ARITHMETIC REPRESENTATIONS OF REAL NUMBERS IN TERMS OF SELF-SIMILAR SETS

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Abstract. Suppose $n \geq 2$ and $\mathcal{A}_i \subset \{0, 1, \cdots, (n-1)\}$ for $i = 1, \cdots, l$, let $K_i = \bigcup_{a \in \mathcal{A}_i} n^{-1}(K_i + a)$ be self-similar sets contained in $[0, 1]$. Given $m_1, \cdots, m_l \in \mathbf{Z}$ with $\prod_i m_i \neq 0$, we let

$$S_x = \{(y_1, \cdots, y_l) \colon m_1 y_1 + \cdots + m_l y_l = x \text{ with } y_i \in K_i \; \forall i\}.$$

In this paper, we analyze the Hausdorff dimension and Hausdorff measure of the following set

$$U_r = \{x \colon \operatorname{Card}(S_x) = r\},$$

where $\operatorname{Card}(S_x)$ denotes the cardinality of S_x, and $r \in \mathbf{N}^+$. We prove under the so-called covering condition that the Hausdorff dimension of U_1 can be calculated in terms of some matrix. Moreover, if $r \geq 2$, we also give some sufficient conditions such that the Hausdorff dimension of U_r takes only finite values, and these values can be calculated explicitly. Furthermore, we come up with some sufficient conditions such that the dimensional Hausdorff measure of U_r is infinity. Various examples are provided. Our results can be viewed as the exceptional results for the classical slicing problem in geometric measure theory.

1. Introduction

Representation of real numbers is a topic of great interest in number theory. There are many approaches which can represent real numbers, for instance, the β-expansions [18, 1, 28, 3, 6, 25], the continued fractions [11, 10], multiplication (division, quadratic sum) on fractal sets [31, 2], the Lüroth expansions [5], and so forth. These representations are related to many different mathematical aspects, for instance, the matrix theory, ergodic theory, fractal geometry, Diophantine approximation, combinatorics, and so on. Different representations have distinct properties from various aspects. Investigating the ergodic, fractal and combinatorial properties of these representations consists of one of the main directions in number theory.

Expansions in non-integer bases were pioneered by Rényi [25] and Parry [23]. Let $1 < \beta < 2$. Given any $x \in [0, (\beta - 1)^{-1}]$, a sequence $(a_n) \in \{0, 1\}^\infty$ is called a β-expansion of x if $x = \sum_{n=1}^\infty a_n \beta^{-n}$. Sidorov [26] proved that given any $1 < \beta < 2$, then almost every point in $[0, (\beta-1)^{-1}]$ has uncountably many expansions. In fact, Sidorov [27], Dajani and de Vries [7] proved that given any $1 < \beta < 2$, then Lebesgue almost every point has uncountably many universal expansions. Erdős et al. [13] proved that there exist some $1 < \beta < 2$ and $x \in [0, (\beta - 1)^{-1}]$ such that x has precisely k different β-expansions. However, up to now, there are few results concerning with the

set of points with exactly k different β-expansions, see [9, 8] and references therein. In particular, if x has a unique expansion (such point is called the univoque point), then there are many results [12, 14, 19]. The main reason is that we may give an criterion of the unique expansions. As such we can discuss the set of points with unique expansions from the symbolic point of view.

Representation of real numbers also arises naturally from Diophantine approximation. Let x be a real number. We say that x is badly approximable if there exists a positive integer n such that for any rational number p/q,

$$|x - p/q| > (nq^2)^{-1}.$$

Clearly, the set of badly approximable points is of Lebesgue measure zero. However, Hall [15] proved that every real number can be expressed as the sum of two badly approximable numbers. For any $m \geq 2$, let $F(m)$ denote the set of numbers

$$F(m) = \{[t, a_1, a_2, \cdots] : t \in \mathbf{Z}, \ 1 \leq a_i \leq m \text{ for } i \leq 1\},$$

where

$$[t, a_1, a_2, \cdots] = t + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}.$$

Hall [15] proved that $F(4) + F(4) = \mathbf{R}$, where $A \pm B = \{x \pm y : x \in A, \ y \in B\}$. Astels [30] proved that

$$F(2) + F(5) = \mathbf{R}, \quad F(2) - F(5) = \mathbf{R}, \quad F(3) - F(4) = \mathbf{R}.$$

There are many other related results for the arithmetic sum of $F(m)$, see the references in [15, 30].

The above two representations are prevalent in number theory. In this paper we shall consider the so-called arithmetic representation of the real numbers. First, we introduce some background of this representation. Given two non-empty sets $A, B \subset \mathbf{R}$. Define $A * B = \{x * y : x \in A, \ y \in B\}$, where $*$ is $+, -, \times$ or $\div$ (when $* = \div, y \neq 0$). We call $a = x * y, \ x \in A, \ y \in B$ an arithmetic representation in terms of A and B. Shortly, we may say $a = x * y$ is an arithmetic representation if there is no fear of ambiguity. It is well-known that for the middle-third Cantor set C,

$$C - C = \{x - y : x, y \in C\} = [-1, 1].$$

Therefore, for any $a \in [-1, 1]$, there are some $x, y \in C$ such that $a = x - y$. The first proof of this result is due to Steinhaus [29] in 1917. Recently, Athreya, Reznick, and Tyson [2] considered the multiplication on the middle-third Cantor set, and proved that $17/21 \leq \mathcal{L}(C \cdot C) \leq 8/9$, where $\mathcal{L}$ denotes the Lebesgue measure. Moreover, they also proved that the division on C, denoted by $\frac{C}{C}$, is exactly the union of some closed sets. In [31], Tian et al. defined a class of overlapping self-similar sets as follows: let K be the attractor of the IFS

$$\{f_1(x) = \lambda x, \ f_2(x) = \lambda x + c - \lambda, \ f_3(x) = \lambda x + 1 - \lambda\},$$

where $f_1(I) \cap f_2(I) \neq \emptyset, (f_1(I) \cup f_2(I)) \cap f_3(I) = \emptyset$, and $I = [0, 1]$ is the convex hull of K. Then $K \cdot K = [0, 1]$ if and only if $(1 - \lambda)^2 \leq c$. Equivalently, they gave a necessary and sufficient condition such that for any $x \in [0, 1]$ there exist some $y, z \in K$ such that $x = yz$.

Motivated by the multiple β-expansions, generally it is natural to analyze the set of points in $A * B = \{x * y : x \in A, \ y \in B\}$ such that these points have exactly r

different representations, i.e. we want to analyze the following set

(1.1) $U_r = \{x \in A * B \colon x \text{ has exactly } r \text{ arithmetic representations}\}$,

where $r \in \mathbf{N}^+$. In this paper, we assume the above algorithm $*$ is $+$ or $-$. For the classical middle-third Cantor set, take $1/3 \in C - C = [-1, 1]$, it is not difficult to prove that $1/3$ has only three arithmetic representations in $C - C$, i.e.

$$\frac{1}{3} = \frac{1}{3} - 0 = \frac{2}{3} - \frac{1}{3} = 1 - \frac{2}{3}.$$

We may give a new explanation for Steinhaus' result from the projectional perspective. Note that $C - C$ is conjugated to $\sqrt{2}\mathrm{Proj}_\theta(C \times C)$, where $\theta = \frac{3\pi}{4}$ and Proj_θ denotes the orthogonal projection onto L_θ which is the line through the origin in direction θ. Since $C - C = [-1, 1]$, it follows that

$$\dim_H(\mathrm{Proj}_\theta(C \times C)) = \min\{\dim_H(C) + \dim_H(C), 1\}.$$

In other words, the orthogonal projection of $C \times C$ to the line $y = -x$ does not drop the expected dimension. Indeed, similar result is still correct for a general class of self-similar sets. Peres and Shmerkin [24], Hochman and Shmerkin [16] proved the following elegant result.

Let K_1 and K_2 be two self-similar sets with IFS's $\{f_i(x) = r_i x + a_i\}_{i=1}^n$ and $\{g_j(x) = r'_j x + b_j\}_{j=1}^m$, respectively. If there are some r_i, r'_j such that

$$\frac{\log |r_i|}{\log |r'_j|} \notin \mathbf{Q},$$

then

$$\dim_H(K_1 + K_2) = \min\{\dim_H(K_1) + \dim_H(K_2), 1\},$$

and $\dim_H(K_1 + K_2) = \dim_P(K_1 + K_2) = \dim_B(K_1 + K_2)$. The condition in the above result is called the irrationality condition. The result above indeed states that under the irrationality condition, the Hausdorff dimension of the projection of two self-similar sets through the angle $\pi/4$ does not drop.

Now we go back to the middle-third Cantor set, and consider a slicing problem, i.e. given $t \in [-1, 1]$, then the set U_r in (1.1), is

$$U_r = \{t \in [-1, 1] \colon \mathrm{Card}(\{y - x = t\} \cap (C \times C)) = r\},$$

where $r \in \mathbf{N}^+$. In other words, the multiple representational problem is indeed a slicing problem in geometric measure theory. In this paper, we shall consider the arithmetic addition or subtraction for more than two Cantor sets. First, we give some basic definitions.

