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REGULARITY OF STABLE SOLUTIONS TO QUASILINEAR ELLIPTIC EQUATIONS ON RIEMANNIAN MODELS

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Abstract. We investigate the regularity of semi-stable, radially symmetric, and decreasing solutions for a class of quasilinear reaction-diffusion equations in the inhomogeneous context of Riemannian manifolds. We prove uniform boundedness, Lebesgue and Sobolev estimates for this class of solutions for equations involving the p-Laplace Beltrami operator and locally Lipschitz non-linearity. We emphasize that our results do not depend on the boundary conditions and the specific form of the non-linearities and metric. Moreover, as an application, we establish regularity of the extremal solutions for equations involving the p-Laplace Beltrami operator with zero Dirichlet boundary conditions.

1. Introduction

Let $(\mathcal{M}, g)$ be a Riemannian model of dimension $N \geq 2$, that is, a manifold $\mathcal{M}$ admitting a pole $\mathcal{O}$ and whose metric g is given, in polar coordinates around $\mathcal{O}$, by

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \psi(r)^2\,\mathrm{d}\theta^2 \quad \text{for } r \in (0,R) \text{ and } \theta \in \mathbf{S}^{N-1}, \tag{1.1}$$

where r is by construction the Riemannian distance between the point $P = (r,\theta)$ to the pole $\mathcal{O}$, $\mathrm{d}\theta^2$ is the canonical metric on the unit sphere $\mathbf{S}^{N-1}$ and ψ is a smooth function in $[0,R)$ and positive in $(0,R)$ for some $R \in (0,+\infty]$, and $\psi(0) = \psi''(0) = 0$ and $\psi'(0) = 1$. As examples we have the important cases of space forms, i.e., the unique complete and simply connected Riemannian manifold of constant sectional curvature K_ψ corresponding to the choice of ψ namely,

$$\begin{array}{rlll} \text{(i)} & \psi(r) = \sinh r, & K_\psi = -1 & \text{(hyperbolic space)},\\ \text{(ii)} & \psi(r) = r, & K_\psi = 0 & \text{(Euclidean space)},\\ \text{(iii)} & \psi(r) = \sin r, & K_\psi = 1 & \text{(elliptic space)}. \end{array} \tag{1.2}$$

Let us denote the geodesic ball of radius r with center at the pole $\mathcal{O}$ by $\mathcal{B}_r$ and $W^{1,p}_r(\mathcal{B}_1)$ the elements of the Sobolev space which are radially symmetric with respect to the pole $\mathcal{O}$. For $u \in W^{1,p}_r(\mathcal{B}_1)$, let us consider the energy functional

$$J_\delta(u) = \frac{1}{p}\int_{\mathcal{B}_1\setminus\overline{\mathcal{B}_\delta}} |\nabla_g u|^p\,\mathrm{d}v_g - \int_{\mathcal{B}_1\setminus\overline{\mathcal{B}_\delta}} F(u)\,\mathrm{d}v_g, \text{ where } F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s. \tag{1.3}$$

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Definition 1.1. We say that a decreasing function $u \in W_r^{1,p}(\mathcal{B}_1)$ is a radial local minimizer of (1.3) if for any $0 < \delta < 1$ there exists $\epsilon = \epsilon(\delta) > 0$ such that for all radial functions $\phi \in C_c^1\left(\mathcal{B}_1 \setminus \overline{\mathcal{B}_\delta}\right)$ satisfying $\|\phi\|_{C^1} \leq \epsilon$, we have

$$J_\delta(u) \leq J_\delta(u + \phi).$$

Definition 1.2. Let $u \in W_r^{1,p}(\mathcal{B}_1)$ be a critical point of (1.3). We say that u is semi-stable if $u_r(r) < 0$ for all $r \in (0, 1)$ and for all radially symmetric function $\xi \in C_c^1(\mathcal{B}_1 \setminus \{\mathcal{O}\})$ it holds

$$\int_{\mathcal{B}_1} \left[(p-1)|\nabla u|^{p-2}|\nabla \xi|^2 - f'(u)\xi^2\right] \mathrm{d}v_g \geq 0. \tag{1.4}$$

We note that critical points of the functional J_δ correspond to weak solutions of the singular problem

$$-\mathrm{div}(|\nabla_g u|^{p-2}\nabla_g u) = f(u) \quad \text{in} \quad \mathcal{B}_1 \setminus \{\mathcal{O}\}\,. \tag{$\mathcal{S}$}$$

In particular, if u is a radial local minimizer of J_δ, then u is a semi-stable solution of ($\mathcal{S}$). We are going to focus our analysis on the important case $1 < p \leq N$, since for $p > N$, it holds $W^{1,p}(\mathcal{B}_1) \hookrightarrow L^\infty(\mathcal{B}_1)$. Precisely, if $N < p \leq +\infty$, from Morrey's inequality we have $\|u\|_{C^{0,\gamma}(\mathcal{B}_1)} \leq C\|u\|_{W^{1,p}(\mathcal{B}_1)}$, where C is a constant which depends on p and N.

1.1. Main results and comments. The aim of the paper is twofold. Firstly, to establish a priori estimates for radial semi-stable classical solutions of ($\mathcal{S}$). Precisely, we establish L^∞, L^q and $W^{1,q}$ estimates for semi-stable, radially symmetric, and decreasing solutions of ($\mathcal{S}$) without assuming Dirichlet boundary condition or any other kind of boundary conditions. It should be an interesting question to study similar results for non-radial solutions. We stress that our results hold for any locally Lipschitz non-linearity $f(s)$ and metric g satisfying (1.1).

Since the celebrated paper by Gidas, Ni and Nirenberg [24] the question of symmetry in non-linear partial differential equations has been the subject of intensive investigations. In [16], by using a variant of moving planes method, it was established radial symmetry for non-negative solutions of quasilinear elliptic equations defined in geodesic balls of the hyperbolic space $\mathbf{H}^n$ with homogeneous Dirichlet boundary condition.

Theorem 1.1. *For any $f(s)$ locally Lipschitz function and u semi-stable solution of ($\mathcal{S}$), it holds that:*

(a) *If $N < p + 4p/(p-1)$, then $u \in L^\infty(\mathcal{B}_1)$ and*

$$\|u\|_{L^\infty(\mathcal{B}_1)} \leq C_{N,p,\alpha,\psi}\|u\|_{L^p(\mathcal{B}_1)}.$$

(b) *If $N \geq p + 4p/(p-1)$, then $u \in L^q(\mathcal{B}_1)$ and*

$$\|u\|_{L^q(\mathcal{B}_1)} \leq C_{N,\psi,p,q}\|u\|_{L^p(\mathcal{B}_1)} \quad \textit{for any} \quad q < q_0 := \frac{Np}{N - p - 2 - 2\sqrt{\frac{N-1}{p-1}}}.$$

Moreover, $u \in W^{1,q}(\mathcal{B}_1)$ and

$$\|u\|_{W^{1,q}(\mathcal{B}_1)} \leq C_{N,\psi,p,q}\|u\|_{L^p(\mathcal{B}_1)} \quad \textit{for any} \quad q < q_1 := \frac{Np}{N - 2 - 2\sqrt{\frac{N-1}{p-1}}}.$$

Remark 1.1. For our argument in the proof of Theorem 1.1 it was crucial the following key estimate for semi-stable solutions of $(\mathcal{S})$

$$\int_0^\delta |u_r|^p \psi^{N-1-2\alpha}\,\mathrm{d}r \le C_{N,p,\alpha,\psi}\|u\|^p_{L^p(\mathcal{B}_1)}, \tag{1.5}$$

where ψ is the polar decomposition of $\mathrm{d}s^2$ given in (1.1) (see Proposition 3.1 below). We have proved this estimate by using the radial symmetry of the solution u and by choosing an appropriated test function in the semi-stability inequality (1.4),

Remark 1.2. From Theorem 1.1 (a), one can see that Problem $(\mathcal{S})$ does not have any singular solution.

