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DISTORTION THEOREMS, LIPSCHITZ CONTINUITY AND THEIR APPLICATIONS FOR BLOCH TYPE MAPPINGS ON BOUNDED SYMMETRIC DOMAINS IN $\mathbf{C}^n$

Hidetaka Hamada

Kyushu Sangyo University, Faculty of Science and Engineering
3-1 Matsukadai, 2-Chome, Higashi-ku, Fukuoka 813-8503, Japan; h.hamada@ip.kyusan-u.ac.jp

Abstract. Let $\mathbf{B}_X$ be a bounded symmetric domain realized as the unit ball of an n-dimensional JB*-triple $X = (\mathbf{C}^n, \|\cdot\|_X)$. In this paper, we give a new definition of Bloch type mappings on $\mathbf{B}_X$ and give distortion theorems for Bloch type mappings on $\mathbf{B}_X$. When $\mathbf{B}_X$ is the Euclidean unit ball in $\mathbf{C}^n$, this new definition coincides with that given by Chen and Kalaj or by the author. As a corollary of the distortion theorem, we obtain the lower estimate for the radius of the largest schlicht ball in the image of f centered at $f(0)$ for α-Bloch mappings f on $\mathbf{B}_X$. Next, as another corollary of the distortion theorem, we show the Lipschitz continuity of $(\det B(z,z))^{1/2n} |\det Df(z)|^{1/n}$ for Bloch type mappings f on $\mathbf{B}_X$ with respect to the Kobayashi metric, where $B(z,z)$ is the Bergman operator on X, and use it to give a sufficient condition for the composition operator C_φ to be bounded from below on the Bloch type space on $\mathbf{B}_X$, where φ is a holomorphic self mapping of $\mathbf{B}_X$. In the case $\mathbf{B}_X = \mathbf{B}^n$, we also give a necessary condition for C_φ to be bounded from below which is a converse to the above result. Finally, as another application of the Lipschitz continuity, we obtain a result related to the interpolating sequences for the Bloch type space on $\mathbf{B}_X$.

1. Introduction

Let f be a Bloch function on the unit disc $\mathbf{U}$ in $\mathbf{C}$. Ghatage, Yan and Zheng [6] showed that $(1-|z|^2)|f'(z)|$ is Lipschitz with respect to the hyperbolic distance on $\mathbf{U}$ and used it to give a sufficient condition for the composition operator C_φ to be bounded from below on the Bloch space on $\mathbf{U}$, where φ is a holomorphic self mapping of $\mathbf{U}$. Chen and Kalaj [2] generalized the above results to the Euclidean unit ball $\mathbf{B}^n$ in $\mathbf{C}^n$. They showed the Lipschitz continuity of $(1-\|z\|^2)^{\frac{n+1}{2n}} |\det Df(z)|^{1/n}$ for Bloch type mappings f on $\mathbf{B}^n$ with respect to the Bergman distance on $\mathbf{B}^n$ and used it to give a sufficient condition for the composition operator C_φ to be bounded from below on the Bloch type space on $\mathbf{B}^n$, where φ is a holomorphic self mapping of $\mathbf{B}^n$. In the proof of Lipschitz continuity, the distortion theorem for Bloch type mappings on $\mathbf{B}^n$ due to Chen, Ponnusamy and Wang [3] played an important role.

Bounded symmetric domains in $\mathbf{C}^n$ can be realized as the unit ball $\mathbf{B}_X$ of a JB*-triple $X = (\mathbf{C}^n, \|\cdot\|_X)$. In this paper, we give a new definition of Bloch type mappings on $\mathbf{B}_X$ and give distortion theorems for Bloch type mappings on $\mathbf{B}_X$ (Theorem 3.1). When $\mathbf{B}_X$ is the Euclidean unit ball in $\mathbf{C}^n$, this new definition coincides with that given by Chen and Kalaj [2] and the author [10]. Note that the author

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[10] (cf.[5], [13]) gave distortion theorems for other Bloch type mappings on bounded symmetric domains in $\mathbf{C}^n$. As in [10], [13], as a corollary of the distortion theorem, we obtain the lower estimate for the radius of the largest schlicht ball in the image of f centered at $f(0)$ for α-Bloch mappings f on $\mathbf{B}_X$ (Theorem 3.2). Next, as another corollary of the distortion theorem, we show the Lipschitz continuity of $(\det B(z,z))^{1/2n}|\det Df(z)|^{1/n}$ for Bloch type mappings f on $\mathbf{B}_X$ with respect to the Kobayashi metric (Theorem 4.1), where $B(z,z)$ is the Bergman operator on X, and use it to give a sufficient condition for the composition operator C_φ to be bounded from below on the Bloch type space on $\mathbf{B}_X$ (Theorem 5.1), where φ is a holomorphic self mapping of $\mathbf{B}_X$. In the case $\mathbf{B}_X = \mathbf{B}^n$, we also give a necessary condition for C_φ to be bounded from below which is a converse to Theorem 5.1 (Theorem 5.2). Finally, as another application of the Lipschitz continuity, we obtain a result related to the interpolating sequences for the Bloch type space on $\mathbf{B}_X$ (Corollary 5.3).

2. Preliminaries

Let X and Y be complex Banach spaces. Let $L(X,Y)$ denote the set of continuous linear operators from X into Y. Let I_X be the identity in $L(X)=L(X,X)$. For a linear operator $A\in L(X,Y)$, let

$$\|A\|_{X,Y} = \sup\{\|Az\|_Y\colon \|z\|_X = 1\},$$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms on X and Y, respectively. In the case $Y=\mathbf{C}^n$ is the Euclidean space, we write $\|A\|_{X,e}$ for $A\in L(X,\mathbf{C}^n)$.

