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CONSTRUCTING QUASICONFORMAL MAPS USING CIRCLE PACKINGS AND BROOK'S PARAMETRIZATION OF QUADRILATERALS

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Abstract. Circle packings have deep and well-established connections to conformal maps. Some methods for using circle packings to approximate quasiconformal maps have been studied, but they are not directly tied to the circle geometry. We present here a means to construct quasiconformal maps using Brooks's parameterization of quadrilateral regions bounded by circles. The Brooks parameter acts as a sort of circle packing module, allowing us to directly affect the complex dilatation of our quasiconformal maps.

1. Introduction

Quasiconformal maps play an important role in the geometric theory of functions of a complex variable and a central role in the study of deformations of Riemann surfaces [10, 17]. The "quasi-ness" of a quasiconformal map is determined by its complex dilatation μ. Maps with complex dilatation zero are the classical conformal maps.

The Measureable Riemann Mapping Theorem states that given a measurable function μ with $\|\mu\|_\infty < 1$ on a simply connected domain $\Omega \subsetneq \mathbf{C}$, there exists a unique (suitably normalized) quasiconformal map F_μ from Ω onto the unit disc $\mathbf{D}$ having μ as its complex dilatation. This generalization of the classical Riemann Mapping Theorem plays a key role in Teichmüller theory.

Various numerical schemes for approximating F_μ have been introduced including He's use of circle packings to produce the mapping onto $\mathbf{D}$ once Ω has been quasiconformally deformed [11] and Williams's use of conformal welding on circle packings [24], providing a discrete analog of Lehto and Virtanen's classical conformal welding proof of the Measurable Riemann Mapping Theorem [17].

However, as both conformal and quasiconformal maps are deeply geometric phenomena, one might expect that a direct method of constructing quasiconformal maps using the geometric properties of circle packings would be possible. Our goal in this paper is to present such a construction.

The key to this construction is Brooks's parameterization of quadrilateral regions bounded by circles [6, 7, 8]. Brooks's parameter counts the number and type of circles which are required to fill the quadrilateral. It thus uses the circles themselves to act as combinatorial analog of the conformal module.

After a brief introduction to quasiconformal maps in Section 2 and circle packings in Section 3, we describe how to use subdivision and the Brooks parameter to construct quasiconformal maps of $\mathbf{C}$ with constant complex dilatation in Section 4.

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Finally we extend these results to nonconstant dilatations with values in a certain range in Section 5.

2. Quasiconformal Maps

2.1. Definitions.

Definition 2.1. A homeomorphism $f\colon \Omega \to \Omega'$ is *quasiconformal* if the complex partial derivatives f_z and $f_{\bar z}$ exist and f satisfies the Beltrami equation

$$f_{\bar z} = \mu f_z$$

for some measurable *Beltrami differential* μ with $\|\mu\|_\infty < 1$.

This Beltrami differential μ is called the *complex dilatation* of the quasiconformal map f. Notice that if $\mu \equiv 0$, then the Beltrami equation reduces to the Cauchy-Riemann equations. Thus $\|\mu\|_\infty$ provides a measure of how "quasi" a quasi-conformal map is.

2.2. Constant dilatation. If μ is constant on $\mathbf{C}$ with modulus less than 1, it is easy to show there is unique solution to the Beltrami equation on $\mathbf{C}$ normalized to fix 1 and 0, namely

$$F_\mu(z) = \frac{\mu \bar z + z}{\mu + 1}.$$

If we write $\mu = a+ib$, we can after a straightforward, if somewhat tedious calculation, express F_μ in matrix form as

$$F_\mu(x+iy) = \begin{pmatrix} 1 & \frac{b(1-a^2)}{(1+a)^2+b^2} \\ 0 & \frac{1-a^2-b^2}{(1+a)^2+b^2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In the language of linear algebra, this is the change of basis on $\mathbf{R}^2$ that changes the basis $\{<1,0>, <0,1>\}$ to a new basis $\{<1,0>, <c,d>\}$ with

$$c = \frac{b(1-a^2)}{(1+a)^2+b^2}, \quad d = \frac{1-a^2-b^2}{(1+a)^2+b^2} = \frac{1-|\mu|^2}{(1+a)^2+b^2} > 0.$$

In the language of complex variables, the map is a quasiconformal automorphism of $\mathbf{C}$ which leaves the upper half-plane $\mathbf{H}$ invariant. Moreover, we can easily recover the dilatation μ from the image of the point i, as

$$\mu = \frac{(F_\mu)_{\bar z}}{(F_\mu)_z} = \frac{1-d+ic}{1+d-ic} = \frac{i-(c+id)}{i+(c+id)} = \frac{i-F_\mu(i)}{i+F_\mu(i)}.$$

We will return to this example and this notation in Section 4.