Remark 1.3. Note that $q_0 > p^* = (Np)/(N-p)$ (the critical Sobolev exponent) and $q_1 > p$. Under the hypotheses of Theorem 1.1 if $N \ge p+4p/(p-1)$ then u belongs to $L^q(\mathcal{B}_1)$ for all $q < q_0$. Since q_0 is greater than the critical Sobolev exponent, from Theorem 1.1 (ii) we conclude that semi-stable radially symmetric and decreasing weak solutions of $(\mathcal{S})$ have a better regularity than the one expected by using the classical Sobolev embedding. Moreover, we established better regularity than $W^{1,p}$ for semi-stable solutions to Problem $(\mathcal{S})$, since our estimates shows an improvement in the Sobolev space $W^{1,q}$ for $q < q_1$.

Our second purpose of this work is to apply the elliptic estimates obtained in Theorem 1.1 to prove regularity results for the following class of quasilinear elliptic problems

$$\begin{cases} -\mathrm{div}(|\nabla_g u|^{p-2}\nabla_g u) = \lambda h(u) & \text{in } \mathcal{B}_1, \\ u > 0 & \text{in } \mathcal{B}_1, \\ u = 0 & \text{on } \partial\mathcal{B}_1, \end{cases} \tag{$\mathcal{P}_\lambda$}$$

where λ is a positive parameter and $h(s)$ is an increasing C^1-function such that $h(0) > 0$ and

$$\lim_{t\to+\infty} \frac{h(t)}{t^{p-1}} = +\infty. \tag{H_1}$$

The study of this class of problems with various boundaries conditions has received considerable attention in recent years under the influence of the pioneering works of Gelfand [22], Joseph and Lundgren [28], Keener and Keller [31], Crandall and Rabinowitz [12], Mignot and Puel [37]. First, we would like to mention the progress involving Laplacian

$$\begin{cases} -\Delta u = \lambda h(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where Ω is a bounded domain of $\mathbf{R}^N$. Non-linear elliptic problems like (1.6) appear naturally in several physical phenomena, just to mention some applications, it arises in the theory of non-linear diffusion generated by non-linear sources [26, 27, 29], thermal ignition of a chemically active mixture of gases [22], membrane buckling [7] and gravitation equilibrium [11]. We refer the reader to [13, 17, 18] for a recent survey on this subject.

In recent years, regularity issues about this class of singular elliptic problems have been the focus of an active research area. The parameter λ measure the non-dimensional strength of the non-linearity. It is well known that if h is super-linear, there exists $\lambda^* \in (0,+\infty)$ such that if $\lambda \in (0,\lambda^*)$, then problem (1.6) admits a semi-stable solution u_λ and if $\lambda > \lambda^*$, then problem (1.6) admits no regular solution. This allows one to define the extremal solution $u^* := \lim_{\lambda \nearrow \lambda^*} u_\lambda$, which is a weak solution of (1.6). In [40], G. Nedev proved regularity results for extremal solutions of (1.6) in dimensions 2 and 3 and L^q estimates for every $q < N/(N-4)$ when $N \geq 4$ just assuming that $h(s)$ is a positive convex function with $h(0) > 0$ and $h'(0) \geq 0$. For a related problem still in the Euclidean case see [3], where X. Cabré assuming $h(s)$ to be a C^1 non-decreasing super-linear non-linearity with $h(0) > 0$, proved boundedness of the extremal solution for Problem (1.6) in dimension $N \leq 4$. In dimension 2 the domain Ω can be general but, in contrast with Nedev's result, in dimensions 3 and 4 the domain is assumed to be convex. After that, Cabré and Sanchón [6] completed the analysis in [3] when they proved that if $N \geq 5$ and Ω is a convex bounded domain of Euclidean space $\mathbf{R}^N$ then the extremal solution belongs to $L^{\frac{2N}{N-4}}$.

Recently, there has been growing interest on singular elliptic partial differential equations on Riemannian manifolds. The problem involving the Laplace–Beltrami operator

$$\begin{cases} -\Delta_g u = \lambda h(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain was studied recently by Castorina and Sanchón in [8] for the inhomogeneous context. They proved qualitative properties for semi-stable solutions and they established L^∞, L^q and $W^{1,q}$ estimates which do not depends on the non-linearity $h(s)$. Furthermore, the authors obtained regularity results for the extremal solution for exponential and power non-linearities. A similar setting has been considered by Berchio, Ferrero and Grillo [2] in order to study uniqueness and qualitative properties of radial entire solutions of the Lane–Emden–Fowler equation $-\Delta u = |u|^{m-1}u$ with $m > 1$ on certain classes of Cartan–Hadamard manifolds where the so-called Joseph-Lundgren exponent is involved in the stability of solutions. The existence of a stable solution to the semi-linear equation $-\Delta_g u = f(u)$ on a complete, non-compact, boundaryless Riemannian manifold with non-negative Ricci curvature and $f \in C^1$ was studied by Farina, Mari and Valdinoci [19]. They classify both the solution and the manifold and also discuss the classification of monotone solutions with respect to the direction of some Killing vector field, in the spirit of a conjecture of De Giorgi. In [39], Morabito investigated the existence and uniqueness of positive radial solutions of the problem

$$\begin{cases} \Delta_g u + \lambda u + u^p = 0 & \text{in } \mathcal{A}, \\ u = 0 & \text{on } \mathcal{A}, \end{cases}$$

when $\lambda < 0$, $\mathcal{A}$ is an annular domain in a Riemannian manifold of dimension N endowed with the metric $\mathrm{d}s^2 = \mathrm{d}r^2 + S^2(r)\mathrm{d}\theta^2$ under suitable assumptions on the function $S^2(r)$. He also show that there exist positive non-radial solutions arising by bifurcation from the radial solution, where λ and p are the bifurcation parameters.

Many non-linear problems in physics and mechanics are formulated in equations that contain the p-Laplacian, for example on non-Newtonian fluids, glaceology and non-linear elasticity (see [14]). For some problems of non-linear partial differential equations on Riemannian manifold we refer to [25, 30, 23]. Gelfand type problems involving the p-Laplacian in the homogeneous case of the form

$$\begin{cases} -\Delta_p u = \lambda h(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

was studied by García-Azorero, Peral and Puel [20, 21] where Ω is a smooth bounded domain of $\mathbf{R}^N$. They proved that for every $p > 1$ and $h(s) = e^s$, the extremal solution u^* is an energy solution for every dimension and that it is bounded in some range of dimensions. For a more general non-linearity, Cabré and Sanchón [5] proved that every semi-stable solution is bounded for a explicit exponent which is optimal for the boundedness of semi-stable solutions and, in particular, it is bigger than the critical Sobolev exponent $p^* - 1$. For general $h(s)$ and $p > 1$ the interested reader can see [4, 10, 41, 43] for more regularity results about the extremal solution. In [4], Cabré, Capella and Sanchón treated the delicate issue about regularity of extremal solutions u^* of (1.7) at $\lambda = \lambda^*$ when Ω is the unit ball of $\mathbf{R}^N$. Among other results, they established pointwise, L^q and $W^{1,q}$ estimates which are optimal and do not depend on the non-linearity $h(s)$.

Furthermore, Castorina and Sanchón [9] obtain a priori estimates for semi-stable solutions of the reaction-diffusion problem $-\Delta_p u = h(u)$ in Ω while the reaction term is driven by any positive C^1 non-linearity h and, as a main tool, they develop Morrey-type and Sobolev-type inequalities that involve the functional

$$I_{p,q}(v;\Omega) = \left(\int_\Omega \left[\left(\frac{1}{p'} |\nabla_{T,v} |\nabla v|^{p/q}| \right)^q + |H_v|^q |\nabla v|^p \right] dx \right)^{1/p}, \quad p, q \geq 1, \tag{1.8}$$

where $v \in C_0^\infty(\overline{\Omega})$. In (1.8), $H_v(x)$ denotes the mean curvature at x of the hypersurface $\{y \in \Omega \colon |v(y)| = |v(x)|\}$ and $\nabla_{T,v}$ is the tangential gradient along a level set of $|v|$. In addition to being of independent interest, these geometric inequalities are used, together with judicious choice of test functions in the semi-stability condition, to obtain their a priori estimates for semi-stable solutions.

