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EXISTENCE OF POSITIVE SOLUTION FOR THE NONLINEAR KIRCHHOFF TYPE EQUATIONS IN THE HALF SPACE WITH A HOLE

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Abstract. In this paper, we study the following nonlinear problem of Kirchhoff type

$$-\left(a+b\int_{\Omega_{r,\rho}}|\nabla u|^2\right)\Delta u+u=|u|^{p-1}u,\quad u>0,\ x\in\Omega_{r,\rho},\ u\in H_0^1(\Omega_{r,\rho}), \tag{0.1}$$

where $\Omega_{r,\rho}$ is a half space with a hole which is related to r,ρ in $\mathbf{R}^3$, $a,b>0$ are constants and $3<p<5$. We prove that problem (0.1) has a positive high energy solution by using a linking argument with barycenter map restricted on a Nehari manifold.

1. Introduction

Let $\mathbf{R}^3_+=\{(x_1,x_2,x_3)\in\mathbf{R}^3 \mid x_3>0\}$ be the upper half space. We consider the following nonlinear problem of Kirchhoff type:

$$-\left(a+b\int_{\Omega_{r,\rho}}|\nabla u|^2\right)\Delta u+u=|u|^{p-1}u,\quad u>0,\ x\in\Omega_{r,\rho},\ u\in H_0^1(\Omega_{r,\rho}), \tag{1.1}$$

where $a,b>0$ are constants, $3<p<5$, $r>0$, $\Omega_{r,\rho}$ is an unbounded smooth domain such that

$$\overline{\Omega_{r,\rho}}\subset\mathbf{R}^3_+,$$

and its complement

$$\overline{\Omega_{r,\rho}}^c\subset B_\rho(a_r)\quad\text{and}\quad\overline{\Omega_{r,\rho}}^c\subset\mathbf{R}^3_+,$$

where $B_\rho(a_r)$ is the open ball in the Euclidean space with center $a_r=(a,r)\notin\overline{\Omega_{r,\rho}}$ and radius $\rho>0$. Indeed, $\Omega_{r,\rho}$ is a half space with a hole.

Problem (1.1) has two features. Firstly, we consider the following elliptic problem

$$-\Delta u+u=|u|^{p-1}u,\quad x\in\Omega,\quad u\in H_0^1(\Omega), \tag{1.2}$$

where $1<p<5$. When Ω is a bounded domain, by applying the compactness of the embedding $H_0^1(\Omega)\hookrightarrow L^p(\Omega), 1<p<6$, there is a positive solution of (1.2). If Ω is an unbounded domain, we can not obtain a solution for problem (1.2) by using Mountain Pass Theorem directly because the embedding $H_0^1(\Omega)\hookrightarrow L^p(\Omega), 1<p<6$ is not compact. However, if $\Omega=\mathbf{R}^3$, Berestycki–Lions [3] proved that there is a radial positive solution of equation (1.2) by applying the compactness of the embedding $H_r^1(\mathbf{R}^3)\hookrightarrow L^p(\mathbf{R}^3), 2<p<6$, where $H_r^1(\mathbf{R}^3)$ consists of the radially symmetric functions in $H^1(\mathbf{R}^3)$. By the Lions's Concentration-Compactness Principle [13], there

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exists an unique positive solution for problem (1.2) in $\mathbf{R}^3$. By the moving plane method, Gidas–Ni–Nirenberg [5] also proved that every positive solution of equation

$$-\Delta u + u = |u|^{p-1}u, \quad x \in \mathbf{R}^3, \quad u \in H^1(\mathbf{R}^3) \tag{1.3}$$

is radially symmetric with respect to some point in $\mathbf{R}^3$ satisfying

$$u(r)re^r = \gamma + o(1) \quad \text{as } r \to \infty. \tag{1.4}$$

Kwong [9] proved that the positive solution of (1.3) is unique up to translations.

From the above researches, we believed that the existence of the solution to the equation (1.2) will be affected by the topological property of the domain Ω. In fact, Esteban and Lions [4] proved that there is no any nontrivial solution of equation (1.2) when Ω is an Esteban–Lions domain (for example $\mathbf{R}^3_+$). Thus, we want to change the topological property of the domain Ω to look for a solution of problem (1.2). Wang [15] proved that if ρ is sufficiently small and $z_{0N} \to \infty$, then equation (1.2) admits a positive higher energy solution in half space with a hole. Such a problem has been extensively studied in recent years, see for instance, [2, 12] and references therein.

Secondly, we consider the following Kirchhoff problem

$$-\left(a + b\int_\Omega |\nabla u|^2\right)\Delta u + u = |u|^{p-1}u, \quad x \in \Omega, \quad u \in H_0^1(\Omega), \tag{1.5}$$

which is related to the stationary analogue of the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial t}\right|^2 dx\right)\frac{\partial^2 u}{\partial t^2} = 0$$

presented by Kirchhoff in [8].

When $\Omega = \mathbf{R}^3$, Li and Ye [11] proved that

$$-\left(a + b\int_{\mathbf{R}^3} |\nabla u|^2\right)\Delta u + u = |u|^{p-1}u, \quad x \in \mathbf{R}^3, \quad u \in H^1(\mathbf{R}^3), \tag{1.6}$$

has a positive ground state solution by using a monotonicity trick and a new version of global compactness lemma. There are many works about the existence of nontrivial solutions to (1.6) by using variational methods, see [6, 7, 14, 17, 18] etc. Li–Peng–Xiang [10] proved that every positive solution of (1.6) is unique up to translations. Indeed, they obtained that the positive solutions of problem (1.6) will be expressed as

$$u(x) = Q\left(\frac{x-t}{\sqrt{c}}\right), \quad x \in \mathbf{R}^3$$

for some $t \in \mathbf{R}^3$, where $Q(x) \in H^1(\mathbf{R}^3)$ is the unique positive radial function of problem (1.3), and $c > 0$ satisfies

$$\sqrt{c} = \frac{1}{2}\left(b\|\nabla Q\|_2^2 + \sqrt{b^2\|\nabla Q\|_2^4 + 4a}\right),$$

which implies that

$$\mathcal{M} = \left\{ Q\left(\frac{2(x-t)}{b\|\nabla Q\|_2^2 + \sqrt{b^2\|\nabla Q\|_2^4 + 4a}}\right) \,\middle|\, x, t \in \mathbf{R}^3 \right\}$$

consists of all the positive solutions of equation (1.6). Therefore, by the similar method of [10], we get that the following problem

$$-\left(a+b\int_{\mathbf{R}^3_+}|\nabla u|^2\right)\Delta u+u=|u|^{p-1}u,\quad x\in\mathbf{R}^3_+,\ u\in H^1_0(\mathbf{R}^3_+), \tag{1.7}$$

has no any nontrivial solution. In fact, if there exists a solution $\tilde{u}$ of problem (1.7), it implies that Equation (1.2) will also have a solution

$$\tilde{Q}(x)=\tilde{u}(x\sqrt{c})\quad\text{and}\quad c=a+b\int_{\mathbf{R}^3_+}|\nabla\tilde{u}|^2\,dx$$

in $\mathbf{R}^3_+$. However, by Esteban–Lions' conclusion, we know that $\mathbf{R}^3_+$ is an Esteban–Lions domain and there is no nontrivial solution of problem (1.2). Thus we get a contradiction and it is interesting to consider the existence of the high energy equation for the problem (1.1) in the half space with a hole in $\mathbf{R}^3$.

