Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 44, 2019, 693–721

A REIFENBERG TYPE CHARACTERIZATION FOR m-DIMENSIONAL C^1-SUBMANIFOLDS OF $\mathbf{R}^n$

Bastian Käfer

RWTH Aachen University, Institut für Mathematik
Templergraben 55, D-52062 Aachen, Germany; kaefer@instmath.rwth-aachen.de

Abstract. We provide a Reifenberg type characterization for m-dimensional C^1-submanifolds of $\mathbf{R}^n$. This characterization is also equivalent to Reifenberg-flatness with vanishing constant combined with suitably converging approximating m-planes. Moreover, a sufficient condition can be given by the finiteness of the integral of the quotient of $\theta(r)$-numbers and the scale r, and examples are presented to show that this last condition is not necessary.

1. Introduction

It is often useful to control local geometric properties of a subset $\Sigma \subset \mathbf{R}^n$ to obtain topological and analytical information about that set. One of these geometric properties is the local flatness of a set, first introduced and studied by Reifenberg in [12] for his solution of the Plateau problem in arbitrary dimensions. The content of his so-called Topological-Disk Theorem is that δ-Reifenberg-flatness ensures that Σ is locally a topological $C^{0,\alpha}$-disk if $\delta < \delta_0$, where $\delta_0 = \delta_0(m, n)$ is a positive constant, which depends only on the dimensions of Σ and n (see e.g. [12, 10, 5]).

Definition 1.1. Let $n, m \in \mathbf{N}$ with $m < n$ and $\Sigma \subset \mathbf{R}^n$. For $x \in \Sigma$ and $r > 0$ set

$$\theta_\Sigma(x, r) := \frac{1}{r} \inf_{L \in G(n,m)} \operatorname{dist}_{\mathcal{H}} \Big(\Sigma \cap B_r(x), (x + L) \cap B_r(x)\Big),$$

where $G(n, m)$ denotes the Grassmannian of all m-dimensional linear subspaces (m-planes) of $\mathbf{R}^n$. For $\delta > 0$, the set Σ is called *δ-Reifenberg-flat of dimension m* if for all compact sets $K \subset \Sigma$ there exists a radius $r_K > 0$ such that

$$\theta_K(r) := \sup_{x \in \Sigma \cap K} \theta_\Sigma(x, r) \le \delta \text{ for all } r \in (0, r_K].$$

Σ is called Reifenberg-flat of dimension m *with vanishing constant* if Σ is δ-Reifenberg-flat of dimension m for all $\delta > 0$.

It is easy to see that δ-Reifenberg-flat sets do not have to be C^1-submanifolds. For example, for each fixed $\delta > 0$, a δ-Reifenberg-flat set of dimension 1 can be constructed as the graph of $u\colon \mathbf{R} \to \mathbf{R}\colon x \mapsto \delta|x|$, which is not a C^1-submanifold of $\mathbf{R}^2$. Moreover, even Reifenberg-flatness with vanishing constant is still not enough to guarantee C^1-regularity. It can be shown that the graph of

$$u\colon \mathbf{R} \to \mathbf{R}, \quad x \mapsto \sum_{k=1}^{\infty} \frac{\cos(2^k x)}{2^k \sqrt{k}},$$

is a Reifenberg-flat set with vanishing constant (see [14]). Nevertheless, although u is continuous, it is nowhere differentiable. Moreover, Toro stated that the graph

https://doi.org/10.5186/aasfm.2019.4443
2010 Mathematics Subject Classification: Primary 28A75, 53A07, 53C40.
Key words: Reifenberg-flatness, submanifolds, Grassmannian.

is not rectifiable in the sense of geometric measure theory, and therefore not a C^1-submanifold. We will show in detail with an indirect argument that graph(u) cannot be represented as a graph of a C^1-function in a neighbourhood of $(0, u(0))$ in Appendix A.

There are a couple of variations to the definition of Reifenberg-flat sets with additional conditions, which guarantee more regularity than Reifenberg's Topological-Disk Theorem. If for a Reifenberg-flat set with vanishing constant there exists in addition, an exponent $\sigma \in (0,1]$ and for each compact set $K \subset \Sigma$ a constant $C_K > 0$, such that the decay of the so-called β-numbers introduced by Jones in [6] can be estimated as

$$\beta_\Sigma(x,r) := \frac{1}{r} \inf_{L \in G(n,m)} \left(\sup_{y \in \Sigma \cap B_r(x)} \operatorname{dist}(y, x+L) \right) \le C_K r^\sigma \tag{1}$$

for all $x \in K$ and $r \le 1$, then David, Kenig and Toro could show in [2, Prop. 9.1], that Σ is an embedded, m-dimensional $C^{1,\sigma}$-submanifold of $\mathbf{R}^n$.

A weaker assumption on $\Sigma \subset \mathbf{R}^n$ was stated by Toro in [13] calling it (δ, ε, R)-*Reifenberg-flat at* $x \in \Sigma$ for $\delta, \varepsilon, R > 0$, if and only if

$$\theta_{B_R(x)}(r) \le \delta \quad \text{for all} \quad r \in (0, R]$$

and

$$\int_0^R \frac{\theta_{B_R(x)}(r)^2}{r}\, dr \le \varepsilon^2. \tag{2}$$

In this setting it can be shown that there exist universal positive constants $\delta_0(m,n)$ and $\varepsilon_0(m,n)$, depending only on the dimensions m and n, such that all sets $\Sigma \subset \mathbf{R}^n$ that are (δ, ε, R)-Reifenberg-flat at all of their points with $0 < \delta < \delta_0$, $0 < \varepsilon < \varepsilon_0$, can be locally parameterized, on a scale determined by R, by bi-Lipschitz-homeomorphisms over open subsets of $\mathbf{R}^m$. In particular, such sets Σ are embedded $C^{0,1}$-submanifolds of $\mathbf{R}^n$.

In search of a characterization of C^1-submanifolds one may consider slightly stronger variants of Toro's integral condition in (2), which on the other hand, need to be weaker than the power-decay (1) of the β-numbers. We will present such a characterization in our main result, Theorem 1.4 below, but first state a corollary of that result that uses an integral condition stronger than (2). This statement was independently proven by Ranjbar–Motlagh in [11].

Theorem 1.2. *Let $\Sigma \subset \mathbf{R}^n$ be closed. If for all $x \in \Sigma$ there exists a radius $R_x > 0$ such that*

$$\int_0^{R_x} \frac{\theta_{B_{R_x}(x)}(r)}{r}\, dr < \infty,$$

then Σ is an embedded, m-dimensional C^1-submanifold of $\mathbf{R}^n$.

Note that the dimension m is encoded in the definition of the θ-numbers; see Definition 1.1. Moreover, Σ is not explicitly claimed to be Reifenberg-flat in Theorem 1.2, but the finite integral will ensure that Σ is Reifenberg-flat with vanishing constant. Nevertheless, Theorem 1.2 does not yet yield a characterization for C^1-submanifolds, since there are graphs of C^1-functions leading to an infinite integral. For example, let $u\colon (-1/2, 1/2) \to \mathbf{R}$ be defined by

$$u(x) = \left| \int_0^x \left(-\frac{2}{\log(y^2)} \right) dy \right| \quad \text{for all} \quad x \in \left(-\frac{1}{2}, \frac{1}{2} \right),$$

then u is of class C^1 on $(-1/2,1/2)$ and can be extended to a function $\tilde{u}\in C^1(\mathbf{R})$. But $\Sigma := \operatorname{graph}(\tilde{u})\subset\mathbf{R}^2$ does *not* satisfy the integral condition in Theorem 1.2 as shown in detail in Appendix B. Moreover, for every fixed $\alpha,\beta>0$ minor modifications of u lead to a C^1-submanifold with

$$\int_0^{R_x}\frac{\theta^\beta_{B_{R_x}(x)}(r)}{r^\alpha}\,dr=\infty.$$

A characterization for C^1-submanifolds using the condition of Reifenberg-flatness needs to allow θ-numbers and the scale r to decay more independently. Roughly speaking, a closed $\Sigma\subset\mathbf{R}^n$ is a C^1-submanifold, if and only if there exists a sequence of radii tending to zero, with controlled decay, such that Σ satisfies the estimate for Reifenberg-flatness at these scales and the planes approximating Σ converge to a limit-plane. We call this condition (RPC) and the precise definition is as follows.

Definition 1.3. (**R**eifenberg-**P**lane-**C**onvergence) For $1\le m<n$, we say $\Sigma\subset\mathbf{R}^n$ satisfies *the condition* (RPC) *with dimension* m if the following holds: For all $x\in\Sigma$ there exist a radius $R_x>0$, a sequence $(r_{x,i})_{i\in\mathbf{N}}\subset(0,R_x]$ and a constant $C_x>1$ with

$$r_{x,i+1}<r_{x,i}\le C_x r_{x,i+1}\quad\text{for all }\ i\in\mathbf{N}\ \text{ and }\ \lim_{i\to\infty}r_{x,i}=0.$$

Furthermore, there exist two sequences $(\delta_{x,i})_{i\in\mathbf{N}},(\varepsilon_{x,i})_{i\in\mathbf{N}}\subset(0,1]$, both converging to zero, such that for all $y\in\Sigma\cap B_{R_x}(x)$ there exist planes $P(y,r_{x,i}),P_y\in G(n,m)$ with

$$\operatorname{dist}_{\mathcal{H}}\Big(\Sigma\cap B_{r_{x,i}}(y),\big(y+P(y,r_{x,i})\big)\cap B_{r_{x,i}}(y)\Big)\le\delta_{x,i}r_{x,i}$$

and

$$\sphericalangle\big(P(y,r_{x,i}),P_y\big)\le\varepsilon_{x,i}.$$

Notice that the Grassmannian $G(n,m)$ equipped with the angle-metric is compact (see Definition 2.3), so that every sequence of m-planes contains a converging subsequence, but the relation between the approximating planes $P(y,r_{x,i})$ and the scale $r_{x,i}$ is crucial in Definition 1.3. Notice also that (RPC) does not explicitly claim that the set is Reifenberg-flat, since the approximation of Σ is postulated only for a specific sequence of radii. Nevertheless, we show that (RPC) is actually equivalent to Reifenberg-flatness with vanishing constant and uniformly converging approximating planes.

Here is our main result.

Theorem 1.4. *For a closed $\Sigma\subset\mathbf{R}^n$ the followings are equivalent:*

(1) *Σ satisfies (RPC) with dimension m;*
(2) *Σ is an embedded, m-dimensional C^1-submanifold of $\mathbf{R}^n$;*
(3) *Σ is Reifenberg-flat with vanishing constant, and for all compact subsets $K\subset\Sigma$ and all $x\in K$ there exists an m-plane $L_x\in G(n,m)$ such that*

$$\sup_{x\in K}\sphericalangle\big(L(x,r),L_x\big)\xrightarrow[r\to0]{}0,$$

for all $L(x,r)\in G(n,m)$ with

$$\sup_{x\in K}\frac{1}{r}\operatorname{dist}_{\mathcal{H}}\Big(\Sigma\cap B_r(x),\big(x+L(x,r)\big)\cap B_r(x)\Big)\xrightarrow[r\to0]{}0.$$

As one can expect intuitively, in this case P_x from condition (RPC) and L_x will coincide with the tangent plane $T_x\Sigma$.

In Section 2 we will review some basic facts about the Grassmannian and about orthogonal projections onto linear as well as onto affine subspaces of $\mathbf{R}^n$. Section 3 is dedicated to the proof of the main theorem and finally, in Section 4 we will prove that the condition of Theorem 1.2 is sufficient to obtain an embedded C^1-submanifold. The detailed structure of the examples mentioned in the introduction is presented in the appendix as well as the proofs of two technical lemmata.

2. Projections and preparations

The aim of this section is to introduce all needed definitions and properties for linear and affine spaces, as well as for the projections onto those planes.

Definition 2.1. For $n, m \in \mathbf{N}$ with $m \leq n$, the *Grassmannian* $G(n,m)$ denotes the set of all m-dimensional linear subspaces of $\mathbf{R}^n$.

Definition 2.2. For $P \in G(n,m)$, the orthogonal projection of $\mathbf{R}^n$ onto P is denoted by π_P. Further $\pi_P^{\perp} := id_{\mathbf{R}^n} - \pi_P$ shall denote the orthogonal projection onto the linear subspace perpendicular to P.

Using orthogonal projections it is possible to define a distance between two elements of $G(n,m)$.

Definition 2.3. For two planes $P_1, P_2 \in G(n,m)$ the *included angle* is defined by

$$\sphericalangle(P_1, P_2) := \|\pi_{P_1} - \pi_{P_2}\| := \sup_{x \in \mathbf{S}^{n-1}} |\pi_{P_1}(x) - \pi_{P_2}(x)|.$$

The angle $\sphericalangle(\cdot,\cdot)$ is a metric on the Grassmannian $G(n,m)$.

Together with this metric, the Grassmannian $(G(n,m), \sphericalangle(\cdot,\cdot))$ is a compact manifold. The following lemma allows to use different useful presentations for the angle between two planes.

Lemma 2.4. [1, 8.9.3] *If* $P_1, P_2 \in G(n,m)$, *then*

$$\|\pi_{P_1} - \pi_{P_2}\| = \|\pi_{P_1}^{\perp} - \pi_{P_2}^{\perp}\| = \|\pi_{P_1}^{\perp} \circ \pi_{P_2}\| = \|\pi_{P_1} \circ \pi_{P_2}^{\perp}\| = \|\pi_{P_2}^{\perp} \circ \pi_{P_1}\| = \|\pi_{P_2} \circ \pi_{P_1}^{\perp}\|.$$

Citing the first part of Lemma 2.2 in [9] we get

Lemma 2.5. *Assume* $P_1, P_2 \in G(n,m)$. *If* $\sphericalangle(P_1, P_2) < 1$, *then the projection* $\pi_{P_1|P_2} \colon P_2 \to P_1$ *is a linear isomorphism.*

Although we use linear spaces most of the time, it is also necessary to define projections onto affine spaces and the angles between those.