In this paper we investigate similar results in the inhomogeneous context of a Riemannian manifold. We use some ideas of [5], comparison principle for $-\Delta_p$ (because it is uniformly elliptic) and the positivity of the first eigenvalue (as well the corresponding eigenfunction) of $-\Delta_p$ on Ω (cf. [1, 36, 38]). We point out that the regularity results achieved in this paper represent a geometrical extension of the ones obtained for the Euclidean case in [4].

Before we state our main result on the regularity of semi-stable solutions for $(\mathcal{P}_\lambda)$, let us introduce some basic definitions. We say that $u \in W_0^{1,p}(\mathcal{B}_1)$ is a weak solution of $(\mathcal{P}_\lambda)$ if $h(u) \in L^1(\mathcal{B}_1)$ and

$$\int_{\mathcal{B}_1} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, \mathrm{d}v_g = \int_{\mathcal{B}_1} h(u) \phi \, \mathrm{d}v_g,$$

for all $\phi \in C_0^\infty(\mathcal{B}_1)$. Furthermore, by minimal solution we mean smaller than any other super-solution of the Problem $(\mathcal{P}_\lambda)$ and regular solution means that a weak solution u of $(\mathcal{P}_\lambda)$ is $C^{1,\beta}(\mathcal{B}_1)$.

Let us state the existence and basic properties of touchdown parameter.

Theorem 1.2. *There exist $\lambda^* \in (0, \infty)$ such that*

(i) *for $0 < \lambda < \lambda^*$, the problem $(\mathcal{P}_\lambda)$ has a regular minimal solution u_λ,*
(ii) *for $\lambda > \lambda^*$, $(\mathcal{P}_\lambda)$ admits no weak solution,*

(iii) *the map $\lambda \to u_\lambda$ is increasing.*

As a consequence of Theorem 1.2, the increasing limit

$$u^* = \lim_{\lambda \nearrow \lambda^*} u_\lambda$$

is well defined by the point-wise increasing property. If u^* is a weak solution of $(\mathcal{P}_{\lambda^*})$, then u^* is called the extremal solution. Since the extremal solutions can be obtained as the limit of classical minimal solutions, our next result is useful in order to prove that u^* has the same regularity properties as the ones stated in Theorem 1.1. For this, we need to bound u^{p-1} and $h(u)$ in $L^1(\mathcal{B}_1)$ uniformly in λ. This is possible because we have the growth condition (H_1) on $h(s)$ and the radially decreasing property of the minimal solutions u_λ. Let us now state precisely our results for $(\mathcal{P}_\lambda)$.

Theorem 1.3. *Suppose that $N < p + 4p/(p-1)$ and let $h(s)$ be a positive and increasing C^1-function satisfying (H_1). Then u^* is a semi-stable solution of (P_{λ^*}) and $u^* \in L^\infty(\mathcal{B}_1)$.*

1.2. Outline. In the next section we bring basic facts about the p-Laplace Beltrami operator which will be used through the paper. In Section 2 we prove existence of extremal parameter λ^* and minimal solutions of $(\mathcal{P}_\lambda)$ for $0 < \lambda < \lambda^*$. In Section 3 we prove that key-estimate (1.5) by using a suitable choice of test functions under the semi-stability property. In Section 4 we use (1.5) to prove our main theorem about regularity for radially symmetric and decreasing semi-stable solutions of Problem $(\mathcal{S})$ and apply this results for the study of the regularity of extremal solutions of $(\mathcal{P}_\lambda)$.

2. Proof of Theorem 1.2

Our first proposition establishes the analog of the classical results for $(\mathcal{P}_\lambda)$ in the Euclidean case. Using some ideas coming from Cabré and Sanchón [5] and Luo, Ye and Zhou [35] we prove the existence of a critical parameter λ^* which is related with the resolvability of $(\mathcal{P}_\lambda)$.

Proof of Theorem 1.2. For (i), let $w \in W_0^{1,p}(\mathcal{B}_1)$ a weak solution of

$$-\mathrm{div}(|\nabla_g w|^{p-2}\nabla_g w) = 1 \quad \text{in} \quad \mathcal{B}_1,$$

that is

$$\int_{\mathcal{B}_1} |\nabla_g w|^{p-2}\nabla_g w \cdot \nabla_g \phi \, \mathrm{d}\sigma = \int_{\mathcal{B}_1} \phi \, \mathrm{d}\sigma, \quad \forall \phi \in C_0^\infty(\mathcal{B}_1).$$

We can see that $w \in C^{1,\alpha}(\overline{\mathcal{B}_1})$ by using $C^{1,\alpha}$-regularity results (see [33, 15, 42]). By the Maximum principle [1, Theorem 3.3] w is non-negative in $\mathcal{B}_1$. It is easy verify that 0 is sub-solution of $(\mathcal{P}_\lambda)$ and if $\lambda \le \lambda_0 := 1/h(\max_{\overline{\mathcal{B}_1}} w)$,

$$\int_{\mathcal{B}_1} |\nabla_g w|^{p-2}\nabla_g w \cdot \nabla_g \phi \, \mathrm{d}\sigma = \int_{\mathcal{B}_1} \phi \, \mathrm{d}\sigma \ge \int_{\mathcal{B}_1} \lambda h(w)\phi \, \mathrm{d}\sigma$$

that is, w is a super-solution of $(\mathcal{P}_\lambda)$. Thus, for any $\lambda \le \lambda_0$, Problem $(\mathcal{P}_\lambda)$ has a weak solution $u \in W_0^{1,p}(\mathcal{B}_1)$ given by Sub and Super-solution Method (see [32]) with $0 \le u \le w$ in $\overline{\mathcal{B}_1}$. This implies that $u \in C^{1,\alpha}(\overline{\mathcal{B}_1})$. As any regular solution u of $(\mathcal{P}_\lambda)$ is also a super-solution for (P_μ) if $\mu \in (0, \lambda)$, the set of λ for which $(\mathcal{P}_\lambda)$ admits a regular solution is an interval. For (ii) we will show that for λ sufficient large, there is no regular solution for $(\mathcal{P}_\lambda)$, so $\lambda^* < +\infty$. It is well known that for the non-linear eigenvalue problem

$$\begin{cases} -\mathrm{div}(|\nabla_g v_1|^{p-2}\nabla_g v_1) = \lambda_1 |v_1|^{p-2} v_1 & \text{in } \mathcal{B}_1, \\ v_1 = 0 & \text{in } \partial\mathcal{B}_1, \end{cases}$$

there exists a smaller positive and simple eigenvalue λ_1 with a positive eigenfunction v_1 in $\mathcal{B}_1$. Now, suppose that $(\mathcal{P}_\lambda)$ admits a regular solution u for $\lambda > \lambda_1$. The regularity result in [33] give that $v_1 \in C^{1,\alpha}(\overline{\mathcal{B}_1})$. By homogeneity, we can assume that $\|v_1\|_\infty < h(0)^{\frac{1}{p-1}}$. Note that

$$-\mathrm{div}(|\nabla_g v_1|^{p-2}\nabla_g v_1) = \lambda_1 v_1^{p-1} \le \lambda_1 h(0) < \lambda h(u) = -\mathrm{div}(|\nabla_g u|^{p-2}\nabla_g u).$$

By the comparison principle [1] we have that $v_1 \le u$. Let us to take v_2 a solution of

$$\begin{cases} -\mathrm{div}(|\nabla_g v_2|^{p-2}\nabla_g v_2) = (\lambda_1 + \epsilon) v_1^{p-1} & \text{in } \mathcal{B}_1, \\ v_2 = 0 & \text{in } \partial\mathcal{B}_1, \end{cases}$$

where ϵ is a positive constant. For $\lambda > \frac{\lambda_1+\epsilon}{h(0)} \max_{\overline{\mathcal{B}_1}} u^{p-1}$ we obtain

$$-\mathrm{div}(|\nabla_g v_2|^{p-2}\nabla_g v_2) = (\lambda_1 + \epsilon) v_1^{p-1} \le (\lambda_1 + \epsilon) u^{p-1} \le \lambda h(u) = -\mathrm{div}(|\nabla_g u|^{p-2}\nabla_g u).$$