Theorem 1.1. *There is $\rho_0>0, r_0>0$ such that if $0<\rho\le\rho_0$ and $r\ge r_0$, then there is a positive solution of equation (1.1).*

From the fact that the ratio

$$\frac{\int_{\Omega_{r,\rho}}a|\nabla u|^2+u^2\,dx+b\left(\int_{\Omega_{r,\rho}}|\nabla u|^2\,dx\right)^2}{\left(\int_{\Omega_{r,\rho}}|u|^{p+1}\,dx\right)^{\frac{2}{p+1}}}$$

is not invariant under scaling, so we can't deal with it in the set

$$\left\{u\in H^1_0(\Omega_{r,\rho})\backslash\{0\}\ \middle|\ \int_{\Omega_{r,\rho}}|u|^{p+1}\,dx=1\right\}$$

by the similar argument in [2, 15]. In this paper, we try to consider it in the Nehari manifold

$$M_{r,\rho}=\left\{u\in H^1_0(\Omega_{r,\rho})\backslash\{0\}\ \middle|\ \int_{\Omega_{r,\rho}}a|\nabla u|^2+u^2\,dx+b\left(\int_{\Omega_{r,\rho}}|\nabla u|^2\,dx\right)^2=\int_{\Omega_{r,\rho}}|u|^{p+1}\,dx\right\}.$$

The paper is organized as follows. In Section 2, we give some preliminary results. The compactness lemma will be given in Section 3. At last, we give the proof of Theorem 1.1.

2. Some preliminary results

It is well known that the solutions for equation (1.1) are the critical points of the energy function $I_{\Omega_{r,\rho}}\colon H^1_0(\Omega_{r,\rho})\to R$ defined as

$$I_{\Omega_{r,\rho}}(u)=\frac{1}{2}\int_{\Omega_{r,\rho}}(a|\nabla u|^2+u^2)\,dx+\frac{b}{4}\left(\int_{\Omega_{r,\rho}}|\nabla u|^2\,dx\right)^2-\frac{1}{p+1}\int_{\Omega_{r,\rho}}|u|^{p+1}\,dx. \tag{2.1}$$

In order to obtain the existence of solution for equation (1.1), we must consider the equation (1.6). Now, we denote the energy function $I\colon H^1(\mathbf{R}^3)\to R$ associated to (1.6) as

$$I(u)=\frac{1}{2}\int_{\mathbf{R}^3}(a|\nabla u|^2+u^2)\,dx+\frac{b}{4}\left(\int_{\mathbf{R}^3}|\nabla u|^2\,dx\right)^2-\frac{1}{p+1}\int_{\mathbf{R}^3}|u|^{p+1}\,dx, \tag{2.2}$$

and we introduce an equivalent norm on $H^1(\mathbf{R}^3)$: the norm of $u \in H^1(\mathbf{R}^3)$ is defined as

$$\|u\| = \left(\int_{\mathbf{R}^3} (a|\nabla u|^2 + u^2)\, dx\right)^{\frac{1}{2}}.$$

Consider the set of solutions to equation (1.6)

$$m = \inf\{I(v)\colon v \in H^1(\mathbf{R}^3) \text{ is a nontrival solution to (1.6)}\}. \tag{2.3}$$

A nontrivial solution u to equation (1.6) is called a ground state solution if

$$I(u) = m.$$

First, we recall some known facts from Li, Peng and Xiang [10].

Lemma 2.1. *Let $a, b > 0$ be positive constants and $1 < p < 5$. Let m be the ground state energy defined as in (2.3). Then, there exists a ground state of (1.6) which is positive, and there holds*

$$m > 0.$$

Moreover, for any positive solution u, there hold

(1) (smoothness) $u \in C^\infty(\mathbf{R}^3)$;
(2) (symmetry) *there exists a decreasing function $v\colon [0,\infty) \to (0,\infty)$ such that $u(x) = v(|x - x_0|)$ for a point $x_0 \in \mathbf{R}^3$;*
(3) (asymptotics) *for any multi-index $\alpha \in \mathbf{N}^n$, there exist constants $\delta_\alpha > 0$ and C_α such that*

$$|D^\alpha u(x)| \le C_\alpha e^{-\delta_\alpha |x|} \quad \text{for all } x \in \mathbf{R}^3. \tag{2.4}$$

Lemma 2.2. *Let $a, b > 0$ be positive constants and $1 < p < 5$. Then, the positive solutions of equation (1.6) are unique up to translations* [10].

Now, we define the following Nehari manifolds

$$M_{r,\rho} = \left\{ u \in H_0^1(\Omega_{r,\rho})\backslash\{0\} \;\middle|\; \int_{\Omega_{r,\rho}} a|\nabla u|^2 + u^2\, dx + b\left(\int_{\Omega_{r,\rho}} |\nabla u|^2\, dx\right)^2 = \int_{\Omega_{r,\rho}} |u|^{p+1}\, dx \right\}, \tag{2.5}$$

$$M = \left\{ u \in H^1(\mathbf{R}^3)\backslash\{0\} \;\middle|\; \int_{\mathbf{R}^3} a|\nabla u|^2 + u^2\, dx + b\left(\int_{\mathbf{R}^3} |\nabla u|^2\, dx\right)^2 = \int_{\mathbf{R}^3} |u|^{p+1}\, dx \right\}, \tag{2.6}$$

and

$$m_{\mathbf{R}^3} = \inf_{u\in M} I(u), \qquad m_{\Omega_{r,\rho}} = \inf_{u \in M_{r,\rho}} I_{\Omega_{r,\rho}}(u).$$

Lemma 2.3. (i) *Each critical point of I in M is a critical point of I in $H^1(\mathbf{R}^3)$ and*

$$m_{\mathbf{R}^3} = m.$$

(ii) *Each critical point of $I_{\Omega_{r,\rho}}$ in $M_{r,\rho}$ is a critical point of $I_{\Omega_{r,\rho}}$ in $H_0^1(\Omega_{r,\rho})$.*

Proof. (i) Suppose that u is a critical point of $I|_M$, i.e. $\langle I'(u), u\rangle = 0,\ u \in M$, then there is a Lagrange multiplier $\lambda \in \mathbf{R}$ such that $\langle I'(u), \varphi\rangle - \lambda\langle g'(u), \varphi\rangle = 0,\ \varphi \in H^1(\mathbf{R}^3)$, where $g(u) = \int_{\mathbf{R}^3} a|\nabla u|^2 + u^2\,dx + b(\int_{\mathbf{R}^3} |\nabla u|^2\,dx)^2 - \int_{\mathbf{R}^3} |u|^{p+1}\,dx$ and

$$\begin{aligned}\langle g'(u), u\rangle &= 2\int_{\mathbf{R}^3} a|\nabla u|^2 + u^2\,dx + 4b\left(\int_{\mathbf{R}^3} |\nabla u|^2\,dx\right)^2 - (p+1)\int_{\mathbf{R}^3} |u|^{p+1}\,dx\\ &= (1-p)\int_{\mathbf{R}^3} a|\nabla u|^2 + u^2\,dx + (3-p)b\left(\int_{\mathbf{R}^3} |\nabla u|^2\,dx\right)^2.\end{aligned}$$

Since $\langle g'(u), u\rangle \neq 0$, we get $\lambda = 0$. Thus $\langle I'(u), \varphi\rangle = 0, \forall\varphi \in H^1(\mathbf{R}^3)$ and u is a critical point of I in $H^1(\mathbf{R}^3)$. Moreover, u is also a nontrivial solution to equation (1.6). Then $m_{\mathbf{R}^3} \geq m$. On the other hand, we can get $m_{\mathbf{R}^3} \leq m$. Indeed, if v is a ground state solution to equation (1.6), it holds $v \in M$.