2.3. Measurable Riemann Mapping Theorem. Given a Beltrami differential μ on a simply connected region $\Omega \subsetneq \mathbf{C}$, it is natural to ask whether there exists a quasiconformal map $F_\mu\colon \Omega \to \mathbf{D}$ having μ as its complex dilatation. This is of course a generalization of the Riemann Mapping Theorem which settles the question for the case $\mu \equiv 0$.

Measurable Riemann Mapping Theorem. *Given a measurable μ with $\|\mu\|_\infty < 1$ on Ω, there exists a quasiconformal map $F_\mu\colon \Omega \to \mathbf{D}$ whose complex dilatation is given by μ. This map F_μ is unique up to conformal automorphisms of $\mathbf{D}$. In particular, if a, b are distinct points in Ω and F_μ is normalized so that $F_\mu(a) = 0$ and $F_\mu(b) > 0$, then F_μ is unique.*

The Measurable Riemann Mapping Theorem guarantees the existence of the quasiconformal map F_μ but does not provide directions for constructing it. We will describe how to approximate F_μ by distorting the combinatorial structures of circle packings.

3. Circle packings

3.1. Definitions and examples. Thurston first conjectured the connection between circle packings and analytic functions [22, 21]. Since that time, packings have been widely studied and numerous deep connections between the combinatorics of packings and the geometry of analytic functions have been uncovered [4, 5, 12, 20, 18, 25].

Definition 3.1. A *circle packing* is a locally finite configuration of circles with a specified pattern of tangencies. In particular, if $\mathcal{K}$ is a triangulation of a simply connected region, then a circle packing P for $\mathcal{K}$ is a configuration of circles such that

(1) P contains a circle C_v for each vertex v in $\mathcal{K}$,
(2) C_v is externally tangent to C_u if $[v, u]$ is an edge of $\mathcal{K}$,
(3) $\langle C_v, C_u, C_w\rangle$ forms a positively oriented mutually tangent triple of circles if $\langle v, u, w\rangle$ is a positively oriented face of $\mathcal{K}$
(4) P is locally finite; that is, any compact subset of the ambient space intersects at most finitely many circles of P.

A packing is called *univalent* if none of its circles overlap, that is, if no pair of circles intersect in more than one point.

A univalent circle packing produces an embedding in $\mathbf{C}$ of its underlying triangulation. Vertices can embedded as centers of their corresponding circles, and edges can be added as line segments joining centers of circles. The resulting collection of triangles is called the *carrier* of the packing, written $\operatorname{carr} P$. See Figure 1.

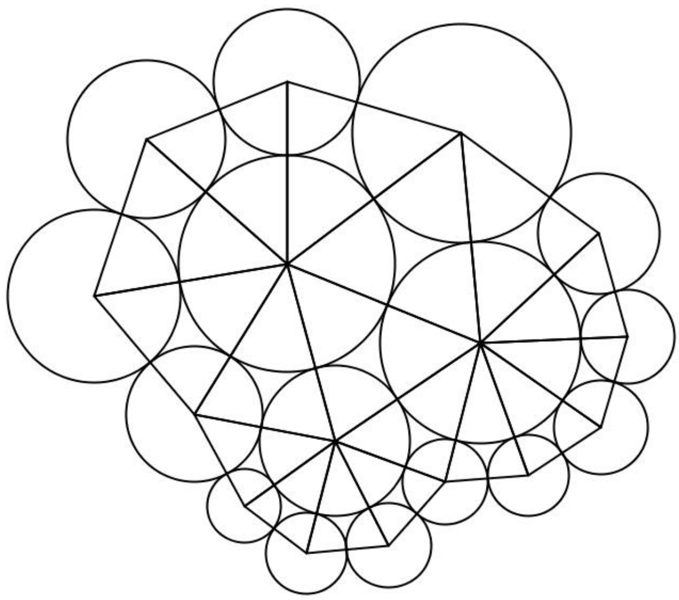

Figure 1. A circle packing with its carrier.