Using the comparison principle again we obtain $v_1 \le v_2 \le u$. Now, let us define recursively u_n as the unique solution of

$$\begin{cases} -\mathrm{div}(|\nabla_g v_n|^{p-2}\nabla_g v_n) = (\lambda_1 + \epsilon) v_{n-1}^{p-1} & \text{in } \mathcal{B}_1, \\ v_n = 0 & \text{in } \partial\mathcal{B}_1. \end{cases}$$

By comparison principle we obtain $v_1 \le \cdots \le v_{n-1} \le v_n \le u \in C^{1,\alpha}(\overline{\mathcal{B}_1})$. This implies that $v_n \rightharpoonup u_\lambda$ in $W_0^{1,p}(\mathcal{B}_1)$ and consequently u_λ satisfies

$$\begin{cases} -\mathrm{div}(|\nabla_g u_\lambda|^{p-2}\nabla_g u_\lambda) = (\lambda_1 + \epsilon) u_\lambda^{p-1} & \text{in } \mathcal{B}_1, \\ u_\lambda = 0 & \text{in } \partial\mathcal{B}_1, \end{cases}$$

which is impossible since the first eigenvalue of p-Laplace Beltrami operator is isolated (see [44, 34]). Define the critical threshold λ^* as the supremum of $\lambda > 0$ for which $(\mathcal{P}_\lambda)$ admits a regular solution. Thus we have that $\lambda^* < +\infty$. Note that, by the construction above, u_λ is independent of the choice of the super-solution. Since any regular solution of $(\mathcal{P}_\lambda)$ is also a super-solution of $(\mathcal{P}_\lambda)$, we can conclude that u_λ is a regular minimal solution of $(\mathcal{P}_\lambda)$. In order to check (iii), let $\lambda \le \mu$. Thus, u_μ is a super-solution of $(\mathcal{P}_\lambda)$, which implies that $u_\lambda \le u_\mu$, that is, the map $\lambda \to u_\lambda$ is increasing. □

3. A priori estimates

In this section we prove the principal estimate (1.5), which as we already mention it is the crucial in our argument to obtain the regularity of the semi-stable solutions in Theorem 1.1. The main idea is to apply an appropriate test function in the stability inequality (see Proposition 3.1). The radial form of $(\mathcal{S})$ can be written as follows

$$-(p-1)|u_r|^{p-2}u_{rr} - \frac{(N-1)\psi'}{\psi}|u_r|^{p-2}u_r = f(u) \quad \text{with } r \in (0,1). \tag{3.1}$$

Let us discuss a few preliminary estimates which will be used in our argument. In the next Lemma we prove that the second variation of energy associated to $(\mathcal{S})$ is independent of the non-linearity $f(s)$.

Lemma 3.1. *Let $u \in W_r^{1,p}(\mathcal{B}_1)$ be a semi-stable solution of $(\mathcal{S})$ satisfying $u_r(r) < 0$ for all $r \in (0,1)$. Then, for all radially symmetric function $\eta \in C_c^1(\mathcal{B}_1 \setminus \{\mathcal{O}\})$ it holds*

$$\int_{\mathcal{B}_1} |u_r|^p \left[(p-1)|\eta_r|^2 + \frac{\partial}{\partial r}\left(\frac{(N-1)\psi'}{\psi}\right)\eta^2\right] \mathrm{d}v_g \geq 0.$$

Proof. We start considering $\eta \in C_c^1(\mathcal{B}_1 \setminus \mathcal{O})$ be a radial function with compact support in $\mathcal{B}_1 \setminus \mathcal{O}$ and choosing $\xi = u_r\eta$ as test function in (1.4) there holds

$$\begin{aligned} 0 &\leq \int_{\mathcal{B}_1} (p-1)|u_r|^{p-2}|\nabla_g(u_r\eta)|^2 - f'(u)|u_r|^2\eta^2 \,\mathrm{d}v_g \\ &= \int_{\mathcal{B}_1} (p-1)|u_r|^p|\nabla_g\eta|^2 + (p-1)|u_r|^{p-2}\nabla_g(\eta^2u_r)\nabla_g(u_r) - f'(u)|u_r|^2\eta^2 \,\mathrm{d}v_g. \end{aligned} \tag{3.2}$$

On the other hand, multiplying (3.1) by $(\eta^2u_r\psi^{N-1})_r$, integrating and using integration by parts we are able to compute

$$\begin{aligned} 0 &= \int_0^1 (p-1)|u_r|^{p-2}u_{rr}\left(\eta^2u_r\psi^{N-1}\right)_r + \left[\frac{(N-1)\psi'}{\psi}|u_r|^{p-2}u_r + g(u)\right]\left(\eta^2u_r\psi^{N-1}\right)_r \mathrm{d}v_g \\ &= \int_0^1 (p-1)|u_r|^{p-2}u_{rr}(\eta^2u_r\psi^{N-1})_r - \left[\frac{(N-1)\psi'}{\psi}|u_r|^{p-2}u_r + g(u)\right]_r \eta^2u_r\psi^{N-1}\,\mathrm{d}v_g, \end{aligned}$$

which together with $\partial_r(|u_r|^{p-2}u_r) = (p-1)|u_r|^{p-2}u_{rr}$ implies

$$\begin{aligned} 0 &= \int_0^1 (p-1)|u_r|^{p-2}u_{rr}\partial_r(\eta^2u_r\psi^{N-1})\,\mathrm{d}r - \int_0^1 \partial_r\left(\frac{(N-1)\psi'}{\psi}\right)|u_r|^{p-2}u_r\eta^2u_r\psi^{N-1}\,\mathrm{d}r \\ &\quad - \int_0^1 \frac{(N-1)\psi'}{\psi}(p-1)|u_r|^{p-2}u_{rr}\eta^2u_r\psi^{N-1}\,\mathrm{d}r - \int_0^1 f'(u)u_r\eta^2u_r\psi^{N-1}. \end{aligned}$$

Thus

$$\begin{aligned} 0 &= \int_0^1 (p-1)|u_r|^{p-2}u_{rr}\partial_r\left(\eta^2u_r\right)\psi^{N-1}\,\mathrm{d}r \\ &\quad - \int_0^1 \partial_r\left(\frac{(N-1)\psi'}{\psi}\right)|u_r|^{p-2}u_r^2\eta^2\psi^{N-1}\,\mathrm{d}r - \int_0^1 f'(u)\eta^2u_r^2\psi^{N-1}\,\mathrm{d}r, \end{aligned}$$

which yields

$$\begin{aligned} \int_{\mathcal{B}_1} \partial_r\left(\frac{(N-1)\psi'}{\psi}\right)|u_r|^p\eta^2\,\mathrm{d}v_g &= \int_{\mathcal{B}_1} (p-1)|u_r|^{p-2}u_{rr}\partial_r(\eta^2u_r)\,\mathrm{d}v_g \\ &\quad - \int_{\mathcal{B}_1} f'(u)\eta^2u_r^2\,\mathrm{d}v_g. \end{aligned} \tag{3.3}$$

Using (3.2) and (3.3) we have

$$\begin{aligned} 0 &\leq \int_{\mathcal{B}_1} (p-1)|u_r|^{p-2}u_r^2|\nabla\eta|^2 + \int_{\mathcal{B}_1} (p-1)|u_r|^{p-2}u_{rr}\nabla(\eta^2u_r)\nabla u_r - f'(u)u_r^2\eta^2\,\mathrm{d}v_g \\ &= \int_{\mathcal{B}_1} (p-1)|u_r|^p|\nabla\eta|^2 + \int_{\mathcal{B}_1} \frac{\partial}{\partial r}\left(\frac{(N-1)\psi'}{\psi}\right)|u_r|^p\eta^2\,\mathrm{d}v_g \\ &= \int_{\mathcal{B}_1} |u_r|^p\left[(p-1)|\eta_r|^2 + \frac{\partial}{\partial r}\left(\frac{(N-1)\psi'}{\psi}\eta^2\right)\right]\mathrm{d}v_g, \end{aligned}$$

which is the desired conclusion. □

We obtain L^p-estimates for the radial derivative of semi-stable solutions of $(\mathcal{S})$ with the help of Lemma 3.1. For that we consider a suitable class of test functions to analyze the inhomogeneous context of a Riemannian manifold assuming that $p \le N$ and $1 \le \alpha < 1 + \sqrt{(N-1)/(p-1)}$. To be more precise,

Proposition 3.1. *Let $u \in W_r^{1,p}(\mathcal{B}_1)$ be a semi-stable solution in $\mathcal{B}_1 \setminus \mathcal{O}$ of $(\mathcal{S})$ satisfying $u_r(r) < 0$ for $r \in (0,1)$ and $\delta = \delta(\psi) \in (0, 1/2)$ such that $\psi' > 0$ in $[0, \delta]$. Then*

$$\int_0^\delta |u_r|^p \psi^{N-1-2\alpha} \,\mathrm{d}r \le C_{N,p,\alpha,\psi} \|u\|^p_{L^p(\mathcal{B}_1)}$$

for every $1 \le \alpha < 1 + \sqrt{(N-1)/(p-1)}$, where $C_{N,p,\alpha,\psi}$ is a constant depending only on N, p, α and ψ.