Definition 2.1. (See e.g. [4] and [17]) A complex Banach space X is called a JB*-triple if X is a complex Banach space equipped with a continuous Jordan triple product

$$X\times X\times X\to X \quad (x,y,z)\mapsto \{x,y,z\}$$

satisfying

- (J_1) $\{x,y,z\}$ is symmetric bilinear in the outer variables, but conjugate linear in the middle variable,
- (J_2) $\{a,b,\{x,y,z\}\} = \{\{a,b,x\},y,z\} - \{x,\{b,a,y\},z\} + \{x,y,\{a,b,z\}\}$,
- (J_3) $x\square x\in L(X)$ is a hermitian operator with spectrum $\geqq 0$,
- (J_4) $\|\{x,x,x\}\| = \|x\|^3$

for $a,b,x,y,z\in X$, where the *box operator* $x\square y\colon X\to X$ is defined by $x\square y(\cdot) = \{x,y,\cdot\}$ and $\|\cdot\|$ is the norm on X.

An element $u\in X$ which satisfies $\{u,u,u\}=u$ is called a tripotent. Two tripotents u and v which satisfy $u\square v=0$ are said to be orthogonal. A tripotent u is said to be maximal if the only tripotent which is orthogonal to u is 0.

For every $x,y\in X$, the Bergman operator $B(x,y)\in L(X)$ is defined by

$$B(x,y)(z) = z - 2(x\square y)(z) + \{x,\{y,z,y\},x\}, \quad z\in X.$$

When $\|x\square y\|<1$, the fractional power $B(x,y)^r\in L(X)$ exists for every $r\in\mathbf{R}$, since the spectrum of $B(x,y)$ lies in $\{\zeta\in\mathbf{C}\colon |\zeta-1|<1\}$ (cf. [14, p. 517]).

Let $\mathbf{B}_X$ be the unit ball of X. We denote by $H(\mathbf{B}_X)$ the family of holomorphic mappings from $\mathbf{B}_X$ into X. Let $Df(x)$ denote the Fréchet derivative of $f\in H(\mathbf{B}_X)$. Also, let $g_a\colon \mathbf{B}_X\to\mathbf{B}_X$ be the Möbius transformation induced by a, given by

$$g_a(x) = a + B(a,a)^{1/2}(I_X + x\square a)^{-1}(x), \quad x\in\mathbf{B}_X.$$

Then g_a is an automorphism of $\mathbf{B}_X$ such that $g_a^{-1} = g_{-a}$, and $g_a(-a) = 0$. We will denote by ρ the Kobayashi metric on $\mathbf{B}_X$, which is the integral form of the infinitesimal Kobayashi metric. For $a, b \in \mathbf{B}_X$, we have $\rho(a, b) = \tanh^{-1} \|g_{-a}(b)\|$, where g_{-a} is the Möbius transformation induced by $-a$.

In the rest of this paper, let $\mathbf{B}_X$ be the unit ball of an n-dimensional JB*-triple $X = (\mathbf{C}^n, \|\cdot\|_X)$. Also, let h_0 be the Bergman metric on $\mathbf{B}_X$ at 0, and let (see [11])

$$c(\mathbf{B}_X) = \frac{1}{2} \sup_{x,y\in\mathbf{B}_X} |h_0(x, y)|.$$

The following lemma is obtained in the proof of [13, Theorem 3.1].

Lemma 2.2. *Let $\alpha > 0$, $c = c(\mathbf{B}_X)$ and let*

$$r(t) = t(1 - t^2)^{c\alpha} \left(\frac{2c\alpha + 1}{2c\alpha}\right)^{c\alpha} \sqrt{2c\alpha + 1}, \quad t \in [0, 1].$$

Then $r(t)$ is increasing on $\left[0, \frac{1}{\sqrt{2c\alpha+1}}\right]$, decreasing on $\left[\frac{1}{\sqrt{2c\alpha+1}}, 1\right]$ and $r\left(\frac{1}{\sqrt{2c\alpha+1}}\right) = 1$.

The following result is a Schwarz–Pick lemma on the open unit ball $\mathbf{B}_X$ of the JB*-triple X [7] (cf. [9, Theorem 3] and [12, Corollary 4.7]; see [8, Lemma 1.1] and [18], in the case $\mathbf{B}_X = \mathbf{B}^n$).

Lemma 2.3. *Let $\mathbf{B}_X$ be the unit ball of a finite dimensional JB*-triple X and let $\varphi\colon \mathbf{B}_X \to \mathbf{B}_X$ be a holomorphic mapping. Then*

$$|\det D\varphi(z)| \le \left[\frac{\det B(\varphi(z), \varphi(z))}{\det B(z, z)}\right]^{1/2}, \quad \forall\, z \in \mathbf{B}_X.$$

The above estimate is sharp and equality at a given point $z \in \mathbf{B}_X$ holds if and only if φ is an automorphism of $\mathbf{B}_X$.

The following lemma was obtained in [10](see also [13, Lemma 2.2]).

Lemma 2.4. *Let $\mathbf{B}_X$ be a bounded symmetric domain realized as the unit ball of an n-dimensional JB*-triple $X = (\mathbf{C}^n, \|\cdot\|_X)$. Then we have*

$$\det B(z, z) \ge (1 - \|z\|^2)^{2c(\mathbf{B}_X)}, \quad z \in \mathbf{B}_X.$$

The equality holds for $z \ne 0$ such that $e = z/\|z\|$ is a maximal tripotent in X.

