)$ *and* $g^*_{k,\alpha;\lambda}(f) \in L^1(\mathbf{R}^d)$ *for* $\alpha < k - (d+1)/2$.

Moreover, for every $f \in H^{1,\alpha}_L(\mathbf{R}^d)$,

$$\|f\|_{H^{1,\alpha}_L} \sim \|f\|_{H^1_L} + \|g_{k,\alpha}(f)\|_{L^1} \sim \|f\|_{H^1_L} + \|G_{k,\alpha}(f)\|_{L^1} \sim \|f\|_{H^1_L} + \|g^*_{k,\alpha;\lambda}(f)\|_{L^1}.$$

The paper is organized as follows. In Section 2, we give some estimates of the kernels associated with e^{-tL} and P^L_t. In Section 3, we give the proof of Theorem 1.4. The proof of Theorem 1.7 will be given in Section 4.

Throughout this article, we will use c and C to denote the positive constants, which are independent of main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $1/C \leq B_1/B_2 \leq C$.

2. Preliminaries

In this section, we give some estimates of the heat kernel and the Poisson kernel associated with L which will be used in the sequel. We first collect some basic facts about the potential V satisfying the reverse Hölder inequality. Obviously, $B_{q_1} \subset B_{q_2}$ if $q_1 > q_2$. But it is important that the B_q class has a property of self improvement; that is, if $V \in B_q$, then $V \in B_{q+\varepsilon}$ for some $\varepsilon > 0$. We have assumed that $V \in B_{d/2}$, and hence $V \in B_{q_0}$ for some $q_0 > d/2$. We also write $\delta_0 = 2 - d/q_0 > 0$ and $\delta = \min(1, \delta_0) \leq 1$, and throughout the paper we keep this assumption and the meanings of q_0, δ_0 and δ.

We first give the following estimates of the kernels $K^L_t(\cdot,\cdot)$ (cf. [17]).

Proposition 2.1. (a) *For every* $N \in \mathbf{N}$, *there is a constant* $C_N > 0$ *such that*

$$0 \leq K^L_t(x,y) \leq C_N t^{-d/2} e^{-|x-y|^2/5t} \Big(1 + \sqrt{t}/\rho(x) + \sqrt{t}/\rho(y)\Big)^{-N}.$$

(b) *There exist* $\delta > 0$ *and* $C > 0$ *such that for every* $N > 0$, *there is a constant* $C_N > 0$ *so that, for all* $|h| \leq \sqrt{t}$,

$$\begin{aligned}&|K^L_t(x+h,y) - K^L_t(x,y)| \\ &\leq C_N \big(|h|/\sqrt{t}\big)^\delta t^{-d/2} e^{-A|x-y|^2/t} \Big(1 + \sqrt{t}/\rho(x) + \sqrt{t}/\rho(y)\Big)^{-N}.\end{aligned}$$

Remark 2.2. By part (a) of Proposition 2.1, it is easy to see that the condition $|h| \leq \sqrt{t}$ can be replaced by $|h| \leq |x-y|/2$ in part (b) of Proposition 2.1.

For $k \in \mathbf{N}$, define

$$Q^k_t(x,y) := t^{2k} \partial^k_s K^L_s(x,y)|_{s=t^2}.$$

The following estimates about the kernel $Q_t^k(\cdot,\cdot)$ is obtained in [15].

Proposition 2.3. *There exist constants $C, 0 < \delta' < \delta$ such that for every N, there exist constants $C_N > 0$ and $C_{k,N} > 0$, so that*

(a) $\left|Q_t^k(x,y)\right| \le C_N t^{-d} e^{-C|x-y|^2/t^2}\Big(1+t/\rho(x)+t/\rho(y)\Big)^{-N}$;

(b) $|Q_t^k(x+h,y)-Q_t^k(x,y)| \le C_{k,N}\left(\frac{|h|}{t}\right)^{\delta'} t^{-d} e^{-C|x-y|^2/t^2}\Big(1+t/\rho(x)+t/\rho(y)\Big)^{-N}$, $|h| \le t$;

(c) $\left|\int_{\mathbf{R}^d} Q_t^k(x,y)\,dy\right| \le C_N \frac{(t/\rho(x))^{\delta'}}{(1+t/\rho(x))^N}$.

By the Bochner's subordination formula, for any $x \in \mathbf{R}^d$ and $t > 0$,

$$P_t^L f(x) = e^{-t\sqrt{L}} f(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} T^L_{t^2/(4u)} f(x)\,du = \frac{t}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-t^2/(4u)}}{u^{3/2}} T_u^L f(x)\,du.$$

The above identity indicates that the Poisson kernel is given by

$$P_t^L(x,y) = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} K^L_{t^2/(4u)}(x,y)\,du = \frac{t}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-t^2/(4u)}}{u^{3/2}} K_u^L(x,y)\,du.$$

The kernel $\partial_t^\alpha P_t^L(\cdot,\cdot)$ satisfies the following estimates.

Proposition 2.4. [Proposition 3.6] *Let $\alpha > 0$. For any $N > 0$ and $0 < \delta' \le \delta$ with $0 < \delta' < \alpha$, there exists a constant C_N such that for any $x, y \in \mathbf{R}^d$ and $t \in (0,\infty)$,*

(a) $0 \le P_t^L(x,y) \le \frac{C_N t}{(t^2+|x-y|^2)^{(d+1)/2}}\Big(1+t/\rho(x)+t/\rho(y)\Big)^{-N}$;

(b) $|t^\alpha \partial_t^\alpha P_t^L(x,y)| \le \frac{C_N t^\alpha}{(t^2+|x-y|^2)^{(d+\alpha)/2}}\Big(1+t/\rho(x)+t/\rho(y)\Big)^{-N}$;

(c) $|t^\alpha \partial_t^\alpha P_t^L(x+h,y) - t^\alpha \partial_t^\alpha P_t^L(x,y)| \le \frac{C_N t^\alpha (|h|/t)^{\delta'}}{(t^2+|x-y|^2)^{(d+\alpha)/2}}\Big(1+t/\rho(x)+t/\rho(y)\Big)^{-N}$, $|h| \le t$;

(d) $\left|\int_0^\infty t^\alpha \partial_t^\alpha P_t^L(x,y)\,dy\right| \le C_N \frac{(t/\rho(x))^{\delta'}}{(1+t/\rho(x))^N}$.

We can express $\partial_t^\alpha P_t^L f$ as follows (cf. [37]).