Definition 2.6. For $x \in \mathbf{R}^n$ and $P \in G(n,m)$, the orthogonal projection onto $Q := x + P$ and the corresponding perpendicular plane are defined by

$$\pi_Q(z) := x + \pi_P(z - x)$$

and

$$\pi_Q^{\perp}(z) = z - \pi_Q(z) = (z - x) - \pi_P(z - x) = \pi_P^{\perp}(z - x).$$

Moreover, for $x_1, x_2 \in \mathbf{R}^n$ and $P_1, P_2 \in G(n,m)$ the angle between $Q_1 := x_1 + P_1$ and $Q_2 := x_2 + P_2$ is defined as

$$\sphericalangle(Q_1, Q_2) := \sphericalangle(P_1, P_2).$$

For a smooth function's graph, [1, 8.9.5] leads to an estimate for the angle between tangent spaces.

Lemma 2.7. *Let $\alpha \geq 0$, $P \in G(n,m)$ and assume $f \in C^1(P, P^\perp)$ satisfies $\|f'\| \leq \alpha$ and $f'(0) = 0$. Let $g(x) := x + f(x)$ and $\Sigma := g(P)$ be the graph of f, then for all $x, y \in P$ the following estimates hold:*

$$\|\pi_{T_{g(y)}\Sigma} - \pi_{T_{g(x)}\Sigma}\| \leq \|f'(x) - f'(y)\| \leq \sqrt{\frac{1+\alpha^2}{1-\alpha^2}}\|\pi_{T_{g(y)}\Sigma} - \pi_{T_{g(x)}\Sigma}\|$$

Lastly there is an estimate for angles between planes, in a more generel setting.

Lemma 2.8. [8, Prop. 2.5] *Let $P_1, P_2 \in G(n,m)$ and let $(e_1, \dots, e_m)$ be some orthonormal basis of P_1. Assume that for each $i = 1, \dots, m$ we have the estimate* $\operatorname{dist}(e_i, P_2) \leq \theta$ *for some $\theta \in (0, 1/\sqrt{2})$. Then there exists a constant $C_1 = C_1(m)$ such that*

$$\sphericalangle(P_1, P_2) \leq C_1\theta.$$

3. Equivalence of (RPC) and C^1-regularity

In this section we prove the main theorem. First we will show that (RPC) is equivalent to Reifenberg-flatness with vanishing constant and a uniform convergence of approximating planes. This allows us to use (RPC) and Reifenberg-flatness to prove that every set, which satisfies (RPC) is an embedded C^1-submanifold. We will approach this by using a different characterization, namely writing Σ locally as the graph of a C^1-function. It turns out, that for an element $x \in \Sigma$ the radius r providing $\Sigma \cap B_r(x)$ can be represented as a graph, can be given depending on the ratio of decay of $\delta_{x,i}, \varepsilon_{x,i}$ and $r_{x,i}$. Lastly we will show the other implication, using that the representation as a graph of a smooth function already provides Reifenberg-flatness.

Notice that we will fix the dimension m of a subset $\Sigma \subset \mathbf{R}^n$ and say that Σ is a δ-Reifenberg-flat set or satisfies (RPC) without mentioning the dimension.

Lemma 3.1. *Assume $\Sigma \subset \mathbf{R}^n$ satisfies (RPC). Then for all $x \in \Sigma$ we get*

$$\operatorname{dist}(z, y + P_y) \leq w_x(|z-y|) \cdot |z-y| \ \text{ for all } \ y \in \Sigma \cap B_{R_x}(x) \ \text{ and } \ z \in \Sigma \cap B_{r_{x,1}}(y),$$

where the function $w_x \colon \mathbf{R} \to \mathbf{R}$ is given by

$$w_x(r) = \varepsilon_{x,i} + C_x\delta_{x,i} \ \text{ for all } \ r \in [r_{x,i+1}, r_{x,i}).$$

Note that w_x is a piecewise constant function with $\lim_{r\to 0} w_x(r) = 0$. It is possible for w_x to be not monotonically decreasing, because (RPC) require this neither for $\delta_{x,i}$ nor for $\varepsilon_{x,i}$.

Proof. Let $x \in \Sigma$ and $y \in \Sigma \cap B_{R_x}(x)$ be fixed. For $z \in \Sigma \cap B_{r_{x,1}}(y)$ there exists an $i \in \mathbf{N}$ with $|z-y| \in [r_{x,i+1}, r_{x,i})$. This yields

$$\begin{aligned}
\operatorname{dist}(z, y + P_y) &= |\pi^\perp_{P_y}(z-y)| \\
&\leq \left|\left(\pi^\perp_{P_y} - \pi^\perp_{P(y,r_{x,i})}\right)(z-y)\right| + \left|\pi^\perp_{P(y,r_{x,i})}(z-y)\right| \\
&\leq \varepsilon_{x,i}|z-y| + \delta_{x,i} r_{x,i} \\
&\leq \varepsilon_{x,i}|z-y| + \delta_{x,i} C_x |z-y|.
\end{aligned}$$ ☐

The idea of Lemma 2.8 will frequently be used for Reifenberg-flat sets Σ while P_1 and P_2 are the approximating planes of Definition 1.1 for either different or the same radii and points of Σ. The following lemma uses Lemma 2.8 to get an estimate in this setting.

Lemma 3.2. *Let $x_1, x_2 \in \Sigma \subset \mathbf{R}^n$, $0 < r_1 \le r_2$, $\delta_1, \delta_2 \in (0, \frac{1}{2})$ and $P_1, P_2 \in G(n,m)$ be given such that*

$$|x_1 - x_2| < \frac{r_1}{2}$$

and

$$\operatorname{dist}_{\mathcal{H}} \Big(\Sigma \cap B_{r_j}(x_j), (x_j + P_j) \cap B_{r_j}(x_j) \Big) \le \delta_j r_j \quad \text{for} \quad j = 1, 2.$$

If

$$\frac{2}{1-2\delta_1} \left(\delta_1 + 2\frac{r_2}{r_1}\delta_2 \right) < \frac{1}{\sqrt{2}},$$

then we get

$$\sphericalangle(P_1, P_2) \le C_1 \frac{2}{1-2\delta_1} \left(\delta_1 + 2\frac{r_2}{r_1}\delta_2 \right).$$

Proof. Let $(e_1, \dots, e_m)$ be an orthonormal basis of P_1. Define

$$y_0 := x_1 \quad \text{and} \quad y_i := x_1 + \frac{1-2\delta_1}{2} r_1 e_i \quad \text{for} \quad i = 1, \dots, m.$$

For all $i = 1, \dots, m$ there exists a $z_i \in \Sigma \cap B_{r_1}(x_1)$ with

$$|z_i - y_i| \le r_1 \delta_1.$$

Note that for $z_0 := y_0 = x_0$, the point z_0 is also an element of $\Sigma \cap B_{r_1}(x_1) \cap B_{r_2}(x_2)$. Further we get

$$|z_i - x_1| \le |z_i - y_i| + |y_i - x_1| \le r_1\delta_1 + r_1 \frac{1-2\delta_1}{2} = \frac{r_1}{2} \quad \text{for all} \quad i = 1, \dots, m.$$

This leads to

$$|z_i - x_2| \le |z_i - x_1| + |x_1 - x_2| < r_1 \left(\frac{1}{2} + \frac{1}{2} \right) = r_1 \le r_2 \quad \text{for all} \quad i = 1, \dots, m.$$

Therefore for every $i = 0, \dots, m$ there exists a $w_i \in (x_2 + P_2) \cap B_{r_2}(x_2)$ with

$$|w_i - z_i| \le r_2 \delta_2.$$

Define $\tilde{y}_i := y_i - y_0$ and $\tilde{w}_i := w_i - w_0$ for $i = 1, \dots, m$. Then $\tilde{y}_i / |\tilde{y}_i| = e_i$ is obviously an orthonormal basis of P_1 and $\tilde{w}_i / |\tilde{y}_i|$ is an element of P_2. The previous estimates yield

$$\begin{aligned} \left| \frac{\tilde{y}_i}{|\tilde{y}_i|} - \frac{\tilde{w}_i}{|\tilde{y}_i|} \right| &= \frac{1}{|\tilde{y}_i|} \Big| y_i - y_0 - w_i + w_0 \Big| \\ &= \frac{2}{(1-2\delta_1) r_1} \Big| y_i - z_i + z_0 - y_0 + z_i - w_i + w_0 - z_0 \Big| \\ &\le \frac{2}{(1-2\delta_1) r_1} (r_1 \delta_1 + 0 + r_2\delta_2 + r_2\delta_2) \\ &\le \frac{2}{1-2\delta_1} \left(\delta_1 + 2\frac{r_2}{r_1}\delta_2 \right) \quad \text{for all} \quad i = 1, \dots, m. \end{aligned}$$

This is assumed to be strictly less than $1/\sqrt{2}$ and therefore Lemma 2.8 leads to

$$\sphericalangle(P_1, P_2) \le C_1(m) \frac{2}{1-2\delta_1} \left(\delta_1 + 2\frac{r_2}{r_1}\delta_2 \right). \qquad \square$$

Now we will show that every set satisfying (RPC) is indeed Reifenberg-flat with vanishing constant. Moreover, we will see that (RPC) is an even stronger assumption and allows to approximate the set for a fixed point with the same plane at each scale.

In fact, we will show the estimation for Reifenberg-flatness only for a ball around $x \in \Sigma$. By a covering argument, we later see, that the estimate holds true for all compact subsets of Σ.

Lemma 3.3. *Assume $\Sigma \subset \mathbf{R}^n$ satisfies (RPC), then for all $x \in \Sigma$ and $k \geq \tilde{k}_x$, where $\tilde{k}_x \in \mathbf{N}$ denotes the index with*

$$\delta_{x,k} < \frac{1}{C_x} \quad \text{for all} \quad k \geq \tilde{k}_x,$$

we get

$$\sup_{y \in B_{R_x}(x) \cap \Sigma} \frac{1}{r} \operatorname{dist}_{\mathcal{H}} \Big(\Sigma \cap B_r(y), (y + P_y) \cap B_r(y) \Big) \leq \sup_{i \geq k} (\varepsilon_{x,i} + 2C_x \delta_{x,i}) =: \tilde{\delta}_{x,r} \quad \text{for all} \quad r \leq r_{x,k}.$$

Note that the existence of $\tilde{k}_x$ is an immediate result of $\delta_{x,k}$ tending to zero. The value of $\tilde{k}_x$ and therefore the scale of the approximation depends highly on the point $x \in \Sigma$.

Proof. Let $x \in \Sigma$ be fixed, $y \in \Sigma \cap B_{R_x}(x)$ and $z \in \Sigma \cap B_r(y)$ for a radius $r \in (0, r_{x,\tilde{k}_x}]$. Then for $y \neq z$ there exists an $i \in \mathbf{N}$ with $r_{x,i+1} \leq |z-y| < r_{x,i}$ and Lemma 3.1 leads to

$$\frac{1}{r} \operatorname{dist} \Big(z, (y + P_y) \cap B_r(y) \Big) \leq \frac{1}{r} w_x(|z-y|) \cdot |z-y| \leq w_x(|z-y|) = \varepsilon_{x,i} + C_x \delta_{x,i}.$$

Let $k \in \mathbf{N}$ such that $r_{x,k+1} < r \leq r_{x,k}$, then this implies

$$\sup_{z \in \Sigma \cap B_r(y)} \frac{1}{r} \operatorname{dist} \Big(z, (y + P_y) \cap B_r(y) \Big) \leq \sup_{i \geq k} (\varepsilon_{x,i} + C_x \delta_{x,i}).$$

Moreover, we have $k \geq \tilde{k}_x$. Using the definition of $\tilde{k}_x$ we have

$$r - r_{x,k} \delta_{x,k} \geq r - r C_x \delta_{x,r} > 0.$$

For $z \in (y + P_y) \cap B_{r - r_{x,k}\delta_{x,k}}(y)$ defining

$$\tilde{z} := y + \pi_{P(y, r_{x,k})}(z - y),$$

leads to

$$|\tilde{z} - y| = |\pi_{P(y,r_{x,k})}(z-y)| \leq |z - y| < r - r_{x,k}\delta_{x,k} < r \leq r_{x,k}.$$

Hence there exists a $w \in \Sigma \cap B_{r_{x,k}}(y)$ with

$$|\tilde{z} - w| \leq r_{x,k} \delta_{x,k}.$$

Moreover,

$$|w - y| \leq |w - \tilde{z}| + |\tilde{z} - y| < r_{x,k}\delta_{x,k} + r - r_{x,k}\delta_{x,k} = r$$

and therefore $w \in \Sigma \cap B_r(y)$. Using $z - y \in P_y$ and Lemma 2.4, we get

$$\begin{aligned} \operatorname{dist} \Big(z, \Sigma \cap B_r(y) \Big) &\leq |z - w| \leq |z - \tilde{z}| + |\tilde{z} - w| = |\pi^{\perp}_{P(y,r_{x,k})}(z-y)| + |\tilde{z} - w| \\ &\leq \varepsilon_{x,k} |z - y| + r_{x,k}\delta_{x,k} \leq r \left(\varepsilon_{x,k} + C_x \delta_{x,k} \right). \end{aligned}$$

Now let $z \in (y + P_y) \cap (B_r(y) \setminus B_{r - r_{x,k}\delta_{x,k}}(y))$, then there exists a $z' \in (y + P_y) \cap B_{r - r_{x,k}\delta_{x,k}}(y)$ such that

$$|z' - z| < r_{x,k}\delta_{x,k}.$$

Therefore we get a $w \in \Sigma \cap B_r(y)$ with

$$|w - z| \leq |w - z'| + |z' - z| \leq r \left(\varepsilon_{x,k} + C_x \delta_{x,k} \right) + r_{x,k}\delta_{x,k} \leq r \left(\varepsilon_{x,k} + 2C_x \delta_{x,k} \right).$$

Finally

$$\frac{1}{r}\operatorname{dist}_{\mathcal{H}}\Big(\Sigma\cap B_r(y),(y+P_y)\cap B_r(y)\Big)\le\max\Big\{\sup_{i\ge k}(\varepsilon_{x,i}+C_x\delta_{x,i}),\varepsilon_{x,k}+2C_x\delta_{x,k}\Big\}$$
$$\le\sup_{i\ge k}(\varepsilon_{x,i}+2C_x\delta_{x,i}),$$

which is independent of $y\in B_{R_x}(x)$ and implies the postulated statement. □

Remark 3.4. Note that $\tilde{\delta}_{x,k}$ is monotonically decreasing and using the convergence of $\delta_{x,i}$ and $\varepsilon_{x,i}$ we get $\tilde{\delta}_{x,k}\to 0$ as $k\to\infty$. Lemma 3.3 then implies that Σ is a δ-Reifenberg-flat set for all $\delta>0$, i.e. it is Reifenberg-flat with vanishing constant. Moreover, the plane which approximates Σ at the point $y\in\Sigma$ with respect to the δ-Reifenberg-flatness can be fixed as $y+P_y$ for all small radii.