(ii) By the similar method, we can prove that each critical point of $I_{\Omega_{r,\rho}}|_{M_{r,\rho}}$ is a critical point of $I_{\Omega_{r,\rho}}$ in $H_0^1(\Omega_{r,\rho})$. □

Take

$$\xi \in C^\infty(\mathbf{R}^+, \mathbf{R}), \quad \eta \in C^\infty(\mathbf{R}, \mathbf{R}), \tag{2.7}$$

such that

$$\xi(t) = \begin{cases} 0, & 0 \leq t \leq \rho,\\ 1, & t \geq 2\rho,\end{cases} \qquad \eta(t) = \begin{cases} 0, & t \leq 0,\\ 1, & t \geq 1,\end{cases}$$

and $0 \leq \zeta \leq 1$, $0 \leq \eta \leq 1$.

Now, we define

$$f_y(x) = \xi(|x - a_r|)\eta(x_3)\overline{u}(x - y),$$

and

$$f_{y,t}(x) = t\xi(|x - a_r|)\eta(x_3)\overline{u}(x - y),$$

where $\overline{u}(x)$ is a solution of problem (1.6).

Lemma 2.4. *Let $y = (y', y_3)$, where $y' = (y_1, y_2)$, then we have*

(1) $\|f_y - \overline{u}(x - y)\|_{L^p(\mathbf{R}^3)} = o(1)$, *$|y - a_r| \to \infty$, and $y_3 \to +\infty$, or $y_3 \to +\infty$ and $\rho \to 0$;*

(2) $\|f_y - \overline{u}(x-y)\| = o(1)$, *$|y - a_r| \to \infty$, and $y_3 \to +\infty$, or $y_3 \to +\infty$ and $\rho \to 0$.*

Proof. By Lemma 2.1, and similarly as Lemma 4 of [2], we have

$$\begin{aligned}\|f_y - \overline{u}(x - y)\|^p_{L^p(\mathbf{R}^3)} &= \int_{\Omega_{r,\rho}} |\xi(|x - a_r|)\eta(x_3) - 1|^p|\overline{u}(x - y)|^p\,dx\\ &\leq 2^p\int_{B_{2\rho(a_r)}\cup\{x_3\leq 1\}} |\overline{u}(x - y)|^p\,dx\\ &\leq 2^p\int_{B_{2\rho(a_r)}} |\overline{u}(x - y)|^p\,dx + 2^p\int_{\{x_3\leq 1\}} |\overline{u}(x - y)|^p\,dx,\end{aligned}$$

and

$$\int_{B_{2\rho(a_r)}} |\overline{u}(x - y)|^p\,dx \leq |B_{2\rho(a_r)}| \max_{x\in\mathbf{R}^3} \overline{u}(x) \to 0 \quad \text{as } \rho \to 0,$$

$$\int_{B_{2\rho(a_r)}} |\overline{u}(x - y)|^p\,dx \leq |B_{2\rho(a_r)}| \max_{x\in B_{2\rho(a_r)}} C_\alpha e^{-\delta_\alpha|x-y|} \to 0 \quad \text{as } |y - a_r| \to \infty,$$

$$
\begin{aligned}
\int_{\{x_3\le 1\}}|\overline{u}(x-y)|^p\,dx &= \int_{\{x_3\le 1\}}|\overline{u}(x-y)|^p\,dx \le \int_{\{x_3\le 1\}}|\overline{u}(x-y)|^p\,dx\\
&\le \int_{\{x_3\le 1\}}\left(C_\alpha e^{-\delta_\alpha\frac{|x-y|}{2}}\right)^p\left(C_\alpha e^{-\delta_\alpha\frac{|x-y|}{2}}\right)^p dx\\
&= \int_{\{x_3\le 1\}}\left(C_\alpha e^{-\delta_\alpha\frac{|x-y|}{2}}\right)^p\left(C_\alpha e^{-\delta_\alpha\frac{((x'-y')^2+(x_3-y_3)^2)^{\frac12}}{2}}\right)^p dx\\
&\le \int_{\{x_3\le 1\}}\left(C_\alpha e^{-\delta_\alpha\frac{|x-y|}{2}}\right)^p\left(C_\alpha e^{-\delta_\alpha\frac{((y_3-1)^2)^{\frac12}}{2}}\right)^p dx\\
&= \left(C_\alpha e^{-\delta_\alpha\frac{((y_3-1)^2)^{\frac12}}{2}}\right)^p\int_{\{x_3\le 1\}}\left(C_\alpha e^{-\delta_\alpha\frac{|x-y|}{2}}\right)^p dx\\
&\to 0 \quad \text{as } y_3\to+\infty.
\end{aligned}
$$

So $\|f_y-\overline{u}(x-y)\|_{L^p(\mathbf{R}^3)}=o(1)$, $|y-a_r|\to\infty$, and $y_3\to+\infty$, or $y_3\to+\infty$ and $\rho\to 0$.

(2) In the same manner we can see that, we have

$$
\begin{aligned}
\|f_y-\overline{u}(x-y)\| &= \|(\xi(|x-a_r|)\eta(x_3)-1)\overline{u}(x-y)\|^2\\
&\le \frac{c}{\rho}\int_{B_{2\rho(a_r)}\cup\{x_3\le 1\}}|\nabla\overline{u}(x-y)|^2+|\overline{u}(x-y)|^2\,dx=o(1),
\end{aligned}
$$

$|y-a_r|\to\infty$, and $y_3\to+\infty$, or $y_3\to+\infty$ and $\rho\to 0$. □

Now, we will prove that for any $y\in\mathbf{R}^3$, there exists $t_y\in\mathbf{R}^+$ such that

$$f_{y,t_y}\in M_{r,\rho},$$

and

$$t_y\to 1,\quad \text{as } |y-a_r|\to\infty, \text{ and } y_3\to+\infty.$$

Lemma 2.5. *Let $\overline{u}(x-y)$, f_y and $f_{y,t}$ be definition as above, then $\exists t_y\in\mathbf{R}^+$ s.t. $f_{y,t_y}\in M_{r,\rho}$ and $t_y\to 1$ as $|y-a_r|\to\infty$ and $y_3\to+\infty$.*

Proof. Firstly, we prove that there exists $t_y\in\mathbf{R}^+$ s.t. $f_{y,t_y}\in M_{r,\rho}$. Indeed, let

$$
\begin{aligned}
g(f_{y,t}) &:= \langle I'_{\Omega_{r,\rho}}(f_{y,t_y}),f_{y,t_y}\rangle=\langle I'_{\Omega_{r,\rho}}(tf_y),tf_y\rangle\\
&= t^2\int_{\Omega_{r,\rho}}a|\nabla f_y|^2+f_y^2\,dx+bt^4\left(\int_{\Omega_{r,\rho}}|\nabla f_y|^2\,dx\right)^2-t^{p+1}\int_{\Omega_{r,\rho}}|f_y|^{p+1}\,dx,
\end{aligned}
$$

then we have $g(f_{y,t})\to-\infty$ *as* $t\to+\infty$ if $3<p<5$. It is easy to prove that there exists $\delta>0$, $\forall 0<t<\delta$, s.t. $g(f_{y,t})>0$, and $\exists t_y\in\mathbf{R}^+$ s.t. $f_{y,t_y}\in M_{r,\rho}$.

Next we prove that t_y is unique. We suppose that there are two points $0<t_y^1<t_y^2$, s.t. $g(f_{y,t_y^1})=g(f_{y,t_y^2})=0$, then

$$(t_y^i)^2\int_{\Omega_{r,\rho}}a|\nabla f_y|^2+f_y^2\,dx+b(t_y^i)^4\left(\int_{\Omega_{r,\rho}}|\nabla f_y|^2\,dx\right)^2-(t_y^i)^{p+1}\int_{\Omega_{r,\rho}}|f_y|^{p+1}\,dx,$$

$i=1,2$, and

$$0<\left(\frac{1}{(t_y^1)^2}-\frac{1}{(t_y^2)^2}\right)\int_{\Omega_{r,\rho}}a|\nabla f_y|^2+f_y^2\,dx=[(t_y^1)^{p-3}-(t_y^2)^{p-3}]\int_{\Omega_{r,\rho}}|f_y|^{p+1}\,dx<0,$$

which is a contradiction.