Suppose $n \geq 2$ and $\mathcal{A}_i \subset \{0, 1, \cdots, (n-1)\}$ for $i = 1, \cdots, l$, let

$$K_i = \bigcup_{a \in \mathcal{A}_i} \frac{K_i + a}{n}$$

be self-similar sets contained in $[0, 1]$. Fix $\mathbf{m} = (m_1, \cdots, m_l) \in \mathbf{Z}^l$ with $\prod_i m_i \neq 0$ denote

$$S_x = \left\{\mathbf{y} \in \prod_{i=1}^l K_i \colon (\mathbf{m}, \mathbf{y}) = m_1 y_1 + \cdots + m_l y_l = x\right\}.$$

In this paper, we will focus on the fractal dimension of

$$U_r = \{x \colon \mathrm{Card}(S_x) = r\} \text{ for } r < \infty,$$

It is worthwhile pointing out that if $\mathrm{Card}(S_x) = 1$ then there is a *unique solution* for the equation

$$x = m_1 y_1 + \cdots + m_l y_l \quad \text{with } y_i \in K_i \; \forall i.$$

Let $m_* = \sum_{m_i<0} m_i$ and $m^* = \sum_{m_i>0} m_i$ where $\sum_{b\in\emptyset} b = 0$. Given a subset B of $\mathbf{R}^l$, we write $(\mathbf{m}, B) = \bigcup_{b\in B}(\mathbf{m}, b)$. For $\mathbf{i} = (i_1, \cdots, i_l) \in \prod_{i=1}^{l} \mathcal{A}_i$ and we write the small cube $\mathbf{c_i} = \frac{(i_1,\cdots,i_l)+[0,1]^l}{n}$. We say that the *covering condition* holds for $\prod_{i=1}^{l} K_i$ with respect to $\mathbf{m}$, if

$$\bigcup_{\mathbf{i}\in\prod_{i=1}^{l}\mathcal{A}_i}(\mathbf{m}, \mathbf{c_i}) = (\mathbf{m}, [0,1]^l),$$

where $(\mathbf{m}, [0,1]^l) = [m_*, m^*]$ and $(\mathbf{m}, \mathbf{c_i}) = n^{-1}((\mathbf{m}, \mathbf{i}) + [m_*, m^*])$.

We call $I = n^{-1}[u, u+1] \subset [m_*, m^*]$ with $u \in \mathbf{Z}$ an integer interval, and $\{J_t = [t, t+1]\}_{t\in[m_*,m^*)\cap\mathbf{Z}}$ working intervals. We say I is of type t (with respect to the small cube $c_\mathbf{i}$), if $u - (\mathbf{m}, \mathbf{i}) = t \in [m_*, m^*-1]\cap\mathbf{Z}$ for some $\mathbf{i} \in \prod_{i=1}^{l} \mathcal{A}_i$, i.e., t is the relative position of I according to the projection interval $(\mathbf{m}, \mathbf{c_i}) = n^{-1}((\mathbf{m}, \mathbf{i}) + [m_*, m^*])$ of the small cube $\mathbf{c_i}$. For $t \in [m_*, m^*) \cap \mathbf{Z}$, the corresponding geometric type is $[0,1]^l \cap \{\mathbf{y}: (\mathbf{m}, \mathbf{y}) \in J_t\}$ or its similar copy. Two integer intervals $I_1 = n^{-1}[u_1, u_1+1]$ and $I_2 = n^{-1}[u_2, u_2+1]$ are said to be congruent modulo n, if $u_1 \equiv u_2(\mathrm{mod}\ n)$, i.e., $I_2 = I_1 + k$ for some $k \in \mathbf{Z}$.

For a directed graph, we give a partial order on its strongly connected components $\{H_i\}_i$, we denote $H \prec H'$, if $H = H'$ or there is a directed path from one vertex of H to another vertex of H'. Let $\rho(H_i)$ be the spectral radius of the matrix with respect to the subgraph restricted in H_i. We say an infinite sequence $v_{i_1} v_{i_2} \cdots v_{i_k} v_{i_{k+1}} \cdots$ of vertexes is admissible if there is a directed edge from v_{i_k} to $v_{i_{k+1}}$ for all k.

In this paper, we need three directed graphs.

(1) The first graph has the vertex set of all integer intervals. For two vertexes (or integer intervals) I_1 and I_2, there is a directed edge from I_1 to I_2, denoted by $I_1 \to I_2$, if and only if there exists an integer $t \in [m_*, m^* - 1]$ such that I_1 is of type t and $I_2 \subset J_t$. We denote $I \rightarrowtail J_t$ if the integer interval I is of type t.

(2) The second graph is a subgraph of the first. Let Ξ denote the collection of integer intervals I such that there is a unique $\mathbf{i} \in \prod_{i=1}^{l} \mathcal{A}_i$ satisfying $I \subset (\mathbf{m}, \mathbf{c_i})$. Then we obtain a directed subgraph G_Ξ of the first one. From this directed graph, we obtain a 0-1 transition matrix M with its spectral radius $\rho(M)$. Denote by $\{\Xi_i\}_i$ the strongly connected components of G_Ξ. In the graph G_Ξ, we say that a strongly connected component Ξ_i can reach J_t, if there is a directed path in G_Ξ from one vertex of Ξ_i to another vertex of type t. We also say that $[m_*, m^*]$ is dominated by Ξ for $d > 0$, if each J_t can be reached by some Ξ_i with $\frac{\log\rho(M_i)}{\log n} \geq d$.

(3) The third graph G^* contains G_Ξ. A subset ω of Ξ is said to be congruent, if any two of ω are congruent. For congruent subset ω, let

$$\mathcal{D}(\omega) = \{t: \text{there is an element of } \omega \text{ contained in } J_t = [t, t+1]\}.$$

Let the vertex set of G^* be the collection of all congruent subsets of Ξ. Then there is a directed edge from ω to ω', if and only if for any $I \in \omega$, there exists some $I' \in \omega'$ such that $I \to I'$ (in the first graph), and for any $I' \in \omega'$, there exists some $I \in \omega$ such that $I \to I'$ (in the first graph). Denote by $\{\Omega_j\}_j$ the strongly connected components of G^*. Note that G_Ξ is a subgraph of G^*, and any strongly connected component of G_Ξ is also a strongly connected component of G^*, hence $\{\rho(\Xi_i)\}_i \subset \{\rho(\Omega_j)\}_j$.

Theorem 1. *Let* $s = \frac{\log\rho(M)}{\log n}$. *Then we have the following results.*

(1) $\dim_H U_1 \geq s$ *and* $\mathcal{H}^s(U_1) > 0$.

(2) *Suppose the covering condition holds for $\prod_{i=1}^{l} K_i$ with respect to $\mathbf{m}$, then*

$$\dim_H U_1 = s.$$

(3) *Suppose the covering condition holds for $\prod_{i=1}^{l} K_i$ with respect to $\mathbf{m}$ and K_i satisfies the strong separation condition holds (for instance, $1 \notin \mathcal{A}_i - \mathcal{A}_i$ implies the strong separation condition) for all i. Suppose $s > 0$, then*

$$\mathcal{H}^s(U_1) = \infty$$

if and only if there are two different strongly connected components Ξ_i and Ξ_j in G_Ξ such that $\rho(\Xi_i) = \rho(\Xi_j) = \rho(M)$ and $\Xi_i \prec \Xi_j$. In particular if M is irreducible then
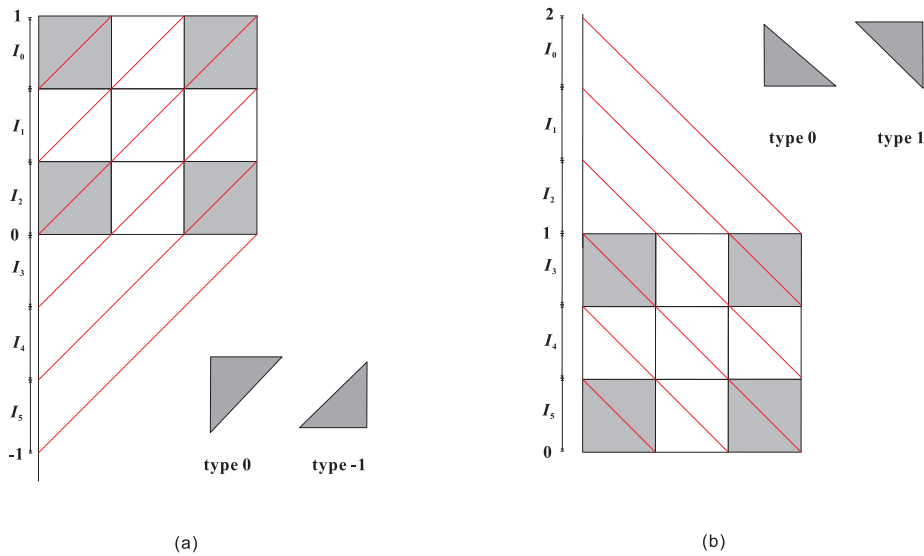
$$0 < \mathcal{H}^s(U_1) < \infty.$$

Figure 1. Example 1.