For a set $\Sigma\subset\mathbf{R}^n$ which satisfies (RPC) and $y\in\Sigma$ the plane P_y arises as a limit of planes $P(y,r_{x,i})$. Up to this point, we did not mention that these planes might also depend on x and that we should have writen P_y^x, but in fact, we are now ready to show, that the P_y^x are the same for all $x\in\Sigma$ with $y\in\Sigma\cap B_{R_x}(x)$. Moreover, we get an estimate for the angle between two planes P_y and P_z, whenever z is an element of $\Sigma\cap B_{R_x}(x)$ with $|y-z|$ small enough.

Lemma 3.5. *Assume $\Sigma\subset\mathbf{R}^n$ satisfies (RPC).*

(1) *For $x,\tilde{x}\in\Sigma$ we get*

$$P_y^x=P_y^{\tilde{x}}\quad\text{for all}\;\; y\in\Sigma\cap B_{R_x}(x)\cap B_{R_{\tilde{x}}}(\tilde{x}).$$

(2) *For $x\in\Sigma$, $k\ge\tilde{k}_x$ and $y,z\in\Sigma\cap B_{R_x}(x)$ with $|z-y|<\frac{r_{x,k}}{2}$ and $\tilde{\delta}_{x,k}<\frac{1}{11}$ we get*

$$\sphericalangle(P_y,P_z)\le\frac{22}{3}C_1(m)\tilde{\delta}_{x,k}=:C_2(m)\tilde{\delta}_{x,k}.$$

Proof. (1) Let $x,\tilde{x}\in\Sigma$ and $y\in\Sigma\cap B_{R_x}(x)\cap B_{R_{\tilde{x}}}(\tilde{x})$. The sequences $\varepsilon_{x,k}$ and $\varepsilon_{\tilde{x},k}$ converge to zero and hence for all $\varepsilon>0$ there exist an $N_1\in\mathbf{N}$ such that

$$\varepsilon_{x,k},\varepsilon_{\tilde{x},k}\le\frac{\varepsilon}{3}\quad\text{for all}\;\; k\ge N_1.$$

Moreover, there exists an $N_2\in\mathbf{N}$ with $N_2>N_1$ and

$$\delta_{x,k}<\min\Big\{\frac{\varepsilon}{24C_1},\frac{1}{4}\Big\}\quad\text{and}\quad\delta_{\tilde{x},k}<\frac{\varepsilon}{48C_1C_x}\quad\text{for all}\;\; k\ge N_2.$$

Define

$$k:=\begin{cases}N_2 & \text{for}\;\; r_{\tilde{x},N_2}\le r_{x,N_2},\\ \min\{l\in\mathbf{N}\mid r_{\tilde{x},l}\le r_{x,N_2}\} & \text{for}\;\; r_{\tilde{x},N_2}>r_{x,N_2},\end{cases}$$

and

$$i:=\min\{l\in\mathbf{N}\mid r_{x,l}\le r_{\tilde{x},k}\}.$$

Then we have $k,i\ge N_2$ and

$$r_{x,i}\le r_{\tilde{x},k}\le r_{x,i-1}.$$

Let ε be sufficiently small, i.e. $\frac{\varepsilon}{3C_1} < \frac{1}{\sqrt{2}}$. Then

$$\frac{2}{1-2\delta_{x,i}}\left(\delta_{x,i} + \frac{r_{\tilde{x},k}}{r_{x,i}}\delta_{\tilde{x},k}\right) \leq 4(\delta_{x,i} + 2C_x\delta_{\tilde{x},k}) \leq 4\left(\frac{\varepsilon}{24C_1} + 2C_x\frac{\varepsilon}{48C_1C_x}\right) = \frac{\varepsilon}{3C_1} < \frac{1}{\sqrt{2}}.$$

Using Lemma 3.2 we get

$$\sphericalangle\big(P(y,r_{x,i}), P(y,r_{\tilde{x},k})\big) \leq C_1\frac{2}{1-2\delta_{x,i}}\left(\delta_{x,i} + 2\frac{r_{\tilde{x},k}}{r_{x,i}}\delta_{\tilde{x},k}\right) \leq \frac{\varepsilon}{3}.$$

Finally

$$\sphericalangle\left(P_y^x, P_y^{\tilde{x}}\right) \leq \sphericalangle\big(P_y^x, P(y,r_{x,i})\big) + \sphericalangle\big(P(y,r_{x,i}), P(y,r_{\tilde{x},k})\big) + \sphericalangle\big(P(y,r_{\tilde{x},k}), P_y^{\tilde{x}}\big) \leq \varepsilon.$$

The limit $\varepsilon \to 0$ implies

$$P_y^x = P_y^{\tilde{x}}.$$

(2) For $y, z \in \Sigma \cap B_{R_x}(x)$, $k \geq \tilde{k}_x$ and $r \leq r_{x,k}$ Lemma 3.3 leads to

$$\mathrm{dist}_{\mathcal{H}}\Big(\Sigma \cap B_r(y), (y+P_y) \cap B_r(y)\Big) \leq r\tilde{\delta}_{x,k}$$

and

$$\mathrm{dist}_{\mathcal{H}}\Big(\Sigma \cap B_r(z), (z+P_z) \cap B_r(z)\Big) \leq r\tilde{\delta}_{x,k}.$$

If $|z-y| < \frac{r_{x,k}}{2}$ and $\tilde{\delta}_{x,k} < \frac{1}{11}$, then

$$\frac{2}{1-2\tilde{\delta}_{x,k}}(\tilde{\delta}_{x,k} + 2\tilde{\delta}_{x,k}) < \frac{22}{3}\tilde{\delta}_{x,k} < \frac{1}{\sqrt{2}}$$

and for $r_1 := r_2 := r_{x,k}$ and $\delta_1 := \delta_2 := \tilde{\delta}_{x,k}$ Lemma 3.2 yields

$$\sphericalangle(P_y, P_z) \leq \frac{22}{3}C_1(m)\tilde{\delta}_{x,k},$$

which completes the proof. □

Lemma 3.6. *For closed $\Sigma \subset \mathbf{R}^n$, the following statements are equivalent:*

(1) *Σ satisfies (RPC).*

(2) *Σ is Reifenberg-flat with vanishing constant and, for all compact subsets $K \subset \Sigma$ and all $x \in K$ there exists a plane $L_x \in G(n,m)$ such that*

$$\sup_{x\in K} \sphericalangle\big(L(x,r), L_x\big) \xrightarrow[r\to 0]{} 0,$$

for all $L(x,r) \in G(n,m)$ with

$$\sup_{x\in K}\frac{1}{r}\,\mathrm{dist}_{\mathcal{H}}\Big(\Sigma \cap B_r(x), \big(x+L(x,r)\big) \cap B_r(x)\Big) \xrightarrow[r\to 0]{} 0.$$

Note that the existence of planes $L(x,r)$, which approximate Σ with respect to the Reifenberg-flatness such that their distances to Σ converges uniformly to zero is already guaranteed by the Reifenberg-flatness with vanishing constant. Only the existence of a limit-plane is an additional condition to the Reifenberg-flatness in Lemma 3.6 (2). Obviously, L_x and P_x will coincide.

Proof. (1) $\implies$ (2): For fixed $x \in \Sigma$ using Lemma 3.3 yields for $k \geq \tilde{k}_x$

$$\sup_{y \in \Sigma \cap B_{R_x}(x)} \frac{1}{r} \operatorname{dist}_{\mathcal{H}} \Big(\Sigma \cap B_r(y), (y + P_y) \cap B_r(y) \Big) \leq \tilde{\delta}_{x,k} \quad \text{for all} \quad r \leq r_{x,k}.$$

For a compact set $K \subset \Sigma$ we have

$$K \subset \bigcup_{x \in K} B_{R_x}(x)$$

and the compactness provides $x_1, \dots, x_N \in K$ with

$$K \subset \bigcup_{i=1}^{N} B_{R_{x_i}}(x_i).$$

Let $\tilde{k} \in \mathbf{N}$ be defined by $\tilde{k} := \max\{\tilde{k}_{x_1}, \dots, \tilde{k}_{x_N}\}$. For given $\delta > 0$ and $i \in \{1, \dots, N\}$ the convergence of $\tilde{\delta}_{x_i,k}$ to zero guarantees that there is a $j(x_i, \delta) \geq \tilde{k}$ such that $\tilde{\delta}_{x_i, j(x_i,\delta)} \leq \delta$. This implies

$$\sup_{y \in \Sigma \cap B_{R_{x_i}}(x_i)} \frac{1}{r} \operatorname{dist}_{\mathcal{H}} \Big(\Sigma \cap B_r(y), (y + P_y) \cap B_r(y) \Big) \leq \tilde{\delta}_{x_i, j(x,\delta)} \leq \delta \ \text{ for all } \ r \leq r_{x_i, j(x_i,\delta)}.$$

Now define $r_0 = r_0(\delta) := \min\{r_{x_1, j(x_1,\delta)}, \dots, r_{x_N, j(x_N,\delta)}\}$. Then we get

$$\begin{aligned} &\sup_{y \in K} \frac{1}{r} \operatorname{dist}_{\mathcal{H}} \Big(\Sigma \cap B_r(y), (y + P_y) \cap B_r(y) \Big) \\ &\leq \max_{i=1,\dots,N} \sup_{y \in \Sigma \cap B_{R_{x_i}}(x)} \frac{1}{r} \operatorname{dist}_{\mathcal{H}} \Big(\Sigma \cap B_r(y), (y + P_y) \cap B_r(y) \Big) \leq \delta \quad \text{for all} \quad r \leq r_0. \end{aligned}$$

This holds true for every arbitrary $\delta > 0$, implying that Σ is a Reifenberg-flat set with vanishing constant and fixed approximating plane.

Now let $x \in K$ and $L(x, r) \in G(n, m)$ be a plane, depending on x and r, such that

$$\frac{1}{r} \operatorname{dist}_{\mathcal{H}} \Big(\Sigma \cap B_r(x), \big(x + L(x, r)\big) \cap B_r(x) \Big) =: \delta(x, r) \xrightarrow[r \to 0]{} 0.$$

We have to show that $L(x, r)$ converges to a limit plane $L_x \in G(n, m)$ and in fact we will show $L_x = P_x$.

For $x_1 = x_2 = x$, $r_1 = r_2 = r$, $P_1 = L(x, r)$, $P_2 = P_y$, $\delta_1 = \delta(x, r)$ and $\delta_2 = \tilde{\delta}_{x,k(r)}$, where $k(r)$ is defined such that $r_{x,k(r)+1} < r \leq r_{x,k(r)}$, we have $\delta_1, \delta_2 < \frac{1}{2}$ for r small enough, as well as

$$\frac{2}{1 - 2\delta(x, r)} \Big(\delta(x, r) + 2\tilde{\delta}_{x,k(r)} \Big) < \frac{1}{\sqrt{2}},$$

Lemma 3.2 leads to

$$\lim_{r \to 0} \sphericalangle\big(L(x, r), P_y\big) \leq \lim_{r \to 0} C_1(m) \frac{2}{1 - 2\tilde{\delta}_{x,k(r)}} \Big(\delta(r) + 2\tilde{\delta}_{x,k(r)} \Big) = 0.$$

(2) $\implies$ (1): For $x \in \Sigma$ define $R_x := 1$, $C_x > 1$ arbitrary and a sequence $r_{x,i} \subset (0, 1]$ with $r_{x,i+1} \leq r_{x,i} \leq C_x r_{x,i+1}$ and $r_{x,i} \xrightarrow[i \to \infty]{} 0$. The compactness of $(G(n, m), \sphericalangle(\cdot, \cdot))$ implies that for $y \in \Sigma \cap B_{R_x}(x)$ there exists a minimizer of

$$L \mapsto \frac{1}{r_{x,k}} \operatorname{dist}_{\mathcal{H}} \Big(\Sigma \cap B_{r_{x,k}}(y), (y + L) \cap B_{r_{x,k}}(y) \Big).$$

Let $P(y, r_{x,k})$ denote this minimizer. Define

$$\delta_{x,k} := \sup_{y \in \Sigma \cap \overline{B_{R_x}(x)}} \frac{1}{r_{x,k}} \operatorname{dist}_{\mathcal{H}} \Big(\Sigma \cap B_{r_{x,k}}(y), \big(y + P(y, r_{x,k})\big) \cap B_{r_{x,k}}(y) \Big).$$

The Reifenberg-flatness with vanishing constant guarantees $\delta_{x,k} \xrightarrow[k\to\infty]{} 0$. Finally, the assumptions imply that for all $y \in \Sigma \cap B_{R_x}(x)$ there exists a $P_y := L_y \in G(n,m)$ with

$$\sup_{y \in \Sigma \cap B_{R_x}(x)} \sphericalangle\big(P(y, r_{x,k}), P_y\big) =: \varepsilon_{x,k} \xrightarrow[k\to\infty]{} 0. \qquad \square$$

Σ being a C^1-submanifold, is equivalent to Σ locally being a graph of a C^1-function. Therefore it is a necessary condition, that for each $x \in \Sigma$ there exists a plane $P \in G(n,m)$ such that the orthogonal projection $\pi_{x+P|\Sigma}$ is locally bijective onto an open subset of $x + P$. Both, the injectivity and surjectivity will be results of the Reifenberg-flatness of Σ. (RPC) guarantees for Σ to be Reifenberg-flat with vanishing constant, which allows us to use Lemma 3.8, stated for codimension 1 in [2] and ensuring the surjectivity. Although the main argument of [2] does not depend on the dimension, we will present the proof of Lemma 3.8 and 3.7, which is also part of [2], in Appendix C to make sure, that this result still holds for higher codimension.

Lemma 3.7 yields a parameterization for Reifenberg-flat sets, which is often used to achieve more results for Reifenberg-flat sets. Here we will need this parameterization only to prove Lemma 3.8.