Definition 2.5. Let $\mathbf{B}_X$ be a bounded symmetric domain realized as the unit ball of an n-dimensional JB*-triple $X = (\mathbf{C}^n, \|\cdot\|_X)$ and let $\alpha > 0$. The prenorm $\|f\|_{\mathcal{P}(X,\alpha)}$ of $f \in H(\mathbf{B}_X)$ is given by

$$\|f\|_{\mathcal{P}(X,\alpha)} = \sup_{z\in\mathbf{B}_X} D_f^{X,\alpha}(z),$$

where

$$D_f^{X,\alpha}(z) = (\det B(z, z))^{\alpha/2n} |\det Df(z)|^{1/n}.$$

Let $\mathcal{B}_{\mathcal{P}(X,\alpha)}$ be the class of all mappings $f \in H(\mathbf{B}_X)$ which satisfy $\|f\|_{\mathcal{P}(X,\alpha)} < +\infty$. A mapping $f \in \mathcal{B}_{\mathcal{P}(X,\alpha)}$ will be called a Bloch type mapping on $\mathbf{B}_X$.

Remark 2.6. (i) In [13, Definition 2.5], another prenorm $\|f\|_{0,\alpha}$ is defined as follows:

$$\|f\|_{0,\alpha} = \sup\left\{(1 - \|z\|^2)^{\alpha c(\mathbf{B}_X)/n} |\det Df(z)|^{1/n} \colon z \in \mathbf{B}_X\right\}.$$

Then, by Lemma 2.4, we have $\|f\|_{0,\alpha} \le \|f\|_{\mathcal{P}(X,\alpha)}$.

(ii) When $\mathbf{B}_X$ is the Euclidean unit ball $\mathbf{B}^n$ in $\mathbf{C}^n$, then $c(\mathbf{B}^n) = (n+1)/2$ ([11]) and $\det B(z,z) = (1-\|z\|^2)^{n+1}$ for all $z \in \mathbf{B}^n$ ([10], see also [13, Lemma 2.2]). Therefore, if $\mathbf{B}_X$ is the Euclidean unit ball $\mathbf{B}^n$ in $\mathbf{C}^n$, then we have $\|f\|_{0,\alpha} = \|f\|_{\mathcal{P}(X,\alpha)}$.

(iii) When $\mathbf{B}_X$ is the unit polydisc $\mathbf{U}^n$ in $\mathbf{C}^n$, we have $\det B(z,z) = \Pi_{i=1}^n (1-|z_i|^2)^2$. Therefore, we have

$$\|f\|_{\mathcal{P}(X,\alpha)} = \sup_{z\in\mathbf{U}^n} \Pi_{i=1}^n (1-|z_i|^2)^{\alpha/n} |\det Df(z)|^{1/n}.$$

When $\alpha = 1$, $\|f\|_{\mathcal{P}(X,1)}$ coincides with the prenorm $\|f\|_0$ defined by Wang and Liu [19].

As in [13, Lemma 2.6], we obtain the following lemma.

Lemma 2.7. *Let $\mathbf{B}_X$ be a bounded symmetric domain realized as the unit ball of an n-dimensional JB*-triple $X = (\mathbf{C}^n, \|\cdot\|_X)$ and let $\alpha > 0$.*

(i) *If $\|f\|_{\mathcal{P}(X,\alpha)} < +\infty$, then*

$$|\det Df(z)| \le \frac{\|f\|^n_{\mathcal{P}(X,\alpha)}}{(1-\|z\|^2)^{\alpha c(\mathbf{B}_X)}}, \quad z \in \mathbf{B}_X.$$

(ii) *If $\|f\|_{\mathcal{P}(X,\alpha)} = 1$ and $\det Df(0) = 1$, then $|\det Df(z)| = 1 + o(\|z\|)$.*

Proof. (i) This is a consequence of the definition of $\|f\|_{\mathcal{P}(X,\alpha)}$ and Lemma 2.4.
(ii) This follows from (i). □

For $x \in X \setminus \{0\}$, we define

$$T(x) = \{l_x \in X^* \colon l_x(x) = \|x\|,\ \|l_x\| = 1\},$$

where X^* is the dual space of X. Then $T(x) \neq \emptyset$ in view of the Hahn–Banach theorem. Let $H(\mathbf{U})$ denote the set of holomorphic functions on the unit disc $\mathbf{U}$ in $\mathbf{C}$. The following lemma was proved by the author [10] (see also [13, Lemma 2.9]).

Lemma 2.8. *Let $\mathbf{B}_X$ be a bounded symmetric domain realized as the unit ball of an n-dimensional JB*-triple $X = (\mathbf{C}^n, \|\cdot\|_X)$. Let $u \in \partial\mathbf{B}_X$ be fixed and let*

$$f_{\psi,l_u}(z) = \left(\int_0^{l_u(z)} \psi(\zeta)\,d\zeta\right) u + z - l_u(z)u, \quad z \in \mathbf{B}_X,$$

where $l_u \in T(u)$ and $\psi \in H(\mathbf{U})$. Then $f_{\psi,l_u} \in H(\mathbf{B}_X)$, $f_{\psi,l_u}(0) = 0$ and $\det Df_{\psi,l_u}(z) = \psi(l_u(z))$ for $z \in \mathbf{B}_X$.

Let d_h denote the hyperbolic distance on $\mathbf{U}$:

$$d_h(x,y) = \frac{1}{2}\log\frac{1+\left|\frac{x-y}{1-\overline{x}y}\right|}{1-\left|\frac{x-y}{1-\overline{x}y}\right|} = \operatorname{arctanh}\left|\frac{x-y}{1-\overline{x}y}\right|, \quad x,y \in \mathbf{U}.$$

Denote by $D_h(a,r)$ the hyperbolic disc in $\mathbf{U}$ with center a and radius r. Also, let $\mathbf{U}(a,r) = \{\zeta \in \mathbf{C} \colon |\zeta - a| < r\}$ for $a \in \mathbf{C}$ and $r > 0$. Then the following lemma holds. The relation (i) was proved by Wang and Liu [20, Lemma 1], while the relation (ii) was recently proved by the author [10].