Secondly, we will prove $t_y \to 1$ as $|y-a_r| \to \infty$ and $y_3 \to +\infty$. Since $\overline{u}(x-y) \in M$ and $f_{y,t_y} \in M_{r,\rho}$, we get

$$\begin{aligned}(2.8)\quad \langle I'(\overline{u}(x-y)), \overline{u}(x-y)\rangle = &\int_{\mathbf{R}^3} a|\nabla \overline{u}(x-y)|^2 + \overline{u}(x-y)^2\,dx \\ &+ b\left(\int_{\mathbf{R}^3}|\nabla \overline{u}(x-y)|^2\,dx\right)^2 - \int_{\mathbf{R}^3}|\overline{u}(x-y)|^{p+1}\,dx = 0,\end{aligned}$$

$$\begin{aligned}(2.9)\qquad \langle I'_{\Omega_{r,\rho}}(f_{y,t_y}), f_{y,t_y})\rangle = t_y^2 &\int_{\Omega_{r,\rho}} a|\nabla f_y|^2 + (f_y)^2\,dx + bt_y^4\left(\int_{\Omega_{r,\rho}}|\nabla f_y|^2\,dx\right)^2 \\ &- t_y^{p+1}\int_{\Omega_{r,\rho}}|f_y|^{p+1}\,dx = 0.\end{aligned}$$

From $f_{y,t_y} \in M_{r,\rho}$ and $\int_{\mathbf{R}^3}|f_y|^{p+1}\,dx \neq 0$, we obtain $t_y \nrightarrow +\infty$ and there exists a constant $C > 0$ such that $t_y \to C$. Thus

$$\begin{aligned}(2.10)\quad &C^2\int_{\Omega_{r,\rho}} a|\nabla f_y|^2\,dx + (f_y)^2\,dx + bC^4\left(\int_{\Omega_{r,\rho}}|\nabla f_y|^2\,dx\right)^2 - C^{p+1}\int_{\Omega_{r,\rho}}|f_y|^{p+1}\,dx \\ &= o(1),\end{aligned}$$

and

$$\begin{aligned}(2.11)\qquad &\int_{\mathbf{R}^3} a|\nabla \overline{u}(x-y)|^2 + (\overline{u}(x-y))^2\,dx + b\left(\int_{\mathbf{R}^3}|\nabla \overline{u}(x-y)|^2\,dx\right)^2 \\ &- \int_{\mathbf{R}^3}|\overline{u}(x-y)|^{p+1}\,dx = 0.\end{aligned}$$

By (2.10), it holds

$$(2.12)\quad \int_{\Omega_{r,\rho}} a|\nabla f_y|^2 + (f_y)^2\,dx + bC^2\left(\int_{\Omega_{r,\rho}}|\nabla f_y|^2\,dx\right)^2 - C^{p-1}\int_{\Omega_{r,\rho}}|f_y|^{p-1}\,dx = o(1).$$

By (2.11), (2.12) and Lemma 2.4, there hold

$$(2.13)\qquad (1-C^2)b\left(\int_{\mathbf{R}^3}|\nabla \overline{u}(x-y)|^2\,dx\right)^2 = (1-C^{p-1})\int_{\mathbf{R}^3}|\overline{u}(x-y)|^{p+1}\,dx,$$

and

$$(2.14)\qquad b\left(\int_{\mathbf{R}^3}|\nabla \overline{u}(x-y)|^2\,dx\right)^2 = \int_{\mathbf{R}^3}|\overline{u}(x-y)|^{p+1}\,dx - \|\overline{u}(x-y)\|^2.$$

Then we have

$$\begin{aligned}(C^2-1)\|\overline{u}(x-y)\| &= (C^2 - C^{p-1})\int_{\mathbf{R}^3}|\overline{u}(x-y)|^{p+1}\,dx + o(1) \\ &\le A(C^2 - C^{p-1})\|\overline{u}(x-y)\|^{p+1},\end{aligned}$$

which implies that

$$(2.15)\qquad (C^2-1) \le A(C^2 - C^{p-1})\|\overline{u}(x-y)\|^{p-1}.$$

By (2.15) and $3 < p < 5$, we get

$$C \le 1.$$

If $C<1$, from (2.10) and Lemma 2.4, it holds

$$\begin{aligned}C^2\int_{\mathbf{R}^3} a|\nabla\overline{u}(x-y)|^2\,dx+bC^4\left(\int_{\mathbf{R}^3}|\nabla\overline{u}(x-y)|^2\,dx\right)^2\\ -C^{p+1}\int_{\mathbf{R}^3}\overline{u}(x-y)^{p+1}\,dx=0.\end{aligned}\tag{2.16}$$

By (2.6) and (2.16), we have

$$(C^2-C^{p+1})\int_{\mathbf{R}^3} a|\nabla\overline{u}(x-y)|^2\,dx+b(C^4-C^{p+1})\left(\int_{\mathbf{R}^3}|\nabla\overline{u}(x-y)|^2\,dx\right)^2=0.\tag{2.17}$$

As $3<p<5$ and

$$\int_{\mathbf{R}^3} a|\nabla\overline{u}(x-y)|^2\,dx>0,\quad \text{and}\quad \left(\int_{\mathbf{R}^3}|\nabla\overline{u}(x-y)|^2\,dx\right)^2>0,\tag{2.18}$$

it implies that there is a contradiction, so $C=1$. □

Lemma 2.6. *Problem (1.1) has no ground state solution, that is $m_{\Omega_{r,\rho}}$ is not a critical value of $I_{\Omega_{r,\rho}}$.*

Proof. The proof is similar to that of Theorem 2.4 of [15], and we only give a sketch here. Note that $m_{\Omega_{r,\rho}}\ge m_{\mathbf{R}^3}$ since each function in $H_0^1(\Omega_{r,\rho})$ can be extended by 0 outside $\Omega_{r,\rho}$. Take a sequence y^n in $\Omega_{r,\rho}$ such that

$$|y^n-a_r|\to\infty,\quad\text{and}\quad y_3^n\to+\infty\quad\text{as}\ \ n\to\infty.$$

Then by Lemma 2.4 and Lemma 2.5, we have

$$\begin{aligned}\|f_y-\overline{u}(x-y)\|_{L^p(\mathbf{R}^3)}&=o(1),\ |y-a_r|\to\infty,\ \text{and}\ y_3\to+\infty,\\ \|f_y-\overline{u}(x-y)\|&=o(1),\ |y-a_r|\to\infty,\ \text{and}\ y_3\to+\infty,\\ I_{\Omega_{r,\rho}}(f_{y,t_{y^n}})-I(\overline{u}(x-y_n))&=o(1).\end{aligned}$$

So

$$m_{\mathbf{R}^3}=I(\overline{u}(x-y))=I_{\Omega_{r,\rho}}(f_{y,t_{y^n}})+o(1)\ge m_{\Omega_{r,\rho}}.$$

We then conclude that $m_{\mathbf{R}^3}=m_{\Omega_{r,\rho}}$.