Lemma 2.5. *Let $\alpha > 0$. For every $f \in L^2(\mathbf{R}^d)$,*

$$\partial_t^\alpha P_t^L f = e^{i\pi\alpha}\int_0^\infty \lambda^{\alpha/2} e^{-t\sqrt{\lambda}}\,dE_L(\lambda) f, \quad t > 0.$$

At the end of this section, we give some basic facts on the tent spaces which will be used in the sequel(cf. [10]). Let $0 < p < \infty$, and $1 \le q \le \infty$. The tent space T_q^p is defined as the space of all functions $u(\cdot,\cdot)$ on $\mathbf{R}_+^{d+1}$ such that

$$\begin{cases} \left(\int_{\Gamma(x)} |u(y,t)|^q \frac{dy\,dt}{t^{d+1}}\right)^{1/q} \in L^p(\mathbf{R}^d), & 1 \le q < \infty, \\ \sup\limits_{(y,t)\in\Gamma(x)} |u(y,t)| \in L^p(\mathbf{R}^d), & q = \infty, \end{cases}$$

where $\Gamma(x)$ is the standard cone whose vertex is $x \in \mathbf{R}^d$, i.e.,

$$\Gamma(x) = \{(y,t)\colon |y-x| < t\}.$$

Assume $B(x_0, r)$ is a ball in $\mathbf{R}^d$, its tent $\widehat{B}$ is defined by $\widehat{B} = \{(x,t)\colon |x-x_0| \le r-t\}$. A function $a(\cdot,\cdot)$ that supported in a tent $\widehat{B}$, where B is a ball in $\mathbf{R}^d$, is said to be an atom in the tent space T_2^p if it satisfies

$$\Big(\int_{\widehat{B}} \big|a(x,t)\big|^2 \frac{dxdt}{t}\Big)^{1/2} \le |B|^{1/2-1/p}.$$

The atomic decomposition of T_2^p is stated as follows (cf. [10]).

Proposition 2.6. *Let $0 < p \le 1$. Any $u \in T_2^p$ can be written as $u = \sum \lambda_k a_k$, where $\{a_k\}$ are atoms and $\sum |\lambda_k|^p \le C\|u\|_{T_2^p}^p$.*

3. The characterization of the Hardy space $H_L^1(\mathbf{R}^d)$

In this section, we will give the characterizations of Hardy space $H_L^1(\mathbf{R}^d)$ by the fractional square functions. Firstly, we give the following lemmas, which will be used in the sequel (cf. [15] or [31]).

Lemma 3.1. *Let $\alpha > 0$. The operator $t^\alpha \partial_t^\alpha P_t^L$ defines an isometry from $L^2(\mathbf{R}^d)$ into $L^2(\mathbf{R}_+^{d+1}, dx\, dt/t)$. Moreover,*

$$f(x) = \frac{4^\alpha}{\Gamma(2\alpha)} \lim_{\epsilon\to 0} \lim_{N\to\infty} \int_\epsilon^N (t^\alpha \partial_t^\alpha P_t^L)^2 f(x) \frac{dt}{t} \quad \textit{in } L^2(\mathbf{R}^d).$$

By the spectral theorem, we can prove

Lemma 3.2. *Let $\alpha > 0$. The operators g_α, G_α and $g_{\alpha,\lambda}^*$ are bounded on $L^2(\mathbf{R}^d)$. Moreover, there exist constants C, C_1 and C_2 such that*

$$\|g_\alpha f\|_{L^2} = C\|f\|_{L^2}, \quad \|G_\alpha f\|_{L^2} \le C_1\|f\|_{L^2}, \quad \|g_{\alpha,\lambda}^* f\|_{L^2} \le C_2\|f\|_{L^2}.$$

The following proposition has been proved in [31].

Proposition 3.3. [31, Lemma 5.3] *Let $\alpha > 0$. There exists a constant C such that for any function f which is a linear combination of H_L^1-atoms, we have $\|G_\alpha f\|_{L^1} \le C\|f\|_{H_L^1}$.*

Similar to the proof of Proposition 3.3, we can prove

Corollary 3.4. *Let $\alpha > 0$. There exists a constant C such that for any function f which is a linear combination of H_L^1-atoms, we have $\|g_\alpha f\|_{L^1} \le C\|f\|_{H_L^1}$ and $\|g_{\alpha,\lambda}^* f\|_{L^1} \le C\|f\|_{H_L^1}$.*

In order to prove the main result, we need the following Lemma (cf. [21]).

Lemma 3.5. *$f \in H_L^1(\mathbf{R}^d)$ if and only if $f \in L^1(\mathbf{R}^d)$ and $M_P(f) = \sup_{s>0} P_s^L(f) \in L^1(\mathbf{R}^d)$. Moreover, we have*

$$\|f\|_{H_L^1} \sim \|M_P(f)\|_{L^1}.$$

Now, we can give the proof of Theorem 1.4.

Proof of Theorem 1.4. By Proposition 3.3 and Corollary 3.4, for $f \in H_L^1(\mathbf{R}^d)$, we know that $g_\alpha f \in L^1(\mathbf{R}^d)$, $G_\alpha f \in L^1(\mathbf{R}^d)$ and $g_{\alpha,\lambda}^* f \in L^1(\mathbf{R}^d)$, respectively.

For the reverse, we first show that for $G_\alpha f \in L^1(\mathbf{R}^d)$, we have $f \in H_L^1(\mathbf{R}^d)$. In the following, for the sake of convenience, we use $D_t^\alpha(\cdot,\cdot)$ to denote the kernel $t^\alpha \partial_t^\alpha P_t(\cdot,\cdot)$. Assume that $f \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$. When $G_{k,\alpha} f \in L^1(\mathbf{R}^d)$, we can see that

$$\int_{\mathbf{R}^d} |G_\alpha f(x)|\, dx = \int_{\mathbf{R}^d} \Big(\int_0^\infty \int_{B(x,t)} |D_t^\alpha f(y)|^2 \frac{dy\, dt}{t^{d+1}}\Big)^{1/2} dx,$$

which implies that $D_t^\alpha f(x)$ belongs to the tent space T_2^1, where

$$D_t^\alpha f(x) := \int_{\mathbf{R}^d} D_t^\alpha(x,y) f(y)\,dy.$$

By the atomic decomposition of T_2^1, we have $D_t^\alpha f(x) = \sum_i \lambda_i a_i(x,t)$, where $a_i(x,t)$ are T_2^1-atoms and $\sum_i |\lambda_i| < \infty$. Assume that the atom $a(x,t)$ is supported on $\widehat{B}(x_0,r)$. By Lemma 3.1,