Lemma 2.9. *Assume that $0 < a < 1$ and $0 \in D_h(a,R)$. If $g \in H(\mathbf{U})$ satisfies $g(0) = \lambda > 0$ and $g(D_h(a,R)) \subset \mathbf{U}\left(0, \frac{\lambda}{a}\tanh R\right)$, then*

(i) *$\Re g(x) \ge G(x)$ for $x \in \left(0, \frac{a+\tanh(R)}{1+a\tanh(R)}\right]$ with equality for some x if and only if $g = G$, where $G(\zeta) = \lambda(a-\zeta)/a(1-a\zeta)$;*

(ii) $|g(-x)| \le G(-x)$ *for* $x \in \left(0, \frac{\tanh(R)-a}{1-a\tanh(R)}\right]$ *with equality for some* x *if and only if* $g = G$.

3. Distortion theorems

We begin this section with the distortion theorems for Bloch type mappings $f \in \mathcal{B}_{\mathcal{P}(X,\alpha)}$. The following theorem is a generalization of [3, Theorem 1.2] to any finite dimensional bounded symmetric domain. In [13, Theorem 3.1], the following theorem was obtained under the condition $\|f\|_{0,\alpha} = 1$ instead of $\|f\|_{\mathcal{P}(X,\alpha)} = 1$. (When $\alpha = 1$, see also [1, Theorem 2], [10, Theorem 3.1], [15, Theorem 5], [19, Theorem 3.1] and [20, Theorem 1]).

Theorem 3.1. *Let* $\mathbf{B}_X$ *be a bounded symmetric domain realized as the unit ball of an* n*-dimensional* JB^**-triple* $X = (\mathbf{C}^n, \|\cdot\|_X)$. *Let* $\alpha > 0$, $\lambda \in (0,1]$, *and let* $m(\lambda)$ *be the unique root of the equation*

$$t(1-t^2)^{\alpha c(\mathbf{B}_X)}\left(\frac{2\alpha c(\mathbf{B}_X)+1}{2\alpha c(\mathbf{B}_X)}\right)^{\alpha c(\mathbf{B}_X)}\sqrt{2\alpha c(\mathbf{B}_X)+1} = \lambda$$

in $\left[0, \frac{1}{\sqrt{2\alpha c(\mathbf{B}_X)+1}}\right]$. *Let* $f \in \mathcal{B}_{\mathcal{P}(X,\alpha)}$ *be such that* $\det Df(0) = \lambda$ *and* $\|f\|_{\mathcal{P}(X,\alpha)} = 1$. *Then*

(i)

$$|\det Df(z)| \ge \Re \det Df(z) \ge \frac{\lambda}{m(\lambda)}\frac{m(\lambda)-\|z\|}{(1-m(\lambda)\|z\|)^{2\alpha c(\mathbf{B}_X)+1}} \tag{3.1}$$

for $\|z\| \le \frac{1+\sqrt{2\alpha c(\mathbf{B}_X)+1}m(\lambda)}{\sqrt{2\alpha c(\mathbf{B}_X)+1}+m(\lambda)}$,

(ii)

$$|\det Df(z)| \le \frac{\lambda}{m(\lambda)}\frac{m(\lambda)+\|z\|}{(1+m(\lambda)\|z\|)^{2\alpha c(\mathbf{B}_X)+1}}$$

for $\|z\| \le \frac{1-\sqrt{2\alpha c(\mathbf{B}_X)+1}m(\lambda)}{\sqrt{2\alpha c(\mathbf{B}_X)+1}-m(\lambda)}$.

Proof. It suffices to use arguments similar to those in the proof of [13, Theorem 3.1] (cf. [3, Theorem 1.2], [10, Theorem 3.1]). For completeness, we give a proof for (i). Let $c = c(\mathbf{B}_X)$. Note that there exists a unique $m(\lambda) \in \left(0, \frac{1}{\sqrt{2\alpha c+1}}\right]$ such that

$$m(\lambda)(1-m(\lambda)^2)^{\alpha c}\left(\frac{2\alpha c+1}{2\alpha c}\right)^{\alpha c}\sqrt{2\alpha c+1} = \lambda \tag{3.2}$$

by Lemma 2.2. Let $z \in \mathbf{B}_X\backslash\{0\}$ be fixed and let $u = z/\|z\|$.

First, we consider the case $\lambda \in (0,1)$. In this case, we have $m(\lambda) \in \left(0, \frac{1}{\sqrt{2\alpha c+1}}\right)$. Let

$$g(\zeta) = (1-m(\lambda)\zeta)^{2\alpha c}\det Df(\zeta u), \quad \zeta \in \mathbf{U}. \tag{3.3}$$

Then $g \in H(\mathbf{U})$ and $g(0) = \lambda \in (0,1)$. Let $\zeta \in \mathbf{U}$ be such that $d_h(m(\lambda),\zeta) = \operatorname{arctanh}\frac{1}{\sqrt{2\alpha c+1}}$. This is equivalent to

$$\left|\frac{m(\lambda)-\zeta}{1-m(\lambda)\zeta}\right| = \frac{1}{\sqrt{2\alpha c+1}}. \tag{3.4}$$

By the relation $\|f\|_{\mathcal{P}(X,\alpha)} = 1$, Lemma 2.7 (i), (3.2), (3.3) and (3.4), we have

$$|g(\zeta)| \le \frac{|1-m(\lambda)\zeta|^{2\alpha c}\|f\|_{\mathcal{P}(X,\alpha)}}{(1-|\zeta|^2)^{\alpha c}} = \left(\frac{1-m(\lambda)^2}{1-\left|\frac{m(\lambda)-\zeta}{1-m(\lambda)\zeta}\right|^2}\right)^{\alpha c}$$
$$= (1-m(\lambda)^2)^{\alpha c}\left(\frac{2\alpha c+1}{2\alpha c}\right)^{\alpha c} = \frac{\lambda}{m(\lambda)}\frac{1}{\sqrt{2\alpha c+1}}.$$