Lemma 3.7. *There exists a $\delta_0 > 0$ such that for every closed, m-dimensional δ-Reifenberg-flat set $\Sigma \subset \mathbf{R}^n$ with $\delta \le \delta_0$ and $x \in \Sigma$ there is a $R_0 = R_0(x, \delta, \Sigma) > 0$ such that for all $L \in G(n,m)$ with*

$$\operatorname{dist}_{\mathcal{H}} \Big(\Sigma \cap B_r(x), (x + L) \cap B_r(x) \Big) \le r\delta \ \text{ for } \ r \le R_0$$

exists a continuous function

$$\tau\colon (x + L) \cap \overline{B_{\frac{15}{16}r}(x)} \to \Sigma \cap \overline{B_r(x)}$$

with

$$|\tau(y) - y| \le Cr\delta \le \frac{5}{144} r \quad \text{for all } \ y \in (x + L) \cap \overline{B_r(x)}.$$

The constants δ_0 and R_0 can be set as $\delta_0 < (48(3C_1(m)+2))^{-1}$ and $R_0(x, \delta, \Sigma) > 0$ small enough, such that

$$\frac{1}{r} \inf_{L \in G(n,m)} \operatorname{dist}_{\mathcal{H}} \Big(\Sigma \cap B_r(y), (y + L) \cap B_r(y) \Big) \le \delta \quad \text{for all } \ y \in \Sigma \cap \overline{B_{R_0}(x)}.$$

Such an $R_0(x, \delta, \Sigma)$ exists, because of the Reifenberg-flatness.

Lemma 3.8. *For all closed, δ-Reifenberg-flat sets $\Sigma \subset \mathbf{R}^n$ with $\delta \le \delta_0$, all $x \in \Sigma$ and $L \in G(n,m)$ with*

$$\frac{1}{r} \operatorname{dist}_{\mathcal{H}} \Big(\Sigma \cap B_r(x), (x + L) \cap B_r(x) \Big) \le \delta \quad \text{for } \ r \le R_0,$$

we get

$$(x + L) \cap B_{\frac{r}{4}}(x) \subset \pi_{x+L} \left(\Sigma \cap B_{\frac{r}{2}}(x) \right),$$

where δ_0 and R_0 are as stated in Lemma 3.7.

We are now ready to prove Theorem 1.4 in two steps. First we will see that if Σ satisfies (RPC), it is locally a graph of a C^1 function, i.e. it is an embedded C^1-submanifold. Finally we prove that every embedded C^1-submanifold satisfies the (RPC) condition.

Lemma 3.9. *Assume $\Sigma \subset \mathbf{R}^n$ is closed and satisfies (RPC) with dimension m, then for all $x \in \Sigma$ there exist a radius r_x and a function $u_x \in C^1(P_x, P_x^\perp)$ with*

$$(\Sigma \cap B_{r_x}(x)) - x = \operatorname{graph}(u_x) \cap B_{r_x}(0),$$

i.e. Σ is an embedded, m-dimensional C^1-submanifold of $\mathbf{R}^n$.

Note that the radius r_x can be given explicitly by $\frac{1}{3} r_{x,k}$ for $k \in \mathbf{N}_{>1}$ such that $\tilde{\delta}_{x,k-1} < \min\{(48(3C_1(m)+2))^{-1}, (6C_2(m)+2C_x)^{-1}\}$. Therefore, the radius for the neighbourhood, where Σ can be represented as a C^1-graph depends only on the dimension of Σ and the ratio of decay between the sequences $\delta_{x,i}, \varepsilon_{x,i}$ and $r_{x,i}$.

Proof. Let x be fixed and $k \in \mathbf{N}$ be sufficiently large, such that

$$\tilde{\delta}_{x,k-1} < \min\left\{\delta_0, (6C_2(m)+2C_x)^{-1}\right\}.$$

Note that $\tilde{\delta}_{x,k-1} < \min\{\delta_0, (6C_2(m)+2C_x)^{-1}\}$ already implies $\delta_{x,i} \le \tilde{\delta}_{x,k-1} < C_x^{-1}$ for all $i \ge k$, i.e. $k \ge \tilde{k}_x$. The δ_0 stated in the remark after Lemma 3.7 already guarantees $\delta_0 < \frac{1}{11}$. Moreover, we have for all $r \in (0, r_{x,k}]$

$$\frac{1}{r} \operatorname{dist}_{\mathcal{H}}\Big(\Sigma \cap B_r(y), (y+P_y) \cap B_r(y)\Big) \le \tilde{\delta}_{x,k-1} < \delta_0$$

for all $y \in \Sigma \cap \overline{B_{r_{x,k}}}(x) \subset \Sigma \cap B_{r_{x,k-1}}(x)$. This implies $r_{x,k} \le R_0(x, \tilde{\delta}_{x,k-1}, \Sigma)$. Therefore we have

$$k \ge \tilde{k}_x, \quad r_{x,k} < R_0(x, \tilde{\delta}_{x,k-1}, \Sigma) \quad \text{and} \quad \tilde{\delta}_{x,k-1} < \min\left\{\frac{1}{11}, \delta_0, (6C_2(m)+2C_x)^{-1}\right\}.$$

Lemma 3.8 implies

$$(x+P_x) \cap B_{\frac{r}{2}}(x) \subset \pi_{x+P_x}(\Sigma \cap B_r(x)) \quad \text{for all} \quad r \le \frac{r_{x,k}}{2}.$$

Because of $\tilde{\delta}_{x,k} < \frac{1}{11}$, Lemma 3.5 yields for $r \le \frac{r_{x,k}}{2}$

$$\sphericalangle(P_x, P_y) \le C_2(m)\tilde{\delta}_{x,k} \ \text{ for all } \ y \in B_r(x).$$

For $y \ne y' \in \Sigma \cap B_r(x)$, there exist an $i \ge k$ with $r_{x,i+1} \le |y'-y| < r_{x,i}$ and therefore $y' \in \Sigma \cap B_{r_{x,k}}(x) \cap B_{r_{x,i}}(y)$. This implies

$$\begin{aligned}|\pi_{P_x}^\perp(y-y')| &\le \sphericalangle(P_x,P_y)|y-y'| + |\pi_{P_y}^\perp(y-y')| \le C_2(m)\tilde{\delta}_{x,k}|y-y'| + \tilde{\delta}_{x,i} r_{x,i} \\ &\le \Big(C_2(m)\tilde{\delta}_{x,k} + C_x\tilde{\delta}_{x,i}\Big)|y-y'| < \frac{1}{2}|y-y'|.\end{aligned}$$

Here we have used $\tilde{\delta}_{x,i} \le \tilde{\delta}_{x,k} < (6C_2(m)+2C_x)^{-1} \le (2C_2(m)+2C_x)^{-1}$. Then for $\Sigma_1 := \Sigma \cap B_r(x) \cap \pi_{x+P_x}^{-1}(B_{\frac{r}{2}}(x))$, the projection $\pi_{P_x|\Sigma_1}$ is injective and

$$\pi_{x+P_x|\Sigma_1} \colon \Sigma_1 \to (x+P_x) \cap B_{\frac{r}{2}}(x)$$

is bijective. We move x to zero and let $\tilde{\Sigma}_1 := (\Sigma - x) \cap B_r(0) \cap \pi_{P_x|\Sigma-x}^{-1}(B_{\frac{r}{2}}(0))$, then the projection

$$\pi_{P_x|\tilde{\Sigma}_1} \colon \tilde{\Sigma}_1 \to P_x \cap B_{\frac{r}{2}}(0)$$

is also a bijection and invertible. Especially, for all $y \in \Sigma_1$, there exists exactly one $z = z(y) \in P_x \cap B_{\frac{r}{2}}(0)$ with

$$\pi_{P_x}(y-x) = z.$$

Moreover, we have

$$y = x + \pi_{P_x}(y-x) + \pi_{P_x}^{\perp}(y-x) = x + z + \pi_{P_x}^{\perp}(y-x).$$

Defining

$$f\colon P_x \cap B_{\frac{r}{2}}(0) \to P_x^{\perp}; \quad z \mapsto \pi_{P_x}^{\perp} \circ \left(\pi_{P_x|\tilde{\Sigma}_1}\right)^{-1}_{|P_x\cap B_{\frac{r}{2}}(0)}(z),$$

then we get

$$\pi_{P_x}^{\perp}(y-x) = f(z) \quad \text{and} \quad f(0) = 0,$$

because $z(x) = 0$.

For $z, z' \in P_x \cap B_{\frac{r}{2}}(0)$ define

$$\left(\pi_{P_x|\tilde{\Sigma}_1}\right)^{-1}(z) =: y \quad \text{and} \quad \left(\pi_{P_x|\tilde{\Sigma}_1}\right)^{-1}(z') =: y'.$$

Now we have

$$\begin{aligned}\left|\left(\pi_{P_x|\tilde{\Sigma}_1}\right)^{-1}(z) - \left(\pi_{P_x|\tilde{\Sigma}_1}\right)^{-1}(z')\right| &= |y-y'| \le |\pi_{P_x}(y-y')| + |\pi_{P_x}^{\perp}(y-y')| \\ &\le |z-z'| + \frac{1}{2}|y-y'|.\end{aligned}$$

This leads to

$$|y-y'| \le 2|z-z'|,$$

which implies the continuity of $(\pi_{P_x|\tilde{\Sigma}_1})^{-1}$ and therefore also of f.

For $z \in P_x \cap B_{\frac{r}{2}}(0)$ the definition of f and Lemma 3.1 lead to

$$|f(z)| = |\pi_{P_x}^{\perp}(y(z)-x)| = \operatorname{dist}(y(z), x+P_x) \le w_x(|y(z)-x|) \cdot |y(z)-x|,$$

where $y(z)$ denotes the unique element of Σ_1 with $\pi_{P_x}(y(z)-x) = z$. We further get

$$\begin{aligned}|y(z)-x| &= |x+z+f(z)-x| = |z+f(z)| \le |z| + |f(z)| \\ &\le |z| + w_x(|y(z)-x|) \cdot |y(z)-x|.\end{aligned}$$

Note that $w_x(|y(z)-x|) \le \tilde{\delta}_{x,k} < \frac{1}{11}$ and therefore

$$|y(z)-x| \le \frac{11}{10}|z|.$$

Finally, this leads to

$$|f(z)| \le \frac{11}{10} w_x(|y(z)-x|) \cdot |z| = o(|z|),$$

because $y(z) \xrightarrow[z\to 0]{} x$ and $w_x(r) \xrightarrow[r\to 0]{} 0$. This yields the existence of $Df(0)$ and $Df(0) = 0$.

Let $z \in P_x \cap B_{\frac{r}{2}}(0)$ and F be defined as $F(z) = x + z + f(z)$, as well as

$$L := \left(\pi_{P_x|P_{F(z)}}\right)^{-1} : P_x \to P_{F(z)}.$$

Note that $F(z) \in B_r(x)$ and

$$\sphericalangle\left(P_x, P_{F(z)}\right) < C_2(m)\tilde{\delta}_{x,k} < \frac{1}{6} < 1,$$

then Lemma 2.5 implies, that L is well-defined. For $z, z+h \in P_x \cap B_{\frac{r}{2}}(0)$, we get

$$F(z+h) - F(z) = L(h) + F(z+h) - F(z) - L(h).$$

Using $e := F(z+h) - F(z) - L(h)$ leads to

$$\begin{aligned}\pi_{P_x}(e) &= \pi_{P_x}(x + z + h + f(z+h) - x - z - f(z) - L(h)) \\ &= \pi_{P_x}(h + f(z+h) - f(z) - L(h)) \\ &= h - \pi_{P_x}(f(z+h)) - \pi_{P_x}(f(z)) - \pi_{P_x}(L(h)) = h - h = 0,\end{aligned}$$

since $f(\cdot) \in P_x^\perp$ and $\pi_{P_x} \circ L = id_{P_x}$. This implies

$$|e| = |\pi_{P_x}^\perp(e)| \le \sphericalangle\left(P_x, P_{F(z)}\right)|e| + |\pi_{P_{F(z)}}^\perp(e)| \le C_2(m)\tilde{\delta}_{x,k}|e| + |\pi_{P_{F(z)}}^\perp(e)|.$$

Transforming this inequality and using $C_2(m)\tilde{\delta}_{x,k} < \frac{1}{6}$ yield

$$\begin{aligned}|e| &< \frac{6}{5}|\pi_{P_{F(z)}}^\perp(e)| = \frac{6}{5}|\pi_{P_{F(z)}}^\perp(F(z+h) - F(z) - L(h))| \\ &= \frac{6}{5}|\pi_{P_{F(z)}}^\perp(F(z+h) - F(z))| = \frac{6}{5}\operatorname{dist}(F(z+h), F(z) + P_{F(z)}) \\ &\le \frac{6}{5}w_x(|F(z+h) - F(z)|) \cdot |F(z+h) - F(z)|.\end{aligned}$$

For the last inequality we used Lemma 3.1 and the fact that $F(z), F(z+h) \in B_{r_{x,k}}(x)$, as well as $F(z+h) \in B_{r_{x,k}}(F(z))$ for all $h \in P_x$ such that $z + h \in P_x \cap B_r(0)$.

To estimate $|F(z+h) - F(z)|$ note

$$|L(h) - h| = |\pi_{P_{F(z)}}(L(h)) - \pi_{P_x}(L(h))| \le \sphericalangle\left(P_{F(z)}, P_x\right)|L(h)| < \frac{1}{6}|L(h)|.$$

Therefore we get

$$\frac{5}{6}|L(h)| < |h| < \frac{7}{6}|L(h)|.$$

Using these estimates yields

$$\begin{aligned}|F(z+h) - F(z)| &= |L(h) + e| \le |L(h)| + |e| \\ &\le \frac{6}{5}|h| + \frac{6}{5}w_x(|F(z+h) - F(z)|) \cdot |F(z+h) - F(z)|.\end{aligned}$$

The fact that $F(z+h) \in B_{r_{x,k}}(F(z))$ for $z + h \in P_x \cap B_{\frac{r}{2}}(0)$ leads to

$$w_x(|F(z+h) - F(z)|) \le \tilde{\delta}_{x,k} < \frac{1}{11}.$$

This implies

$$|F(z+h) - F(z)| < \frac{66}{49}|h|.$$

Finally we get with the continuity of F

$$\begin{aligned}|F(z+h) - F(z) - L(h)| = |e| &\le \frac{6}{5}w_x(|F(z+h) - F(z)|) \cdot |F(z+h) - f(z)| \\ &\le 2w_x(|F(z+h) - F(z)|) \cdot |h| = o(|h|).\end{aligned}$$

This is the differentiability of F with $DF(z) = (\pi_{P_x|P_{F(z)}})^{-1}$ and, equivalent to this, the differentiability of f with $Df(z) = DF(z) - \mathrm{id}$.