$$f(x) = C\int_0^\infty D_t^\alpha\left(\sum_{i=1}^\infty \lambda_i a_i(x,t)\right)\frac{dt}{t} := \sum_{i=1}^\infty \lambda_i \tau_i(x),$$

where $\tau_i(x) = \int_0^\infty D_t^\alpha a_i(x,t)\frac{dt}{t}$. For simplicity, we use $\tau(x)$ to denote $\tau_i(x)$ for $i = 1, 2, \cdots$. We write

$$\begin{aligned}\|\sup_{t>0}|e^{-t\sqrt{L}}\tau(x)|\|_{L^1} &\leq \|(\sup_{t>0}|e^{-t\sqrt{L}}\tau(x)|)\chi_{B^*}\|_{L^1} + \|(\sup_{t>0}|e^{-t\sqrt{L}}\tau(x)|)\chi_{(B^*)^c}\|_{L^1}\\ &:= I_1 + I_2,\end{aligned}$$

where $B^* = B(x_0, 2r)$. For I_1, we use Hölder's inequality to deduce that

$$\begin{aligned}\|\tau\|_2 &= \sup_{\|\phi\|_2\leq 1}\int_{\mathbf{R}^d}\left(\int_0^\infty D_t^\alpha a(x,t)\frac{dt}{t}\right)\overline{\phi}(x)\,dx\\ &\leq \sup_{\|\phi\|_2\leq 1}\left(\int_0^\infty\int_{\mathbf{R}^d}|a(x,t)|^2\frac{dt\,dx}{t}\right)^{1/2}\left(\int_0^\infty\int_{\mathbf{R}^d}|D_t^\alpha\overline{\phi}(x)|^2\frac{dt\,dx}{t}\right)^{1/2}\\ &\leq \sup_{\|\phi\|_2\leq 1}|B|^{-1/2}\|\phi\|_2 \leq |B|^{-1/2},\end{aligned}$$

which gives $I_1 \leq |B^*|^{1/2}|B|^{-1/2} \leq C$.

Now we deal with I_2. For $s > 0$, by functional calculus and Proposition 2.4, we have

$$\begin{aligned}\left|e^{-s\sqrt{L}}\left(\int_0^\infty D_t^\alpha a(x,t)\frac{dt}{t}\right)\right| &= C\left|e^{-s\sqrt{L}}\int_0^\infty\int_0^\infty t^\alpha\partial_t P_{t+\lambda}^L a(x,t) r^{1-\alpha}\frac{d\lambda}{\lambda}\frac{dt}{t}\right|\\ &= \left|\int_0^\infty\int_0^\infty t^\alpha\partial_t P_{s+t+\lambda}^L a(x,t)\lambda^{1-\alpha}\frac{d\lambda}{\lambda}\frac{dt}{t}\right|\\ &= \left|\int_0^\infty t^\alpha\partial_t^\alpha P_{s+t}^L a(x,t)\frac{dt}{t}\right|\\ &\leq C\int_0^\infty\frac{t^\alpha}{((s+t)^2+|x-y|^2)^{\frac{d+\alpha}{2}}}|a(y,t)|\frac{dy\,dt}{t}.\end{aligned}$$

When $y \in B(x_0,r)$ and $x \in (B^*)^c$, we have $|x-y| \sim |x-x_0|$. Then

$$\begin{aligned}&\left|e^{-s\sqrt{L}}\left(\int_0^\infty D_t^\alpha a(x,t)\frac{dt}{t}\right)\right|\\ &\leq C|x-x_0|^{-(d+\alpha)}\left(\int_0^r\int_B t^{2\alpha-1}dy\,dt\right)^{1/2}\left(\int_0^r\int_B|a(y,t)|^2\frac{dy\,dt}{t}\right)^{1/2}\\ &\leq C|B|^{-\frac{1}{2}}|x-x_0|^{-(d+\alpha)}\left(\int_0^r\int_B t^{2\alpha-1}dy\,dt\right)^{1/2} \leq Cr^\alpha|x-x_0|^{-(d+\alpha)}.\end{aligned}$$

Finally, we get

$$I_2 \leq \int_{B^c(x_0,r)}\frac{r^\alpha}{|x-x_0|^{d+\alpha}}\,dx \leq C.$$

When $f \in H^1_L(\mathbf{R}^d)$, let $\widetilde{G_\alpha}$ be the bounded extension of $G_\alpha(f)$ from $L^2 \cap H^1_L(\mathbf{R}^d)$ to $H^1_L(\mathbf{R}^d)$. Since $L^2 \cap H^1_L(\mathbf{R}^d)$ is dense in $H^1_L(\mathbf{R}^d)$, there exists a sequence $\{f_n\} \subset L^2 \cap H^1_L(\mathbf{R}^d)$ such that $f_n \to f$ as $n \to \infty$ in $H^1_L(\mathbf{R}^d)$. By Proposition 2.4, we conclude that $G_\alpha(f_n) \to G_\alpha(f)$ as $n \to \infty$. By the definition of $\widetilde{G_\alpha}$, we know that $G_\alpha(f_n) \to \widetilde{G_\alpha}(f)$ as $n \to \infty$. Therefore, for $f \in H^1_L(\mathbf{R}^d)$, $G_\alpha(f) = \widetilde{G_\alpha}(f)$. This gives

$$\|f\|_{H^1_L} = \|\lim_{n\to\infty} f_n\|_{H^1_L} \le \lim_{n\to\infty} \|G_\alpha(f_n)\|_{L^1} = \|\widetilde{G_\alpha}(f)\|_{L^1} = \|G_\alpha(f)\|_{L^1}.$$

For the Littlewood–Paley g-function, it is sufficient to prove $\|G_\alpha(f)\|_{L^1} \le C\|g_\alpha(f)\|_{L^1}$. For $\beta > 0$, we define $\widetilde{G_\beta}(f)$ by

$$\widetilde{G_\beta}(f)(x) = \left(\int_0^{+\infty}\int_{|x-y|<\beta t} \left|D^\alpha_t f(y)\right|^2 \frac{dydt}{t^{d+1}}\right)^{1/2}.$$

Then same as the above proof, we can prove that $f \in H^1_L(\mathbf{R}^d)$ if and only if $\widetilde{G_\beta}(f) \in L^1(\mathbf{R}^d)$ and $f \in L^1(\mathbf{R}^d)$. Moreover, $\|f\|_{H^1_L} \sim \|\widetilde{G_\beta}(f)\|_{L^1}$.