Example 1. As in part (a) of Fig. 1, let us consider the equation $x = -b_1 + b_2$ with $b_i \in C$, where C is the middle-third Cantor set and $(m_1, m_2) = (-1, 1)$ with $m_* = -1$ and $m^* = 1$. Now, the covering condition and the strong separation condition hold. Using the lines $\bigcup_{d \in 3^{-1}\{-3,\cdots,3\}} \{(x, y) : y - x = d\}$, we obtain six integer intervals $I_0, \cdots, I_5$ and working intervals $[0, 1]$ and $[-1, 0]$. Four small squares $\mathbf{c}_{(0,0)}, \mathbf{c}_{(0,2)}, \mathbf{c}_{(2,0)}$ and $\mathbf{c}_{(2,2)}$ are shown in part (a) of Fig. 1. We have $\Xi = \{I_0, I_1, I_4, I_5\}$ and the transition matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{with } \rho(M) = 2,$$

then $\dim_H U_1 = \frac{\log 2}{\log 3} = s$ and $0 < \mathcal{H}^s(U_1) < \infty$ since M is irreducible. In this example, $\dim_H U_1 = \dim_H C = \frac{\log 2}{\log 3}$.

For part (b) of Fig. 1, let us consider the equation $x = b_1 + b_2$ with $b_i \in C$, where C is the middle-third Cantor set and $(m_1, m_2) = (1, 1)$ with $m_* = 0$ and $m^* = 2$. We also have $\Xi = \{I_0, I_1, I_4, I_5\}$ and the same transition matrix M with $\rho(M) = 2$. Hence in this case $\dim_H U_1 = \dim_H C = \frac{\log 2}{\log 3}$.

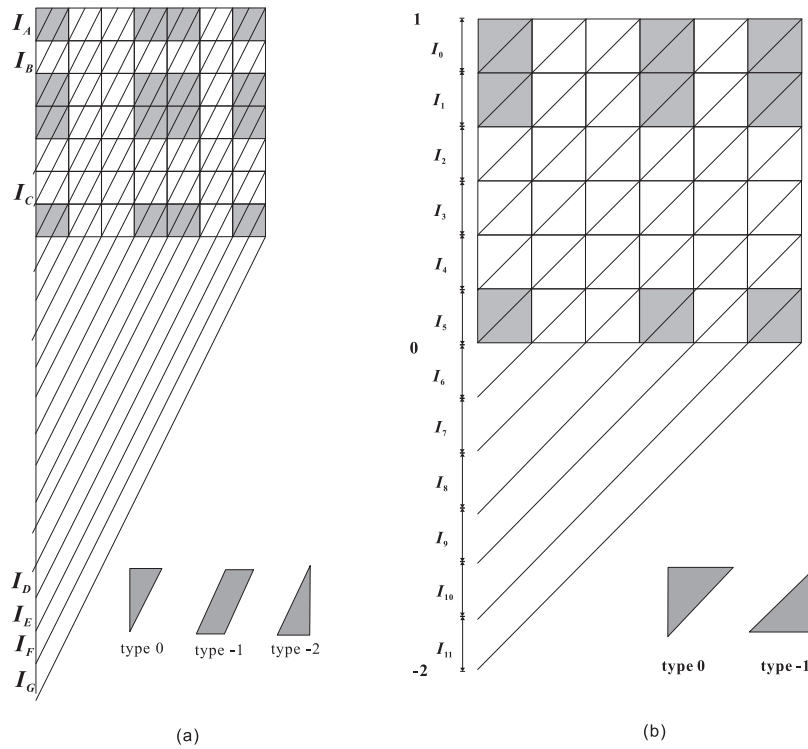In the following examples, we have $\dim_H U_1 < \min_i(\dim_H K_i)$ or $\dim_H U_1 > \max_i(\dim_H K_i)$.

Figure 2. Examples 2 and 3.

Example 2. As in part (a) of Fig. 2, let us consider the equation $x = -2b_1 + b_2$ with $b_i \in K = \bigcup_{a \in \{0,3,4,6\}} \frac{K+a}{7}$, where $(m_1, m_2) = (-2, 1)$ with $m_* = -2$ and $m^* = 1$. The covering condition holds. We have $\Xi = \{I_A, I_B, I_C, I_D, I_E, I_F, I_G\}$ and the transition matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{with } \rho(M) = \frac{\sqrt{5}+3}{2}.$$

Then $\dim_H U_1 = \frac{\log \frac{\sqrt{5}+3}{2}}{\log 7} = s$. In this example, the spectral radius is not an integer and $\dim_H U_1 < \dim_H K = \frac{\log 4}{\log 7}$.

Example 3. As in part (b) of Fig. 2, let us consider the equation $x = -b_1 + b_2$ with $b_1 \in K_1 = \bigcup_{a \in \{0,3,5\}} \frac{K_1+a}{6}$ and $b_2 \in K_2 = \bigcup_{a \in \{0,4,5\}} \frac{K_2+a}{6}$, where $(m_1, m_2) = (-1, 1)$ with $m_* = -1$ and $m^* = 1$. The covering condition and the strong separation condition hold. We have $\Xi = \{I_0, I_2, I_3, I_7, I_8, I_9, I_{10}, I_{11}\}$ and the transition matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{with } \rho(M) = 4.$$

Then $\dim_H U_1 = \frac{\log 4}{\log 6} = s$ and $0 < \mathcal{H}^s(U_1) < \infty$ since M is irreducible. Notice that in this example $\dim_H U_1 > \max\{\dim_H K_1, \dim_H K_2\} = \frac{\log 3}{\log 6}$.

Given $j \in \{0, \cdots, (n-1)\}$, a $\|\mathbf{m}\|_1 \times \|\mathbf{m}\|_1$ matrix $T_j = (b_{uv})_{m_* \leq u, v \leq m^* - 1}$ is defined by

$$b_{uv} = \mathrm{Card}\{\mathbf{i} \in \prod_{i=1}^{l} \mathcal{A}_i \colon (nu + j) - (\mathbf{m}, \mathbf{i}) = v\}.$$

In fact, $b_{uv} > 0$ if and only if $I_{(u,j)} \rightarrowtail J_v$, where $I_{(u,j)} = n^{-1}[nu + j, nu + j + 1]$ and $J_v = [v, v+1]$. We also note that b_{uv} is the number of small cubes $\mathbf{c_i}$ such that $I_{(u,j)}$ has the relative position v according to the projection interval $(\mathbf{m}, \mathbf{i})$ of the small cube $\mathbf{c_i}$. Please see the following two examples 4 and 5 for this definition.
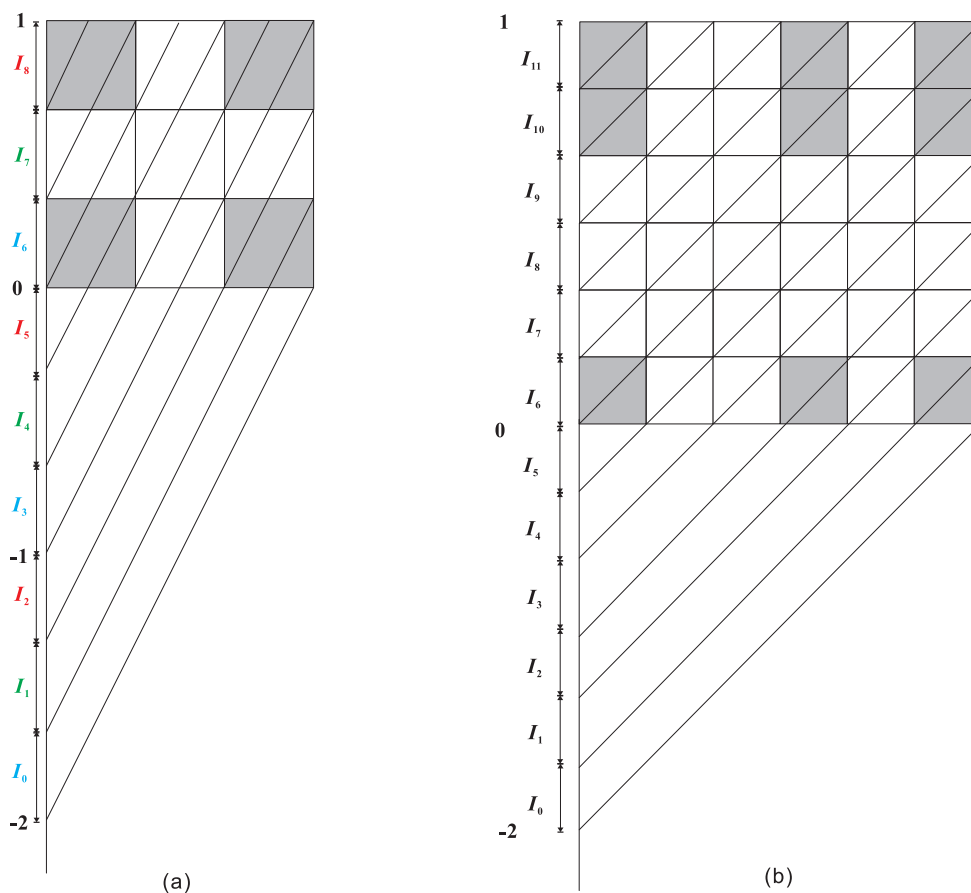

Figure 3. Examples 4 and 5.