Proof. Using the semi-stability condition of u and applying Lemma 3.1 with $\psi\eta$ as test function we obtain

$$(N-1)\int_{\mathcal{B}_1} \left[-\psi''\psi + (\psi')^2\right] |u_r|^p \eta^2 \,\mathrm{d}v_g \le (p-1)\int_\Omega |u_r|^p |(\psi\eta)_r|^2 \,\mathrm{d}v_g \tag{3.4}$$

Now, take α satisfying $1 \le \alpha < 1 + \sqrt{(N-1)/(p-1)}$, $\epsilon \in (0,1)$ sufficiently small and

$$\eta_\epsilon(r) = \begin{cases} \psi^{-\alpha}(\epsilon) - \psi^{-\alpha}(\delta) & \text{for } 0 \le r \le \epsilon, \\ \psi^{-\alpha}(r) - \psi^{-\alpha}(\delta) & \text{for } \epsilon < r \le \delta, \\ 0 & \text{for } \delta < r \le 1, \end{cases}$$

a Lipschitz function which vanishes on $\partial\mathcal{B}_1$. Choosing $\eta = \eta_\epsilon$ in the inequality (3.4) we have

$$\begin{aligned} &(N-1)\left(\int_0^\epsilon \left[-\psi''\psi + (\psi')^2\right] \eta_\epsilon^2 |u_r|^p \psi^{N-1} \,\mathrm{d}r + \int_\epsilon^\delta \left[-\psi''\psi + (\psi')^2\right] \eta_\epsilon^2 |u_r|^p \psi^{N-1} \,\mathrm{d}r\right) \\ &\le (p-1)\left(\int_\epsilon^\delta \left[(1-\alpha)\psi^{-\alpha} - \psi^{-\alpha}(\delta)\right]^2 |u_r|^p (\psi')^2 \psi^{N-1} \,\mathrm{d}r + \int_0^\epsilon \eta_\epsilon^2 |u_r|^p (\psi')^2 \psi^{N-1} \,\mathrm{d}r\right), \end{aligned}$$

which can be written as

$$\begin{aligned} &(N-p)\int_0^\epsilon \eta_\epsilon^2 |u_r|^p (\psi')^2 \psi^{N-1} \,\mathrm{d}r + (N-1)\int_\epsilon^\delta (\psi')^2 \eta_\epsilon^2 |u_r|^p \psi^{N-1} \,\mathrm{d}r \\ &\le (p-1)\int_\epsilon^\delta \left[(1-\alpha)\psi^{-\alpha} - \psi^{-\alpha}(\delta)\right]^2 |u_r|^p (\psi')^2 \psi^{N-1} \,\mathrm{d}r \\ &\quad + (N-1)\int_0^\delta \psi''\psi |u_r|^p \eta_\epsilon^2 \psi^{N-1} \,\mathrm{d}r. \end{aligned}$$

Since $(N-p)\eta_\epsilon^2 |u_r|^p (\psi')^2 \psi^{N-1} \,\mathrm{d}r \ge 0$ we obtain

$$\begin{aligned} (N-1)\int_\epsilon^\delta |u_r|^p (\psi')^2 \eta_\epsilon^2 \psi^{N-1} \,\mathrm{d}r &\le (p-1)\int_\epsilon^\delta |u_r|^p \left((1-\alpha)\psi^{-\alpha} - \psi^{-\alpha}\right)^2 (\psi')^2 \psi^{N-1} \,\mathrm{d}r \\ &\quad + (N-1)\int_0^\delta \psi''\psi |u_r|^p \eta_\epsilon^2 \psi^{N-1} \,\mathrm{d}r \end{aligned}$$

Throughout the proof, $\tilde{C}_{n,p,\alpha}$ (respectively $\tilde{C}_{n,p,\alpha,\psi}$) denote different positive constants depending only on n, p and α (respectively on n, p, α, ψ). Rewritten the above equation follows

$$\int_\epsilon^\delta (\psi')^2 |u_r|^p \psi^{-2\alpha}\psi^{N-1}\,\mathrm{d}r \le \tilde{C}_{n,p,\alpha}\left\{\int_\epsilon^\delta (\psi')^2 |u_r|^p \psi^{-2\alpha}(\delta)\psi^{N-1}\,\mathrm{d}r\right.$$
$$\left. + \int_0^\delta \psi^{N-1}|u_r|^p|\psi''|\psi\left(\psi^{-2\alpha}+\psi^{-2\alpha}(\delta)\right)\mathrm{d}r + \int_\epsilon^\delta (\psi')^2 |u|^p\psi^{-\alpha}\psi^{-\alpha}(\delta)\psi^{N-1}\,\mathrm{d}r\right\}.$$

Observe that, by assumption, $\inf_{(0,\delta)}\psi'$ and $\sup_{(0,\delta)}\psi'$ are positive. Now, we can rearrange the terms in the integrals to obtain

$$\begin{aligned}\int_\epsilon^\delta |u_r|^p\psi^{-2\alpha}\psi^{N-1}\,\mathrm{d}r &\le \tilde{C}_{N,p,\alpha}\int_0^\delta \psi^{N-1}|u_r|^p|\psi''|\psi\psi^{-2\alpha}\left(1+\frac{\psi^{2\alpha}}{\psi^{2\alpha}(\delta)}\right)\mathrm{d}r \\ &\quad + \tilde{C}_{N,p,\alpha}\int_0^\delta (\psi')^2|u|^p\psi^{-\alpha}\psi^{-\alpha}(\delta)\psi^{N-1}\left(1+\frac{\psi^\alpha}{\psi^\alpha(\delta)}\right)\mathrm{d}r \\ &\le \tilde{C}_{N,p,\alpha,\psi}\int_0^\delta \psi^{N-1}|u_r|^p\psi^{-\alpha}\left\{1+\psi^{1-\alpha}\right\}\mathrm{d}r.\end{aligned}$$

Taking $\epsilon \to 0$, follows that

$$\int_0^\delta |u_r|^p\psi^{-2\alpha}\psi^{N-1}\mathrm{d}r \le \tilde{C}_{N,p,\alpha,\psi}\int_0^\delta \psi^{N-1}|u_r|^p\psi^{-\alpha}\left\{1+\psi^{1-\alpha}\right\}\mathrm{d}r. \tag{3.5}$$

If we define

$$\zeta(t) = \frac{\tilde{C}_{N,p,\alpha,\psi}t^{-\alpha}(1+t^{1-\alpha}) - \frac{t^{-2\alpha}}{2}}{t^{\frac{N-1}{p-1}}},$$

using that $1 \le p \le N$ and α satisfying $1 \le \alpha < 1+\sqrt{(N-1)/(p-1)}$, we can check that $\lim_{t\to+\infty}\zeta(t) = 0$ and $\lim_{t\to 0^+}\zeta(t) = -\infty$. Thus, by a compactness argument, $\zeta(t)$ is bounded from above. This implies that there exists $C_{N,p,\alpha,\psi} > 0$ such that

$$\tilde{C}_{N,p,\alpha,\psi}t^{-\alpha}(1+t^{1-\alpha}) \le \frac{t^{-2\alpha}}{2} + C_{N,p,\alpha,\psi}t^{\frac{N-1}{p-1}} \quad \forall t > 0$$

and (3.5) leads to

$$\int_0^\delta |u_r|^p\psi^{-2\alpha}\psi^{N-1}\,\mathrm{d}r \le C_{N,p,\alpha,\psi}\int_0^\delta |u_r|^p\psi^{(N-1)p/(p-1)}\,\mathrm{d}r. \tag{3.6}$$