Let $F(x)(t) := \big(\partial^\alpha_t e^{-t\sqrt{L}} f\big)(x)$ and $V(x,s) := e^{-s\sqrt{L}}F(x)$. Then

$$V(x,s)(t) = e^{-s\sqrt{L}}\big(\partial^\alpha_t e^{-t\sqrt{L}} f\big)(x) = \big(\partial^\alpha_t e^{-(s+t)\sqrt{L}} f\big)(x).$$

Therefore,

$$\begin{aligned}\int_0^{+\infty} \left|V(x,s)(t)\right|^2 t^{2\alpha-1}\,dt &= \int_0^{+\infty} \left|\big(\partial^\alpha_t e^{-(s+t)\sqrt{L}} f\big)(x)\right|^2 t^{2\alpha-1}\,dt\\ &= \int_s^{+\infty} \left|\big(\partial^\alpha_t e^{-t\sqrt{L}} f\big)(x)\right|^2 (t-s)^{2\alpha-1}\,dt.\end{aligned}$$

When $\alpha \ge \frac{1}{2}$, we have $(t-s)^{2\alpha-1} \le t^{2\alpha-1}$. Hence,

$$\sup_{s>0}\int_0^{+\infty} \left|V(x,s)(t)\right|^2 t^{2\alpha-1}\,dt \le \int_0^{+\infty} \left|\big(\partial^\alpha_t e^{-t\sqrt{L}} f\big)(x)\right|^2 t^{2\alpha-1}\,dt = \big(g_\alpha f(x)\big)^2.$$

Let $\mathbf{X} = L^2\big((0,\infty), t^{2\alpha-1}dt\big)$. Then

$$\sup_{s>0}\left\|e^{-sL}F(x)\right\|_{\mathbf{X}} \le g_\alpha f(x) \in L^1(\mathbf{R}^d).$$

Therefore, $F \in H^1_{\mathbf{X}}(\mathbf{R}^d)$, where $H^1_{\mathbf{X}}(\mathbf{R}^d)$ can be seen as a vector-valued Hardy space (cf. [19]). This shows that $\tilde{G}^{\mathbf{X}}_2 F(x) \in L^1(\mathbf{R}^d)$, where

$$\tilde{G}^{\mathbf{X}}_2 F(x) = \left(\int_0^{+\infty}\int_{|x-y|<2t} \left\|D^\alpha_t F(y)\right\|^2_{\mathbf{X}} \frac{dy\,dt}{t^{d+1}}\right)^{1/2}.$$

We can assume that $\frac{1}{2} \le \alpha < 1$. Then the identity

$$\partial^\alpha_t P^L_t = C\int_0^\infty \partial_s P^L_{s+t} s^{-\alpha}\,ds$$

gives

$$\begin{aligned}
\partial_t^\alpha P_t^L \partial_s^\alpha P_s^L &= C\int_0^\infty \int_0^\infty \partial_a P_{a+t}^L \partial_b P_{s+b}^L a^{-\alpha} da b^{-\alpha}\, db \\
&= C\int_0^\infty \int_0^\infty \partial_a^2 P_{a+b+s+t}^L a^{-\alpha} b^{-\alpha} da\, db \\
&= C\int_0^\infty \int_a^\infty \partial_\lambda^2 P_{\lambda+s+t}^L a^{-\alpha}(\lambda-a)^{-\alpha} d\lambda\, da \\
&= C\int_0^\infty \partial_\lambda^2 P_{\lambda+s+t}^L \Big(\int_0^\lambda a^{-\alpha}(\lambda-a)^{-\alpha} da\Big) d\lambda \\
&= C\int_0^\infty \partial_\lambda^2 P_{\lambda+s+t}^L \left[\left(\int_0^{\lambda/2} + \int_{\lambda/2}^\lambda\right) a^{-\alpha}(\lambda-a)^{-\alpha} da\right] d\lambda \\
&= C\int_0^\infty \partial_\lambda^2 P_{\lambda+s+t}^L \lambda^{1-2\alpha} d\lambda.
\end{aligned}$$

When $\alpha \geq 1/2$, we get

$$\partial_t^\alpha P_t^L \partial_s^\alpha P_s^L = \partial_t^{2\alpha} P_{s+t}^L.$$

Then

$$\begin{aligned}
\big(\tilde{G}_2^{\mathbf{X}} F(x)\big)^2 &= \int_0^{+\infty} \int_{|x-y|<2t} \big\|D_t^\alpha F(y)\big\|_{\mathbf{X}}^2 \frac{dy\, dt}{t^{d+1}} \\
&= \int_0^{+\infty} \int_{|x-y|<2t} \int_0^{+\infty} \big|t^\alpha \partial_t^{2\alpha} e^{-t\sqrt{L}} F(y)(s)\big|^2 s^{2\alpha-1} ds \frac{dy\, dt}{t^{d+1}} \\
&= \int_0^{+\infty} \int_0^{+\infty} \int_{|x-y|<2t} \big|\partial_t^{2\alpha} e^{-(s+t)\sqrt{L}} f(y)\big|^2 t^{2\alpha-1-d} s^{2\alpha-1} dy\, dt\, ds \\
&= \int_0^{+\infty} \int_s^{+\infty} \int_{|x-y|<2(t-s)} \big|\partial_t^{2\alpha} e^{-t\sqrt{L}} f(y)\big|^2 (t-s)^{2\alpha-1-d} s^{2\alpha-1} dy\, dt\, ds \\
&= \int_0^{+\infty} \int_0^{t} \int_{|x-y|<2(t-s)} \big|\partial_t^{2\alpha} e^{-t\sqrt{L}} f(y)\big|^2 (t-s)^{2\alpha-1-d} s^{2\alpha-1} dy\, ds\, dt \\
&\geq \int_0^{+\infty} \int_0^{t/2} \int_{|x-y|<2(t-s)} \big|\partial_t^{2\alpha} e^{-t\sqrt{L}} f(y)\big|^2 (t-s)^{2\alpha-1-d} s^{2\alpha-1} dy\, ds\, dt \\
&\geq \int_0^{+\infty} \int_0^{t/2} \int_{|x-y|<t} \big|\partial_t^{2\alpha} e^{-t\sqrt{L}} f(y)\big|^2 t^{2\alpha-1-d} s^{2\alpha-1} dy\, ds\, dt \\
&= \frac{2\alpha}{8} \int_0^{+\infty} \int_{|x-y|<t} \big|\partial_t^{2\alpha} e^{-t\sqrt{L}} f(y)\big|^2 t^{4\alpha-1-d} dy\, dt \\
&= \frac{2\alpha}{8} \int_0^{+\infty} \int_{|x-y|<t} \big|t^{2\alpha}\partial_t^{2\alpha} e^{-t\sqrt{L}} f(y)\big|^2 \frac{dy\, dt}{t^{d+1}} = \frac{2\alpha}{8}\big(\widetilde{S_L^1} f(x)\big)^2,
\end{aligned}$$

which implies $G_\alpha(f) \in L^1(\mathbf{R}^d)$. Hence, $f \in H_L^1(\mathbf{R}^d)$.