Example 4. Consider $-2C + C$ as in part (a) of Fig. 3. We have

$$
T_0 = \begin{matrix} & \begin{matrix} J_{-2} & J_{-1} & J_0 \end{matrix} \\ \begin{matrix} I_0 \\ I_3 \\ I_6 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{matrix}, \quad
T_1 = \begin{matrix} & \begin{matrix} J_{-2} & J_{-1} & J_0 \end{matrix} \\ \begin{matrix} I_1 \\ I_4 \\ I_7 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{matrix}
$$

$$
\text{and} \quad T_2 = \begin{matrix} & \begin{matrix} J_{-2} & J_{-1} & J_0 \end{matrix} \\ \begin{matrix} I_2 \\ I_5 \\ I_8 \end{matrix} & \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix}.
$$

Example 5. Consider $-K_1 + K_2$ for $K_1 = \bigcup_{a \in \{0,3,5\}} \frac{K_1 + a}{6}$ and $K_2 = \bigcup_{a \in \{0,4,5\}} \frac{K_2 + a}{6}$ as in part (b) of Fig. 3. We have

$$
T_0 \;=\; \begin{matrix} \\ I_0 \\ I_6 \end{matrix} \begin{matrix} J_{-1} \; J_0 \\ \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \end{matrix}, \quad
T_1 \;=\; \begin{matrix} \\ I_1 \\ I_7 \end{matrix} \begin{matrix} J_{-1} \; J_0 \\ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \end{matrix},
$$

$$
T_2 \;=\; \begin{matrix} \\ I_2 \\ I_8 \end{matrix} \begin{matrix} J_{-1} \; J_0 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{matrix}, \quad
T_3 \;=\; \begin{matrix} \\ I_3 \\ I_9 \end{matrix} \begin{matrix} J_{-1} \; J_0 \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{matrix},
$$

$$
T_4 \;=\; \begin{matrix} \\ I_4 \\ I_{10} \end{matrix} \begin{matrix} J_{-1} \; J_0 \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \end{matrix}, \quad
T_5 \;=\; \begin{matrix} \\ I_5 \\ I_{11} \end{matrix} \begin{matrix} J_{-1} \; J_0 \\ \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \end{matrix}.
$$

In fact, under the covering condition, by the method of [20], we can obtain that for Lebesgue almost all $x \in [m_*, m^*]$,

$$
\dim_H S_x = \dim_B S_x = \frac{\xi}{\log n},
$$

where ξ is the Lyapunov exponent for the symmetric independent random product of $T_0, \cdots, T_{n-1}$, i.e.,

$$
\xi = \lim_{k \to \infty} \frac{\log \|T_{x_1} T_{x_2} \cdots T_{x_k}\|}{k}
$$

with x_k i.i.d. random variables assuming the values $\{0, 1, \cdots, (n-1)\}$ with equal probabilities. Related results can be found in [4, 21]. When the direction is fixed, some general result on sections of self-similar sets can be found in [32].

A mapping $\psi\colon \bigcup_{I \in \Xi} \mathrm{int}(I) \to (m_*, m^*)$ is defined by

$$
\psi(x) = n(x - a_I) + t(I) \quad \text{for all } x \in \mathrm{int}(I) = (a_I, a_I + n^{-1}),
$$

where $t(I)$ is the type of I. Then $\psi|_{\mathrm{int}(I)}$ is a linear surjection from I to $\mathrm{int}(J_t) = (t, t+1)$ with factor n. Notice that if $x_1, x_2 \in \bigcup_{I \in \Xi} \mathrm{int}(I)$ and $x_1 = x_2 + k$ with $k \in \mathbf{Z}$ and $x_i \in \mathrm{int}(I_i)$ for $i = 1, 2$, then

(1.2) $\psi(x_1) - \psi(x_2) = n(k - a_{I_1} + a_{I_2}) + (t(I_1) - t(I_2)) \in \mathbf{Z}.$

That means the integer intervals containing $\psi(x_1)$ and $\psi(x_2)$ are congruent (modulo n) if x_1, x_2 lie in the interiors of the corresponding integer intervals respectively. In fact, the third graph defined above is based on this observation. Given $z \in [0,1) \backslash \{\frac{q_1}{n^{q_2}}\}_{q_1, q_2 \in \mathbf{Z}}$ and a subset $\mathcal{D}$ of $\mathbf{Z} \cap [m_*, m^* - 1]$, we say that the vector $(z+p)_{p \in \mathcal{D}}$ (with index lying in $\mathbf{Z} \cap [m_*, m^* - 1]$) has an infinite coding $\omega_0 \omega_1 \cdots \omega_k \cdots$ in G^*, if $\psi^i(z + p)$ belongs to the interior of some integer interval of Ξ for each $i \geq 0$ and $p \in \mathcal{D}$, and ω_i is the smallest congruent subset of Ξ containing $\bigcup_{p \in \mathcal{D}} \{\psi^i(z + p)\}$ for each $i \geq 0$ such that $\omega_0 \omega_1 \cdots \omega_k \cdots$ is admissible in G^*. Let e_i be the i-th one of the natural basis on $\mathbf{R}^{\|\mathbf{m}\|_1}$ for $m_* \leq i \leq m^* - 1$.

Theorem 2. *Suppose the covering condition holds and K_i satisfies the strong separation condition for each i. Then*

$$
U_r \backslash \left\{ \frac{q_1}{n^{q_2}} \right\}_{q_1, q_2 \in \mathbf{Z}} \neq \emptyset,
$$

if and only if there exist $i, j_1, \cdots, j_k$ such that

$$
r = \|e_i T_{j_1} \cdots T_{j_k}\|_1
$$

with $e_i T_{j_1} \cdots T_{j_k} = (\beta_{m_*}, \cdots, \beta_{m^*-1})$ and $\mathcal{D} = \{p \colon \beta_p \neq 0\}$ satisfying

$$(1.3) \qquad \left\{ z \in [0, 1) \backslash \left\{ \frac{q_1}{n^{q_2}} \right\}_{q_1, q_2 \in \mathbf{Z}} \colon (z + p)_{p \in \mathcal{D}} \text{ has an infinite coding in } G^* \right\} \neq \emptyset.$$

The condition (1.3) implies that the set $\{\bar{\omega} = \omega_{i_0} \omega_{i_1} \cdots \omega_{i_k} \cdots \colon$ the infinite sequence $\bar{\omega}$ is admissible in G^* with $\mathcal{D}(\omega_{i_0}) = \mathcal{D}\}$ is non-empty. If the latter set is uncountable, then (1.3) follows. In particular, if $\mathcal{D}(\omega_{i_0}) = \mathcal{D}$ and $\omega_{i_0} \in \Omega_i$ such that $\rho(\Omega_j) > 0$ with $\Omega_i \prec \Omega_j$ for some Ω_j, then (1.3) follows.

Theorem 3. *Suppose the covering condition holds. Let $U_r \neq \emptyset$ with $r \geq 2$ and*

$$d_r = \dim_H U_r.$$

(1) *Then we have*

$$d_r \leq \frac{\log \rho(M)}{\log n}.$$

(2) *Moreover, we assume that K_i satisfies the strong separation condition for each i, then either U_r is countable, or*

$$d_r \in \left(\bigcup_i \max \left\{ \frac{\log \rho(\Omega_j)}{\log n} \colon \Omega_i \prec \Omega_j \right\} \right) \backslash \{0\} \text{ and } \mathcal{H}^{d_r}(U_r) > 0.$$

(3) *Suppose K_i satisfies the strong separation condition for all i and $[m_*, m^*]$ is dominated by Ξ for d_r, then $\mathcal{H}^{d_r}(U_r) = \infty$. In particular, if M is irreducible and the elements in Ξ can reach every working interval, then*

$$\mathcal{H}^{d_r}(U_r) = \infty.$$

Example 6. Let us consider $C - C$ again. We note that for $\omega = \{I_0, I_4\}$ we only find the edges from ω to $\{I_0\}$ and $\{I_1\}$, and the edges from $\{I_1, I_5\}$ to $\{I_0\}$ and $\{I_1\}$. The effective part of the matrix is the submatrix with respect to G_{Ξ}. Then the non-empty set U_r is countable or has dimension $\log 2 / \log 3$. We also check that M in Example 1 is irreducible and the elements in Ξ can reach every working interval. By Theorem 3, U_r is countable or $\dim_H U_r = \log 2 / \log 3$ with infinite Hausdorff measure.

Now, we will find r such that U_r is non-empty. Ignoring a countable set $\{\frac{q}{3^p}\}_{p, q \in \mathbf{Z}}$, we can take the infinite coding with digit $\{0, 1, 2, 3, 4, 5\}$ to represent the number in $[-1, 1]$ uniquely, where digit j represents I_j. Now, the coding is not free, we have the following rules:

(1) If the current digit is $0, 2$ or 4, then the next digit shall be taken in $\{0, 1, 2\}$;
(2) If the current digit is $1, 3$ or 5, then the next digit shall be taken in $\{3, 4, 5\}$.