3.2. Existence and uniqueness. The basic existence and uniqueness results were proven independently (and in slightly different forms) by Koebe, Andreev, and Thurston [1, 16, 21] and extended in various directions by many other authors [20]. We will state their results only in the case of simply connected triangulations of a topological disc and in a form best suited to our purposes here.

Theorem 3.2. *Given a finite simply connected triangulation of a closed topological disc $\mathcal{K}$, there exists a packing P for $\mathcal{K}$ filling the unit disc $\mathbf{D}$, that is, having the boundary circles of P internally tangent to the unit circle. This packing is unique up to disc automorphisms.*

In constructing circle packing solutions to the Beltrami equation, we will employ Theorem 3.2 to produce packings in $\mathbf{D}$ which will form the range of our quasiconformal maps. As these packings are unique up to disc automorphisms, we can, for example, choose two circles C_a and C_b and center C_a at 0 and C_b on the positive real axis, thus mirroring the normalization in our statement of the Measureable Riemann Mapping Theorem.

A key step in creating the packings that will form the domains of our quasiconformal maps and in showing that the circle packing maps actually converge to the required map, will rely on the existence and uniqueness of infinite packings.

Theorem 3.3. *Given an infinite simply connected triangulation of an open topological disc $\mathcal{K}$, there exists a packing P for $\mathcal{K}$ filling either the unit disc or the complex plane. This packing is unique up to automorphisms of the underlying space.*

There are numerous results connecting the combinatorial properties of $\mathcal{K}$ with the choice of the underlying space $\mathbf{C}$ or $\mathbf{D}$ in which the packing P lives, but the most directly useful for our purposes is due to He and Schramm [14]. They showed that if the simple random walk on $\mathcal{K}$ is recurrent, then P will fill the plane.

3.3. Circle packings and conformal maps. This rigidity of the infinite packings was what led Thurston to conjecture [22] that circle packings could be used to approximate conformal maps. Conformality is often described as "sending infinitesimal circles to infinitesimal circles" so it was not unreasonable to suggest that circle packings would also reflect this behavior as the size of the circles goes to zero.

Thurston's reasoning (and the eventual proof by Rodin and Sullivan) went somewhat deeper, however. At small scale, conformal maps act like planar automorphisms, that is, maps of the form $re^{i\theta}+c$, with $r > 0$. Thurston showed that as a consequence of the Mostow Rigidity Theorem, infinite univalent hex packings (packings filling $\mathbf{C}$ in which each circle has exactly 6 neighbors) are unique up to planar automorphisms.

On the other hand, large finite univalent hex packings need not show any such rigidity. Theorem 3.2 states that a finite packing that completely fills $\mathbf{D}$ is unique up to disc automorphisms, but the condition that the packing fills $\mathbf{D}$ is crucial. There could be uncountably many other finite packings; for example, if Ω is a proper simply connected subset of $\mathbf{C}$, we could create a finite hex packing P^{Ω} inside Ω by simply throwing away all the circles in an infinite hex packing which lie outside Ω. Theorem 3.2 then guarantees the existence of another suitably normalized packing $P^{\mathbf{D}}$ filling $\mathbf{D}$ with the same underlying triangulation as P^{Ω}.

Now consider a triple of tangent circles deep inside P^{Ω}, surrounded by N generations of circles with 6 neighbors, and the corresponding triple deep inside $P^{\mathbf{D}}$. The uniqueness of the infinite hex packing implies these two triples of circles will be approximately the image of one another under a planar automorphism. More precisely, as $N \to \infty$, the map formed by sending the centers of circles in one triple to the centers of circles in the corresponding triple and extending linearly, will converge to a planar automorphism.

Repeating this process on each triple in P^{Ω}, we construct a map from carr P^{Ω} to carr $P^{\mathbf{D}}$. Repeating the entire construction using smaller and smaller circles to cover Ω produces a sequence of maps which converges uniformly on compact subsets of Ω to the Riemann map $f\colon \Omega \to \mathbf{D}$.

Thus the rigidity of infinite packings and the consequent convergence of maps between packings sharing the same underlying triangulation was soon extended to non-hex packings [13, 19, 20]. In settings ranging from random walks to Teichmüller theory, it has repeatedly been established that maps between circle packings sharing the same underlying triangulation act exactly as one would expect conformal maps to behave [15, 14, 23, 2, 9].