Since in the cone $\Gamma(x) = \{(y,t) \colon |x-y| < t\}$, we have $[t/(|x-y|+t)]^{2\lambda} > 2^{-2\lambda}$. Therefore,

$$G_\alpha(f)(x) \leq \left[\int_{\Gamma(x)} 2^{2\lambda} \left(\frac{t}{|x-y|+t}\right)^{2\lambda} |t^\alpha \partial_t^\alpha P_t^L f(y)|^2 \frac{dy\, dt}{t^{d+1}}\right]^{1/2} \leq 2^\lambda G_{\alpha,\lambda}^*(f)(x).$$

This completes the proof of Theorem 1.4. □

4. The characterization of the Hardy–Sobolev space $H_L^{1,\alpha}(\mathbf{R}^d)$

We denote by E_L the spectral decomposition of the operator L. If M is a bounded function on $(0,\infty)$, the spectral multiplier $M(L)$ is defined by

$$M(L)f = \int_0^\infty M(\lambda)\, dE_L(\lambda)f, \quad f \in D(M(L)),$$

where the domain

$$D(M(L)) = \Big\{ f \in L^2(\mathbf{R}^d) \colon \int_0^\infty |M(\lambda)|^2 \langle dE_L(\lambda)f, f\rangle < \infty \Big\}.$$

We say that a function M on $\mathbf{R}$ belongs to the space $C(s)$, $s \geq 0$, if

$$\begin{cases} \|M\|_{C(s)} := \sum\limits_{k=0}^{s} \sup |M^{(k)}(\lambda)| < \infty, & s \in \mathbf{Z}; \\ \|M\|_{C(s)} := \|M^{([s])}\|_{Lip(s-[s])} + \sum\limits_{k=0}^{[s]} \sup |M^{(k)}(\lambda)| < \infty, & s \in \mathbf{Z}^c. \end{cases}$$

We have the following version of spectral multiplier theorems.

Proposition 4.1. [14, Theorem 1.11] *Let M be a bounded continuous function on $(0,\infty)$. If for some $\epsilon > 0$ and a nonzero function $\phi \in C_c^\infty(0,\infty)$, there exists a constant $C > 0$ such that for every $t > 0$,*

$$\|\phi(\cdot)M(t\cdot)\|_{C(d/2+\epsilon)} \leq C,$$

then the operator $M(L)$ is bounded on $H_L^1(\mathbf{R}^d)$.

For $\alpha, \beta > 0$, let

$$M_1(\lambda) = \frac{\lambda^\alpha}{(1+\lambda)^\alpha}, \quad M_2(\lambda) = \frac{(1+\lambda)^\alpha}{1+\lambda^\alpha}, \quad M_3(\lambda) = (\beta+\lambda)^{-\alpha},$$

where $\lambda > 0$. Then, it is clear that M_j, $j = 1, 2, 3$, are smooth and bounded on $(0,\infty)$. It follows from Proposition 4.1 that

Proposition 4.2. *Let $\alpha, \beta > 0$. Then, for $j = 1, 2, 3$, the operators $M_j(L)$ can be extended to bounded operators on $H_L^1(\mathbf{R}^d)$.*

By Lemma 2.5, we can get

$$\partial_t^{k-\alpha} P_t^L(L^{\alpha/2}f) = L^{(k-\alpha)/2} P_t^L(L^{\alpha/2}f) = L^{k/2} P_t^L(f) = \partial_t^k P_t^L(f).$$

Therefore, we can prove (cf. [5] Proposition 4.1)

Lemma 4.3. *Let $0 < \alpha < k, k \in \mathbf{N}$ and $\lambda > d/2$. For every $f \in D(L^{\alpha/2})$, we have*

$$g_{k-\alpha}(L^{\alpha/2}f) = g_{k,\alpha}(f), \quad G_{k-\alpha}(L^{\alpha/2}f) = G_{k,\alpha}(f), \quad g^*_{k-\alpha,\lambda}(L^{\alpha/2}f) = g^*_{k,\alpha,\lambda}(f).$$

By Lemma 4.3 and Theorem 1.4, we get

Proposition 4.4. *Let $0 < \alpha < k, k \in \mathbf{N}$ and $\lambda > d$. Suppose that*

$$f \in D(L^{\alpha/2}) \cap H_L^1(\mathbf{R}^d) \quad \text{and} \quad L^{\alpha/2}f \in L^2(\mathbf{R}^d) \cap H_L^1(\mathbf{R}^d).$$

Then

$$\|L^{\alpha/2}f\|_{H_L^1} \sim \|g_{k,\alpha}(f)\|_{L^1} \sim \|G_{k,\alpha}(f)\|_{L^1} \sim \|g^*_{k,\alpha,\lambda}(f)\|_{L^1}.$$

Now, we can prove

Theorem 4.5. *Let $0<\alpha<k, k\in \mathbf{N}$ and $\lambda>d$. Suppose that*

$$f\in D((I+L)^{\alpha/2})\cap H^1_L(\mathbf{R}^d) \quad \text{and} \quad (I+L)^{\alpha/2}f\in L^2(\mathbf{R}^d)\cap H^1_L(\mathbf{R}^d).$$

Then

$$\begin{aligned}\|(I+L)^{\alpha/2}f\|_{H^1_L} &\sim \|f\|_{H^1_L}+\|g_{k,\alpha}(f)\|_{L^1}\sim \|f\|_{H^1_L}+\|G_{k,\alpha}(f)\|_{L^1}\\ &\sim \|f\|_{H^1_L}+\|g^*_{k,\alpha,\lambda}(f)\|_{L^1}.\end{aligned}$$

Proof. We give the proof of $\|(I+L)^{\alpha/2}f\|_{H^1_L}\sim \|f\|_{H^1_L}+\|g_{k,\alpha}(f)\|_{L^1}$. The proofs for the cases of $G_{k,\alpha}(f)$ and $g^*_{k,\alpha,\lambda}(f)$ are similar.