Let

$$\eta(i) = \begin{cases} 1 & \text{if } i = 0, 1, 4 \text{ or } 5, \\ 2 & \text{otherwise.} \end{cases}$$

Then

$$\text{Card}\{(b_1, b_2) \in C \times C \colon x = -b_1 + b_2\} = \prod_k \eta(i_k),$$

if x has coding $i_0 i_1 i_2 \cdots i_k \cdots$, when we ignore a countable set $\{\frac{q_1}{3^{q_2}}\}_{q_1, q_2 \in \mathbf{Z}}$.
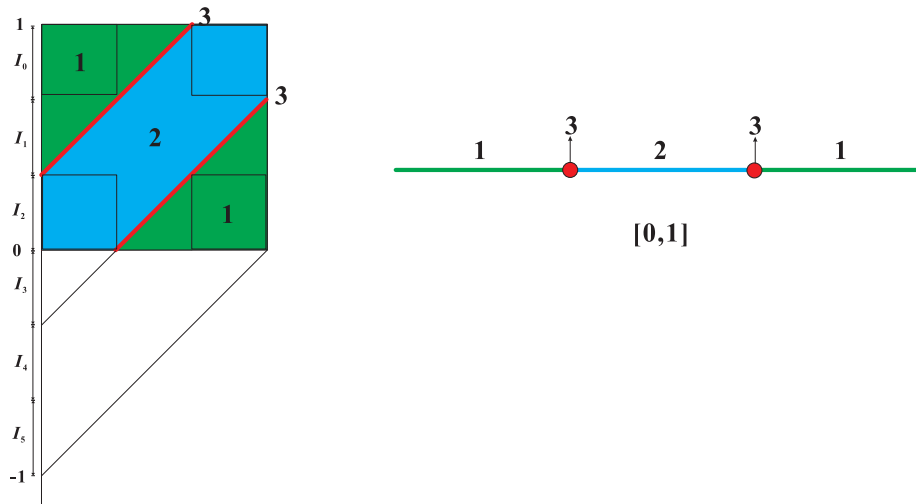
Figure 4. Example 6.

Let $P\colon \{0,1,2,3,4,5\} \to \{0,1,2\}$ be a projection defined by $P(2j) = P(2j+1) = j$, we simplify this above model to the unit interval with 3-adic expansion and take

$$\bar{\eta}(j) = \begin{cases} 1 & \text{if } j = 0,2, \\ 2 & \text{if } j = 1. \end{cases}$$

Then the cardinality is $\prod_k \bar{\eta}(j_k)$ for the 3-adic expansion $0.j_0 j_1 \cdots j_k \cdots$. Using this projection, U_1 becomes the Cantor set, and $\{x\colon \operatorname{Card}(S_x) = 2^u\}$ becomes the set

$$\{0.j_0 j_1 \cdots j_k \cdots \colon \text{digit 1 appears } u \text{ times in the expansion}\}.$$

Let $s = \log 2 / \log 3$. Then one can check that

(1.4) $\dim_H \{x\colon \operatorname{Card}(S_x) = 2^u\} = s,\ \mathcal{H}^s \{x\colon \operatorname{Card}(S_x) = 2^u\} = \infty$

for any $u \in \mathbf{N}^+$. Meanwhile, note red lines of $[0,1]^2$ and red points of $[0,1]$ in Fig. 4, we have

$$U_{3 \cdot 2^u} \subset \left\{ \frac{q_1}{3^{q_2}} \right\}_{q_1, q_2 \in \mathbf{Z}} \quad \text{and} \quad \operatorname{Card}(U_{3 \cdot 2^u}) = \aleph_0,$$

hence

(1.5) $\dim_H(U_{3 \cdot 2^u}) = 0\ \text{ with }\ \mathcal{H}^0(U_{3 \cdot 2^u}) = \infty.$

The paper is organized as follows. In Section 2, we provide the preliminaries, including graph-directed construction [22], multi dynamical systems for sections of self-similar sets [33] and counting formula of sections [20]. Section 3 is devoted to Theorem 1. In Section 4, we will prove Theorems 2 and 3 on U_r with $r \geq 2$. We also give some remarks in Section 5.

2. Preliminaries

Note that

(2.1) $$\prod_{i=1}^l K_i = \bigcup_{\mathbf{i} \in \prod_{i=1}^l \mathcal{A}_i} \frac{\prod_{i=1}^l K_i + \mathbf{i}}{n}.$$

For notational convenience, we write $S_{\mathbf{i}}(x) = \frac{x+\mathbf{i}}{n}$ and $S_{\mathbf{i}_1 \cdots \mathbf{i}_k} = S_{\mathbf{i}_1} \circ \cdots \circ S_{\mathbf{i}_k}$. The cube $S_{\mathbf{i}_1 \cdots \mathbf{i}_k}([0,1]^l)$ with sidelength n^{-k} is said to be a basic cube of rank k. Denote the hyperplane

$$H_x = \{\mathbf{y} \in \mathbf{R}^l \colon (\mathbf{m}, \mathbf{y}) = x\}.$$

Lemma 1. *Suppose the covering condition holds for $\prod_{i=1}^{l} K_i$ with respect to $\mathbf{m}$. Then H_x intersects $S_{\mathbf{i}_1 \cdots \mathbf{i}_k}([0,1]^l)$ if and only it intersects $S_{\mathbf{i}_1 \cdots \mathbf{i}_k}(\prod_{i=1}^{l} K_i)$.*

2.1. Graph-directed construction. Recall the graph-directed construction introduced by Mauldin and Williams [22]. Given a directed graph $G = (V, E)$, suppose $\{H_i\}_i$ are its strongly connected components and $\mathcal{N} = (b_{v,v'})_{v,v' \in V}$ is the matrix respect to the graph, i.e., $b_{v,v'}$ is number of edges from v to v'. Given a directed edge e, we equip a linear mapping $g_e \colon \mathbf{R} \to \mathbf{R}$ with contracting ratio n^{-1} where n^{-1} is also the contracting ratio in (2.1). By the classical result of [22], there is a unique family of compact subsets $\{K_v\}_{v \in V} \subset \mathbf{R}$ such that

$$K_v = \bigcup_{v' \in V} \bigcup_{e \in \mathcal{E}(v,v')} g_e(K_{v'}),$$

where $\mathcal{E}(v, v')$ denotes the set of directed edges from v to v'. We say that the open set condition holds, if there are non-empty open sets $\{U_v\}_{v \in V}$ of $\mathbf{R}$ such that

$$\bigcup_{v' \in V} \bigcup_{e \in \mathcal{E}(v,v')} g_e(U_{v'}) \subset U_v$$

and the left hand of the above formula is a disjoint union for each $v \in V$. By the results in [22], we have the following

Lemma 2. *Suppose the open set condition holds. Then*

(1) $\dim_H(\bigcup_{v \in V} K_v) = \frac{\log \rho(\mathcal{N})}{\log n} = s$;

(2) *Suppose $s > 0$, then $\mathcal{H}^s(\bigcup_{v \in V} K_v) > 0$;*

(3) *Suppose $s > 0$, then $\mathcal{H}^s(\bigcup_{v \in V} K_v) = \infty$ if and only if there are two different strongly connected components H and H' such that $\rho(H) = \rho(H') = \rho(\mathcal{N})$ and $H \prec H'$.*

(4) *Suppose $v \in H_i$, then*

$$\dim_H K_v = \max\left\{ \frac{\log \rho(H_j)}{\log n} : H_i \prec H_j \right\}.$$

Moreover, if $d = \dim_H K_v > 0$, then $\mathcal{H}^d(K_v) > 0$.

For example, let the symbolic system

$$X = \left\{ \bar{v} = v_{i_0} v_{i_1} \cdots v_{i_k} \cdots : \text{the infinite sequence } \bar{v} \text{ is admissible} \right\}$$

and X_v the collection of infinite admissible sequences starting from v. There is a metric on X defined by

$$\mathrm{d}(v_{i_0} \cdots v_{i_k} v_{i_{k+1}} \cdots, v_{i_0} \cdots v_{i_k} v_{i'_{k+1}} \cdots) = n^{-k} \text{ if } v_{i_{k+1}} \neq v_{i'_{k+1}}.$$

As in Lemma 2, by results of [22], we conclude that if $s = \frac{\log \rho(\mathcal{N})}{\log n} > 0$ then $\dim_H(X) = s$ and $\mathcal{H}^s(X) > 0$ and

$$\dim_H X_v = \max\left\{ \frac{\log \rho(H_j)}{\log n} : H_i \prec H_j \right\} \text{ where } v \in H_i.$$

We also have the following

Claim 1. *If $\dim_H X_v = 0$, then X_v is countable.*

Proof. It suffices to show for strongly connected component H_j with $\rho(H_j) \leq 1$ the collection of admissible infinite sequences with letters in H_j is a finite set. Suppose γ is a Perron–Frobenius eigenvector γ of the matrix $\mathcal{N}_j$ w.r.t. H_j such that $\|\gamma\|_1 = 1$, and $\mathcal{D}_k$ denotes the set of admissible sequences of length k such that every letter

lies in H_j. Using the irreducibility, there exists a constant $\varsigma > 0$ such that for all k,
$$\text{Card}(\mathcal{D}_k) \leq \|\mathcal{N}_j^k(1, \cdots, 1)^T\|_1 \leq \varsigma \|\mathcal{N}_j^k \gamma\|_1 = \varsigma \|\rho(H_j)^k \gamma\|_1 \leq \varsigma. \qquad \square$$

2.2. Multi dynamical system. We will construct a *multi dynamical system* including many expanding maps from integer intervals to $[m_*, m^*]$. In fact, for every integer interval $I: n^{-1}[u, u+1] \subset [m_*, m^*]$, we equip the interval with several expanding maps with factor n as follows. Suppose I is of type t with $u - (\mathbf{m}, \mathbf{i}) = t$, we let $f_{\mathbf{i}}: I \to J_t = [t, t+1]$ denote the corresponding linear surjection in the form

$$f_{\mathbf{i}}(x) = nx - u + t: I \to J_t.$$

Then the multi dynamical system consists of $\{f_{\mathbf{i}}\}_{\mathbf{i} \in \prod_{i=1}^l \mathcal{A}_i}$.

The next lemma reveals the connection between the multi dynamical system and intersections of hyperplanes with basic cubes.