As a result, in our quest to create circle packing versions of *quasiconformal* maps, we must find a natural way to distort the underlying triangulation to reflect the distortion we wish to create in our maps. The key to this process is the Brooks parameter.

3.4. The Brooks parameter. Consider a chain of 4 circles tangent to one another in such a way that they surround a quadrilateral interstice. See figure **??**. In his work on Schottky groups, Brooks provided an extremely useful parameterization of these quadrilateral interstices [6, 7, 8].

Label two of the non-tangent circles as the "top" and "bottom" sides and the other two circles as the "left" and "right" sides. If we add a small circle in the corner formed by the top side and the left side, we can increase its radius until it hits either the bottom side or the right side. Label it a "horizontal" circle if it hits the top and bottom sides and a "vertical" circle if it hits the left and right sides. Notice that in general a quadrilateral interstice will remain after adding this new circle and the original labeling of top, left, etc, will extend in an obvious way to the new interstice. We can repeat this procedure, growing a new circle out of the new top left corner.

Let h_1 be the number of horizontal circles which can be added in this way to the original interstice before a vertical circle must be added. Let v_1 denote the number of vertical circles which can then be added before a horizontal circle is produced. Repeat this procedure to define $h_2, h_3, \ldots$ and $v_2, v_3, \ldots$. Notice that if it is ever possible to add a circle which hits all four remaining sides, then only triangular interstices will remain and both h_n and v_n will be zero after that stage. See Figure 2.

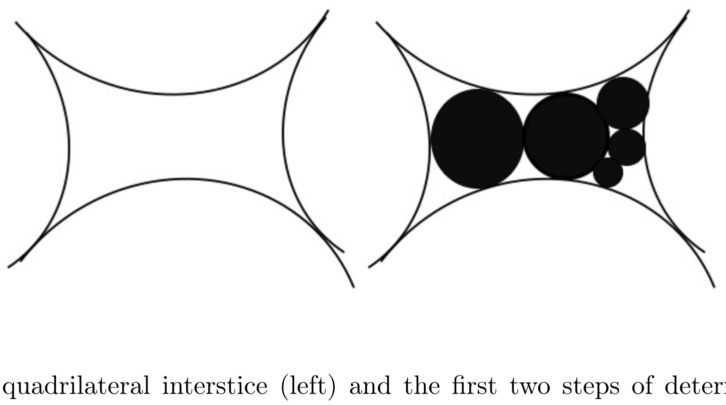

Figure 2. A quadrilateral interstice (left) and the first two steps of determining its Brooks parameter (right). This interstice has $h_1 = 2$ horizontal circles followed by $v_1 = 2$ vertical circles and $h_2 = 1$ horizontal circles.

Definition 3.4. Given a quadrilateral interstice formed by four circles, its Brooks parameter is given by the the continued fraction

$$\beta = h_1 + \cfrac{1}{v_1 + \frac{1}{h_2 + \frac{1}{v_2 + \dots}}}. \tag{3.1}$$

Brooks proved that the parameter depends continuously on the original four circles which form the quadrilateral interstice [7, 8]. Since Möbius transformations preserve circles, the Brooks parameter is Möbius invariant. The Brooks parameter thus provides a kind of combinatorial analog of the conformal module.

Recall that our definition of a circle packing required the underlying combinatorial structure of a packing be formed from triangles, not quadrilaterals. Moreover, the collection of circles in a packing must be locally finite. If a quadrilateral has an irrational Brooks parameter, that is, if the Brooks procedure described above results in infinitely many non-zero terms v_j and h_j and thus infinitely many circles accumulating toward bottom right corner of the quadrilateral, then the resulting collection of circles will not form a packing.

However, if the Brooks parameter is rational, that is, if there are only finitely many non-zero v_j and h_j terms, then the quadrilateral will be filled with a circle packing. Bowers and Stephenson took advantage of this to prove that surfaces which support a circle packing are dense in Teichmüller space [3, 4]. Given a surface, they first covered it with circles, leaving only quadrilateral interstices still to be packed. By making arbitrarily small adjustments to the conformal structure of the surface, they could ensure that all the remaining quadrilateral interstices had rational Brooks parameters and could thus be filled in to produce a circle packing.