This implies that

$$g\left(D_h\left(m(\lambda), \operatorname{arctanh}\frac{1}{\sqrt{2\alpha c+1}}\right)\right) \subset \mathbf{U}\left(0, \frac{\lambda}{m(\lambda)}\frac{1}{\sqrt{2\alpha c+1}}\right).$$

In view of Lemma 2.9 (i), we obtain that

$$\Re g(\|z\|) \ge \frac{\lambda}{m(\lambda)}\frac{m(\lambda)-\|z\|}{1-m(\lambda)\|z\|}$$

for $\|z\| \in \left(0, \frac{1+\sqrt{2\alpha c+1}m(\lambda)}{\sqrt{2\alpha c+1}+m(\lambda)}\right]$. By (3.3), we have

$$\Re \det Df(z) \ge \frac{\lambda}{m(\lambda)}\frac{m(\lambda)-\|z\|}{(1-m(\lambda)\|z\|)^{2\alpha c+1}},$$

for $\|z\| \in \left(0, \frac{1+\sqrt{2\alpha c+1}m(\lambda)}{\sqrt{2\alpha c+1}+m(\lambda)}\right]$, i.e., we obtain the inequality (3.1) for $\lambda \in (0,1)$, as desired.

Next, we consider the case $\lambda = 1$. Let

$$h(\zeta) = (1-m(1)T(\zeta))^{2\alpha c}\det Df(T(\zeta)u), \tag{3.5}$$

where

$$T(\zeta) = \frac{m(1)(1-\zeta)}{1-m(1)^2\zeta}.$$

Since $T(1) = 0$ and

$$T(\mathbf{U}) = \{w\colon |1-m(1)w|^2 < 1-|w|^2\} \subset \mathbf{U}, \tag{3.6}$$

the function h is holomorphic on $\mathbf{U}\cup\{1\}$. Moreover, by (3.5), (3.6), Lemma 2.7 (i), and the relation $\|f\|_{\mathcal{P}(X,\alpha)} = 1$, we have

$$|h(\zeta)| = |1-m(1)T(\zeta)|^{2\alpha c}|\det Df(T(\zeta)u)|$$
$$\le (1-|T(\zeta)|^2)^{\alpha c}|\det Df(T(\zeta)u)| \le \|f\|_{\mathcal{P}(X,\alpha)} = 1$$

for all $\zeta \in \mathbf{U}$. Since

$$h'(1) = 2\alpha c(-m(1))T'(1) = \frac{2\alpha c m(1)^2}{1-m(1)^2} = 1$$

by Lemma 2.7 (ii) and the relation $m(1) = 1/\sqrt{2\alpha c+1}$, we have $h(\mathbf{U}) \subset \mathbf{U}$ by the maximum principle. Since $h(1) = 1$, in view of a classical version of Julia's Lemma [16, p. 327], h maps $\Delta(1,r)$, $r > 0$ of $\mathbf{U}$ into itself, where $\Delta(1,r)$ is a horodisc in $\mathbf{U}$, that is,

$$\Delta(1,r) = \left\{z \in \mathbf{U}\colon \frac{|1-z|^2}{1-|z|^2} < r\right\} = \mathbf{U}\left(\frac{1}{1+r}, \frac{r}{1+r}\right).$$

Therefore, we have

$$\Re h(x) \ge x, \quad \forall x \in [-1,1].$$

By (3.5), we have

$$\Re \det Df(z) \geq \frac{1}{m(1)} \frac{m(1) - \|z\|}{(1 - m(1)\|z\|)^{2\alpha c+1}}$$

for all $\|z\| \in [0, (1+\sqrt{2\alpha c+1}m(1))/(\sqrt{2\alpha c+1}+m(1))]$, i.e., we obtain the inequality (3.1) for $\lambda = 1$, as desired.

(ii) The case $\lambda = 1$ is trivial. The case $\lambda \in (0,1)$ can be proved by using arguments similar those in (i). For the details, see the proof of [13, Theorem 3.1]. This completes the proof. □

For $f \in H(\mathbf{B}_X)$, a Euclidean ball with center $f(p)$ such that f maps an open subset of $\mathbf{B}_X$ containing p biholomorphically onto this ball is called a schlicht ball of f centered at $f(p)$. For a point $p \in \mathbf{B}_X$, let $r(p, f)$ denote the radius of the largest schlicht ball of f centered at $f(p)$.

A mapping $f \in H(\mathbf{B}_X)$ is called an α-Bloch mapping if

$$\|f\|_\alpha < +\infty,$$

where $\|f\|_\alpha$ denotes the α-Bloch semi-norm of f defined by

$$\|f\|_\alpha = \sup_{z \in \mathbf{B}_X} \|Df(z)B(z,z)^{\alpha/2}\|_{X,e}.$$

As a corollary of Theorem 3.1, we obtain the following lower estimate for the radius of the largest schlicht ball in the image of f centered at $f(0)$ for α-Bloch mappings $f \in H(\mathbf{B}_X)$. The following theorem is a generalization of [3, Theorem 1.3] to the unit ball of a finite dimensional JB*-triple. In [13, Theorem 4.5], the following theorem was obtained under the condition $\|f\|_{0,\alpha} = 1$ instead of $\|f\|_{\mathcal{P}(X,\alpha)} = 1$. When $\alpha = 1$, it reduces to the result for Bloch mappings on bounded symmetric domains in $\mathbf{C}^n$ [10, Theorem 4.4] (see also [15, Theorem 6], [19, Theorem 3.3], [20, Theorem 2]). We omit the proof, since it suffices to use the arguments similar to those in the proof of [13, Theorem 4.5].