Now let us suppose that there is a $u_0\in M_{r,\rho}$ such that $I_{\Omega_{r,\rho}}(u_0)=m_{\mathbf{R}^3}$. By putting $u_0=0$ *in* $\mathbf{R}^3\setminus\Omega_{r,\rho}$, we see that u_0 could be regarded as an element of $H^1(\mathbf{R}^3)$. Thus u_0 would be a minimizer for M and a solution of (1.6) strictly positive in $\mathbf{R}^3$ (by the strong maximum principle): a contradiction. In other words, if u_0 is a solution of (1.1) satisfying $u_0\in M_{r,\rho}$, then $I_{\Omega_{r,\rho}}(u_0)>m_{\Omega_{r,\rho}}$. □

3. A compactness lemma

The arguments of the above Lemma 2.6 provide a picture of how the Palais–Smale condition may fail. Now, we will state a decomposition theorem for a $(PS)_\tau$ sequence of problem (1.1).

For $\tau\in\mathbf{R}$, a sequence $\{u_n\}$ is a $(PS)_\tau$ sequence in $H_0^1(\Omega_{r,\rho})$ for $I_{\Omega_{r,\rho}}$ if

$$I_{\Omega_{r,\rho}}(u_n)\to\tau,\quad I_{\Omega_{r,\rho}}(u_n)=o(1)\quad\text{strongly in}\ \ H^{-1}(\Omega_{r,\rho}).$$

Lemma 3.1. (Palais–Smale decomposition lemma for $I_{\Omega_{r,\rho}}$) *Let $\{u_n\}$ be a bounded $(PS)_\tau$ sequence in $H_0^1(\Omega_{r,\rho})$ for $I_{\Omega_{r,\rho}}$ with $\tau > 0$. Then there exists $u \in H_0^1(\Omega_{r,\rho})$ and $A \in \mathbf{R}$ such that $J'_{A,\Omega_{r,\rho}}(u) = 0$, where*

$$J_{A,\Omega_{r,\rho}}(u) = \frac{a+bA^2}{2}\int_{\Omega_{r,\rho}} |\nabla u|^2\,dx + \frac{1}{2}\int_{\Omega_{r,\rho}} u^2\,dx - \frac{1}{p+1}\int_{\Omega_{r,\rho}} |u|^{p+1}\,dx, \tag{3.1}$$

and either

(i) *$u_n \to u$ in $H_0^1(\Omega_{r,\rho})$, or*

(ii) *there exists $l \in N$ and $\{y_n^k\} \subset \mathbf{R}^3$, $|y_n^k| \to +\infty$ as $n \to +\infty$ for each $1 \le k \le l$, nontrivial solutions $w^1, \cdots, w^l$ of the following problem*

$$-(a+bA^2)\Delta u + u = |u|^{p-1}u, \quad u > 0,\ x \in \mathbf{R}^3, \tag{3.2}$$

with its related functional

$$J_{A,\mathbf{R}^3}(u) = \frac{a+bA^2}{2}\int_{\mathbf{R}^3} |\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbf{R}^3} u^2\,dx - \frac{1}{p+1}\int_{\mathbf{R}^3} |u|^{p+1}\,dx, \tag{3.3}$$

such that

$$\tau + \frac{bA^4}{4} = J_{A,\Omega_{r,\rho}}(u) + \sum_{k=1}^{l} J_{A,\mathbf{R}^3}(w^k), \tag{3.4}$$

and

$$\left\| u_n - u - \sum_{k=1}^{l} w^k(x - y_n^k) \right\| \to 0, \tag{3.5}$$

$$A^2 = \int_{\mathbf{R}^3} |\nabla u|^2\,dx + \sum_{k=1}^{l}\int_{\mathbf{R}^3} |\nabla w^k|^2\,dx. \tag{3.6}$$

Moreover,

$$J_{A,\mathbf{R}^3}(w^1) = m_{\Omega_{r,\rho}} + \frac{bA^4}{4}. \tag{3.7}$$

Proof. Note that each function in $H_0^1(\Omega_{r,\rho})$, by extending it to be 0 outside $\Omega_{r,\rho}$, can be considered as a function in $H^1(\mathbf{R}^3)$. Since $\{u_n\}$ is bounded in $H_0^1(\Omega_{r,\rho})$, there exists a $u \in H_0^1(\Omega_{r,\rho})$ and $A \in \mathbf{R}$ such that

$$u_n \rightharpoonup u \quad \text{in} \quad H_0^1(\Omega_{r,\rho}), \tag{3.8}$$

and

$$\int_{\Omega_{r,\rho}} |\nabla u_n|^2\,dx \to A^2. \tag{3.9}$$

Then $I'_{\Omega_{r,\rho}}(u) \to 0$ implies that

$$\int_{\Omega_{r,\rho}} a\nabla u\nabla\varphi + u\varphi\,dx + bA^2\int_{\Omega_{r,\rho}} \nabla u\nabla\varphi\,dx - \int_{\Omega_{r,\rho}} |u|^{p-1}u\varphi\,dx = 0 \tag{3.10}$$

for all $\varphi \in H_0^1(\Omega_{r,\rho})$, i.e. $J'_{A,\Omega_{r,\rho}}(u) = 0$, where

$$J_{A,\Omega_{r,\rho}}(u) = \frac{a+bA^2}{2}\int_{\Omega_{r,\rho}} |\nabla u|^2\,dx + \frac{1}{2}\int_{\Omega_{r,\rho}} u^2\,dx - \frac{1}{p+1}\int_{\Omega_{r,\rho}} |u|^{p+1}\,dx. \tag{3.11}$$

Since

$$
\begin{aligned}
J_{A,\Omega_{r,\rho}}(u_n) &= \frac{a+bA^2}{2}\int_{\Omega_{r,\rho}}|\nabla u_n|^2\,dx + \frac{1}{2}\int_{\Omega_{r,\rho}}u_n^2\,dx - \frac{1}{p+1}\int_{\Omega_{r,\rho}}|u_n|^{p+1}\,dx\\
&= \frac{a}{2}\int_{\Omega_{r,\rho}}|\nabla u_n|^2\,dx + \frac{1}{2}\int_{\Omega_{r,\rho}}u_n^2\,dx + \frac{b}{4}(\int_{\Omega_{r,\rho}}|\nabla u_n|^2\,dx)^2\\
&\quad - \frac{1}{p+1}\int_{\Omega_{r,\rho}}|u_n|^{p+1}\,dx + \frac{bA^2}{4}\int_{\Omega_{r,\rho}}|\nabla u_n|^2\,dx + o(1)\\
&= I_{\Omega_{r,\rho}}(u_n) + \frac{bA^4}{4} + o(1),
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
\langle J'_{A,\Omega_{r,\rho}}(u_n),\varphi\rangle &= (a+bA^2)\int_{\Omega_{r,\rho}}\nabla u_n\nabla\varphi + u_n\varphi\,dx - \int_{\Omega_{r,\rho}}|u_n|^{p-1}u_n\varphi\,dx\\
&= \int_{\Omega_{r,\rho}}a\nabla u_n\nabla\varphi + u_n\varphi\,dx + b\int_{\Omega_{r,\rho}}|\nabla u_n|^2\int_{\Omega_{r,\rho}}\nabla u_n\nabla\varphi\,dx\\
&\quad - \int_{\Omega_{r,\rho}}|u_n|^{p-1}u_n\varphi\,dx + o(1)\\
&= \langle I'_{\Omega_{r,\rho}}(u_n),\varphi\rangle + o(1).
\end{aligned}
\tag{3.13}
$$

We conclude that

$$
J_{A,\Omega_{r,\rho}}(u_n)\to\tau+\frac{bA^4}{4},\quad J'_{A,\Omega_{r,\rho}}(u_n)\to 0 \ in\ H_0^{-1}(\Omega_{r,\rho}).
$$

We next show that either (i) or (ii) holds. The argument is similar to [11].