Lemma 3. *If* $x \in I = n^{-1}[u, u+1]$ $(u \in \mathbf{Z})$ *and* $\mathbf{i} \in \prod_{i=1}^l \mathcal{A}_i$ *such that* $I \subset (\mathbf{m}, \mathbf{c_i})$, *then*

$$(2.2) \qquad H_x \bigcap S_{\mathbf{i}}([0,1]^l) = S_{\mathbf{i}}(H_{f_{\mathbf{i}}(x)} \cap [0,1]^l).$$

Using (2.2) again and again, we obtain

Lemma 4. *If* $x \in (\mathbf{m}, \mathbf{c_{i_1}})$ *and* $(f_{\mathbf{i}_j} \circ \cdots \circ f_{\mathbf{i}_1})(x) \in (\mathbf{m}, \mathbf{c_{i_{j+1}}})$ *for all* $1 \leq j \leq k-1$, *then*

$$H_x \bigcap S_{\mathbf{i_1 i_2 \cdots i_k}}([0,1]^l) = S_{\mathbf{i_1 i_2 \cdots i_k}}(H_{(f_{\mathbf{i}_k} \circ \cdots \circ f_{\mathbf{i}_1})(x)} \cap [0,1]^l)$$

When $I \in \Xi$, since there is only one $\mathbf{i} \in \prod_{i=1}^l \mathcal{A}_i$ such that $I \subset (\mathbf{m}, \mathbf{c_i})$ and thus the type t is uniquely determined by I, we can write $f_{\mathbf{i}}$ to be f_I for convenience.

If there is an admissible sequence $I_{i_0} I_{i_1} \cdots I_{i_{k-1}} I_{i_k} \cdots$ of G_Ξ such that

$$(2.3) \qquad x \in \text{int}(I_{i_0}) \quad \text{and} \quad g_k(x) = (f_{I_{i_{k-1}}} \circ \cdots \circ f_{I_{i_0}})(x) \in \text{int}(I_{i_{k+1}}) \quad \text{for all } k,$$

then we say that $x \in [m_*, m^*]$ has an *infinite coding* $I_{i_0} I_{i_1} \cdots I_{i_{k-1}} I_{i_k} \cdots$ of G_Ξ. By Lemma 4 we have

Lemma 5. *Let* $g_k(x) = f_{I_{i_{k-1}}} \circ \cdots \circ f_{I_{i_0}}(x)$ *be defined as above. Then*

$$(2.4) \qquad H_x \bigcap S_{\mathbf{i_1 i_2 \cdots i_k}}([0,1]^l) = S_{\mathbf{i_1 i_2 \cdots i_k}}(H_{g_k(x)} \cap [0,1]^l).$$

Let

$$\Lambda = \{x \in [m_*, m^*]: x \text{ has an infinite coding in } \Xi\}.$$

On the other hand, given an admissible sequence $I_{i_0} I_{i_1} \cdots I_{i_{k-1}} I_{i_k} \cdots$ in G_Ξ, by the theorem of nested interval, there exists a unique $x \in [m_*, m^*]$ such that

$$(2.5) \qquad x \in I_{i_0} \quad \text{and} \quad g_k(x) \in I_{i_{k+1}} \quad \text{for all } k.$$

That means

$$\bar{\Lambda} = \{x \in [m_*, m^*]: I_{i_0} I_{i_1} \cdots I_{i_{k-1}} I_{i_k} \cdots \text{ is admissible in } G_\Xi \text{ such that}$$
$$x \in I_{i_0} \text{ and } g_k(x) \in I_{i_{k+1}} \text{ for all } k\}$$

and $\bar{\Lambda} \backslash \Lambda \subset \{\frac{q_1}{n^{q_2}}\}_{q_1, q_2 \in \mathbf{Z}}$.

We will consider a graph-directed construction induced by Ξ. Suppose $I_1, I_2 \in \Xi$ with $I_1 \to I_2$, the mapping

$$g_{(I_1, I_2)} = (f_{I_1})^{-1}|_{I_2}: I_2 \to I_1$$

with contracting ratio n^{-1}. Note that the open set condition holds for this graph-directed construction. By the classical result of [22], there is a unique family of compact subsets $\{F_I\}_I$ such that $F_I \subset I$ for all $I \in \Xi$ and

$$F_I = \bigcup_{J \in \mathcal{E}(I)} g_{(I,J)}(F_J),$$

where $\mathcal{E}(I)$ denotes the set of ending vertex of the edge starting from I. Now, we have

$$(2.6) \qquad \bar{\Lambda} = \bigcup_{I \in \Xi} F_I \quad \text{with } \dim_H(\bar{\Lambda}) = s.$$

Moreover, if $s > 0$ then $\mathcal{H}^s\left(\bar{\Lambda}\right) > 0$ under the open set condition. Note that

$$\bar{\Lambda} \backslash \Lambda \subset \{\frac{q_1}{n^{q_2}}\}_{q_1,q_2 \in \mathbf{Z}}$$

and $\dim_H(\Lambda) = \dim_H(\bar{\Lambda})$, by Lemma 2, we have

Lemma 6. *Let* $s = \frac{\log \rho(M)}{\log n}$. *We have*

(1) $\dim_H \Lambda = s$;
(2) *If* $s > 0$, *then* $\mathcal{H}^s(\Lambda) > 0$, *moreover,* $\mathcal{H}^s(\Lambda) = \infty$ *if and only if there are two strongly connected components* Ξ_i *and* Ξ_j *in* Ξ *such that* $\rho(M_i) = \rho(M_j) = \rho(M)$ *and* $\Xi_i \prec \Xi_j$. *In particular, if* M *is irreducible, then* $0 < \mathcal{H}^s(\Lambda) < \infty$.

One can check the following lemma directly in terms of Lemma 2.

Lemma 7. *Suppose* ω *is a congruent subset of* Ξ *and* Ω_i *is the strongly connected component of* G^* *containing* ω. *Let*

$$A_\omega = \{z \colon \text{vector } (z + p)_{p \in \mathcal{D}(\omega)} \text{ has an infinite coding in } G^* \text{ starting from } \omega\}.$$

Then $d_\omega = \dim_H A_\omega = \max\left\{\frac{\log \rho(\Omega_j)}{\log n} \colon \Omega_i \prec \Omega_j\right\}$. *If* $d_\omega > 0$, *then* $\mathcal{H}^{d_\omega}(A_\omega) > 0$.

2.3. Counting formula. For $z \in [0, 1)$, we let

$$\alpha(z) = (\mathrm{Card}(S_{z+m_*}), \mathrm{Card}(S_{z+m_*+1}), \cdots, \mathrm{Card}(S_{z+m^*-1}))^T,$$

and

$$\sigma \colon [0, 1) \to [0, 1) \quad \text{such that } \sigma z = nz(\text{mod } 1).$$

Let $\{x\} \in [0, 1)$ denote the decimal part of $x = [x] + \{x\}$, e.g. $\{-3.4\} = 0.6$.

Using these notations and matrix T_j in Section 1 and results in [20], we have

Lemma 8. *Suppose the covering condition holds. Let* $x \in [m_*, m^*] \backslash \{\frac{q_1}{n^{q_2}}\}_{q_1,q_2 \in \mathbf{Z}}$. *If*

$$x = i + n^{-1}j_1 + n^{-2}j_2 + \cdots + n^{-k}j_k + \cdots,$$

i.e., $x - i$ *has* n*-adic expansion* $0.j_1 j_2 \cdots j_k \cdots$, *then* $\|e_i T_{j_1} \cdots T_{j_k}\|_1$ *is the number of basic cubes of rank* k *intersecting the hyperplane* H_x. *Moreover, if the strong separation condition holds for each* K_i, *then*

$$(2.7) \qquad \mathrm{Card}(S_x) = e_i T_{j_1} \cdots T_{j_k} \alpha(\sigma^k \{x\}) = \|e_i T_{j_1} \cdots T_{j_k}\|_1.$$

3. Dimension of set with unique solution

Proof of Theorem 1. (1) We can show that $\dim_H U_1 \geq s$. By Lemma 6, we only need to verify that

$$(3.1) \qquad \Lambda \backslash \left\{ \frac{q_1}{n^{q_2}} \right\}_{q_1, q_2 \in \mathbf{Z}} \subset U_1.$$

In fact, suppose $x \in \Lambda \backslash \{\frac{q_1}{n^{q_2}}\}_{q_1, q_2 \in \mathbf{Z}}$ has an infinite coding $I_{i_1} \cdots I_{i_k} \cdots$ in G_Ξ, then $g_k(x) \in \mathrm{int}(I_{i_{k+1}})$ for all k as in (2.3)–(2.4). From the definition of coding and (2.2)–(2.4), we obtain a family of nested cubes $\{Q_k\}_k$ such that Q_k is a basic cube of rank k and $\left(\prod_{i=1}^{l} K_i \right) \cap H_x \subset Q_k$ for all k, which implies that the intersection $\left(\prod_{i=1}^{l} K_i \right) \cap H_x$ is a singleton. Hence (3.1) follows. It follows from (3.1) and Lemma 6 that

$$\mathcal{H}^s(U_1) \geq \mathcal{H}^s(\Lambda) > 0 \quad \text{if } s > 0.$$

If $s = 0$, for getting $\mathcal{H}^s(U_1) > 0$, we only need to show that U_1 is non-empty. In fact, we find $\mathrm{Card}(S_x) = 1$ for $x = \sum_{m_i < 0} m_i (\min_{x_i \in K_i} x_i) + \sum_{m_i > 0} m_i (\max_{x_i \in K_i} x_i)$ which implies $U_1 \neq \emptyset$.