By Proposition 4.2, we know that the operators $L^{\alpha/2}(I+L)^{-\alpha/2}$ and $(I+L)^{\alpha/2}(I+L^{\alpha/2})^{-1}$ are bounded on $H^1_L(\mathbf{R}^d)$. Then following from Proposition 4.4, we have

$$\begin{aligned}\|(I+L)^{\alpha/2}f\|_{H^1_L} &= \|(I+L)^{\alpha/2}(I+L^{\alpha/2})^{-1}(I+L^{\alpha/2})f\|_{H^1_L}\\ &\le \|(I+L^{\alpha/2})f\|_{H^1_L}\le C(\|f\|_{H^1_L}+\|L^{\alpha/2}f\|_{H^1_L})\\ &\le C\Big[\|f\|_{H^1_L}+\|g_{k,\alpha}(f)\|_{H^1_L}\Big].\end{aligned}$$

For the reverse, we take the function $M_1(\lambda)=\lambda^{\alpha/2}(1+\lambda)^{-\alpha/2}, \lambda>0$. For any $r\in\mathbf{R}^+$, we get

$$\begin{aligned}\int_0^r \lambda^{\alpha/2}\,dE_L(\lambda)f &= \int_0^r \frac{\lambda^{\alpha/2}}{(1+\lambda)^{\alpha/2}}(1+\lambda)^{\alpha/2}\,dE_L(\lambda)f\\ &= M_1(L)\int_0^r (1+\lambda)^{\alpha/2}\,dE_L(\lambda)f.\end{aligned}$$

Letting $r\to\infty$, we get $L^{\alpha/2}(f)=M_1(\lambda)(I+L)^{\alpha/2}(f)$. Using Proposition 4.2 again, we obtain

$$\|L^{\alpha/2}f\|_{H^1_L}\le C\|(I+L)^{\alpha/2}f\|_{H^1_L}$$

and

$$\|f\|_{H^1_L}=\|(I+L)^{-\alpha/2}(I+L)^{\alpha/2}f\|_{H^1_L}\le C\|(I+L)^{\alpha/2}f\|_{H^1_L}.$$

Theorem 4.5 follows from Proposition 4.4. □

Denote by

$$S_{\alpha,L}=\{f\in H^1_L(\mathbf{R}^d)\colon (I+L)^{\alpha/2}f\in C^\infty_c(\mathbf{R}^d)\}.$$

Since $C^\infty_c(\mathbf{R}^d)$ is dense in $H^1_L(\mathbf{R}^d)$, $S_{\alpha,L}$ is dense in $H^{1,\alpha}_L(\mathbf{R}^d)$. Note that

$$S_{\alpha,L}\subset D((I+L)^{\alpha/2})\cap H^1_L(\mathbf{R}^d)$$

and

$$(I+L)^{\alpha/2}S_{\alpha,L}=C^\infty_c(\mathbf{R}^d)\subset L^2(\mathbf{R}^d)\cap H^1_L(\mathbf{R}^d).$$

By Theorem 4.5, $g_{k,\alpha}$, $G_{k,\alpha}$ and $g^*_{k,\alpha,\lambda}$ can be extended to $H^{1,\alpha}_L(\mathbf{R}^d)$ as bounded operators from $H^{1,\alpha}_L(\mathbf{R}^d)$ to $L^1(\mathbf{R}^d)$. Let $\widetilde{g_{k,\alpha}}$ be the extension of $g_{k,\alpha}$ to $H^{1,\alpha}_L(\mathbf{R}^d)$ as a bounded operator from $H^{1,\alpha}_L(\mathbf{R}^d)$ to $L^1(\mathbf{R}^d)$. Then, there exists $C>0$ such that for $f\in H^{1,\alpha}_L(\mathbf{R}^d)$,

$$\|f\|_{H^1_L}+\|\widetilde{g_{k,\alpha}}(f)\|_1\le C\|f\|_{H^{1,\alpha}_L}. \tag{4.1}$$

Now, we give the proof of Theorem 1.7.

Proof of Theorem 1.7. We first prove

$$\|f\|_{H^1_L}+\|g_{k,\alpha}(f)\|_1\le C\|f\|_{H^{1,\alpha}_L}.$$

By (4.1), it is sufficient to prove $g_{k,\alpha}(f) = \widetilde{g_{k,\alpha}}(f)$. For $N \in \mathbf{N}$ and $h \in H^1_L(\mathbf{R}^d)$, by the subordination formula, we have

$$\left|\frac{\partial^k}{\partial t^k} P^L_t(h)(x)\right| \le C \int_0^\infty \frac{e^{-t^2/8u}}{u^{(k+2)/2}} |T^L_u(h)(x)|\, du \le C t^{-k} \sup_{u>0} |T^L_u(h)(x)|. \tag{4.2}$$

Then

$$\begin{aligned}\left(\int_{1/N}^\infty \left|t^{k-\alpha}\frac{\partial^k}{\partial t^k} P^L_t(h)(x)\right|^2 \frac{dt}{t}\right)^{1/2} &\le C\left(\int_{1/N}^\infty t^{-1-2\alpha}\, dt\right)^{1/2} \sup_{u>0} |T^L_u(h)(x)| \\ &\le C N^\alpha \sup_{u>0} |T^L_u(h)(x)|.\end{aligned}$$

By the definition of $H^1_L(\mathbf{R}^d)$, we conclude that the operator

$$h \to \left(\int_{1/N}^\infty \left|t^{k-\alpha}\frac{\partial^k}{\partial t^k} P^L_t(h)(x)\right|^2 \frac{dt}{t}\right)^{1/2}$$

is bounded from $H^1_L(\mathbf{R}^d)$ to $L^1(\mathbf{R}^d)$.