Theorem 3.2. *Let $\mathbf{B}_X$ be a bounded symmetric domain realized as the unit ball of an n-dimensional JB*-triple $X = (\mathbf{C}^n, \|\cdot\|_X)$ such that $\mathbf{B}_X \supset \mathbf{B}^n$, where $\mathbf{B}^n$ is the Euclidean unit ball of $\mathbf{C}^n$. Let $\alpha > 0$ and $K > 0$. If f is an α-Bloch mapping on $\mathbf{B}_X$ such that $\|f\|_\alpha \leq K$, $\|f\|_{\mathcal{P}(X,\alpha)} = 1$ and $\det Df(0) = \lambda \in (0,1]$, then*

$$r(0,f) \geq K^{1-n} \frac{\lambda}{m(\lambda)} \int_0^{m(\lambda)} \frac{(1-t^2)^{\alpha(n-1)}(m(\lambda)-t)}{(1-m(\lambda)t)^{2\alpha c(\mathbf{B}_X)+1}} dt,$$

where $m(\lambda)$ is the unique root of the equation

$$t(1-t^2)^{\alpha c(\mathbf{B}_X)} \left(\frac{2\alpha c(\mathbf{B}_X)+1}{2\alpha c(\mathbf{B}_X)} \right)^{\alpha c(\mathbf{B}_X)} \sqrt{2\alpha c(\mathbf{B}_X)+1} = \lambda$$

in the interval $\left[0, \frac{1}{\sqrt{2\alpha c(\mathbf{B}_X)+1}}\right]$.

4. Lipschitz continuity

Chen and Kalaj [2, Theorem 1] obtained the Lipschitz continuity of

$$(1-\|z\|^2)^{\frac{n+1}{2n}} |\det Df(z)|^{1/n}$$

for Bloch type mappings f on the Euclidean unit ball in $\mathbf{C}^n$ (see [6, Theorem 1] in the case $n=1$). The following theorem is a generalization to Bloch type mappings on bounded symmetric domains in $\mathbf{C}^n$.

Theorem 4.1. *Let $\mathbf{B}_X$ be a bounded symmetric domain realized as the unit ball of an n-dimensional JB*-triple $X=(\mathbf{C}^n, \|\cdot\|_X)$. Let $f \in \mathcal{B}_{\mathcal{P}(X,1)}$. Then we have*

$$|D_f^{X,1}(z_1) - D_f^{X,1}(z_2)| \leq M(X)\|f\|_{\mathcal{P}(X,1)}[\tanh\rho(z_1,z_2)]^{\frac{1}{n}},$$

for $z_1, z_2 \in \mathbf{B}_X$, where

$$M(X) = (2c(\mathbf{B}_X)+1)^{\frac{1}{2n}} \left(\frac{2c(\mathbf{B}_X)+1}{2c(\mathbf{B}_X)}\right)^{\frac{c(\mathbf{B}_X)}{n}}.$$

Proof. Let $z_1, z_2 \in \mathbf{B}_X$ be fixed. We may assume that $\|f\|_{\mathcal{P}(X,1)} = 1$ and $D_f^{X,1}(z_2) \leq D_f^{X,1}(z_1)$. Let $w = g_{z_1}^{-1}(z_2)$. Then, by the invariance of ρ by automorphisms of $\mathbf{B}_X$, we have

$$\tanh\rho(z_1,z_2) = \tanh\rho(g_{z_1}^{-1}(z_1), g_{z_1}^{-1}(z_2)) = \tanh\rho(0,w) = \|w\|.$$

Therefore, it suffices to show that

$$D_f^{X,1}(z_1) - D_f^{X,1}(z_2) \leq M(X)\|w\|^{\frac{1}{n}}. \tag{4.1}$$

Let $F = f \circ g_{z_1}$. Since

$$|\det Dg_{z_1}(z)| = \left|\frac{\det B(g_{z_1}(z), g_{z_1}(z))}{\det B(z,z)}\right|^{1/2},$$

by Lemma 2.3, we have

$$D_F^{X,1}(0) = |\det DF(0)|^{\frac{1}{n}} = |\det Df(z_1)|^{\frac{1}{n}} |\det B(z_1,z_1)|^{\frac{1}{2n}} = D_f^{X,1}(z_1)$$

and

$$\begin{aligned} D_F^{X,1}(w) &= |\det DF(w)|^{\frac{1}{n}} |\det B(w,w)|^{\frac{1}{2n}} \\ &= |\det Df(z_2)|^{\frac{1}{n}} |\det B(z_2,z_2)|^{\frac{1}{2n}} = D_f^{X,1}(z_2). \end{aligned}$$

By using similar calculations, we also have $\|F\|_{\mathcal{P}(X,1)} = 1$.

If $|\det DF(0)| = 0$, then by the inequality $D_f^{X,1}(z_2) \leq D_f^{X,1}(z_1)$, we have $D_f^{X,1}(z_2) = 0$, Therefore, the inequality (4.1) holds. So, we may assume that $\det DF(0) = \lambda e^{i\theta}$ for some $\lambda \in (0,1]$ and $\theta \in [0,2\pi]$. Let $c = c(\mathbf{B}_X)$ and let $m(\lambda)$ be the constant in Theorem 3.1 for $\alpha = 1$. Then by (3.2), we have

$$m(\lambda)(1-m(\lambda)^2)^c M(X)^n = \lambda. \tag{4.2}$$

Case 1. Assume that $\|w\| \leq m(\lambda)$. Then, we have

$$(1-\|w\|^2)^{c/n} \geq (1-\|w\|m(\lambda))^{(2c+1)/n}$$

and

$$m(\lambda)^{1/n} - \|w\|^{1/n} \leq (m(\lambda)-\|w\|)^{1/n}.$$

These inequalities imply that

$$m(\lambda)^{1/n} - \frac{(1-\|w\|^2)^{c/n}(m(\lambda)-\|w\|)^{1/n}}{(1-m(\lambda)\|w\|)^{(2c+1)/n}} \leq \|w\|^{1/n}. \tag{4.3}$$