Conversely, given $\beta > 0$ we can write it uniquely as a continued fraction of the form (3.1). If $\beta \in \mathbf{Q}^+$, this fraction encodes a (locally finite) triangulation $\mathcal{K}_\beta$ of a rectangle which can be realized by a circle packing P_β.

3.5. Brooks packings of the plane. Consider the infinite simply connected graph S corresponding to a tiling of the plane by squares. For future normalization purposes we will identify one vertex, labeling it v_0, and chose one of its neighbors, labeling it v_1. If we think of the direction from v_0 to v_1 as being a step toward the right, then we have an orientation on B in which the Brooks notions of "top," "bottom", "right", and "left" are also well defined. Thus we will think of B as a sort of base graph from which we can construct triangulations by filling in the squares in various ways as encoded by Brooks parameters.

For example, if inside every square of S we add one vertex with edges connecting the new vertex to the four existing vertices of its square, then we will have created a triangulation S_1 of the plane. Moreover, the triangulation inside each square is exactly the triangulation encoded by a Brooks parameter of 1. A packing for this triangulation is illustrated on the left in Figure 3. This packing is well-known and is often called the "ball bearing" packing since the vertices added inside each square correspond to circles much smaller than the circles at the corners of the squares and thus have the appearance of little ball bearings inside larger wheels.

We could repeat this process for any positive rational Brooks parameter $\beta \in \mathbf{Q}^+$ to create a new triangulation S_β. Theorem 3.3 guarantees the existence of a circle packing P_β for S_β filling either $\mathbf{C}$ or $\mathbf{D}$. Clearly however, P_β must fill the plane as can be seen by either sliding the large circles of the ball bearing packing to create quadrilateral interstices of Brooks parameter β and then filling in the interstices to directly create P_β or by noting that the simple random walk on S, and hence on S_β, is recurrent.

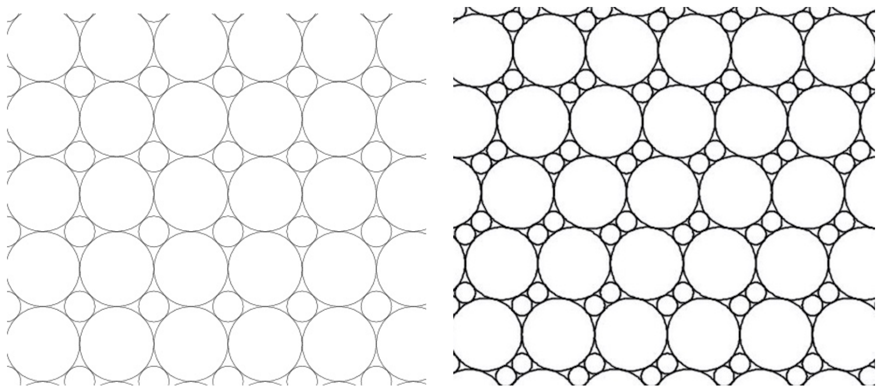

Figure 3. Two infinite Brooks packings in which every interstice has parameter 1 (left) and 2 (right).

The uniqueness of packings for a given triangulation implies that the symmetries of the triangulation S_β are reflected in the packing P_β. For example, all the circles corresponding to vertices of the original squares of S must have the same radius.

To ease the next steps in our construction, we will normalize P_β so that the circles corresponding to v_0 and v_1 are centered at 0 and 1, respectively.

4. Construction of quasiconformal maps with constant dilatation

4.1. Manipulation of the Brooks parameter. We next consider how to construct maps of the entire plane with constant complex dilatation. If we normalize our maps so that 0 and 1 are fixed, these are precisely the linear maps of Section 2.2.

The key step in constructing quasiconformal circle packing maps will be to introduce distortion by subdividing S_β and by manipulating the β parameter.

Consider for a moment the effect of increasing β. As the number h_1 of initial horizontal circles increases, the size of those horizontal circles inside each quadrilateral must decrease toward 0. This will cause P_β to "lean" to the right as the top and bottom circles move closer together.

The limiting situation as $\beta \to \infty$ would have the top and bottom circles in actual contact, producing the well-known infinite hex packing. The large circles corresponding to the original vertices of S would now lean at an angle of $\pi/6$ from their original positions in the P_1 ball bearing packing.

Similarly, as $\beta \to 0$, the packing P_β will "lean" to the left at an angle of $\pi/6$. Thus by manipulating β, we can slide the circles of P_β through a full range of $\pi/3$ radians, thereby creating our first level of quasiconformal distortion.