Therefore, if $f = (I+L)^{-\alpha/2}h$, where $h \in H^1_L(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$, we have

$$\begin{aligned}&\left\|\left(\int_{1/N}^\infty \left|t^{k-\alpha}\frac{\partial^k}{\partial t^k} P^L_t((I+L)^{-\frac{\alpha}{2}}h)(x)\right|^2 \frac{dt}{t}\right)^{1/2}\right\|_{L^1} \\ &= \left\|\left(\int_{1/N}^\infty \left|t^{k-\alpha}\frac{\partial^k}{\partial t^k} P^L_t(L^{-\frac{\alpha}{2}}M(L)h)(x)\right|^2 \frac{dt}{t}\right)^{1/2}\right\|_{L^1} \\ &= \left\|\left(\int_{1/N}^\infty \left|t^{k-\alpha}\frac{\partial^{k-\alpha}}{\partial t^{k-\alpha}} P^L_t(M(L)h)(x)\right|^2 \frac{dt}{t}\right)^{1/2}\right\|_{L^1} \\ &\le C\|M(L)h\|_{H^1_L} \le C\|h\|_{H^1_L},\end{aligned}$$

where $C > 0$ does not depend on $N \in \mathbf{N}$ and $M(\lambda) = \lambda^{\alpha/2}(1+\lambda)^{-\alpha/2},\ \lambda > 0$. Let $N \to \infty$, we get

$$\|\widetilde{g_{k,\alpha}}(f)\|_{L^1} = \left\|\left(\int_0^\infty \left|t^{k-\alpha}\frac{\partial^k}{\partial t^k} P^L_t(f)(x)\right|^2 \frac{dt}{t}\right)^{1/2}\right\|_{L^1} \le C\|f\|_{H^{1,\alpha}_L}.$$

By $S_{\alpha,L}$ is dense in $H^{1,\alpha}_L(\mathbf{R}^d)$, for $f \in H^{1,\alpha}_L(\mathbf{R}^d)$, we have

$$\|g_{k,\alpha}(f)\|_{L^1} = \left\|\left(\int_0^\infty \left|t^{k-\alpha}\frac{\partial^k}{\partial t^k} P^L_t(f)(x)\right|^2 \frac{dt}{t}\right)^{1/2}\right\|_{L^1} = \|\widetilde{g_{k,\alpha}}(f)\|_{L^1} \le C\|f\|_{H^{1,\alpha}_L}.$$

The proofs for the cases of $G_{k,\alpha}$ and $g^*_{k,\alpha,\lambda}$ are similar.

For the reverse, we prove

$$\|\partial^\alpha_t P^L_t(f)\|_{H^1_L} \le C t^{-\alpha}\|f\|_{H^1_L}. \tag{4.3}$$

For $m \in \mathbf{N}$ and $m > \alpha$, by (4.2), we have

$$\begin{aligned}\left|\sup_{\beta>0} T^L_\beta\Big(\partial^\alpha_t P^L_t(f)(x)\Big)\right| &\le \left|\sup_{\beta>0} T^L_\beta\left(\int_0^\infty s^{m-\alpha-1}\frac{\partial^m}{\partial t^m}P^L_{t+s}(f)(x)\,ds\right)\right| \\ &\le C\left|\sup_{\beta>0}\int_0^\infty s^{m-\alpha-1}\int_0^\infty \frac{e^{-(t+s)^2/8u}}{u^{(m+2)/2}}|T^L_{u+\beta}(f)(x)|\,du\,ds\right| \\ &\le C\sup_{u>0}|T^L_u(f)(x)|\int_0^\infty \frac{s^{m-\alpha-1}}{(t+s)^m}\,ds\int_0^\infty \frac{e^{-1/v}}{v^{(m+2)/2}}\,dv \\ &\le Ct^{-\alpha}\sup_{u>0}|T^L_u(f)(x)|.\end{aligned}$$

Therefore, (4.3) follows from the definition of $H^1_L(\mathbf{R}^d)$.

Assume that $f \in H^1_L(\mathbf{R}^d)$ and $g_{k,\alpha}(f) \in L^1(\mathbf{R}^d)$. Let $\{f_n\}$ be a sequence in $C^\infty_c(\mathbf{R}^d)$ such that $f_n \to f$ as $n \to \infty$ in $H^1_L(\mathbf{R}^d)$. Fix $t > 0$, denote $f^t = P^L_t(f)$ and $f^t_n = P^L_t(f_n),\ n \in \mathbf{N}$. Then f^t and f^t_n in $H^1_L(\mathbf{R}^d)$.

By Lemma 2.5 and (4.3), we have

$$\partial^\alpha_t f^t_n = L^{\alpha/2} f^t_n \in H^1_L(\mathbf{R}^d). \tag{4.4}$$

Then, by Proposition 4.1, we get

$$g^t_n = M(L)(I + L^{\alpha/2})f^t_n \in H^1_L(\mathbf{R}^d),$$

where $M(\lambda) = (1+\lambda)^{\alpha/2}(1+\lambda^{\alpha/2})^{-1},\ \lambda > 0$.

Note that $(I + L)^{-\alpha/2} g^t_n = f^t_n$, we have $f^t_n \in H^{1,\alpha}_L(\mathbf{R}^d)$ and $\|f^t_n\|_{H^{1,\alpha}_L} = \|g^t_n\|_{H^1_L}$. By (4.3) again,

$$\partial^\alpha_t f^t_n \to \partial^\alpha_t f^t, \quad \text{as } n \to \infty \quad \text{in } H^1_L(\mathbf{R}^d).$$

Proposition 4.1 and (4.4) imply that

$$g^t_n = M(L)(I + L^{\alpha/2})f^t_n = M(L)(I + \partial^\alpha_t)f^t_n \to M(L)(I + \partial^\alpha_t)f^t$$

as $n \to \infty$ in $H^1_L(\mathbf{R}^d)$. Therefore, there exists $F^t \in H^{1,\alpha}_L(\mathbf{R}^d)$ such that $f^t_n \to F^t$ as $n \to \infty$ in $H^{1,\alpha}_L(\mathbf{R}^d)$. So $f^t_n \to F^t$ as $n \to \infty$ in $H^1_L(\mathbf{R}^d)$. Since $f^t_n \to f^t$ as $n \to \infty$ in $H^1_L(\mathbf{R}^d)$, we get $f^t = F^t \in H^{1,\alpha}_L(\mathbf{R}^d)$ and $f^t_n \to f^t$ as $n \to \infty$ in $H^{1,\alpha}_L(\mathbf{R}^d)$.