On the other hand, since u is radially decreasing follows that

$$u^p(\delta) \le C_{N,\psi}\int_0^\delta u^p\psi^{N-1}\,\mathrm{d}r \le C_{N,\psi}\|u\|^p_{L^p(\mathcal{B}_1)} \tag{3.7}$$

and using Mean value theorem for some $\tilde{\delta} \in (\delta, 2\delta)$ it holds

$$-u_r(\tilde{\delta}) = \frac{u(\delta)-u(2\delta)}{\delta} \le \frac{u(\delta)}{\delta}. \tag{3.8}$$

Thus, integrating (3.1) from $r \in (0,\delta)$ to $\tilde{\delta}$ and using (3.8) we obtain

$$\begin{aligned}-|u_r(r)|^{p-2}u_r(r)\psi^{N-1}(r) &= -|u_r(\tilde{\delta})|^{p-2}u_r(\tilde{\delta})\psi^{N-1}(\tilde{\delta}) - \int_s^{\tilde{\delta}} f(u)\psi^{N-1} \\ &\le \frac{u^{p-1}(\delta)}{\delta^{p-1}}\psi^{N-1}(\tilde{\delta}),\end{aligned}$$

which together (3.7) implies

$$|u_r|^p\psi^{(N-1)p/(p-1)} \le \frac{u^p(\delta)}{\delta^p}\psi^{p(N-1)/(p-1)}(\tilde{\delta}) \le C_{N,\psi}\|u\|^p_{L^p(\mathcal{B}_1)}.$$

Integrating from 0 to δ to obtain

$$\int_0^\delta |u_r|^p \psi^{(N-1)p/(p-1)}\,\mathrm{d}r \leq C_{N,\psi}\|u\|^p_{L^p(\mathcal{B}_1)}$$

and going back to (3.6) we conclude that

$$\int_0^\delta |u_r|^p \psi^{N-1-2\alpha}\mathrm{d}r \leq C_{N,p,\alpha,\psi}\|u\|^p_{L^p(\mathcal{B}_1)},$$

which completes the proof. □

4. Proof of main theorems

Proof of Theorem 1.1. Let $\delta \in (0, 1/2)$ as in Proposition 3.1. Since u is radially symmetric and positive we can check that

$$u(\delta) \leq \int_0^\delta u\psi^{N-1}, \mathrm{d}r \leq C_{N,\psi}\|u\|_{L^1(\mathcal{B}_1)} \tag{4.1}$$

Using Hölder inequality we can estimate

$$\begin{aligned} |u(t)| &= \left| u(\delta) - \int_t^\delta u_r \psi^{(N-1-2\alpha)/p}\psi^{(-N+1+2\alpha)/p}\mathrm{d}r \right| \\ &\leq C_{N,\psi}\|u\|_{L^1(\mathcal{B}_1)} + \left(\int_0^\delta |u_r|^p\psi^{N-1-2\alpha}\right)^{\frac{1}{p}} \left(\int_t^\delta \psi^{(-N+1+2\alpha)/(p-1)}\right)^{\frac{p-1}{p}} \\ &\leq C_{N,\psi}\|u\|_{L^1(\mathcal{B}_1)} + C_{N,p,\alpha,\psi}\|u\|_{L^p(\mathcal{B}_1)} \left(\int_t^\delta \psi^{(-N+1+2\alpha)/(p-1)}\right)^{\frac{p-1}{p}} \end{aligned} \tag{4.2}$$

(i) In order to prove L^∞ estimate, observe that, by monotonicity, for all $\delta \leq t < 1$ we have

$$u^p(t) \leq u^p(\delta) \leq C_{N,\psi}\|u\|^p_{L^p(\mathcal{B}_1)}.$$

Taking $t = 0$ in (4.2), we can analyze the integral and check that

$$\int_0^\delta \psi^{(2\alpha-N+1)/(p-1)} < +\infty$$

when $(2\alpha - N + 1)/(p - 1) > -1$, that is, $\alpha > (N - p)/2$. Thus, for all $0 < t < \delta$ we have

$$|u(t)| \leq C_{N,\psi}\|u\|_{L^1(\mathcal{B}_1)} + C_{N,p,\alpha,\psi}\|u\|_{L^p(\mathcal{B}_1)},$$

whenever $\max\{(N-p)/2, 1\} < \alpha < 1 + \sqrt{(N-1)/(p-1)}$. This occurs if, and only if, $N < p + 4p/(p-1)$. Therefore, the desired L^∞ estimate (1.1) holds true.

(ii) On the other hand, since u is decreasing, using (4.1) we have

$$\left(\int_\delta^1 |u|^q\psi^{N-1}\,\mathrm{d}t\right)^{\frac{1}{q}} \leq C_{N,\psi,q}u(\delta) \leq C_{N,\psi,q}\|u\|_{L^1(\mathcal{B}_1)}. \tag{4.3}$$

Taking $t \in (0, \delta)$ and using (4.2) we have

$$\int_t^\delta |u|^q\psi^{N-1}\,\mathrm{d}t \leq C_{N,\psi,q}\|u\|^q_{L^p(\mathcal{B}_1)} \int_0^\delta \left[1 + \left(\int_t^\delta \psi^{(-N+1+2\alpha)/(p-1)}\right)^{\frac{p-1}{p}}\right]^q \psi^{N-1}\,\mathrm{d}t.$$

Therefore, if $q < Np/(N - p - 2 - 2\sqrt{(N-1)/(p-1)})$, choosing suitable α such that

$$\left(\int_0^\delta |u|^q \psi^{N-1}\,\mathrm{d}t\right)^{\frac{1}{q}} \leq C_{N,\psi,q}\|u\|_{L^p(\mathcal{B}_1)}.$$

Taking this last inequality combined with (4.3) we obtain the desired L^q estimate. Now, we are looking for $W^{1,q}$ estimate. We use similar idea as can be found in the proof of Theorem 1.2 in [8]. For this, observe that every function $u \in W^{1,p}_r(\mathcal{B}_1)$ also belongs to the Sobolev space $W^{1,p}(\delta, 1)$. Thus, we have

$$\int_\delta^1 |u_r|^q \psi^{N-1}\,\mathrm{d}r \leq C_{N,q,\psi}\int_\delta^1 |u_r|^q \leq C_{N,q,\psi} u^q(\delta) \leq C_{N,q,\psi}\|u\|^q_{L^q(\mathcal{B}_1)}. \tag{4.4}$$

Now, using equation (3.1) we have

$$u_{rr} \leq -\frac{(N-1)\psi'}{\psi}u_r \quad \text{in } (0,1).$$

Now, let $\tilde{\delta} \in (\delta, 2\delta)$ such that (3.8) holds. Integrating the last inequality,

$$\int_t^{\tilde{\delta}} u_{rr}\,\mathrm{d}r \leq -(N-1)\int_t^{\tilde{\delta}} \frac{\psi'}{\psi}u_r\,\mathrm{d}r$$

and using (4.1),

$$\begin{aligned} -\frac{u_r(t)}{N-1} &\leq -\frac{u_r(\tilde{\delta})}{N-1} - \int_t^{\tilde{\delta}} \frac{|\psi'|}{\psi}u_r\,\mathrm{d}r \\ &\leq C_{N,\psi}\|u\|_{L^1(\mathcal{B}_1)} + \int_t^{2\delta} \frac{\psi'}{\psi}\psi^{(-N+1+2\alpha)/p}\psi^{(N-1-2\alpha)/p}. \end{aligned}$$