Step 1. Set $u_n^1 = u_n - u$, by (3.8) and the Brezis–Lieb Lemma of [16] we get that

a.1 $|\nabla u_n^1|_2^2 = |\nabla u_n|_2^2 - |\nabla u|_2^2 + o(1)$;
b.1 $|u_n^1|_2^2 = |u_n|_2^2 - |u|_2^2 + o(1)$;
c.1 $J_{A,\mathbf{R}^3}(u_n^1)\to m_{\mathbf{R}^3} + \frac{bA^4}{4} - J_{A,\Omega_{r,\rho}}(u)$;
d.1 $J'_{A,\mathbf{R}^3}(u_n^1)\to 0$ in $H^{-1}(\mathbf{R}^3)$.

Decompose $\mathbf{R}^3$ into nonoverlapping countable cubes Q_i with centres $(x', m+1/2)$ for integers m and side length 1. Let

$$
h_n = \sup_{|i|=0,1,2,\cdots}\int_{Q_i}|u_n^1|^2\,dx,
$$

and

$$
\sigma^1 = \limsup_{n\to+\infty} h_n.
$$

Vanishing: If $\sigma^1 = 0$, then it follows from the Brezis–Lieb Lemma that $u_n^1\to 0$ in $L^s(\mathbf{R}^3)$ for $s\in(2,2^*)$. Since $J'_{A,\mathbf{R}^3}(u^1)\to 0$ in $H^{-1}(\mathbf{R}^3)$, we see that $u_n^1\to 0$ in $H^1(\mathbf{R}^3)$ and the proof is completed.

Non-Vanishing: If $\sigma^1 > 0$, then we can find a sequence $\{Q_n^1\}$ with centre y_n^1 of the form $\{x'_n, m_n+\frac{1}{2}\}$, such that

$$
\int_{Q_n^1}|u_n^1|^2\,dx > \frac{\sigma^1}{2}.
$$

Set $w_n^1 = u_n^1(x + y_n^1)$, then $\{w_n^1\}$ is bounded in $H^1(\mathbf{R}^3)$ and we may assume that $w_n^1 \rightharpoonup w^1$ in $H^1(\mathbf{R}^3)$. It implies $J'_{A,\mathbf{R}^3}(w^1) = 0$. By

$$\int_Q |w_n^1|^2 \, dx > \frac{\sigma^1}{2},$$

where $Q = \{(x', x_3) \in \mathbf{R}^2 \times \mathbf{R} \mid |x'| < \frac{1}{2}, -1/2 < x_3 < 1/2\}$, we see that $w^1 \neq 0$. Moreover, $w_n^1 \rightharpoonup 0$ in $H^1(\mathbf{R}^3)$ implies that $\{y_n^1\}$ is unbounded. Hence, we may assume that $|y_n^1| \to \infty$.

Step 2: Set $u_n^2 = u_n - u - w^1(x - y_n^1)$, we can similarly check that

a.2 $|\nabla u_n^2|_2^2 = |\nabla u_n|_2^2 - |\nabla u|_2^2 - |\nabla w^1|_2^2 + o(1)$;
b.2 $|u_n^2|_2^2 = |u_n|_2^2 - |u|_2^2 - |w^1|_2^2 + o(1)$;
c.2 $J_{A,\mathbf{R}^3}(u_n^1) \to m_{R^3} + \frac{bA^4}{4} - J_{A,\Omega_{r,\rho}}(u) - J_{A,\mathbf{R}^3}(w^1)$;
d.2 $J'_{A,\mathbf{R}^3}(u_n^2) \to 0$ in $H^{-1}(\mathbf{R}^3)$.

Similar to the arguments in Step 1, let

$$\sigma^2 = \limsup_{n\to+\infty} \sup_{|i|=0,1,2,\cdots} \int_{Q_i} |u_n^2|^2 \, dx.$$

If vanishing occurs, then$\|u_n^2\| \to 0$, i.e. $\|u_n - u - w^1(x - y_n^1)\| \to 0$. Moreover, by (3.9) and (a.2) (c.2), we see that

$$A^2 = |\nabla u|_2^2 + |\nabla w^1|_2^2 \quad \text{and} \quad \tau + \frac{bA^4}{4} = J_{A,\mathbf{R}^3}(w^1) + J_{A,\Omega_{r,\rho}}(u).$$

If non-vanishing occurs, then there exists a sequence $\{y_n^2\} \subset \mathbf{R}^3$ and a nontrivial $w^2 \in H^1(\mathbf{R}^3)$ such that $w_n^2 = u_n^2(x + y_n^1) \rightharpoonup w^2$ in $H^1(\mathbf{R}^3)$. Then by (d.2), we have $J'_{A,\mathbf{R}^3}(w^2) = 0$. Furthermore, $u_n^2 \rightharpoonup 0$ in $H^1(\mathbf{R}^3)$ implies that $|y_n^1| \to \infty$ and $|y_n^2 - y_n^1| \to \infty$.

We next proceed by iteration. Recall that if w^k is a nontrivial solution of $J_{A,\mathbf{R}^3}$, then $J_{A,\mathbf{R}^3}(w^k) > 0$. So there exists some finite $l \in \mathbf{N}$ such that only the vanishing case occurs in Step l. Then the lemma is proved.

Next, we prove that

$$J_{A,\mathbf{R}^3}(w^1) = m_{\Omega_{r,\rho}} + \frac{bA^4}{4}.$$

Let $\{u_n\} \subset M_{r,\rho}$ be a minimizing sequence for $m_{\Omega_{r,\rho}}$, then by the Ekelands variational principle (see Theorem 8.5 in [16]), there exists a sequence $\{v_n\} \subset M_{r,\rho}$ such that

$$I_{\Omega_{r,\rho}} \to m_{\Omega_{r,\rho}}, \; (I_{\Omega_{r,\rho}}|_{M_{r,\rho}})'(v_n) \to 0 \quad \text{in} \quad H^{-1}(\Omega_{r,\rho}), \; \|v_n - u_n\| \to 0.$$

By Lemma 2.3(ii), $(I_{\Omega_{r,\rho}})'(v_n) \to 0 \; in \; H^{-1}(\Omega_{r,\rho})$. Since $\{v_n\}$ is a bounded $(PS)_{m_{\Omega_{r,\rho}}}$ sequence for $I_{\Omega_{r,\rho}}$, either (i) or (ii) holds. However, since Lemma 2.6 has showed that $m_{r,\rho}$ is not a critical value of $I_{\Omega_{r,\rho}}$, (ii) holds.

If $u \neq 0$, then $J'_{A,\Omega_{r,\rho}}(u) = 0$ and

$$\begin{aligned} 0 = \langle J'_{A,\Omega_{r,\rho}}(u), u\rangle &= (a + bA^2) \int_{\Omega_{r,\rho}} |\nabla u|^2 \, dx + \int_{\Omega_{r,\rho}} u^2 \, dx - \int_{\Omega_{r,\rho}} |u|^{p+1} \, dx \\ &\geq \int_{\mathbf{R}^3} a|\nabla u|^2 + u^2 \, dx + b\left(\int_{\mathbf{R}^3} |\nabla u|^2 \, dx\right)^2 - \int_{\mathbf{R}^3} |u|^{p+1} \, dx = g(u). \end{aligned}$$