Noting that $f^t_n \in L^2(\mathbf{R}^d) \cap H^1_L(\mathbf{R}^d)$ and $(I + L)^{\alpha/2} f^t_n \in L^2(\mathbf{R}^d) \cap H^1_L(\mathbf{R}^d)$, by Theorem 4.5, we get

$$\|f^t_n\|_{H^1_L} + \|g_{k,\alpha}(f^t_n)\|_1 \sim \|f^t_n\|_{H^{1,\alpha}_L}.$$

Letting $n \to \infty$, we have

$$\|f^t\|_{H^{1,\alpha}_L} \le C(\|f^t\|_{H^1_L} + \|g_{k,\alpha} f^t\|_1).$$

Since

$$\begin{aligned} g_{k,\alpha}(f^t)(x) &= \left(\int_0^\infty \left|s^{k-\alpha}\frac{\partial^k}{\partial s^k}P^L_s(f^t)(x)\right|^2\frac{ds}{s}\right)^{1/2} \\ &= \int_0^\infty \left|P^L_t\Big(s^{k-\alpha}\frac{\partial^k}{\partial s^k}P^L_s(f)(x)\right|^2\frac{ds}{s}\Big)^{1/2} \\ &\le P^L_t\left[\left(\int_0^\infty \left|s^{k-\alpha}\partial s^k P^L_s(f)(\cdot)\right|^2\frac{ds}{s}\right)^{1/2}\right](x), \end{aligned}$$

we get $\|g_{k,\alpha}(f^t)\|_1 \le \|g_{k,\alpha}(f)\|_1$. Furthermore, this gives

$$\|f^t\|_{H^{1,\alpha}_L} \le C(\|f^t\|_{H^1_L} + \|g_{k,\alpha}(f^t)\|_1) \le C(\|f\|_{H^1_L} + \|g_{k,\alpha}(f)\|_1), \tag{4.5}$$

where $C > 0$ does not depend on t.

By (4.5), we know $\{f^t\}$ are uniformly bounded in $H_L^{1,\alpha}(\mathbf{R}^d)$, i.e., $\{(I+L)^{\alpha/2}(f^t)\}$ are uniformly bounded in $H_L^1(\mathbf{R}^d)$. Since $H_L^1(\mathbf{R}^d)$ is a Banach space, we can find $g \in H_L^1(\mathbf{R}^d)$ such that

$$(I+L)^{\alpha/2}(f_j^t) \to g \quad \text{as} \quad j \to \infty,$$

where $\{f_j^t\}$ is a subsequence of $\{f^t\}$.

Since $H_L^1(\mathbf{R}^d)$ is the dual space of $VMO_L(\mathbf{R}^d)$ and $C_c^\infty(\mathbf{R}^d)$ is dense in $VMO_L(\mathbf{R}^d)$ with norm of $VMO_L(\mathbf{R}^d)$ (cf. [27]), we get

$$\lim_{j\to\infty} \langle (I+L)^{\alpha/2}(f_j^t),\ \phi\rangle = \langle g,\ \phi\rangle, \quad \phi \in C_c^\infty(\mathbf{R}^d).$$

By the fact that $(I+L)^{-\alpha/2}$ is self-adjoint and bounded on $C_c^\infty(\mathbf{R}^d)$, we have

$$\lim_{j\to\infty} \langle (f_j^t),\ \phi\rangle = \langle (I+L)^{-\alpha/2} g,\ \phi\rangle, \quad \phi \in C_c^\infty(\mathbf{R}^d).$$

Let $h = (I+L)^{-\alpha/2} g$. Then $h \in H_L^{1,\alpha}(\mathbf{R}^d)$ and

$$\lim_{j\to\infty} \langle (f_j^t),\ \phi\rangle = \langle h,\ \phi\rangle, \quad \phi \in C_c^\infty(\mathbf{R}^d).$$

By the arguments analogous to p. 776 in [2], which relay on the decay of the kernel of P_t^L, we can get

$$\lim_{t\to 0} \langle f^t, \phi\rangle = \langle f, \phi\rangle, \quad \phi \in C_c^\infty(\mathbf{R}^d). \tag{4.6}$$

It follows that $f = h$ and

$$\|f\|_{H_L^{1,\alpha}} \le C(\|f\|_{H_L^1} + \|g_{k,\alpha}(f)\|_{L^1}).$$

This completes the proof of Theorem 1.7. □

Remark 4.6. If we define the following Hardy–Sobolev space $\mathcal{H}_L^{1,\alpha}(\mathbf{R}^d)$ as the set of all functions $f \in H_L^1(\mathbf{R}^d)$ such that $L^{\alpha/2} f \in H_L^1(\mathbf{R}^d)$, with the norm

$$\|f\|_{\mathcal{H}_L^{1,\alpha}} = \|L^{\alpha/2} f\|_{H_L^1} + \|f\|_{H_L^1}.$$

Theorem 4.7. *Let $\alpha > 0$, $k \in \mathbf{N} \setminus \{0\}$ and $\lambda > d$. The following assertions are equivalent:*

(i) *$f \in \mathcal{H}_L^{1,\alpha}(\mathbf{R}^d)$;*
(ii) *$f \in H_L^1(\mathbf{R}^d)$ and $g_{k,\alpha}(f) \in L^1(\mathbf{R}^d)$ for $k > \alpha$;*
(iii) *$f \in H_L^1(\mathbf{R}^d)$ and $G_{k,\alpha}(f) \in L^1(\mathbf{R}^d)$ for $\alpha < k - (d+1)/2$;*
(iv) *$f \in H_L^1(\mathbf{R}^d)$ and $g^*_{k,\alpha;\lambda}(f) \in L^1(\mathbf{R}^d)$ for $\alpha < k - (d+1)/2$.*

Moreover, for every $f \in \mathcal{H}_L^{1,\alpha}(\mathbf{R}^d)$,

$$\|f\|_{\mathcal{H}_L^{1,\alpha}} \sim \|f\|_{H_L^1} + \|g_{k,\alpha}(f)\|_{L^1} \sim \|f\|_{H_L^1} + \|G_{k,\alpha}(f)\|_{L^1} \sim \|f\|_{H_L^1} + \|g^*_{k,\alpha;\lambda}(f)\|_{L^1}.$$

Since the above proof is similar to that of Theorem 1.7, we omit the details. Theorems 1.7 and 4.7 imply that the Hardy–Sobolev spaces $H_L^{1,\alpha}(\mathbf{R}^d)$ and $\mathcal{H}_L^{1,\alpha}(\mathbf{R}^d)$ are equivalent.

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