To see that $z \mapsto Df(z)$ is continuous, let $a \in P_x \cap \mathbf{S}^{m-1}$ and $w, z \in P_x \cap B_r(0)$, then

$$\begin{aligned}&|(Df(z) - Df(w))a| = |(DF(z) - DF(w))a| = |\pi_{P_{F(z)}}(DF(z)a) - \pi_{P_{F(w)}}(DF(w)a)|\\&\leq |\pi_{P_{F(z)}}(DF(z)a) - \pi_{P_{F(w)}}(DF(z)a)| + |\pi_{P_{F(w)}}(DF(z)a - DF(w)a)|\\&\leq \sphericalangle\left(P_{F(z)}, P_{F(w)}\right)|DF(z)a| + |\pi_{P_{F(w)}}(DF(z)a - DF(w)a)|.\end{aligned}$$

First we get

$$\sphericalangle\left(P_{F(z)}, P_{F(w)}\right)|DF(z)a| \leq 2C_2(m)\tilde{\delta}_{x,k}|Df(z)a + a|$$

and since $Df(\cdot)a \in P_x^\perp$

$$\begin{aligned}|\pi_{P_{F(w)}}(DF(z)a - DF(w)a)| &= |\pi_{P_{F(w)}}(Df(z)a - Df(w)a)|\\&= |(\pi_{P_{F(w)}} - \pi_{P_x})(Df(z)a - Df(w)a)|\\&\leq C_2(m)\tilde{\delta}_{x,k}|Df(z)a - Df(w)a|.\end{aligned}$$

In the case $w = 0$ we get $Df(0) = 0$ which leads to

$$\begin{aligned}|Df(z)a| &\leq 2C_2(m)\tilde{\delta}_{x,k}|Df(z)a + a| + C_2(m)\tilde{\delta}_{x,k}|Df(z)a|\\&\leq 3C_2(m)\tilde{\delta}_{x,k}|Df(z)a| + 2C_2(m)\tilde{\delta}_{x,k}.\end{aligned}$$

Using $3C_2(m)\tilde{\delta}_{x,k} < \frac{1}{2}$ yields

$$|Df(z)a| < 1 \quad \text{and} \quad |DF(z)a| < 2.$$

Let $\varepsilon > 0$ be arbitrary. There exists an $i \in \mathbf{N}$ such that $\tilde{\delta}_{x,i} < \frac{5}{12C_2(m)}\varepsilon$. Using the continuity of F yields the existence of an $r' > 0$, such that for $w \in P_x \cap B_r(0)$ with $|z - w| < r'$, we get

$$|F(z) - F(w)| \leq \frac{1}{2}r_{x,i}, \quad \text{for} \;\; i \in \mathbf{N}_{\geq k}.$$

This allows to improve the estimate of the angle, using Lemma 3.5 yields

$$\sphericalangle\left(P_{F(z)}, P_{F(w)}\right) \leq C_2(m)\delta_{x,i}.$$

Then the previous estimates imply

$$\begin{aligned}|Df(z)a - Df(w)a| &\leq C_2(m)\tilde{\delta}_{x,i}|DF(z)a| + C_2(m)\tilde{\delta}_{x,k}|Df(z)a - Df(w)a|\\&< 2C_2(m)\tilde{\delta}_{x,i} + \frac{1}{6}|Df(z)a - Df(w)a|.\end{aligned}$$

Finally this gives

$$|Df(z)a - Df(w)a| < \frac{12}{5}C_2(m)\tilde{\delta}_{x,i} < \varepsilon.$$

Since we can choose $\varepsilon > 0$ arbitrary, this is the continuity of $z \mapsto Df(z)$.

To finish the proof let $\varphi \in C_0^\infty(P_x \cap B_{\frac{r}{2}}(0))$ be a cut-off function with $0 \leq \varphi \leq 1$ and $\varphi_{|P_x \cap B_{\frac{r}{3}}(0)} \equiv 1$. Define

$$\tilde{f}\colon P_x \to P_x^\perp\colon z \mapsto \begin{cases}\varphi(z)f(z) & \text{for} \;\; z \in P_x \cap B_{\frac{r}{2}}(0),\\ 0 & \text{otherwise.}\end{cases}$$

Then for all $z \in P_x \cap B_{\frac{r}{3}}$ we have $\tilde{f}(z) = f(z)$. Moreover, for $y \in \Sigma \cap B_{\frac{r}{3}}(x)$ we have

$$|\pi_{x+P_x}(y) - x| = |x + \pi_{P_x}(y - x) - x| < \frac{r}{3} < \frac{r}{2},$$

which implies

$$\Sigma \cap B_{\frac{r}{3}}(x) = x + \big(\mathrm{graph}(f) \cap B_{\frac{r}{3}}(0)\big) = x + \Big(\mathrm{graph}(\tilde{f}) \cap B_{\frac{r}{3}}(0)\Big). \qquad \square$$

To prove that every C^1-submanifold satisfies (RPC) we will first state, that every graph of a function with bounded Lipschitz-constant can be locally approximated by planes, with respect to the Hausdorff-distance, i.e. it is Reifenberg-flat. The quality of this approximation is given by the Lipschitz-constant.

Lemma 3.10. *Let $\Sigma \subset \mathbf{R}^n$. Assume for $x \in \Sigma$ exist a plane $P \in G(n,m)$, a radius $R > 0$ and a function $u_x \colon P \to P^\perp$ with $u_x(0) = 0$, $\mathrm{Lip}(u_{x|B_R(x)}) \le \alpha$, such that*

$$(\Sigma \cap B_R(x)) - x = \mathrm{graph}(u_x) \cap B_R(0),$$

then for all $y \in \Sigma \cap B_{\frac{R}{2}}(x)$ we have

$$\mathrm{dist}_{\mathcal{H}}\Big(\Sigma \cap B_r(y), (y+P) \cap B_r(y)\Big) \le r\alpha \quad \text{for all} \;\; r \in (0, R/2].$$

Proof. For all $y \in \Sigma \cap B_r(x)$ and $z(y) = \pi_P(y-x)$ we have

$$y = x + \pi_P(y-x) + \pi_P^\perp(y-x) = x + z(y) + u_x(z(y)).$$

Let $r \in (0, \frac{R}{2}]$ be fixed. For $y \in \Sigma \cap B_{\frac{R}{2}}(x)$ and $\tilde{y} \in \Sigma \cap B_r(y)$ we get with $\pi_P(\tilde{y} - y) + y \in (y+P) \cap B_r(y)$

$$\begin{aligned}\mathrm{dist}\Big(\tilde{y}, (y+P) \cap B_r(y)\Big) &\le \left|\pi_P^\perp(\tilde{y}-y)\right| = \left|\pi_P^\perp(\tilde{y}-x) - \pi_P^\perp(y-x)\right| \\ &= |u_x(z(\tilde{y})) - u_x(z(y))| \le \alpha r.\end{aligned}$$

Note that

$$y + P = x + z(y) + u_x(z(y)) + P = x + u_x(z(y)) + P.$$

Using $P \cap (B_r(y) - y) \subset P \cap B_R(0)$ we can write $\Sigma \cap B_r(y) = x + \mathrm{graph}(u_x) \cap B_r(y)$. For $x + \tilde{z} + u_x(z(y)) \in (y+P) \cap B_{\frac{r}{\sqrt{1+\alpha^2}}}(y)$, i.e. $\tilde{z} \in P \cap B_{\frac{r}{\sqrt{1+\alpha^2}}}(z(y))$ we have

$$\begin{aligned}|x + \tilde{z} + u_x(\tilde{z}) - y| &= |\tilde{z} + u_x(\tilde{z}) + z(y) + u_x(z(y))| \\ &= \sqrt{|\tilde{z} - z(y)|^2 + |u_x(\tilde{z}) - u_x(z(y))|^2} \\ &\le \sqrt{1+\alpha^2} \cdot |\tilde{z} - z(y)| < r.\end{aligned}$$

This implies

$$\begin{aligned}\mathrm{dist}\Big(x + \tilde{z} + u_x\big(z(y)\big), \Sigma \cap B_r(y)\Big) &\le |x + \tilde{z} + u_x(z(y)) - x - \tilde{z} - u_x(\tilde{z})| \\ &= |u_x(z(y)) - u_x(\tilde{z})| \le \frac{\alpha r}{\sqrt{1+\alpha^2}}.\end{aligned}$$

For $z' \in P \cap (B_r(z(y)) \setminus B_{\frac{r}{\sqrt{1+\alpha^2}}}(z(y))$ there exists a $\hat{z} \in P \cap B_{\frac{r}{\sqrt{1+\alpha^2}}}(z(y))$ with

$$|z' - \hat{z}| < \left(1 - \frac{1}{\sqrt{1+\alpha^2}}\right) r.$$

This leads to

$$\mathrm{dist}\Big(x + z' + u_x(z(y)), \Sigma \cap B_r(y)\Big) \le \sqrt{\left(1 - \frac{1}{\sqrt{1+\alpha^2}}\right)^2 + \left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)^2}\; r \le \alpha r.$$

Finally this guarantees

$$\operatorname{dist}_{\mathcal{H}}\Big(\Sigma \cap B_r(y), (y+P) \cap B_r(y)\Big) \leq \alpha r. \qquad \square$$

Lemma 3.11. *An embedded C^1-submanifold Σ of $\mathbf{R}^n$ satisfies (RPC). Moreover, we get $P_x = T_x\Sigma$.*

Proof. For all $x \in \Sigma$ and $\alpha > 0$ there is a radius $\tilde{R}_x(\alpha) > 0$ such that $(\Sigma \cap B_{\tilde{R}_x(\alpha)}(x)) - x$ is the graph of a C^1-function $u_x \colon T_x\Sigma \to T_x\Sigma^\perp$ with $u_x(0) = 0$ and $Du_x(0) = 0$ as well as $\|Du_x\|_{C^0(B_{\tilde{R}_x(\alpha)}(0))} \leq \alpha$. Especially $\operatorname{Lip}(u_{x|B_{\tilde{R}_x(\alpha)}}) \leq \alpha$.

Define $R_x := r_{x,1} := \frac{1}{2}\tilde{R}_x(\alpha)$. For $y \in \Sigma \cap B_{R_x}(x)$ let the plane $P(y, r_{x,1})$ be defined by

$$P(y, r_{x,1}) := T_x\Sigma.$$

Lemma 3.10 implies for all $y \in \Sigma \cap B_{R_x}(x)$

$$\operatorname{dist}_{\mathcal{H}}\Big(\Sigma \cap B_r(y), \big(y + P(y, r_{x,1})\big) \cap B_r(y)\Big) \leq \alpha r \quad \text{for all} \quad r \leq r_{x,1}.$$

Now define

$$\delta'_{x,i} := \frac{\delta_{x,1}}{2^{i-1}} := \frac{\alpha}{2^{i-1}}.$$

For all $i \in \mathbf{N}_{>0}$ we have

$$\Sigma \cap \overline{B_{R_x}(x)} \subset \bigcup_{y \in \Sigma \cap \overline{B_{R_x}(x)}} B_{\frac{\tilde{R}_y(\delta'_{x,i})}{2}}(y).$$

Then there exists an $N \in \mathbf{N}$ and $y_1, \dots, y_N \in \Sigma \cap \overline{B_{R_x}(x)}$ with

$$\Sigma \cap \overline{B_{R_x}(x)} \subset \bigcup_{j=1}^{N} B_{\frac{\tilde{R}_{y_j}(\delta'_{x,i})}{2}}(y_j).$$

Define $r'_{x,1} := r_{x,1}$ and recursively

$$r'_{x,i} := \min\left\{ \min_{j \in \{1,\dots,N(i)\}} \left\{ \frac{\tilde{R}_{y_j}(\delta'_{x,i})}{2} \right\}, \frac{r'_{x,i-1}}{2} \right\},$$

as well as $P(y, r'_{x,i}) := T_{y_j}\Sigma$ for an arbitrary $j \in \{1, \dots, N(i)\}$ with $y \in B_{\frac{\tilde{R}_{y_j}(\delta'_{x,i})}{2}}(y_j)$. Using Lemma 3.10 for $R = \tilde{R}_{y_j}(\delta'_{x,i})$, we get for all $y \in B_{r'_{x,i}}(y_j)$

$$\operatorname{dist}_{\mathcal{H}}\Big(\Sigma \cap B_r(y), \big(y + P(y, r'_{x,i})\big) \cap B_r(y)\Big) \leq \delta'_{x,i} r \quad \text{for all} \quad r \leq r'_{x,i}.$$

The $B_{\tilde{R}_{y_j}(\delta'_{x,i})}(y_j)$ cover $\Sigma \cap B_{R_x}(x)$ and therefore we have

$$\operatorname{dist}_{\mathcal{H}}\Big(\Sigma \cap B_r(y), \big(y + P(y, r'_{x,i})\big) \cap B_r(y)\Big) \leq \delta'_{x,i} r \ \text{ for all } \ r \leq r'_{x,i} \ \text{ and } \ y \in \Sigma \cap B_{R_x}(x).$$

This holds for all $i \in \mathbf{N}$. Moreover, for all $\delta > 0$ there exists an $i \in \mathbf{N}$ with $\delta'_{x,i} < \delta$, which implies that Σ is Reifenberg-flat with vanishing constant. Note that it is important, that the $r'_{x,i}$ are independent of $y \in \Sigma \cap B_{R_x}(x)$.