(2) Suppose the covering condition holds. Assume that $x \notin \{\frac{q_1}{n^{q_2}}\}_{q_1, q_2 \in \mathbf{Z}}$ with $\mathrm{Card}(S_x) = 1$ and $\mathbf{y} \in \prod_{i=1}^{l} K_i$ is the unique solution such that $(\mathbf{m}, \mathbf{y}) = x$. Denote by N_k the number of basic cubes with rank k which contain $\mathbf{y}$. By the covering condition and Lemma 1, we obtain

$$N_{k+1} \geq N_k \quad \text{for all } k.$$

Note that $N_k \leq 2^l$ for all k. Therefore, there is an integer k_0 such that $N_{k_0} = \max_k N_k$. Fix a basic cube $S_{\mathbf{i}_1 \cdots \mathbf{i}_{k_0}}([0,1]^l)$ of rank k_0 containing $\mathbf{y}$, by Lemmas 4 and 1 we obtain that

$$(f_{\mathbf{i}_{k_0}} \circ \cdots \circ f_{\mathbf{i}_1})(x) \quad \text{has an infinite coding of } G_\Xi,$$

Hence

$$U_1 \subset \{x : (f_{\mathbf{i}_k} \circ \cdots \circ f_{\mathbf{i}_1})(x) \in \Lambda \text{ for some } f_{\mathbf{i}_k} \circ \cdots \circ f_{\mathbf{i}_1}\},$$

Since $\{f_{\mathbf{i}_k} \circ \cdots \circ f_{\mathbf{i}_1}\}_{\mathbf{i}_1 \cdots \mathbf{i}_k}$ is a countable family, we obtain that

$$(3.2) \qquad \dim_H U_1 \leq \dim_H \Lambda.$$

It follows from (3.1)-(3.2) and $\dim_H \Lambda = s$ (Lemma 6) that

$$\dim_H U_1 = s$$

under the covering condition.

(3) Suppose the covering condition and the strong separation condition hold. It follows from the strong separation condition that basic cubes of rank k are pairwise disjoint. If $x \notin \{\frac{q_1}{n^{q_2}}\}_{q_1, q_2 \in \mathbf{Z}}$ and $\mathrm{Card}(S_x) = 1$ with the unique solution $\mathbf{y}$, then for each k there is a unique basic cube Q_k of rank k containing $\mathbf{y}$ which means $x \in \Lambda$. Therefore

$$(3.3) \qquad (U_1 \setminus \Lambda) \cup (\Lambda \setminus U_1) \subset \left\{ \frac{q_1}{n^{q_2}} \right\}_{q_1, q_2 \in \mathbf{Z}}.$$

The other part of Theorem 1 follows from Lemma 6 and (3.3). $\qquad \square$

4. Number of solutions

Proof of Theorem 2. Suppose

$$x = i + n^{-1}j_1 + n^{-2}j_2 + \cdots + n^{-k}j_k + \cdots \in [m_*, m^*] \setminus \left\{ \frac{q_1}{n^{q_2}} \right\}_{q_1, q_2 \in \mathbf{Z}}.$$

Let $\mathrm{Card}(S_x) = r$ with solution set $\{\mathbf{y}_1, \cdots, \mathbf{y}_r\}$. Denote by N_k the number of basic cubes of rank k intersecting $\{\mathbf{y}_1, \cdots, \mathbf{y}_r\}$. It follows from the covering condition that

$$N_{k+1} \geq N_k \quad \text{for all } k.$$

Note that $N_k \leq r$ for all k due to the strong separation condition. Then there exists an integer k_0 such that $N_{k_0} = r$ and thus

$$\mathrm{Card}(S_x) = e_i T_{j_1} \cdots T_{j_{k_0}} \alpha(\sigma^{k_0}\{x\}) = \|e_i T_{j_1} \cdots T_{j_{k_0}}\|_1 = r$$

due to (2.7) in Lemma 8. Let $(\beta_{m_*}, \cdots, \beta_{m^*-1}) = e_i T_{j_1} \cdots T_{j_{k_0}}$ and $\mathcal{D} = \{p \colon \beta_p \neq 0\}$. Then $\sigma^{k_0}\{x\} + p \in U_1$ for all $p \in \mathcal{D}$, that means $\sigma^{k_0}\{x\} + p$ has an infinite coding in G_{Ξ} for all $p \in \mathcal{D}$. Hence

$$\left\{ z \in [0,1) \setminus \left\{ \frac{q_1}{n^{q_2}} \right\}_{q_1, q_2 \in \mathbf{Z}} : (z+p)_{p \in \mathcal{D}} \text{ has an infinite coding in } G^* \right\} \neq \emptyset.$$

On the other hand, if $r = \|e_i T_{j_1} \cdots T_{j_k}\|_1$ with $e_i T_{j_1} \cdots T_{j_k} = (\beta_{m_*}, \cdots, \beta_{m^*-1})$ and $\mathcal{D} = \{p \colon \beta_p \neq 0\}$ satisfying

$$\left\{ z \in [0,1) \setminus \left\{ \frac{q_1}{n^{q_2}} \right\}_{q_1, q_2 \in \mathbf{Z}} : (z+p)_{p \in \mathcal{D}} \text{ has an infinite coding in } G^* \right\} \neq \emptyset.$$

Then we can construct

$$x = \left(i + \sum_{k=1}^{k_0} n^{-k} j_k + n^{-k_0} z \right) \in [0,1) \setminus \left\{ \frac{q_1}{n^{q_2}} \right\}_{q_1, q_2 \in \mathbf{Z}}$$

and obtain that $\mathrm{Card}(S_x) = r$ using (2.7) again. $\qquad\square$

Proof of Theorem 3. (1) First we will show that $\dim_H U_r \leq \frac{\log \rho(M)}{\log n}$. Let $\mathrm{Card}(S_x) = r$ with solution set $\{\mathbf{y}_1, \cdots, \mathbf{y}_r\}$. Denote by N_k the number of basic cubes of rank k intersecting $\{\mathbf{y}_1, \cdots, \mathbf{y}_r\}$. It follows from the covering condition that

$$N_{k+1} \geq N_k \quad \text{for all } k.$$

Note that $N_k \leq 2^l r$ for all k. Therefore, there exists an integer k_0 such that $N_{k_0} = \max_k N_k$. Fix a basic cube $S_{\mathbf{i}_1 \cdots \mathbf{i}_{k_0}}([0,1]^l)$ of rank k_0 intersecting $\{\mathbf{y}_1, \cdots, \mathbf{y}_r\}$, by Lemma 4, we obtain that

$$(f_{\mathbf{i}_{k_0}} \circ \cdots \circ f_{\mathbf{i}_1})(x) \quad \text{has an infinite coding of } G_{\Xi}.$$

Hence

$$U_r \subset \{x \colon (f_{\mathbf{i}_k} \circ \cdots \circ f_{\mathbf{i}_1})(x) \in \Lambda \text{ for some } f_{\mathbf{i}_k} \circ \cdots \circ f_{\mathbf{i}_1}\},$$

Since $\{f_{\mathbf{i}_k} \circ \cdots \circ f_{\mathbf{i}_1}\}_{\mathbf{i}_1 \cdots \mathbf{i}_k}$ is a countable family, we obtain

$$\dim_H U_r \leq \dim_H \Lambda = \frac{\log \rho(M)}{\log n}.$$

(2) We will obtain the dimension of U_r. Suppose U_r is uncountable. Let $x = i + n^{-1}j_1 + n^{-2}j_2 + \cdots + n^{-k}j_k + \cdots \in [m_*, m^*] \setminus \{\frac{q_1}{n^{q_2}}\}_{q_1, q_2 \in \mathbf{Z}}$. Suppose N_k is defined as above, by the above discussion, there exists an integer k_0 such that $N_{k_0} = r$ and thus

(4.1) $$\mathrm{Card}(S_x) = e_i T_{j_1} \cdots T_{j_{k_0}} \alpha(\sigma^{k_0}\{x\}) = \|e_i T_{j_1} \cdots T_{j_{k_0}}\|_1 = r$$

due to (2.7) in Lemma 8. Let $(\beta_{m_*}, \cdots, \beta_{m^*-1}) = e_i T_{j_1} \cdots T_{j_{k_0}}$ and $\mathcal{D}_x = \{p \colon \beta_p \neq 0\}$. When x is fixed, we use $\mathcal{D}$ to replace $\mathcal{D}_x$ for notational convenience.