Since $m(\lambda) \leq 1$, we have

$$\|w\| \leq m(\lambda) \leq \frac{1+\sqrt{2c+1}\,m(\lambda)}{\sqrt{2c+1}+m(\lambda)}.$$

Also, $|\det DF(0)| = \lambda \le 1$. Then, applying Theorem 3.1 to $e^{-i\theta/n}F$, we have

$$\Re(e^{-i\theta}\det DF(w)) \ge \frac{\lambda}{m(\lambda)}\frac{m(\lambda)-\|w\|}{(1-m(\lambda)\|w\|)^{2c+1}}. \tag{4.4}$$

Therefore, by (4.2), (4.3), (4.4) and Lemma 2.4, we have

$$\begin{aligned} D_f^{X,1}(z_1) - D_f^{X,1}(z_2) &= |\det DF(0)|^{1/n} - |\det DF(w)|^{\frac{1}{n}}|\det B(w,w)|^{\frac{1}{2n}} \\ &\le \left(\frac{\lambda}{m(\lambda)}\right)^{\frac{1}{n}}\left(m(\lambda)^{1/n} - \frac{(1-\|w\|^2)^{c/n}(m(\lambda)-\|w\|)^{1/n}}{(1-m(\lambda)\|w\|)^{(2c+1)/n}}\right) \\ &\le \left(\frac{\lambda}{m(\lambda)}\right)^{\frac{1}{n}}\|w\|^{\frac{1}{n}} = M(X)(1-m(\lambda)^2)^{\frac{c}{n}}\|w\|^{\frac{1}{n}} \\ &\le M(X)\|w\|^{\frac{1}{n}}. \end{aligned}$$

Case 2. Assume that $\|w\| > m(\lambda)$. Then, by (4.2), we have

$$\begin{aligned} D_f^{X,1}(z_1) - D_f^{X,1}(z_2) &\le |\det DF(0)|^{1/n} = \lambda^{1/n} \\ &= m(\lambda)^{\frac{1}{n}}(1-m(\lambda)^2)^{\frac{c}{n}}M(X) \le M(X)\|w\|^{\frac{1}{n}}. \end{aligned}$$

This completes the proof. □

5. Applications of the Lipschitz continuity

As an application of Theorem 4.1, we first show a result related to a lower bound for the composition operator between the Bloch type spaces on bounded symmetric domains in $\mathbf{C}^n$. The following theorem is a generalization of [2, Theorem 3] to bounded symmetric domains in $\mathbf{C}^n$ (see [6, Theorem 2] in the case $n = 1$).

For a holomorphic mapping $\varphi\colon \mathbf{B}_X \to \mathbf{B}_X$, let

$$\tau_\varphi(z) = \frac{\det B(z,z)^{\frac{1}{2n}}}{\det B(\varphi(z),\varphi(z))^{\frac{1}{2n}}}|\det D\varphi(z)|^{\frac{1}{n}},$$

where $B(z,z)$ is the Bergman operator and n is the dimension of X. By Lemma 2.3, we always have $0 \le \tau_\varphi(z) \le 1$.

Theorem 5.1. *Let $\mathbf{B}_X$ be a bounded symmetric domain realized as the unit ball of an n-dimensional JB*-triple $X = (\mathbf{C}^n, \|\cdot\|_X)$. Let φ be a holomorphic mapping of $\mathbf{B}_X$ into $\mathbf{B}_X$. Assume that there are positive constants r, ε with $0 < r < M(X)^{-1}$ such that, for each $w \in \mathbf{B}_X$, there exists a point $z_w \in \mathbf{B}_X$ such that $\tanh\rho(\varphi(z_w), w) < r^n$ and $\tau_\varphi(z_w) > \varepsilon$. Then, we have*

$$\|C_\varphi(f)\|_{\mathcal{P}(X,1)} \ge \frac{(1-rM(X))\varepsilon}{2}\|f\|_{\mathcal{P}(X,1)}, \quad f \in \mathcal{B}_{\mathcal{P}(X,1)}.$$

Proof. We may assume that $\|f\|_{\mathcal{P}(X,1)} = 1$. Then, there exists a point $w \in \mathbf{B}_X$ such that

$$D_f^{X,1}(w) > 1-\sigma,$$

where

$$\sigma = \frac{1-rM(X)}{2} \in (0, 1/2).$$

By the assumption, there exists a point $z_w \in \mathbf{B}_X$ such that

$$[\tanh\rho(\varphi(z_w), w)]^{\frac{1}{n}} < r$$

and $\tau_\varphi(z_w) > \varepsilon$. Therefore, by applying Theorem 4.1, we have

$$\begin{aligned} D_f^{X,1}(\varphi(z_w)) &\geq D_f^{X,1}(w) - M(X)[\tanh \rho(\varphi(z_w), w)]^{\frac{1}{n}} \\ &\geq 1 - \sigma - rM(X) = \frac{1 - rM(X)}{2} > 0. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|C_\varphi(f)\|_{\mathcal{P}(X,1)} &\geq |\det Df(\varphi(z_w)) \det D\varphi(z_w) \det B(z_w, z_w)^{\frac{1}{2}}|^{\frac{1}{n}} \\ &= D_f^{X,1}(\varphi(z_w))\tau_\varphi(z_w) > \frac{(1 - rM(X))\varepsilon}{2}. \end{aligned}$$

This completes the proof. □

In the case $\mathbf{B}_X$ is the Euclidean unit ball $\mathbf{B}^n$ of $\mathbf{C}^n$, we obtain the following theorem which is a converse to Theorem 5.1. We prepare some notations on $\mathbf{B}^n$. Fix $a \in \mathbf{B}^n$. Let

$$P_a(z) = \begin{cases} \frac{\langle z,a\rangle}{\langle a,a\rangle} a, & a \neq 0, \\ 0, & a = 0. \end{cases}$$

and let $Q_a = I - P_a$. Put $s_a = (1 - \|a\|^2)^{1/2}$ and define the automorphism φ_a of $\mathbf{B}^n$ by

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}.$$

Also, we denote $\|\cdot\|_{\mathcal{P}(X,1)}$ by $\|\cdot\|_{\mathcal{P}(n,1)}$, $\mathcal{B}_{\mathcal{P}(X,1)}$ by $\mathcal{B}_{\mathcal{P}(n,1)}$ and $D_f^{X,1}(z)$ by $D_f^{n,1}(z)$.