Hence there exists an unique $t \in (0,1]$ such that $u_t \in M_{\Omega_{r,\rho}}$. So

$$\begin{aligned} J_{A,\Omega_{r,\rho}}(u) &= [J_{A,\Omega_{r,\rho}}(u) - \frac{\langle J'_{A,\Omega_{r,\rho}}(u), u\rangle}{4} - \frac{bA^2}{4}\int_{\Omega_{r,\rho}} |\nabla u|^2\,dx] + \frac{bA^2}{4}\int_{\Omega_{r,\rho}} |\nabla u|^2\,dx \\ &= \frac{a}{4}\int_{\Omega_{r,\rho}} |\nabla u|^2\,dx + \int_{\Omega_{r,\rho}} \frac{1}{4}|u|^{p+1}\,dx - \frac{1}{p+1}|u|^{p+1}\,dx + \frac{bA^2}{4}\int_{\Omega_{r,\rho}} |\nabla u|^2\,dx \\ &\geq \frac{at^2}{4}\int_{\Omega_{r,\rho}} |\nabla u|^2\,dx + \int_{\Omega_{r,\rho}} \frac{1}{4}|tu|^{p+1} - \frac{1}{p+1}|tu|^{p+1}\,dx + \frac{bA^2}{4}\int_{\Omega_{r,\rho}} |\nabla u|^2\,dx \\ &= I_{\Omega_{r,\rho}}(u_t) - \frac{1}{4}\langle J'_{A,\Omega_{r,\rho}}(u_t), u_t\rangle + \frac{bA^2}{4}\int_{\Omega_{r,\rho}} |\nabla u|^2\,dx \\ &\geq m_{\Omega_{r,\rho}} + \frac{bA^2}{4}\int_{\Omega_{r,\rho}} |\nabla u|^2\,dx. \end{aligned}$$

Since $J'_{A,\mathbf{R}^3}(w^i) = 0$ $(i = 1, 2, \cdots, l)$, similar to the above process, we have

$$J_{A,\mathbf{R}^3}(w^i) \geq m_{\mathbf{R}^3} + \frac{bA^2}{4}\int_{\mathbf{R}^3} |\nabla w^i|^2\,dx = m_{\Omega_{r,\rho}} + \frac{bA^2}{4}\int_{\mathbf{R}^3} |\nabla w^i|^2\,dx. \tag{3.14}$$

Then

$$\begin{aligned} m_{\Omega_{r,\rho}} + \frac{bA^2}{4} &= J_{A,\Omega_{r,\rho}}(u) + \sum_{k=1}^{l} J_{A,\mathbf{R}^3}(w^k) \\ &\geq m_{\Omega_{r,\rho}} + \frac{bA^2}{4}\int_{\Omega_{r,\rho}} |\nabla u|^2\,dx + l m_{\Omega_{r,\rho}} + \frac{bA^2}{4}\sum_{k=1}^{l}\int_{\mathbf{R}^3} |\nabla w^k|^2\,dx \\ &\geq 2m_{\Omega_{r,\rho}} + \frac{bA^4}{4}, \end{aligned}$$

which is a contradiction. So $u \equiv 0$, by (3.14) we have

$$\begin{aligned} m_{\Omega_{r,\rho}} + \frac{bA^2}{4} = \sum_{k=1}^{l} J_{A,\mathbf{R}^3}(w^k) &\geq m_{\Omega_{r,\rho}} + l m_{\Omega_{r,\rho}} + \frac{bA^2}{4}\sum_{k=1}^{l}\int_{\mathbf{R}^3} |\nabla w^k|^2\,dx \\ &\geq m_{\Omega_{r,\rho}} + \frac{bA^4}{4}, \end{aligned}$$

which implies $l = 1$. Thus $J_{A,\mathbf{R}^3}(w^1) = m_{\Omega_{r,\rho}} + \frac{bA^4}{4}$. $\square$

Lemma 3.2. *Let $\{u_n\} \subset M_{r,\rho}$ be $(PS)_\tau$ sequence for $I_{\Omega_{r,\rho}}$ with $\tau \in (m_{\Omega_{r,\rho}}, 2m_{\Omega_{r,\rho}})$, then up to a subsequence, $u_n \to u$ in $H_0^1(\Omega_{r,\rho})$ for some $u \in H_0^1(\Omega_{r,\rho})\backslash\{0\}$.*

Proof. Let $\{u_n\} \subset M_{r,\rho}$ be $(PS)_\tau$ sequence for $I_{\Omega_{r,\rho}}$ with $\tau \in (m_{\Omega_{r,\rho}}, 2m_{\Omega_{r,\rho}})$, then $\{u_n\}$ is a bounded in $H_0^1(\Omega_{r,\rho})$. Applying Lemma 3.1, either (i) or (ii) holds. We next show that (i) holds with $u \neq 0$ by discussing the following two cases.

Case 1. $u \equiv 0$. If $u \equiv 0$, then (i) dose not occur since $\tau > 0$, hence (ii) holds. Similar to (3.15), we have

$$\tau + \frac{bA^4}{4} = \sum_{k=1}^{l} J_{A,\mathbf{R}^3}(w^k) \geq l m_{\Omega_{r,\rho}} + \frac{bA^4}{4}.$$

Then $l = 1$. Since $\tau < 2c_{\Omega_{r,\rho}}$, and by (3.7), we have

$$\tau + \frac{bA^4}{4} = J_{A,\mathbf{R}^3}(w^1) = m_{\Omega_{r,\rho}} + \frac{bA^4}{4},$$

which contradicts to $\tau > m_{\Omega_{r,\rho}}$. So Case 1 does not occur.

Case 2. $u \neq 0$. If (ii) holds, then we still have (3.14) and (3.14) hold. Hence if $l \neq 0$, similar to (3.15), we have

$$\tau + \frac{bA^2}{4} = J_{A,\Omega_{r,\rho}}(u) + \sum_{k=1}^{l} J_{A,\mathbf{R}^3}(w^k) \geq 2m_{\Omega_{r,\rho}} + \frac{bA^4}{4},$$

which contradicts to $\tau < 2c_{\Omega_{r,\rho}}$. Thus, $l = 0$, (i) holds and the lemma is proved. □

4. Proof of Theorem 1

In order to get such $(PS)_\tau$ sequence given in Lemma 3.1, we try to use linking arguments with a barycenter map restricted to the Nehari manifold $M_{r,\rho}$.

Recall the definition and some properties of the barycenter map, which can be also found in [1]. The barycenter map $\beta\colon H^1(\mathbf{R}^3)\backslash\{0\} \to \mathbf{R}^3$ is defined as: for any $u \in H^1(\mathbf{R}^3)\backslash\{0\}$,

$$\beta(u) = \frac{1}{|V|_{L^1}} \int_{\mathbf{R}^3} xV(x)\, dx,$$

where $V(x) = (\mu(x) - \frac{1}{2}\max_{x\in\mathbf{R}^3} \mu(x))^+$ and $\mu(x) = \frac{1}{|B_1(x)|}\int_{B_1(x)} |u(y)|\, dy$.

Since $V(x)$ has compact support and is continuous, $\beta(u)$ is well defined and has the following properties:

(a) $\beta(u)$ is continuous;
(b) If u is radial, then $\beta(u) = 0$;
(c) For any $y \in \mathbf{R}^3$, then $\beta(u(\cdot - y)) = \beta(u) + y$;
(d) If u is radial, then for any $y \in \mathbf{R}^3$ and $t > 0$, $\beta(tu(\cdot - y)) = y$.

Set

$$\tilde{M}_{r,\rho} = \{u \in M_{r,\rho} \mid \beta(u) = a_r\}$$

and

$$\tilde{m}_{\Omega_{r,\rho}} = \inf_{\tilde{M}_{r,\rho}} I(u),$$

we have

Lemma 4.1. $\tilde{m}_{\Omega_{r,\rho}} > m_{\Omega_{r,\rho}}$.