Using Hölder inequality and observing that we can use Proposition 3.1 because we can take our δ sufficiently small, follows

$$-u_r(t) \leq C_{N,p,\psi}\|u\|_{L^p(\mathcal{B}_1)}\left(1 + \left(\int_t^{2\delta}\left(\frac{\psi'}{\psi}\right)^{p'}\psi^{p'(-N+1+2\alpha)/p}\right)^{\frac{1}{p'}}\right)$$

for all $\alpha \in [1, 1 + \sqrt{(N-1)/(p-1)}]$. Thus, for $s \in (0, \delta)$,

$$\int_s^\delta |u_r|^q\psi^{N-1}\,\mathrm{d}r \leq C_{N,p,\psi}\|u\|^q_{L^p(\mathcal{B}_1)}\left(1 + \left(\int_t^{2\delta}\left(\frac{\psi'}{\psi}\right)^{p'}\psi^{p'(-N+1+2\alpha)/p}\right)^{\frac{1}{p'}}\right)^q$$

Now, observe that

$$\left(\left(\int_t^{2\delta}(\psi')^{\frac{1}{p'}}\psi^{\frac{1}{p-1}(-N+1+2\alpha-p)}\right)^{\frac{1}{p'}}\right)^q < C_{N,p,q,\psi} < +\infty$$

since $q < Np/(N - 2 - 2\sqrt{(N-1)/(p-1)})$. Thus

$$\int_s^\delta |u_r|^q\psi^{N-1}\,\mathrm{d}r \leq C_{N,p,q,\psi}\|u\|^q_{L^p(\mathcal{B}_1)}. \tag{4.5}$$

Using equations (4.4) and (4.5) we finish the proof. □

Proof of Theorem 1.3. Let $\lambda \in (0, \lambda^*)$. There exists $\rho_\lambda \in (1/2, 1)$ such that mean value property holds, that is,

$$\frac{\partial u_\lambda}{\partial r}(\rho_\lambda) = \frac{u_\lambda(1/2) - u_\lambda(1)}{1/2}.$$

Since u_λ is decreasing, (see proof of Theorem 1.2), we have

$$\left[\frac{\partial u_\lambda}{\partial r}(\rho_\lambda)\right]^{p-1} = [2u_\lambda(1/2)]^{p-1} \le C_{N,p,\psi} \|u_\lambda^{p-1}\|_{L^1(B_{1/2})}.$$

Thus

$$\left\| \psi^{N-1} \left| \frac{\partial u_\lambda}{\partial r} \right|^{p-1} \right\|_{L^\infty(B_{1/2})} \le C_{N,p,\psi} \|u_\lambda^{p-1}\|_{L^1(B_{1/2})}. \tag{4.6}$$

follows by monotonicity. By using $\phi(r) = \min\{1, (2-4r)^+\}$ as test function and (4.6) we obtain

$$\|\lambda h(u_\lambda)\|_{L^1(B_{1/4})} \le C_{N,p,\psi} \int_{1/4}^{1/2} \psi^{N-1} \left| \frac{\partial u_\lambda}{\partial r} \right|^{p-1} \mathrm{d}r \le C_{N,p,\psi} \|u_\lambda^{p-1}\|_{L^1(B_{1/2})}. \tag{4.7}$$

Using the assumption (H_1), given $\delta > 0$ we have for any $\lambda \in (\lambda^*/2, \lambda^*)$ and for all $t > 0$,

$$\lambda h(t) \ge \frac{1}{\delta} t^{p-1} - C_\delta,$$

where C_δ does not depends on λ. With this

$$\|u_\lambda^{p-1}\|_{L^1(B_{1/4})} \le C_{N,p,\psi} \delta \|u_\lambda^{p-1}\|_{L^1(B_{1/2})} + C_\delta. \tag{4.8}$$

Since u_λ is decreasing follows that

$$\|u_\lambda^{p-1}\|_{L^1(B_{1/2} \setminus \overline{B}_{1/4})} \le C_{N,p,\psi} u_\lambda^{p-1}(1/4) \le C_{N,p,\psi} \|u_\lambda^{p-1}\|_{B_{1/4}}. \tag{4.9}$$

Now, take δ sufficiently small and combine (4.8) with (4.9) to obtain

$$\|u_\lambda^{p-1}\|_{L^1(B_{1/4})} \le C,$$

where C is a constant independent of λ. Repeating the argument in (4.9) we are able to obtain an estimate uniform in λ for $\|u_\lambda^{p-1}\|_{L^1(B_1)}$. Using this in (4.7) we obtain a estimate for $\|h(u_\lambda)\|_{L^1(B_{1/4})}$. Again by monotonicity we can apply the same argument used above to control $\|h(u_\lambda)\|_{L^1(B_1)}$ uniformly in λ. Thus

$$\|u_\lambda^{p-1}\|_{L^1(B_1)} + \|h(u_\lambda)\|_{L^1(B_1)} \le C, \tag{4.10}$$

where C is a constant independent of λ. Observe that every radial function $u \in W^{1,p}(\mathcal{B}_1)$ also belongs to the Sobolev space $W^{1,p}(\delta, 1)$ in one dimension for a given $\delta \in (0,1)$. Using the Sobolev embedding in one dimension, u becomes a continuous function of $r = \mathrm{dist}(x, \mathcal{O}) \in [\delta, 1]$ and

$$|u(1)| \le C_{N,p} \|u\|_{W^{1,p}(\mathcal{B}_1)}.$$

In view of this estimate, we can assume that $u > 0 = u(1)$ in $\mathcal{B}_1$. Take α satisfying $1 \le \alpha < 1 + \sqrt{(N-1)/(p-1)}$ and using Proposition 3.1,

$$\int_{\mathcal{B}_1} |u_r|^p \psi^{-2\alpha} \mathrm{d}v_g \le C_{N,p,\psi} \int_{\mathcal{B}_1} |u_r|^p \mathrm{d}x = C_{N,p,\psi} \int_{\mathcal{B}_{r_0}} |u_r|^p \mathrm{d}x + C_{N,p,\psi} \int_{\mathcal{B}_1 \setminus \overline{\mathcal{B}_{r_0}}} |u_r|^p \mathrm{d}x.$$

Now, choose r_0 such that $2C_{N,p,\psi} \le \psi^{-2\alpha}$ in $r \in (0, r_0)$ to obtain

$$C_{N,p,\psi} \int_{\mathcal{B}_{r_0}} |u_r|^p \,\mathrm{d}x \le \frac{1}{2} \int_{\mathcal{B}_1} \psi^{-2\alpha} |u_r|^p \,\mathrm{d}x,$$

which implies

$$C_\psi \int_{\mathcal{B}_1} |u_r|^p \,\mathrm{d}x \le \int_{\mathcal{B}_1} \psi^{-2\alpha} |u_r|^p \,\mathrm{d}x \le C_{N,p,\psi} \int_{\mathcal{B}_1 \setminus \overline{\mathcal{B}_{r_0}}} |u_r|^p \,\mathrm{d}x.$$

Since u is decreasing we have that

$$u(r_0)^{p-1} \le C_{N,p} \|u^{p-1}\|_{L^1(\mathcal{B}_{r_0})}.$$

Thus,

$$\begin{aligned}
\int_{\mathcal{B}_1 \setminus \overline{\mathcal{B}_{r_0}}} |u_r|^p \,\mathrm{d}x &= C_{N,\psi} \int_{r_0}^1 |u_r|^p \psi^{N-1} \,\mathrm{d}r \\
&\le C_{N,\psi} \|\psi^{N-1} |u_r|^{p-1}\|_{L^\infty(\mathcal{B}_1)} \int_{r_0}^1 -u_r \,\mathrm{d}r \\
&\le C_{N,p,\psi} \|h(u)\|_{L^1(\mathcal{B}_1)} \|u^{p-1}\|_{L^1(\mathcal{B}_1)}^{\frac{1}{p-1}}.
\end{aligned}$$

We can conclude that

$$\int_{\mathcal{B}_1} |u_r|^p \,\mathrm{d}x \le C_{N,p,\psi} \int_{\mathcal{B}_1 \setminus \overline{\mathcal{B}_{r_0}}} |u_r|^p \,\mathrm{d}x \le C_{N,p,\psi} \|h(u)\|_{L^1(\mathcal{B}_1)} \|u^{p-1}\|_{L^1(\mathcal{B}_1)}^{\frac{1}{p-1}}.$$

By (4.10) we deduce a bound for $\|u_\lambda\|_{W^{1,p}(B_1)}$. By using the compactness and since $u_\lambda \to u^*$ as $\lambda \to \lambda^*$ follows that $u^* \in W_0^{1,p}(B_1)$. We can pass to the limit and conclude that u^* is a weak solution of $(\mathcal{P}_\lambda)$. It is clear that u^* is radially symmetric and decreasing. By Fatou's Lemma we obtain that u^* is semi-stable. Finally, we can pass to the limit and the regularity statement follows as a consequence of Theorem 1.1. □

Corollary 4.1. *The extremal solution u^* has the same regularity stated in Theorem 1.1.*

Proof. The proof follows straightforward by using above estimates and passing to the limit as $\lambda \to \lambda^*$. □

References

[1] Antonini, P., and D. Mugnai, P. Pucci: Quasilinear elliptic inequalities on complete Riemannian manifolds. - J. Math. Pures Appl. 87, 2007, 582–600.