It remains to show that we can define a sequence of radii $r_{x,i}$ which is controlled by a constant C_x, as well as the convergence of the planes $P(y, r_{x,i})$ to $P_y = T_y\Sigma$. To see this, note that Lemma 2.7 implies

$$\sphericalangle\big(T_y\Sigma, P(y, r'_{x,i})\big) = \sphericalangle\big(T_y\Sigma, T_{y_j}\Sigma\big) \leq \delta'_{x,i} \quad \text{for all} \quad y \in \Sigma \cap B_{R_x}(x).$$

This yields

$$\sup_{y\in B_{R_x}(x)} \sphericalangle\left(T_y\Sigma, P(y, r'_{x,i})\right) \le \delta'_{x,i} \xrightarrow[i\to\infty]{} 0.$$

Now let $C_x > 1$ be fixed. For all $i \in \mathbf{N}$, there exists an $l = l(i) \in \mathbf{N}_0$ with

$$C_x^l r'_{x,i+1} < r'_{x,i} \le C_x^{l+1} r'_{x,i+1}.$$

If $r_{x,s} = r'_{x,i}$ and $\delta_{x,s} = \delta'_{x,i}$ are defined, set recursively

$$r_{x,s+k} := \frac{1}{C_x^k} r_{x,s} \quad \text{for} \quad k \in \{1,\dots,l(i)\}, \quad r_{x,s+l+1} := r'_{x,i+1},$$
$$P(y, r_{x,s+k}) := P(y, r_{x,s}) = P(y, r'_{x,i}) \quad \text{for} \quad k \in \{1,\dots,l(i)\}$$

and

$$\delta_{x,s+k} := \delta_{x,i} \quad \text{for} \quad k \in \{1,\dots,l(i)\}, \quad \delta_{x,s+l(i)+1} := \delta'_{x,i+1}.$$

These definitions lead to

$$\sup_{y\in B_{R_x}(x)} \operatorname{dist}_{\mathcal{H}}\Big(\Sigma \cap B_{r_{x,s}}(y), \big(y + P(y, r_{x,s})\big) \cap B_{r_{x,s}}(y)\Big) \le \delta_{x,s} r_{x,s} \quad \text{for all} \quad s \in \mathbf{N}$$

with $\lim_{s\to\infty} \delta_{x,s} = 0$ and

$$\sup_{y\in B_{R_x}(x)} \sphericalangle\left(T_y\Sigma, P(y, r_{x,s})\right) \le \varepsilon_{x,i} := \delta_{x,s}.$$

Moreover, if $s \in \mathbf{N}$ such that $r_{x,s} = r'_{x,i}$, then the definition of $r_{x,s}$ leads to

$$\frac{r_{x,s+k}}{r_{x,s+k+1}} = C_x \quad \text{for} \quad k \in \{0,\dots,\max\{0, l(i)-1\}\},$$
$$\frac{r_{x,j+l(i)}}{r_{x,j+l(i)+1}} = \frac{r'_{x,i} \cdot \frac{1}{C_x^{l(i)}}}{r'_{x,i+1}} \le \frac{C_x^{l(i)+1}}{C_x^{l(i)}} = C_x.$$

Finally these are all conditions required for Σ to satisfy (RPC). □

4. Proof of Theorem 1.2

Unlikely Toro's condition in (2), the integral condition postulated in Theorem 1.2 does not need a small bound but only to be finite. Note that the important part of this condition is the decay of $\theta_{B_{R_x}(x)}$ near zero, i.e. if for $x \in \Sigma$ there exists an $R_x > 0$ with

$$\int_0^{R_x} \frac{\theta_{B_{R_x}(x)}(r)}{r}\, dr < \infty,$$

then for all r, R with $0 < r \le R_x \le R < \infty$ we get

$$\int_0^r \frac{\theta_{B_{R_x}(x)}(r)}{r}\, dr \le \int_0^R \frac{\theta_{B_{R_x}(x)}(r)}{r},\, dr = \int_0^{R_x} \frac{\theta_{B_{R_x}(x)}(r)}{r}\, dr + \int_{R_x}^R \frac{\theta_{B_{R_x}(x)}(r)}{r}\, dr$$
$$\le \int_0^{R_x} \frac{\theta_{B_{R_x}(x)}(r)}{r}\, dr + \int_{R_x}^R \frac{1}{r}\, dr < \infty.$$

On the other hand, we can not expect R_x to contain any information about the size of the graph patches for Σ.

We will prove Theorem 1.2 by showing that each Σ, which has an finite integral already satisfies (RPC).

Proof of Theorem 1.2. Let $C > 1$ be arbitrary. For every $k \in \mathbf{N}$ there exist an $r_{x,k} \in (R_x/C^{\frac{k+1}{2}}, R_x/C^{\frac{k}{2}})$ with

$$\frac{\theta_{B_{R_x}(x)}(r_{x,k})}{r_{x,k}} \leq \int_{R_x/C^{\frac{k+1}{2}}}^{R_x/C^{\frac{k}{2}}} \frac{\theta_{B_{R_x}(x)}(r)}{r}\, dr \cdot \frac{1}{R_x\left(C^{-\frac{k}{2}} - C^{-\frac{k+1}{2}}\right)},$$

otherwise we would get

$$\begin{aligned}\int_{R_x/C^{\frac{k+1}{2}}}^{R_x/C^{\frac{k}{2}}} \frac{\theta_{B_{R_x}(x)}(r)}{r}\, dr &> \int_{R_x/C^{\frac{k+1}{2}}}^{R_x/C^{\frac{k}{2}}} \frac{1}{R_x\left(C^{-\frac{k}{2}} - C^{-\frac{k+1}{2}}\right)} \int_{R_x/C^{\frac{k+1}{2}}}^{R_x/C^{\frac{k}{2}}} \frac{\theta_{B_{R_x}(x)}(r')}{r'}\, dr'\, dr \\ &= \int_{R_x/C^{\frac{k+1}{2}}}^{R_x/C^{\frac{k}{2}}} \frac{\theta_{B_{R_x}(x)}(r')}{r'}\, dr',\end{aligned}$$

which is a contradiction. Therefore, we have

$$r_{x,k+1} < r_{x,k} \leq C r_{x,k+1} \quad \text{and} \quad \lim_{k\to\infty} r_{x,k} = 0.$$

Moreover,

$$\begin{aligned}\theta_{B_{R_x}(x)}(r_{x,k}) &\leq \frac{r_{x,k}}{R_x\left(C^{-\frac{k}{2}} - C^{-\frac{k+1}{2}}\right)} \cdot \int_{R_x/C^{\frac{k+1}{2}}}^{R_x/C^{\frac{k}{2}}} \frac{\theta_{B_{R_x}(x)}(r)}{r}\, dr \\ &\leq \frac{R_x C^{-\frac{k}{2}}}{R_x C^{-\frac{k}{2}}\left(1 - C^{-\frac{1}{2}}\right)} \cdot \int_{R_x/C^{\frac{k+1}{2}}}^{R_x/C^{\frac{k}{2}}} \frac{\theta_{B_{R_x}(x)}(r)}{r}\, dr \\ &= \frac{C^{\frac{1}{2}}}{C^{\frac{1}{2}} - 1} \cdot \int_{R_x/C^{\frac{k+1}{2}}}^{R_x/C^{\frac{k}{2}}} \frac{\theta_{B_{R_x}(x)}(r)}{r}\, dr.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{k=0}^{\infty} \theta_{B_{R_x}(x)}(r_{x,k}) &\leq \frac{C^{\frac{1}{2}}}{C^{\frac{1}{2}} - 1} \cdot \sum_{k=0}^{\infty} \int_{R_x/C^{\frac{k+1}{2}}}^{R_x/C^{\frac{k}{2}}} \frac{\theta_{B_{R_x}(x)}(r)}{r}\, dr \\ &\leq \frac{C^{\frac{1}{2}}}{C^{\frac{1}{2}} - 1} \int_0^{R_x} \frac{\theta_{B_{R_x}(x)}(r)}{r}\, dr < \infty.\end{aligned}$$

For $\delta_{x,k} := \theta_{B_{R_x}(x)}(r_{x,k})$, this implies

$$\delta_{x,k} \xrightarrow[k\to\infty]{} 0.$$

Then we get for all sufficiently large $k \in \mathbf{N}$

$$\frac{2}{1 - 2\delta_{x,k+1}}(\delta_{x,k+1} + 2C\delta_{x,k}) < \tilde{C}(\delta_{x,k+1} + 2C\delta_{x,k}) < \frac{1}{\sqrt{2}}.$$

Let $P(y, r_{x,k})$ denote a plane which approximates Σ at $y \in \Sigma \cap B_{R_x}(x)$ and scale $r_{x,k}$, corresponding to $\delta_{x,k}$. Then Lemma 3.2 leads to

$$\sphericalangle\big(P(y, r_{x,k}), P(y, r_{x,k+1})\big) \leq \tilde{C} C_1(m)(\delta_{x,k+1} + 2C\delta_{x,k}).$$

For $i \in \mathbf{N}$ we get

$$\begin{aligned}\sphericalangle\big(P(y,r_{x,k}),P(y,r_{x,k+i})\big) &\le \sum_{l=0}^{i-1}\sphericalangle\big(P(y,r_{x,k+l}),P(y,r_{x,k+l+1})\big)\\ &\le \tilde{C}C_1(m)\sum_{l=0}^{i-1}(\delta_{x,k+l+1}+2C\delta_{x,k+l}) \xrightarrow[k\to\infty]{} 0,\end{aligned}$$

since $\sum_{k=1}^{\infty}\delta_{x,k}<\infty$. This yields the existence of a plane $P_y \in G(n,m)$ such that

$$\sphericalangle\big(P(y,r_{x,k}),P_y\big) \xrightarrow[k\to\infty]{} 0.$$

In particular, for all $\varepsilon>0$ there exist a $J_y \in \mathbf{N}$ such that

$$\sphericalangle\big(P(y,r_{x,k}),P_y\big) < \varepsilon \text{ for all } k \ge J_y.$$

For $i \in \mathbf{N}$ and $k > \max\{i, J_y\}$ we get

$$\begin{aligned}\sphericalangle\big(P(y,r_{x,i}),P_y\big) &\le \sphericalangle\big(P(y,r_{x,i}),P(y,r_{x,k})\big)+\sphericalangle\big(P(y,r_{x,k}),P_y\big)\\ &\le \sum_{l=0}^{k-i-1}\sphericalangle\big(P(y,r_{x,i+l}),P(y,r_{x,i+l+1})\big)+\varepsilon\\ &\le \sum_{l=0}^{\infty}\sphericalangle\big(P(y,r_{x,i+l}),P(y,r_{x,i+l+1})\big)+\varepsilon.\end{aligned}$$

The limit $\varepsilon \to 0$ yields

$$\sphericalangle\big(P_y,P(y,r_{x,i})\big) \le \sum_{l=0}^{\infty}\sphericalangle\big(P(y,r_{x,i+l}),P(y,r_{x,i+l+1})\big) \le \tilde{C}C(m)\sum_{l=i}^{\infty}(\delta_{x,l+1}+2C\delta_{x,l}),$$

if $i \ge N$ and $N \in \mathbf{N}$ such that

$$\frac{2}{1-2\delta_{x,k+1}}(\delta_{x,k+1}+2C\delta_{x,k}) < \tilde{C}(\delta_{x,k+1}+2C\delta_{x,k}) < \frac{1}{\sqrt{2}} \text{ for all } k \ge N.$$

Then

$$\varepsilon_{x,k} := \begin{cases}\tilde{C}C(m)\sum_{l=k}^{\infty}(\delta_{x,l+1}+2C\delta_{x,l}) & \text{for } k \ge N,\\ 1 & \text{otherwise,}\end{cases}$$

is independent of $y \in B_{R_x}(x)$ with

$$\sphericalangle\big(P_y,P(y,r_{x,k})\big) \le \varepsilon_{x,k} \xrightarrow[k\to\infty]{} 0.$$

This is the condition of (RPC) for $C = C_x$ and Lemma 3.9 finishes the proof. □

Remark 4.1. An immediat result of the proof is that if there exist a constant $C>0$ and a monotonically decreasing sequence $(r_{x,k})_k \subset (0,R_x]$ with

$$r_{x,k} \le C r_{x,k+1} \quad \text{and} \quad \lim_{k\to\infty} r_{x,k} = 0$$

such that

$$\sum_{k=1}^{\infty}\theta_{B_{R_x}(x)}(r_{x,k}) < \infty,$$

then Σ is an embedded, m-dimensional C^1-submanifold of $\mathbf{R}^n$. Moreover, the finiteness of the integral in Theorem 1.2 implies this condition.

Appendix A. A Reifenberg-flat set with vanishing constant without C^1-regularity

Let

$$u\colon \mathbf{R} \to \mathbf{R}, \quad u(z) := \sum_{k=1}^{\infty} \frac{\cos(2^k z)}{2^k \sqrt{k}}$$

and

$$U\colon \mathbf{R} \to \mathbf{R}^2, \quad U(z) := \begin{pmatrix} z \\ u(z) \end{pmatrix}.$$

Then $\Sigma := \operatorname{graph}(u) = U(\mathbf{R})$ is Reifenberg-flat with vanishing constant as stated in [14].

Assume Σ is a C^1-submanifold of $\mathbf{R}^2$. Then for all $x \in \Sigma$ and all $\alpha > 0$ there exists a radius $r = r(x, \alpha) > 0$ and a C^1-function $f_x\colon T_x\Sigma \to T_x\Sigma^\perp$ such that

$$\Sigma \cap B_r(x) = (x + \operatorname{graph}(f_x)) \cap B_r(x)$$

and

$$\|f_x'\|_{C^0(T_x\Sigma \cap B_r(0), T_x\Sigma^\perp)} \le \alpha.$$

Due to the symmetry of u, i.e. $u(z) = u(-z)$ for all $z \in \mathbf{R}$, we have for $x_0 = U(0)$

$$T_{x_0}\Sigma \neq \{0\} \times \mathbf{R}.$$

This implies that there exists an $r' > 0$ with

$$(\mathbf{R} \times \{0\}) \cap B_{r'}(0) \subset \pi_{\mathbf{R}\times\{0\}}\left(T_{x_0}\Sigma \cap B_r(0)\right).$$

Without loss of generality let r' be small enough such that $U(z) \in B_r(x_0)$ for all $z \in B_{r'}(0)$.