Because we normalized our packings so that the circles corresponding to v_0 and v_1 are centered at 0 and 1, respectively, the vertex v_i directly "above" v_0 in S will correspond to a circle centered at i. As we distort P_1 to a new packing P_β, the circle for v_i will now correspond to a new circle in P_β. If $f(i)$ is the center of this circle, then we have control over $\arg f(i)$ simply by manipulating β. As $\beta \to \infty$, then $\arg f(i) \to \pi/3$, and as $\beta \to 0$, then $\arg f(i) \to 2\pi/3$.

4.2. Scaling by subdivision. Our second level of distortion arises from subdividing S_β to control $|f(i)|$. Suppose we subdivide each square of S into $m \times n$ smaller squares before assigning the Brooks parameter β to each of these squares.

The new triangulation $S_{\beta,\frac{m}{n}}$ is of course combinatorially equivalent to S_β, except that our intially distinguished vertices v_i and v_1 are no longer neighbors of v_0 but are now separated from v_0 by m and n new vertices, respectively.

The unique packing $P_{\beta,\frac{m}{n}}$ for $S_{\beta,\frac{m}{n}}$ normalized so that the circle corresponding to v_0 and v_1 are again centered at 0 and 1, respectively, is of course just a copy of the packing P_β scaled by a factor of $\frac{1}{n}$.

But observe now the effect this subdivision and subsequent scaling has had on the $f(i)$, the center of the circle in $P_{\beta,\frac{m}{n}}$ corresponding to v_i. This circle is separated in P_β from 0 by m other circles before the scaling by $\frac{1}{n}$. Consequently,

$$|f(i)| = \frac{m}{n}.$$

4.3. The induced map on the plane. Our use of the function notation $f(i)$ to denote the the center of the circle corresponding to v_i was of course not coincidental. Let us take a step back and observe what our subdivisions and Brooks parameter manipulations have wrought.

The carrier of our base packing P_1 covers the entire plane and will act as the domain of our circle packing quasiconformal map. We can then create the map $f_{\beta,\frac{m}{n}}$ by sending the centers of circles in P_1 to the centers of the corresponding circles in $P_{\beta,\frac{m}{n}}$ and extending linearly to the rest of the plane.

Recall that the symmetry of S and the uniqueness of the packings will force the same symmetries in the map $f_{\beta,\frac{m}{n}}$. Since $f_{\beta,\frac{m}{n}}$ is linear on each square, it will be globally linear. In fact, since $f_{\beta,\frac{m}{n}}$ fixes both 0 and 1, we can write an explicit form of $f_{\beta,\frac{m}{n}}$ in terms of the image of i. If $f_{\beta,\frac{m}{n}}(i) = \omega$, then

$$f_{\beta,\frac{m}{n}}(x+iy) = \begin{pmatrix} 1 & \operatorname{Re}\omega \\ 0 & \operatorname{Im}\omega \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Of course, not every linear map can be created directly using one circle packing. Only countably many distinct packings $P_{\beta,\frac{m}{n}}$ can be produced by manipulating the rational parameters β and $\frac{m}{n}$, and thus only countably many linear maps can be directly produced from the packings $P_{\beta,\frac{m}{n}}$.

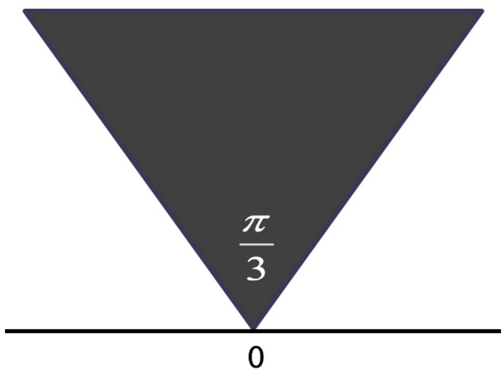

Figure 4. The wedge W in the upper half-plane formed by two rays meeting at the origin at angle $\pi/3$.

Moreover, because manipulating the Brooks parameter β limits $\arg f_{\beta,\frac{m}{n}}(i)$ to the range

$$\frac{\pi}{3} < \arg f_{\beta,\frac{m}{n}}(i) < \frac{2\pi}{3},$$

not every linear map can be created in this way. We are limited to directly building (normalized) maps in which the image w of i lies in the wedge-shaped region W of Figure 4. Moreover our construction also requires $|w|$ to be rational.