Proof. It easy to see that $\tilde{m}_{\Omega_{r,\rho}} \geq m_{\Omega_{r,\rho}}$. By contradiction, we just suppose that $\tilde{m}_{\Omega_{r,\rho}} = m_{\Omega_{r,\rho}}$. Let $\{u_n\} \subset \tilde{M}_{r,\rho}$ be a minimizing sequence for $\tilde{m}_{\Omega_{r,\rho}}$, then $\{u_n\}$ is also a minimizing sequence for $m_{\Omega_{r,\rho}}$. Then there exists a bounded $(PS)_{m_{\Omega_{r,\rho}}}$ sequence $\{v_n\} \subset \tilde{M}_{r,\rho}$ for $I_{m_{\Omega_{r,\rho}}}$ such that

$$\|v_n - u_n\| \to 0. \tag{4.1}$$

Moreover,

$$v_n \to w^1(\cdot - y_n^1) \quad \text{in} \quad H^1(\mathbf{R}^3) \tag{4.2}$$

with $y_n^1 \in \mathbf{R}^3$ and $|y_n^1| \to +\infty$. By the continuity of β and (4.1), we have $\beta(v_n) \to \beta(u_n) = a_r$ However, by (4.2), we see that

$$|a_r| = \lim_{n\to+\infty} |\beta(v_n)| = \lim_{n\to+\infty} |\beta(w^1) + y_n^1| = +\infty,$$

which is impossible. Hence $\tilde{m}_{\Omega_{r,\rho}} > m_{\Omega_{r,\rho}}$. $\square$

Proof of Theorem 1.1. Let $\overline{u}$ be the radial and positive ground state solution of (1.6). By Lemma 2.5, we have

$$\exists t_y \in \mathbf{R}^+ \text{ s.t. } f_{y,t_y} \in M_{r,\rho} \text{ and } t_y \to 1 \text{ as } |y-a_r| \to +\infty \text{ and } y_3 \to +\infty.$$

Then,we define a continuous map $K\colon \mathbf{R}^3 \to M_{r,\rho}$ as

$$K(y)(x) := f_{y,t_y}(x) = \xi(|x-a_r|)\eta(x_3)t_y\overline{u}(x-y).$$

By Lemma 2.4, we have

$$\|f_y - \overline{u}(x-y)\| = o(1), \quad |y-a_r| \to \infty, \quad \text{and} \quad y_3 \to +\infty. \tag{4.3}$$

By the continuity of β and (4.3) and Lemma 2.5,

$$\beta(K(y)) \to \beta(\overline{u}(x-y)) = y, \quad \text{as } |y-a_r| \to +\infty \quad \text{and} \quad y_3 \to +\infty. \tag{4.4}$$

By (4.4) we see $\beta(K(y)) \to y \neq a_r$, as $r \to +\infty$, $for\ y \in \partial B_{\frac{r}{2}}(a_r)$, and there exists $r_1 > 0$, if $r \geq r_1$, then we have

$$\beta(K(y)) \neq a_r \quad \text{for } y \in \partial B_{\frac{r}{2}}(a_r) \tag{4.5}$$

and

$$\langle \beta(K(y)), y \rangle > 0 \quad \text{for } y \in \partial B_{\frac{r}{2}}(a_r). \tag{4.6}$$

From Lemma 2.6, it shows that

$$I_{\Omega_{r,\rho}}(K(y)) \to m_{\Omega_{r,\rho}}, \quad \text{as } |y-a_r| \to +\infty \quad \text{and} \quad y_3 \to +\infty \tag{4.7}$$

and there exists $r_2 > 0$, if $r \geq r_2 > r_1$, it holds

$$I_{\Omega_{r,\rho}}(K(y)) < 2m_{\Omega_{r,\rho}} \quad \text{for } y \in \partial B_{\frac{r}{2}}(a_r). \tag{4.8}$$

Since $\tilde{m}_{\Omega_{r,\rho}} > m_{\Omega_{r,\rho}}$, by (4.7) there exists $r_3 > r_2 > 0$ and $r \geq r_3$ such that

$$\max_{\partial B_{\frac{r}{2}}(a_r)} I_{\Omega_{r,\rho}}(K(y)) < \tilde{m}_{\Omega_{r,\rho}}. \tag{4.9}$$

From Lemma 2.4 and (4.8), fix $\rho_0 > 0$, $r_0 \geq r_3$, if $0 < \rho < \rho_0$, $r \geq r_0$, then

$$I_{\Omega_{r,\rho}}(K(y)) < 2m_{\Omega_{r,\rho}} \quad \text{for } y \in \overline{B_{\frac{r}{2}}(a_r)}.$$

From now on, fixing ρ_0, r_0, for $r \geq r_0$. Let set $Q = K(\overline{B_{\frac{r}{2}}(a_r)})$, and $S = \tilde{M}_{r,\rho}$, which is given in (4.1). Now, we prove ∂Q and S are linking. Indeed, for any $u \in \partial Q$, by (4.5) we see $\beta(u) \neq a_r$, hence $u \notin S$, and $\partial Q \cap S = $. For any $h \in H = \{h \in C(Q,\ M_{r,\rho}) | h|_{\partial Q} = id\}$, we define a map $G\colon \overline{B_{\frac{r}{2}}(a_r)} \to \mathbf{R}^3$, as

$$G(y) = (\beta \circ h \circ K)(y), \quad \forall y \in \overline{B_{\frac{r}{2}}(a_r)}.$$

Then G is continuous since it is a composition of continuous maps. Moreover, consider the homotopy, for $0 \leq t \leq 1$

$$F(t,y) = (1-t)G(y) + ty, \quad \text{for } y \in \mathbf{R}^3. \tag{4.10}$$

If $y \in \partial B_{\frac{r}{2}}(a_r)$, by (4.6),

$$\langle \beta(K(y)), y \rangle > 0.$$

It implies that

$$\langle F(t,y), y \rangle = \langle (1-t)G(y), y \rangle + \langle ty, y \rangle = \langle (1-t)\beta(K(y)), y \rangle + t\langle y, y \rangle > 0$$

Thus $F(t,y) \neq 0$ for $y \in \partial B_{\frac{r}{2}}(a_r)$. By the homotopic invariance of the degree

$$d(K(y), B_{\frac{r}{2}}(a_r), a_r) = d(I, B_{\frac{r}{2}}(a_r), a_r) = 1.$$

There is $\tilde{y} \in \overline{B_{\frac{r}{2}}(a_r)}$ such that $G(\tilde{y}) = a_r$, which implies that $h(K(\tilde{y})) \in S$. Then $K(\tilde{y}) \in h(Q) \cap S$. So S and ∂Q link. By (4.8) and (4.9), we see that

$$\inf_{u\in S} I_{\Omega_{r,\rho}} = \tilde{m}_{\Omega_{r,\rho}} > \max_{u\in\partial Q} I_{\Omega_{r,\rho}} \quad \text{and} \quad \sup_{u\in Q} I_{\Omega_{r,\rho}}(u) < 2m_{\Omega_{r,\rho}}.$$

Applying the Linking Theorem, there exists a $(PS)_\tau$ sequence $\{u_n\}$ for $I_{\Omega_{r,\rho}}$, where

$$\tau = \inf_{h\in H} \max_{u\in Q} I_{\Omega_{r,\rho}}(h(u)).$$

Since $h(Q)\cap S \neq \varnothing$ for any $h \in H$, $\tau \geq \tilde{m}_{\Omega_{r,\rho}}$. On the other hand, $\tau \leq \max_{u\in Q} I_{\Omega_{r,\rho}}(u) < 2m_{\Omega_{r,\rho}}$. So $\{u_n\}$ is a $(PS)_\tau$ sequence for $I_{\Omega_{r,\rho}}$ with $\tau \in (m_{\Omega_{r,\rho}}, 2m_{\Omega_{r,\rho}})$. By Lemma 3.2, there exists $u \in H_0^1(\Omega_{r,\rho})\backslash\{0\}$ such that $u_n \to u$ in $H_0^1(\Omega_{r,\rho})$. So u is a nontrivial solution of problem (1.1). □

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