The representation as a graph of f_{x_0} yields the injectivity of

$$g\colon (\mathbf{R} \times \{0\}) \cap \overline{B_{\frac{r'}{2}}(0)} \to \mathbf{R} \times \{0\}, \quad t \mapsto \pi_{\mathbf{R}\times\{0\}}\left(\pi_{T_{x_0}\Sigma}\left(U(t) - U(0)\right)\right).$$

Together with the continuity of g this implies that g is monotonic. Then for $-\frac{r'}{2} = t_0 < t_1 < \cdots < t_k = \frac{r'}{2}$ and $t_i' := \pi_{T_{x_0}\Sigma}(U(t_i) - U(0))$ for $i = 0, \ldots, k$ we get either

$$\pi_{\mathbf{R}\times\{0\}}\left(t_0'\right) < \pi_{\mathbf{R}\times\{0\}}\left(t_1'\right) < \cdots < \pi_{\mathbf{R}\times\{0\}}\left(t_k'\right),$$

or

$$\pi_{\mathbf{R}\times\{0\}}\left(t_0'\right) > \pi_{\mathbf{R}\times\{0\}}\left(t_1'\right) > \cdots > \pi_{\mathbf{R}\times\{0\}}\left(t_k'\right).$$

Therefore we have $\sum_{i=1}^k |t_i' - t_{i-1}'| = |t_k' - t_0'|$ and

$$\begin{aligned}\sum_{i=1}^k |U(t_i) - U(t_{i-1})| &= \sum_{i=1}^k \left| \begin{pmatrix} t_i' \\ f_{x_0}(t_i') \end{pmatrix} - \begin{pmatrix} t_{i-1}' \\ f_{x_0}(t_{i-1}') \end{pmatrix} \right| \le \sum_{i=1}^k \sqrt{1+\alpha^2} \cdot |t_i' - t_{i-1}'| \\ &= \sqrt{1+\alpha^2} \cdot \left| \pi_{T_{x_0}\Sigma}\left(U\left(-\frac{r'}{2}\right)\right) - \pi_{T_{x_0}\Sigma}\left(U\left(\frac{r'}{2}\right)\right) \right|\end{aligned}$$

which is independent of the partition of the intervall $[-r'/2, r'/2]$. This implies $U \in BV([-r'/2, r'/2], \mathbf{R}^2)$ and $u \in BV([-r'/2, r'/2])$. Then u has to be differentiable for almost all $z \in [-r'/2, r'/2]$ which is a contradiction to u being not differentiable for all $z \in \mathbf{R}$.

Appendix B. Counterexample for integral condition

The finiteness of the integral as well as of the sum in Theorem 1.2 respectively Remark 4.1 imply that Σ is a C^1-submanifold, but the following example will show, that these conditions are not equivalent. Moreover, one can ask if C^1-submanifolds are characterized by

$$\int_0^1 \frac{\theta^\beta_{B_{R_x}(x)}(r)}{r^\alpha}\,dr < \infty \quad \text{for all} \quad x \in \Sigma$$

for any $\alpha, \beta > 0$. Note that as in Theorem 1.2 the upper bound of the integral can be replaced by any $R > 0$ and the case $\alpha = \beta = 1$ leads to the situation of Theorem 1.2. Using $\theta_{B_{R_x}}(r) \leq 1$ for all $x \in \Sigma$ and $r > 0$ leads

$$\int_0^1 \frac{\theta^\beta_{B_{R_x}(x)}(r)}{r^\alpha}\,dr \leq \int_0^1 \frac{1}{r^\alpha}\,dr < \infty \quad \text{for all} \quad 0 < \alpha < 1,$$

which does not depend on Σ. Therefore, if such a condition exists, α has to be greater or equal to one.

Moreover, the finiteness of the integral with $\alpha > 1$ and $\beta < 1$ implies the finiteness for $\alpha, \beta = 1$. For $\alpha = 1$ and fixed $\beta \geq 1$, the following example will provide a set $\Sigma \subset \mathbf{R}^2$, which is a one-dimensional C^1-submanifold, but yields neither a finite integral nor a finite sum of its θ-numbers.

Example B.1. Let $\beta \geq 1$ and

$$f_\beta \colon \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbf{R}, \quad y \mapsto \begin{cases} \left(-\frac{2}{\log(y^2)}\right)^{\frac{1}{\beta}} & \text{for} \;\; y \in \mathbf{R}\setminus\{0\}, \\ 0 & \text{for} \;\; y = 0, \end{cases}$$

and

$$g_\beta \colon \mathbf{R} \to \mathbf{R}, \quad x \mapsto \begin{cases} \int_{-\frac{1}{2}}^0 f_\beta(y)\,dy - \frac{x+\frac{1}{2}}{\log(2)^{\frac{1}{\beta}}} & \text{for} \;\; y \in (-\infty, -\frac{1}{2}), \\ \int_x^0 f_\beta(y)\,dy & \text{for} \;\; y \in [-\frac{1}{2}, 0), \\ \int_0^x f_\beta(y)\,dy & \text{for} \;\; y \in [0, \frac{1}{2}], \\ \int_0^{\frac{1}{2}} f\beta(y)\;dy + \frac{x-\frac{1}{2}}{\log(2)^{\frac{1}{\beta}}} & \text{for} \;\; y \in (\frac{1}{2}, \infty). \end{cases}$$

Then f_β is a continuous function and g_β is C^1, but $g \notin C^{1,\sigma}$ for every $\sigma > 0$. The set $\Sigma := \operatorname{graph}(g_\beta)$ is a C^1-submanifold of $\mathbf{R}^n$. For all $r \leq 2e^{-1} < 1$ we get

$$\left|\log\left(\frac{r^2}{4}\right)\right| \geq 2.$$

Therefore,

$$\left|g_\beta\left(\frac{r}{2}\right)\right| = \int_0^{\frac{r}{2}} \left(\frac{2}{|\log(y^2)|}\right)^{\frac{1}{\beta}} dy \leq \frac{r}{2} \cdot \left(\frac{2}{|\log(\frac{r^2}{4})|}\right)^{\frac{1}{\beta}} \leq \frac{r}{2}$$

and hence $\binom{\frac{r}{2}}{g_\beta(\frac{r}{2})} \in \Sigma \cap B_r(0)$ for all $r \leq 2e^{-1}$. Due to the symmetry of g_β, the planes, which realise $\theta(0, r)$ have to be equal to $T_0\Sigma = \mathbf{R} \times \{0\}$. For all small r we get

$$\theta(0,r) \geq \frac{g_\beta(\frac{r}{2})}{r} = \frac{1}{r}\int_0^{\frac{r}{2}} \left(-\frac{2}{\log(y^2)}\right)^{\frac{1}{\beta}} dy \geq \frac{1}{r}\int_{\frac{r}{4}}^{\frac{r}{2}} \left(-\frac{2}{\log(y^2)}\right)^{\frac{1}{\beta}} dy$$
$$\geq \frac{1}{r}\cdot\frac{r}{4}\cdot\left(-\frac{1}{\log(\frac{r}{4})}\right)^{\frac{1}{\beta}} = \frac{1}{4}\cdot\left(-\frac{1}{\log(\frac{r}{4})}\right)^{\frac{1}{\beta}}.$$

For all $R > 0$ and monotonically decreasing sequences $(r_i)_{i\in\mathbf{N}} \subset (0, \max\{R, 2e^{-1}\}]$ and $C > 1$ with

$$r_i \leq C r_{i+1} \quad \text{for all} \quad i \in \mathbf{N}$$

and therefore

$$r_1 \leq C^{i-1} r_i,$$

we get

$$\theta^\beta_{B_R(0)}(r_i) \geq \frac{1}{4^\beta}\cdot\frac{-1}{\log(\frac{r_i}{4})} \geq \frac{1}{4^\beta}\cdot\frac{-1}{\log(\frac{r_1}{4C^{i-1}})} = \frac{1}{4^\beta}\cdot\frac{-1}{\log(\frac{r_1}{4}) - \log(C^{i-1}))}.$$

Finally

$$\sum_{i=1}^\infty \theta^\beta_{B_R(0)}(r_i) \geq \frac{1}{4^\beta}\sum_{i=1}^\infty \frac{1}{-\log(\frac{r_1}{4}) + \log(C^{i-1})}$$
$$\geq \frac{1}{4^\beta}\sum_{i=1}^\infty \frac{1}{-\log(\frac{r_1}{4}) + (i-1)\log(C)} = \infty.$$

Using the same argument of remark 4.1, this implies that also

$$\int_0^R \frac{\theta^\beta_{B_R(0)}(r)}{r}\, dr = \infty \quad \text{for} \quad R > 0.$$

Appendix C. Proof of Lemma 3.7 and Lemma 3.8

Proof of Lemma 3.7. (1) Notation: Define

$$S_0 := (x + L) \cap \overline{B_r(x)}, \quad \Sigma_x := \Sigma \cap \overline{B_r(x)},$$
$$\tau_0\colon S_0 \to S_0; \quad z \mapsto z, \delta_0 < (48(3C_1(m) + 2))^{-1}$$

and $R_0 > 0$ small enough, that for all $r \in (0, R_0]$ we get

$$\frac{1}{r}\inf_{L\in G(n,m)} \operatorname{dist}_{\mathcal{H}}\Big(\Sigma \cap B_r(y), (y + L) \cap B_r(y)\Big) \leq \delta \quad \text{for all} \quad y \in \Sigma \cap \overline{B_{R_0}(x)}.$$

For $j \in \mathbf{N}_0$ let

$$r_j := \frac{r}{12\cdot 4^j}.$$

For all $j > 0$ we get

$$\Sigma_x \subset \bigcup_{z\in\Sigma_x} B_{r_j}(z).$$

The compactness of Σ_x implies the existence of a $k_j \in \mathbf{N}$ and a set $Z_j := \{z_{j,1}, \dots, z_{j,k_j}\}$ with

$$\Sigma_x \subset \bigcup_{z\in Z_j} B_{r_j}(z).$$

Moreover, there exists a partition of unity $\{\varphi_z\}_{z\in Z_j}$ with

$$\begin{aligned} 0 \le \varphi_z(y) \le 1 &\text{ for all } y \in \mathbf{R}^n \text{ and } z \in Z_j, \\ \varphi_z(y) = 0 &\text{ for all } y \in \mathbf{R}^n \text{ and } z \in \mathbf{Z}_j \text{ with } |y-z| \ge 3r_j, \\ \sum_{z\in Z_j} \varphi_z(y) = 1 &\text{ for all } y \in V_j := \{y \in \mathbf{R}^n \mid \operatorname{dist}(y,\Sigma_x) < r_j\}. \end{aligned}$$

Note that $V_j \subset \bigcup_{z\in Z_j} B_{3r_j}(z)$. Then the existence of this partition is an immediate result of e.g. [3, p. 52].

For $z \in Z_j$ let $L(z,12r_j) \in G(n,m)$ denote a plane with

$$\operatorname{dist}_{\mathcal{H}}\Big(\Sigma \cap B_{12r_j}(z), \big(z + L(z,12r_j)\big) \cap B_{12r_j}(z)\Big) \le 12r_j\delta.$$

The δ-Reifenberg-flatness of Σ and the fact that

$$12r_j \le r \le R_0$$

guarantees the existence of $L(z,12r_j)$. Now define

$$\sigma_j(y) := y - \sum_{z\in Z_j} \varphi_z(y)\cdot \pi^{\perp}_{L(z,12r_j)}(y-z)$$

and

$$\tau_j(y) := (\sigma_j \circ t_{j-1})(y).$$

(2) For $y \in V_j \cap \overline{B_{r-2r_j(1+6\delta)}(x)}$ we get

$$\operatorname{dist}(\sigma_j(y),\Sigma_x) \le (36C_1(m)+24)r_j\delta$$

and

$$|\sigma_j(y) - y| \le \operatorname{dist}(y,\Sigma_x) + (36C_1(m)+24)r_j\delta \le (1 + 36C_1(m)\delta + 24\delta)r_j.$$

Note that

$$r - 2r_j(1+6\delta) \ge r - \frac{1}{6}r\left(1+\frac{1}{16}\right) > 0 \quad \text{for all } j \in \mathbf{N}_0.$$

Let $y \in V_j \cap \overline{B_{r-2r_j(1+6\delta)}(x)}$ and $Z_j(y) := \{z \in Z_j \mid |z-y| < 3r_j\}$. Then we get

$$\sigma_j(y) = y - \sum_{z\in Z_j(y)} \varphi_z(y)\cdot \pi^{\perp}_{L(z,12r_j)}(y-z).$$

For $z, z' \in Z_j(y)$, we have $|z-z'| < 6r_j = \frac{12r_j}{2}$. The definition of δ_0 further yields

$$\frac{6}{1-2\delta}\delta < 12\delta < \frac{1}{\sqrt{2}}.$$

Lemma 3.2 implies for $x_1 = z, x_2 = z'$, $\delta_1 = \delta_2 = \delta$, $r_1 = r_2 = 12r_j$ and $P_1 = L(z,12r_j), P_2 = L(z',12r_j)$ that

$$\sphericalangle\big(L(z,12r_j), L(z',12r_j)\big) \le 12C_1(m)\delta.$$

For fixed $z_0 \in Z_j(y)$ such that $|z_0 - y| < 2r_j$ define

$$\tilde{y} := y - \pi^{\perp}_{L(z_0,12r_j)}(y-z_0)$$

and we get

$$
\begin{aligned}
|\sigma_j(y)-\tilde{y}| &= \left| \sum_{z\in Z_j(y)} \Big(\varphi_z(y)\cdot \pi^{\perp}_{L(z,12r_j)}(y-z)\Big) - \pi^{\perp}_{L(z_0,12r_j)}(y-z_0)\right| \\
&= \left| \sum_{z\in Z_j(y)} \varphi_z(y)\cdot \Big(\pi^{\perp}_{L(z,12r_j)}(y-z) - \pi^{\perp}_{L(z_0,12r_j)}(y-z_0)\Big)\right| \\
&= \left| \sum_{z\in Z_j(y)} \varphi_z(y)\cdot \Big(\pi^{\perp}_{L(z,12r_j)}(y-z) - \pi^{\perp}_{L(z_0,12r_j)}(y-z) - \pi^{\perp}_{L(z_0,12r_j)}(z-z_0)\Big)\right| \\
&\le \sum_{z\in Z_j(y)} \varphi_z(y)\cdot \Big(\left|\pi^{\perp}_{L(z,12r_j)}(y-z) - \pi^{\perp}_{L(z_0,12r_j)}(y-z)\right| + \left|\pi^{\perp}_{L(z_0,12r_j)}(z-z_0)\right|\Big) \\
&\le \sum_{z\in Z_j(y)} \varphi_z(y)\cdot \Big(12C_1(m)\delta\cdot 3r_j + \operatorname{dist}\big(z, z_0+L(z_0,12r_j)\big)\Big) \\
&\le (36C_1(m)+12)\, r_j\delta.
\end{aligned}
$$