On the other hand, the real power of packings has always been in their ability to create maps which converge correctly in the limit. Since the circles corresponding to vertices in S vary continuously with β [8], we can use a sequence of packings to create a sequence of maps which converge locally uniformly to any given linear map normalized to fix 0 and 1 and sending i to a point $\omega \in W$. See Figure 4.

A straightforward calculation shows the dilatation of such a map is given by

$$\mu = \frac{\omega - i}{\omega + i}.$$

The image of the wedge W under $\frac{z-i}{z+i}$ is the region $G \subset \mathbf{D}$ formed by the intersection of two discs which meet at the points -1 and 1 with angle $\pi/3$. See Figure 5.

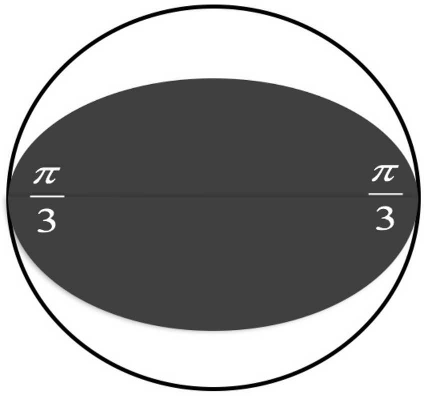

Figure 5. The region G. By manipulating the Brooks parameter, our circle packings can create a quasiconformal map with complex dilatation $\mu \in G$.

Thus we have the following theorem.

Theorem 4.1. *Given a constant complex dilatation μ contained in the region G, the construction described above produces a sequence of maps of $\mathbf{C}$ onto itself converging to quasiconformal map F_μ of Section 2.2.*

4.4. Implications for finite packings. The real power of our construction thus far lies not in its ability to create approximations to perfectly good linear maps for which an explicit expression is already known. Rather, the uniqueness of these infinite packings approximating F_μ implies that any large finite piece sharing the same combinatorics must be almost identical (up to planar automorphisms) to a corresponding piece of the infinite packing. Thus such a finite piece must induce a circle packing map whose dilatation on that piece nearly matches μ.

Lemma 4.2. *Suppose K is a compact subset of $\mathbf{C}$ and for each positive integer N, P_N is a finite packing whose underlying triangulation is combinatorially equivalent to a subcomplex of $S_{\beta,\frac{m}{n}}$ and so that K is surrounded by N generations of squares in the packing P_N. Create a map f_N from $\operatorname{carr} P_N$ to the corresponding subset of $\operatorname{carr} P_{\beta,\frac{m}{n}}$ by sending centers of circles to the centers of corresponding circles and extending linearly. Then as $N \to \infty$, f_N converges uniformly to a planar automorphism.*

Proof. Suppose not. Then as $N \to \infty$ we would have in the limit an infinite packing for $S_{\beta,\frac{m}{n}}$ which is not the image of $P_{\beta,\frac{m}{n}}$ under a planar automorphism, thus violating the uniqueness Theorem 3.3. □

Lemma 4.3. *Suppose K is a compact subset of $\mathbf{C}$ and for each positive integer N, P_N is a finite packing whose underlying triangulation is combinatorially equivalent to a subcomplex of the ball bearing packing S_1 and so that K is surrounded by N generations of squares in the packing P_N. Let β and $\frac{m}{n}$ be the parameters which produce a circle packing map from the ball bearing packing to $P_{\beta,\frac{m}{n}}$ with constant complex dilatation $\mu \in G$. Modify the triangulation for P_N by subdividing and changing the Brooks parameter to obtain a subcomplex of $S_{\beta,\frac{m}{n}}$. Let $\widehat{P}_N$ be a packing for this subcomplex and $f_N\colon \operatorname{carr} P_N \to \operatorname{carr} \widehat{P}_N$ be the associated circle packing map. Then as $N \to \infty$, the dilatation of f_N converges to μ.*

Proof. Since composition by planar automorphisms does not affect the complex dilatation, the result follows form Lemma 4.2. □

Recall that in general, we cannot directly create a single packing which will produce a quasiconformal map of the plane with a given complex dilatation $\mu \in G$. Theorem 4.1 provides a sequence of packings P_j and associated maps $f_j\colon \mathbf{C} \to \mathbf{C}$ converging uniformly to the normalized map $F_\mu\colon \mathbf{C} \to \mathbf{C}$ with dilatation $\mu \in G$.