Theorem 5.2. *Let φ be a holomorphic mapping of $\mathbf{B}^n$ into $\mathbf{B}^n$. Assume that there exists a constant $k \in (0,1)$ such that*

$$\|C_\varphi(f)\|_{\mathcal{P}(n,1)} \geq k\|f\|_{\mathcal{P}(n,1)}, \quad f \in \mathcal{B}_{\mathcal{P}(n,1)}.$$

Then there exist positive constants r, ε with $R < r < 1$ such that, for each $w \in \mathbf{B}^n$, there exists a point $z_w \in \mathbf{B}^n$ such that $\tanh \rho(\varphi(z_w), w) < r^n$ and $\tau_\varphi(z_w) > \varepsilon$, where

$$R = \left(1 - k^{\frac{2n}{n+1}}\right)^{\frac{1}{2n}}.$$

Proof. Let $w \in \mathbf{B}^n$ be fixed. Let $f_{\psi,l_u} \in H(\mathbf{B}^n)$ be as in Lemma 2.8, where

$$\psi(\zeta) = \frac{(1 - \|w\|^2)^{\frac{n+1}{2}}}{(1 - \|w\|\zeta)^{n+1}}, \quad \zeta \in \mathbf{U},$$

and $u = \frac{w}{\|w\|}$ for $w \in \mathbf{B}^n \setminus \{0\}$ and $u \in \partial\mathbf{B}^n$ is arbitrary for $w = 0$. By Lemma 2.8, we have

$$\begin{aligned} D_{f_{\psi,l_u}}^{n,1}(z) &= |\det Df_{\psi,l_u}(z)|^{\frac{1}{n}}(1 - \|z\|^2)^{\frac{n+1}{2n}} = \left(\frac{(1 - \|w\|^2)(1 - \|z\|^2)}{|1 - \langle z, w\rangle|^2}\right)^{\frac{n+1}{2n}} \\ &= (1 - \|\varphi_w(z)\|^2)^{\frac{n+1}{2n}}, \quad z \in \mathbf{B}^n. \end{aligned}$$

Therefore, we have $\|f_{\psi,l_u}\|_{\mathcal{P}(n,1)} = 1$. By the assumption, we have

$$k = k\|f_{\psi,l_u}\|_{\mathcal{P}(n,1)} \leq \|C_\varphi(f_{\psi,l_u})\|_{\mathcal{P}(n,1)}.$$

Then, for any $c \in (0,1)$, there exists $z_w \in \mathbf{B}^n$ such that

$$D_{f_{\psi,l_u}\circ\varphi}^{n,1}(z_w) \geq ck.$$

Since

$$D_{f_{\psi,l_u}\circ\varphi}^{n,1}(z_w) = \tau_\varphi(z_w) D_{f_{\psi,l_u}}^{n,1}(\varphi(z_w)) = \tau_\varphi(z_w)(1 - \|\varphi_w(\varphi(z_w))\|^2)^{\frac{n+1}{2n}},$$

we have

$$ck \leq \tau_\varphi(z_w)(1 - \|\varphi_w(\varphi(z_w))\|^2)^{\frac{n+1}{2n}}.$$

Since $\tau_\varphi(z_w) \leq 1$, we have

$$ck \leq \tau_\varphi(z_w)$$

and

$$ck \leq (1 - \|\varphi_w(\varphi(z_w))\|^2)^{\frac{n+1}{2n}}.$$

Let

$$r = \left(1 - (ck)^{\frac{2n}{n+1}}\right)^{\frac{1}{2n}}$$

and let $\varepsilon = ck$. Then we have $r \in (R, 1)$,

$$\tanh \rho(\varphi(z_w), w) = \|\varphi_w(\varphi(z_w))\| \leq r^n$$

and $\tau_\varphi(z_w) \geq \varepsilon$. This completes the proof. □

As another application of Theorem 4.1, we obtain the following result related to the interpolating sequences for the Bloch type space on bounded symmetric domains in $\mathbf{C}^n$. A sequence (z_ν) in $\mathbf{B}_X$ is said to be separated in the Kobayashi metric if there exists a constant $\varepsilon > 0$ such that $\rho(z_\nu, z_\mu) \geq \varepsilon$ whenever $\nu \neq \mu$. The following corollary generalizes [6, Corollary 1]. We omit the proof, since it suffices to use arguments similar to those in the proof of [6, Corollary 1].

Corollary 5.3. *Let $\mathbf{B}_X$ be a bounded symmetric domain realized as the unit ball of an n-dimensional JB*-triple $X = (\mathbf{C}^n, \|\cdot\|_X)$. If a sequence $\{z_\nu\} \subset \mathbf{B}_X$ satisfies the property that the map $S\colon \mathcal{B}_{\mathcal{P}(X,1)} \to \ell^\infty$ defined by $S(f) = \{\det Df(z_\nu) \det B(z_\nu, z_\nu)^{\frac{1}{2}}\}$ is onto, then $\{z_\nu\}$ is separated in the Kobayashi metric.*

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