Then the above formula (4.1) implies that

$$\sigma^{k_0}\{x\} + p \in U_1 \quad \text{if } p \in \mathcal{D}.$$

Let $\sigma^{k_0}\{x\} \in n^{-1}[h_x, h_x + 1)$. Then $\sigma^{k_0}\{x\} + p \in \Lambda$ has an infinite coding in G_Ξ for any $p \in \mathcal{D}$. Let

$$\mathcal{F}_\mathcal{D} = \{\omega_h \colon \omega_h = \{n^{-1}[np + h, np + h + 1]\}_{p \in \mathcal{D}} \in \Xi \text{ with } h \in [0, (n-1)] \cap \mathbf{Z}\}.$$

Suppose $\omega_h \in \mathcal{F}_\mathcal{D}$ and $\Omega_{i(h,\mathcal{D})}$ is the strongly connected components containing ω_h. If $\sigma^{k_0}\{y\} = z \in n^{-1}[h, h+1)$, using the formula in Lemma 8 we have

$$\mathrm{Card}(S_y) = e_i T_{j_1} \cdots T_{j_{k_0}} \alpha(z) = (\beta_{m_*}, \cdots, \beta_{m^*-1})\alpha(z) = r,$$

i.e.,

$$\left\{ y = i + \sum_{k=1}^{k_0} n^{-k} j_k + n^{-k_0} z \colon z \in \Lambda \cap [0,1) \text{ whenever } p \in \mathcal{D} \right\} \subset U_r.$$

In fact, $\sigma^{k_0}\{y\} + p \in \Lambda$ for all $p \in \mathcal{D}$ implies that $(z + p)_{p \in \mathcal{D}}$ has an infinite coding in G^* starting from ω_h, i.e., $z \in A_{\omega_h}$, where A_{ω_h} is defined in Lemma 7. Using Lemma 7 we have

$$\dim_H U_r \geq \dim_H A_{\omega_h} = \max\left\{ \frac{\log \rho(\Omega_j)}{\log n} \colon \Omega_{i(h,\mathcal{D})} \prec \Omega_j \right\}$$

for any $\omega_h \in \mathcal{F}_\mathcal{D} = \mathcal{F}_{\mathcal{D}_x}$. In particular, for $d' = \max\left\{ \frac{\log \rho(\Omega_j)}{\log n} \colon \Omega_{i(h,\mathcal{D})} \prec \Omega_j \right\}$, using Lemma 7 again we have

(4.2) $$\mathcal{H}^{d'}(U_r) \geq (n^{-k_0})^{d'} \mathcal{H}^{d'}(A_{\omega_h}) > 0 \quad \text{if } d' > 0.$$

Let $\mathcal{G} = \{\mathcal{D}_x \colon \mathrm{Card}(S_x) = r\}$. The above discussion shows that

$$\text{if } x \in U_r \backslash \{\tfrac{q_1}{n^{q_2}}\}_{q_1, q_2 \in \mathbf{Z}}, \quad \text{then } \sigma^{k_0}\{x\} \in A_{\omega_{h_x}} \text{ for some } k_0.$$

Hence

$$\dim_H(U_r) = \dim_H\left(U_r \backslash \{\tfrac{q_1}{n^{q_2}}\}_{q_1,q_2\in\mathbf{Z}}\right) \leq \max_{\mathcal{D}\in\mathcal{G}} \dim_H\left(\bigcup_{\omega\in\mathcal{F}_\mathcal{D}} A_\omega\right)$$

$$\leq \max_{\mathcal{D}\in\mathcal{G}} \max_{\omega_h\in\mathcal{F}_\mathcal{D}} \max\left\{ \frac{\log \rho(\Omega_j)}{\log n} \colon \Omega_{i(h,\mathcal{D})} \prec \Omega_j \right\}.$$

i.e.,

(4.3) $$d_r = \dim_H U_r = \max_{\mathcal{D}\in\mathcal{G}} \max_{\omega_h\in\mathcal{F}_\mathcal{D}} \max\left\{ \frac{\log \rho(\Omega_j)}{\log n} \colon \Omega_{i(h,\mathcal{D})} \prec \Omega_j \right\}.$$

By (4.2) and (4.3), we obtain that

$$\mathcal{H}^{d_r}(U_r) > 0 \quad \text{if } d_r > 0.$$

It suffices to show that $d_r > 0$. In fact, by the above discussion we have

(4.4) $$\mathcal{U}_r \subset \bigcup_{\mathcal{D}\in\mathcal{G}} \bigcup_{\omega_h\in\mathcal{F}_\mathcal{D}} \bigcup_{(i,j_1,\cdots j_{k_0})} \left\{ y = i + \sum_{k=1}^{k_0} n^{-k} j_k + n^{-k_0} z \colon z \in A_{\omega_h} \right\},$$

where $\mathcal{U}_r = U_r \backslash \{\tfrac{q_1}{n^{q_2}}\}_{q_1,q_2\in\mathbf{Z}}$. Note that

$$d_r = \max_{\mathcal{D}\in\mathcal{G}} \max_{\omega_h\in\mathcal{F}_\mathcal{D}} \dim_H(A_{\omega_h}).$$

If $d_r = 0$, then $\dim_H(A_{\omega_h}) = 0$ for all ω_h which implies that A_{ω_h} is countable for all ω_h due to Claim 1. Using (4.4) we obtain that U_r is countable, this is a contradiction.

(3) Now we will obtain the infinity of Hausdorff measure $\mathcal{H}^{d_r}(U_r)$ when

$$\mathcal{U}_r = U_r \backslash \{\frac{q_1}{n^{q_2}}\}_{q_1,q_2 \in \mathbf{Z}} \neq \emptyset \quad \text{with } r \geq 2$$

and $[m_*, m^*]$ is dominated by Ξ for d_r. Take $x \in \mathcal{U}_r$ and $\omega_h \in \mathcal{F}_{\mathcal{D}_x}$ such that

$$\max \left\{ \frac{\log \rho(\Omega_j)}{\log n} : \Omega_{i(h,\mathcal{D})} \prec \Omega_j \right\} = d_r,$$

where $\Omega_{i(h,\mathcal{D})}$ is the strongly connected component containing ω_h. It follows from Lemma 7 that

$$\mathcal{H}^{d_r}(A_{\omega_h}) > 0.$$

Since $[m_*, m^*]$ is dominated by Ξ for d_r, we note that for $x = i + \sum_{k=1}^{k_0} n^{-k} j_k + \cdots \in J_i$ as above, we take a strongly connected component Ξ_j of G_Ξ with $\frac{\log \rho(\Xi_j)}{\log n} \geq d_r$ such that there is a path $\mathbf{P}'$ in G_Ξ from some $I \in \Xi_j$ to an integer interval $I' \in \Xi$ of type i. Let

$$\mathcal{C}_k = \{\mathbf{P} \colon \mathbf{P} \text{ is a path in } \Xi_j \text{ ending at } I \text{ and } |\mathbf{P}| = k\},$$

where $|\mathbf{P}|$ denotes the length of the path. Then we obtain a family $\{\mathcal{B}_k\}_k$ of pairwise disjoint subsets of U_r as follows

$$\mathcal{B}_k = \left\{ \mathbf{P}\mathbf{P}'(i + \sum_{k=1}^{k_0} n^{-k} j_k + n^{-k_0} z) \colon \mathbf{P} \in \mathcal{C}_k \text{ and } z \in A_{\omega_h} \right\} \quad \text{for } k = 1, 2, \cdots,$$

where the coding $\mathbf{P}\mathbf{P}'x_1$ denotes a real number which is turned to be $x_1 \in [t, t+1]$ through a series of expanding maps with respect to the path $\mathbf{P}\mathbf{P}'$ in Ξ and the expanding map according to $I' \rightarrowtail J_i$. Hence

(4.5) $$\mathcal{H}^{d_r}(\mathcal{B}_k) \geq \text{Card}(\mathcal{C}_k)(n^{-(k+k_0)d_r} \mathcal{H}^{d_r}(A_{\omega_h})),$$

and there is a constant $c > 0$ such that

(4.6) $$\text{Card}(\mathcal{C}_k) \geq c\rho(\Xi_j)^k \geq cn^{d_r k}.$$

Here we can check $\text{Card}(\mathcal{C}_k) \geq c\rho(\Xi_j)^k$ in (4.6), because by the irreducibility we can take a Perron–Frobenius eigenvector γ of the matrix M_j w.r.t. Ξ_j such that $\|\gamma\|_1 = 1$, and thus

$$\text{Card}(\mathcal{C}_k) \geq c\|M_j^k(1, \cdots, 1)^T\|_1 \geq c\|M_j^k \gamma\|_1 = c\|\rho(\Xi_j)^k \gamma\|_1 \geq c\rho(\Xi_j)^k.$$

It follows from (4.5) and (4.6) that there is a constant $c' > 0$ such that

$$\mathcal{H}^{d_r}(\mathcal{B}_k) \geq c' \quad \text{for all } k.$$

Hence

$$\mathcal{H}^{d_r}(U_r) \geq \mathcal{H}^{d_r} \left(\bigcup_k \mathcal{B}_k \right) \geq \sum_k \mathcal{H}^{d_r}(\mathcal{B}_k) = +\infty. \qquad \square$$

5. Final remarks

From the slicing point of view, our results can be restated. For instance, the results concerning with the set $U_r = \{x \colon \text{Card}(S_x) = r\}$ ($r < \infty$) in Theorems 1, 2 and 3 are still correct for the following systems: let $K \subset [0,1]^l$ be a self-similar set with IFS $\{\mathbf{y} \to n^{-1}(\mathbf{y} + \mathbf{a})\}_{\mathbf{a} \in \mathcal{A}}$, where $\mathcal{A} \subset \{0, 1, \cdots, (n-1)\}^l$. We do not need to modify the definitions of the directed graphs and the proofs.

The covering condition is essential in this paper. Without this condition, it is much more difficult to analyze the set U_r. We shall discuss this case in another paper. Our main ideas may be implemented in the setting of some overlapping self-similar sets. Nevertheless, the discussion is more complicated. We consider only the

addition or subtraction on the Cantor sets. It is natural to consider similar problems for the multiplication or division on self-similar sets, for instance, the set of points with unique representations.

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