A standard diagonalization argument allows us to extend Lemma 4.3 to handle the case when a sequence of maps is required to produce a map with dilatation μ.

Lemma 4.4. *Let $\mu \in G$ and $f_j\colon \operatorname{carr} P_j \to \operatorname{carr} \widehat{P}_j$ be the sequence of circle packing maps constructed in Theorem 4.1 to converge to F_μ. Suppose $K \subset \mathbf{C}$ is compact and $\epsilon > 0$. Then there exists $N_j \in \mathbf{N}$ so that if P^{N_j} is a finite simply connected portion of the packing P_j with triangulation S^{N_j} and with K surrounded by N_j generations of squares in P^{N_j} and if $\widehat{P^{N_j}}$ is any other packing for S^{N_j}, then the difference between μ and the complex dilatation of the induced circle packing map from $\operatorname{carr} P^{N_j}$ to $\operatorname{carr} \widehat{P^{N_j}}$ and μ is less than ϵ.*

This lemma will play the same role for us that Rodin and Sullivan's Hex Packing Lemma played in their proof of Thurston's Conjecture [18].

5. Construction of quasiconformal maps of proper simply connected subsets of C

Having shown how circle packings can be used to create maps of $\mathbf{C}$ with constant complex dilatation, we will now develop our own version of the Measurable Riemann Mapping Theorem and create quasiconformal maps with any complex dilatation μ whose values lie in G.

Consider a simply connected region $\Omega \subsetneq \mathbf{C}$ and a measurable Beltrami differential $\mu\colon \Omega \to G$, where G is the subset $\mathbf{D}$ depicted in Figure 5. For normalization purposes, choose distinct points $a, b \in \Omega$.

Let P_1^n be the infinite ball bearing packing scaled so that the radius of the large circles is $\frac{1}{n}$. Remove circles from P_1^n which lie outside Ω to create a finite simply connected packing $P^{\Omega,n}$ whose underlying triangulation S_1^n is a subcomplex of the ball bearing triangulation.

On each square s in $\operatorname{carr} P^{\Omega,n}$, let μ_s denote the average of μ over s. Consider the compact subset $K_s \subset s$ formed by shrinking s by a factor of $(1 - \frac{1}{n})$ about its center. Apply Lemma 4.4 to subdivide s into as many smaller squares as necessary and apply the subdivision and Brooks parameter manipulations of Section 4 so that the dilatation of the induced circle packing map on the compact subset K_s is within $\frac{1}{n}$ of μ_s.

This new finite triangulation K^n will have a packing $P^{\mathbf{D},n}$ filling $\mathbf{D}$ by Theorem 3.2. It will be unique if we normalize $P^{\mathbf{D},n}$ by centering at 0 and on the positive real axis the circles in $P^{\mathbf{D},n}$ which correspond to the circles in $P^{\Omega,n}$ closest to a and b, respectively.

Now construct a circle packing map $f_n\colon \operatorname{carr} P^{\Omega,n} \to \operatorname{carr} P^{\mathbf{D},n}$ by mapping the centers of corresponding circles and extending linearly on each square.

Theorem 5.1. *As $n \to \infty$, f_n converges locally uniformly to the quasiconformal map $F_\mu\colon \Omega \to \mathbf{D}$ normalized by $F_\mu(a) = 0$ and $F_\mu(b) > 0$.*

Proof. First note that as $n \to \infty$, the domain of f_n fills out all of Ω. Similarly, it is an easy application of Rodin and Sullivan's Length-Area Lemma [18, 20] to show that the range of f_n expands to fill $\mathbf{D}$ as $n \to \infty$.

By our construction, the dilatation of f_n converges to μ. Since the range of each f_n lies in $\mathbf{D}$, the sequence $\{f_n\}$ forms a normal family [17], and thus has a subsequence f_{n_j} which converges uniformly on compact subsets of Ω. Our normalization and the uniqueness part of the Measurable Riemann Mapping Theorem ensures that this limit function is in fact F_μ.

Finally, we note that since any subsequence of $\{f_n\}$ would likewise form a normal family and the limit of any subsequence must be the same function F_μ, the entire sequence f_n must converge to F_μ. □

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