[2] Berchio, E., A. Ferrero, and G. Grillo: Stability and qualitative properties of radial solutions of the Lane–Emden–Fowler equation on Riemannian models. - J. Math. Pures Appl. 102, 2014, 1–35.

[3] Cabré, X.: Regularity of minimizers of semilinear elliptic problems up to dimension 4. - Comm. Pure Appl. Math. 63, 2010, 1362–1380.

[4] Cabré, X., A. Capella, and M. Sanchón: Regularity of radial minimizers of reaction equations involving the p-Laplacian. - Calc. Var. Partial Differential Equations 34, 2009, 475–494.

[5] Cabré, X., and M. Sanchón: Semi-stable and extremal solutions of reaction equations involving the p-Laplacian. - Commun. Pure Appl. Anal. 6, 2007, 43–67.

[6] Cabré, X., and M. Sanchón: Geometric-type Sobolev inequalities and applications to the regularity of minimizers. - J. Funct. Anal. 264, 2013, 303–325.

[7] Callegari, A., E. Reiss, and H. Keller: Membrane buckling. - Comm. Pure Appl. Math. 24, 1971, 499–527.

[8] Castorina, D., and M. Sanchón: Regularity of stable solutions to semilinear elliptic equations on Riemannian models. - Adv. Nonlinear Anal. 4, 2015, 295–309.

[9] Castorina, D., and M. Sanchón: Regularity of stable solutions of p-Laplace equations through geometric Sobolev type inequalities. - J. Eur. Math. Soc. (JEMS) 17, 2015, 2949–2975.

[10] Castorina, D.: Regularity of the extremal solution for singular p-Laplace equations. - Manuscripta Math. 146, 2015, 519–529.

[11] Chandrasekhar, S.: An introduction to the study of stellar structure. - Dover Publ. Inc., New York, 1985.

[12] Crandall, M., and P. Rabinowitz: Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems. - Arch. Ration. Mech. Anal. 58, 1975, 207–218.

[13] Dávila, J.: Singular solutions of semi-linear elliptic problems. - In: Handbook of differential equations: stationary partial differential equations. Vol. VI, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008, 83–176.

[14] Díaz, J. I.: Nonlinear partial differential equations and free boundaries. Vol. I. Elliptic equations. - Res. Notes Math. 106, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[15] DiBenedetto, E.: $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. - Nonlinear Anal. 7, 1983, 827–850.

[16] do Ó, J. M., and R. da Costa: Symmetry properties for nonnegative solutions of non-uniformly elliptic equations in the hyperbolic space. - J. Math. Anal. Appl. 435, 2016, 1753–1771.

[17] Dupaigne, L.: Stable solutions of elliptic partial differential equations. - Chapman & Hall/CRC, Boca Raton, 2011.

[18] Esposito, P., N. Ghoussoub, and N. Guo: Mathematical analysis of partial differential equations modeling electrostatic MEMS. - Courant Lect. Notes Math. 20, Courant Inst. Math. Sci., New York, Amer. Math. Soc., Providence, RI, 2010.

[19] Farina, A., L. Mari, and E. Valdinoci: Splitting theorems, symmetry results and overdetermined problems for Riemannian manifolds. - Comm. Partial Differential Equations 38, 2013, 1818–1862.

[20] García-Azorero, J., and I. Peral: On an Emden–Fowler type equation. - Nonlinear Anal. 18, 1992, 1085–1097.

[21] García-Azorero, J., I. Peral, and J. P. Puel: Quasilinear problems with exponential growth in the reaction term. - Nonlinear Anal. 22, 1994, 481–498.

[22] Gelfand, I. M.: Some problems in the theory of quasilinear equations. - Amer. Math. Soc. Transl. 29, 1963, 295–381; Russian original: Uspekhi Mat. Nauk. 14, 1959, 87–158.

[23] Grosjean, J.: p-Laplace operator and diameter of manifolds. - Ann. Global Anal. Geom. 28, 2005, 257–270.

[24] Gidas, B., W. M. Ni, and L. Nirenberg: Symmetry and related properties via the maximum principle. - Comm. Math. Phys. 68, 1979, 209–243.

[25] Holopainen, I.: Asymptotic Dirichlet problem for the p-Laplacian on Cartan–Hadamard manifolds. - Proc. Amer. Math. Soc. 130, 2002, 3393–3400.

[26] Joseph, D.: Non-linear heat generation and the stability of the temperature distribution in conducting solids. - Int. J. Heat Mass Transfer 8, 1965, 281–288.

[27] Joseph, D., and D. Cohen: Some positone problems suggested by nonlinear heat generation. - J. Math. Mech. 16, 1967, 1361–1376.

[28] Joseph, D., and T. Lundgren: Quasilinear Dirichlet problem driven by positive sources. - Arch. Ration. Mech. Anal. 49, 1972, 241–269.

[29] Joseph, D., and E. Sparrow: Nonlinear diffusion induced by nonlinear sources. - Quart. Appl. Math. 28, 1970, 327–342.

[30] Kawai, S., and N. Nakauchi: The first eigenvalue of the p-Laplacian on a compact Riemannian manifold. - Nonlinear Anal. 55, 2003, 33–46.

[31] Keener, J., and H. Keller: Positive solutions of convex nonlinear eigenvalue problems. - J. Differential Equations 16, 1974, 103–125.

[32] Kura, T.: The weak supersolution-subsolution method for second order quasilinear elliptic equations. - Hiroshima Math. J. 19, 1989, 1–36.

[33] Lieberman, G. M.: Boundary regularity for solutions of degenerate elliptic equations. - Nonlinear Anal. 12, 1988, 1203–1219.

[34] Lindqvist, P.: On the equation $div(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$. - Proc. Amer. Math. Soc. 109, 1990, 157–164

[35] Luo, X., D. Ye, and F. Zhou: Regularity of the extremal solution for some elliptic problems with singular nonlinearity and advection. - J. Differential Equations 251, 2011, 2082–2099.

[36] Mao, J.: Eigenvalue inequality for the p-Laplacian on a Riemannian manifold and estimates for the heat kernel. - J. Math. Pures Appl. 9, 2014, 372–393.

[37] Mignot, F., and J.-P. Puel: Sur une classe de problèmes non linéaires avec non linéarité positive, croissante, convexe. - Publications de I'UER de Math. de Lille 1, 1978.

[38] Montenegro, M.: Strong maximum principles for super-solutions of quasilinear elliptic equations. - Nonlinear Anal. 37, 1999, 431–448.

[39] Morabito, F.: Radial and non-radial solutions to an elliptic problem on annular domains in Riemannian manifolds with radial symmetry. - J. Differential Equations 258, 2015, 1461–1493.

[40] Nedev, G.: Regularity of the extremal solution of semilinear elliptic equations. - C. R. Acad. Sci. Paris Sér. I Math. 300, 2000, 997–1002.

[41] Sanchón, M.: Boundedness of the extremal solution for some p-Laplacian problems. - Nonlinear Anal. 67, 2007, 281–294.

[42] Tolksdorf, P.: Regularity for a more general class of quasilinear elliptic equations. - J. Differential Equations 51, 1984, 126–150.

[43] Wei, L.: Boundedness of the extremal solution for some p-Laplacian problems. - Math. Slovaca 64, 2014, 379–390.

[44] Zhang, H.: Lower bounds for the first eigenvalue of the p-Laplace operator on compact manifolds with nonnegative Ricci curvature. - Adv. Geom. 7:1, 2007, 145–155.

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