In the last inequalities we used $z\in\Sigma\cap B_{12r_j}(z_0)$ and therefore $\operatorname{dist}(z,z_0+L(z_0,12r_j))\le 12r_j\delta$, as well as the fact that $\sum_{z\in Z_j(y)}\varphi_z(y)=1$ for $y\in V_j$ several times. $\tilde{y}\in L(z_0,12r_j)\cap B_{12r_j}(z_0)$ implies that there exists a $w\in\Sigma\cap B_{12r_j}(z_0)\subset\Sigma_x$ with

$$|\tilde{y}-w|\le 12r_j\delta.$$

Using $|\tilde{y}-x|\le|y-x|+|y-z_0|$, we get

$$|w-x|\le|w-\tilde{y}|+|\tilde{y}-x|<12r_j\delta+r-2r_j(1+6\delta)+2r_j=r.$$

This implies $w\in\Sigma_x$ and

$$\operatorname{dist}(\sigma_j(y),\Sigma_x)\le|\sigma_j(y)-\tilde{y}|+|\tilde{y}-w|\le(36C_1(m)+24)\,r_j\delta.$$

Due to the definition of V_j and the fact that Σ_x is closed,for all $y\in V_j$ we get a $w'\in\Sigma_x$ with

$$\operatorname{dist}(y,\Sigma_x)=|y-w'|<r_j.$$

This yields

$$|z_0-w'|<3r_j$$

and therefore

$$
\begin{aligned}
|\tilde{y}-y| = \left|\pi^{\perp}_{L(z_0,12r_j)}(y-z_0)\right| &\le \left|\pi^{\perp}_{L(z_0,12r_j)}(y-w')\right| + \left|\pi^{\perp}_{L(z_0,12r_j)}(w'-z_0)\right| \\
&\le |y-w'|+12r_j\delta.
\end{aligned}
$$

Finally we get

$$|\sigma_j(y)-y|\le\operatorname{dist}(y,\Sigma_x)+(36C_1(m)+24)\,r_j\delta.$$

(3) For $y\in S_0\cap\overline{B_{r'}(x)}$ with $r':=r-(2+36C_1(m)\delta+24\delta)\sum_{k=1}^{\infty}r_k$ we get

$$\tau_j(y)\in V_{j+1}\cap\overline{B}_{r-(2+36C_1(m)\delta+24\delta)\sum_{k=j+1}^{\infty}r_k}(x)\quad\text{for all}\quad j\in\mathbf{N}_0.$$

Note that

$$r'=r-(2+36C_1(m)\delta+24\delta)\sum_{k=1}^{\infty}r_k>r-\frac{r}{12}\cdot\frac{1}{3}(2+\frac{1}{4})=\frac{15}{16}r$$

and

$$r' \le r - 2r_j(1+6\delta).$$

For $j = 0$ and $y \in S_0 \cap \overline{B_{r'}(x)}$ we have $\tau_0(y) = y$ and the Reifenberg-flatness yields

$$\operatorname{dist}(y, \Sigma_x) \le r\delta < \frac{r}{48} = r_1.$$

This implies $\tau_0(y) = y \in V_1 \cap \overline{B_{r'}(x)}$.

Now we assume that the statement holds for $j-1 \in \mathbf{N}_0$ and let $y \in S_0 \cap \overline{B_{r'}(x)}$. We have

$$\begin{aligned}\tau_{j-1}(y) &\in V_j \cap \overline{B}_{r-(2+36C_1(m)\delta+24\delta)\sum\limits_{k=j}^{\infty} r_k}(x)\\ &\subset V_j \cap \overline{B}_{r-r_j(2+36C_1(m)\delta+24\delta)}(x) \subset V_j \cap \overline{B}_{r-2r_j(1+6\delta)}(x).\end{aligned}$$

Therefore step (2) implies

$$\operatorname{dist}(\tau_j(y), \Sigma_x) = \operatorname{dist}(\sigma_j(\tau_{j-1}(y)), \Sigma_x) \le (36C_1(m)+24)r_j\delta < r_{j+1},$$

which is $\tau_j(y) \in V_{j+1}$. Moreover, step (2) leads to

$$\begin{aligned}|\tau_j(y) - x| &\le |\sigma_j(\tau_{j-1}(y)) - \tau_{j-1}(y)| + |\tau_{j-1}(y) - x|\\ &\le (1+36C_1(m)\delta+24\delta)r_j + r - (2+36C_1(m)\delta+24\delta)\sum_{k=j}^{\infty} r_k\\ &\le r - (2+36C_1(m)\delta+24\delta)\sum_{k=j+1}^{\infty} r_k.\end{aligned}$$

This is the postulated statement for j and inductively it holds for all $j \in \mathbf{N}_0$.

(4) τ_i converges on $S_0 \cap \overline{B_{r'}(x)}$ uniformly to a continuous function τ: For $y \in S_0 \cap \overline{B_{r'}(x)}$ and $i \in \mathbf{N}$ we get

$$\begin{aligned}|\tau_i(y) - \tau_{i-1}(y)| &= |\sigma_i(\tau_{i-1}(y)) - \tau_{i-1}(y)|\\ &\le \operatorname{dist}(\tau_{i-1}(y), \Sigma_x) + (36C_1(m)+24)r_i\delta.\end{aligned}$$

If $i = 1$, then

$$\operatorname{dist}(\tau_0(y), \Sigma_x) \le r\delta < (36C_1(m)+24)r_0\delta$$

and for $i > 1$ we get

$$\operatorname{dist}(\tau_{j-1}(y), \Sigma_x) = \operatorname{dist}(\sigma_{i-1}(\tau_{i-2}(y)), \Sigma_x) \le (36C_1(m)+24)r_{i-1}\delta,$$

because of $\tau_{i-2}(y) \in V_{i-1}$. Using $r_i = \frac{1}{4}r_{i-1}$ yields

$$|\tau_i(y) - \tau_{i-1}(y)| \le \frac{5}{4}\,(36C_1(m)+24)\,r_{i-1}\delta \quad \text{for all} \quad i \in \mathbf{N}.$$

Let $k, j \in \mathbf{N}_0$ then

$$\begin{aligned}|\tau_{j+k}(y) - \tau_j(y)| &\le \sum_{i=1}^{k} |\tau_{j+i}(y) - \tau_{j+i-1}(y)| \le \frac{5}{4}\,(36C_1(m)+24)\,\delta \sum_{i=1}^{k} r_{j+i-1}\\ &= \frac{5}{4}\,(36C_1(m)+24)\,\delta r_j \sum_{i=0}^{k-1} 4^{-i} \xrightarrow[j\to\infty]{} 0.\end{aligned}$$

This is independent of $y \in S_0 \cap \overline{B_{r'}(x)}$ and implies the uniform convergence of τ_i to a function τ. All τ_i are continuous as compositions of continuous functions and therefore τ is as well.

(5) $|\tau(y) - y| < Cr\delta$ and $\tau(S_0 \cap \overline{B_{r'}(x)}) \subset \Sigma_x$: We have $\tau(y) = \lim_{j\to\infty} \tau_j(y)$ for all $y \in S_0 \cap \overline{B_{r'}(x)}$. Therefore, for all $\varepsilon > 0$ there exists a $J = J(\varepsilon) \in \mathbf{N}$ with

$$|\tau(y) - \tau_j(y)| < \varepsilon \quad \text{for all} \quad j \geq J \quad \text{and} \quad y \in S_0 \cap \overline{B_{r'}(x)}.$$

For $k \in \mathbf{N}_0$ there is a $j > \max\{k, J\}$ with

$$|\tau(y) - \tau_k(y)| < \varepsilon + \sum_{i=k}^{j-1} |\tau_{i+1}(y) - \tau_i(y)| \leq \varepsilon + \sum_{i=k}^{\infty} |\tau_{i+1}(y) - \tau_i(y)|.$$

The limit $\varepsilon \to 0$ yields

$$\begin{aligned}|\tau(y) - \tau_k(y)| &\leq \sum_{i=k}^{\infty} |\tau_{i+1}(y) - \tau_i(y)| \leq \frac{5}{4}\,(36C_1(m) + 24)\,\delta r_k \cdot \sum_{i=0}^{\infty} 4^{-i} \\ &= \frac{5}{3}\,(36C_1(m) + 24)\,\delta r_k.\end{aligned}$$

Especially for $k = 0$ we get

$$|\tau(y) - y| \leq \frac{5}{3}\,(36C_1(m) + 24)\,\delta r_0 < \frac{5}{144} r.$$

We have $\tau_j(y) \in V_{j+1}$ for all $j \in \mathbf{N}_0$ and therefore there is a $w_j \in \Sigma_x$ with

$$|\tau_j(y) - w_j| < r_{j+1} \ \text{for all} \ \ j \in N_0.$$

This leads to

$$\begin{aligned}\operatorname{dist}(\tau(y), \Sigma_x) &\leq |\tau(y) - \tau_j(y)| + |\tau_j(y) - w_j| \\ &\leq \frac{5}{3}\,(36C_1(m) + 24)\,\delta r_j + r_{j+1} \xrightarrow[j\to\infty]{} 0,\end{aligned}$$

which implies $\tau(S_0 \cap \overline{B_{r'}(x)}) \subset \Sigma_x$ and finishes the proof. □

Proof of Lemma 3.8. Assume there exists a $\xi \in (x + L) \cap B_{\frac{r}{4}}(x)$ such that $\pi_{x+L}(y) \neq \xi$ for all $y \in \Sigma \cap B_{\frac{r}{2}}(x)$. Using Lemma 3.7 leads to a continuous function $\tau\colon (x + L) \cap \overline{B_{\frac{15}{16}r}(x)} \to \Sigma \cap \overline{B_r(x)}$ with

$$|\tau(y) - y| \leq \frac{5}{144} r.$$

Then for all $z \in (x + L) \cap \overline{B_{\frac{r}{3}}(x)}$ we get

$$|\tau(z) - x| \leq |\tau(z) - z| + |z - x| \leq \frac{5}{144} r + \frac{1}{3} r < \frac{1}{2} r.$$

Therefore,

$$\pi_{x+L}\,(\tau(z)) \neq \xi \quad \text{for all} \ \ z \in (x + L) \cap \overline{B_{\frac{r}{3}}(x)}.$$

Let $h\colon (x + L) \setminus \{\xi\} \to (x + L) \cap \partial B_{\frac{r}{12}}(\xi)$ be defined by

$$h(z) := \xi + \frac{r}{12} \cdot \frac{z - \xi}{|z - \xi|}.$$

h is a continuous projection of $(x + L) \setminus \{\xi\}$ onto $(x + L) \cap \partial B_{\frac{r}{12}}(\xi)$. Define

$$\varphi := h \circ \pi_{x+L} \circ \tau\colon (x + L) \cap B_{\frac{r}{12}}(\xi) \to (x + L) \cap \partial B_{\frac{r}{12}}(\xi).$$

Note that $B_{\frac{r}{12}}(\xi) \subset B_{\frac{r}{3}}(x)$, then we have $\xi \notin \pi_{x+L} \circ \tau((x + L) \cap B_{\frac{r}{12}}(\xi)$ and φ is continuous and well-defined.

For $z \in (x+L) \cap \partial B_{\frac{r}{12}}(\xi)$ we get

$$|\pi_{x+L}(\tau(z)) - z| = |\pi_{x+L}(\tau(z) - z)| \le |\tau(z) - z| \le \frac{5}{144} r.$$

Moreover,

$$\begin{aligned} |h(\pi_{x+L}(\tau(z))) - \pi_{x+L}(\tau(z))| &= \operatorname{dist}\left(\pi_{x+L}(\tau(z)), \partial B_{\frac{r}{12}}(\xi)\right) \\ &\le |\pi_{x+L}(\tau(z)) - z| \le \frac{5}{144} r, \end{aligned}$$

which implies

$$|\varphi(z) - z| \le \frac{10}{144} r \quad \text{for all} \quad z \in (x+L) \cap \partial B_{\frac{r}{12}}(\xi).$$

Define $\tilde{\varphi}\colon L \cap \overline{B_1(0)} \to L \cap \partial B_1(0)$ by

$$\tilde{\varphi}(z) := \frac{12}{r}\left(\varphi\left(\frac{r}{12} z + \xi\right) - \xi\right).$$

The continuity of φ implies that $\tilde{\varphi}$ is also continuous and for $z \in L \cap \overline{B_1(0)}$ we get $\tilde{z} := \frac{r}{12} z + \xi \in (x+L) \cap \partial B_{\frac{r}{12}}(\xi)$, which leads to

$$|\tilde{\varphi}(z) - z| = \frac{12}{r} |\varphi(\tilde{z}) - \tilde{z}| \le \frac{12}{r} \cdot \frac{10}{144} \cdot r = \frac{10}{12} < 1 \quad \text{for all} \quad z \in L \cap \partial B_1(0).$$

But this implies that

$$H\colon L \cap \partial B_1(0) \times [0,1] \cong \mathbf{S}^{m-1} \times [0,1] \to L \cap \partial B_1(0) \cong \mathbf{S}^{m-1},$$
$$H(z,t) := \frac{(1-t)\tilde{\varphi}_{|\mathbf{S}^{m-1}}(z) + tz}{|(1-t)\tilde{\varphi}_{|\mathbf{S}^{m-1}}(z) + tz|}$$

is a homotopy between $id_{\mathbf{S}^{m-1}}$ and $\tilde{\varphi}_{|\mathbf{S}^{m-1}}$. The homotopy equivalence of the degree of a map (see [4, 5.1.6 a]) leads to

$$\deg(\tilde{\varphi}_{|\mathbf{S}^{m-1}}) = \deg(id_{\mathbf{S}^{m-1}}) = 1.$$

This is a contradiction to the continuous extension $\tilde{\varphi}$ of $\tilde{\varphi}_{|\mathbf{S}^{m-1}}$ on $\overline{B_1^m(0)}$, because this would by [4, 5.1.6 b] imply

$$\deg(\tilde{\varphi}_{|\mathbf{S}^{m-1}}) = 0.$$

Therefore, the assumed ξ can not exist. □

Acknowledgement. Part of this work is contained in the author's Ph.D. thesis [7]. The author's work is partially funded by his advisor's DFG Grant no. Mo 966/7-1 Geometric curvature functionals: energy landscape and discrete methods.

References

[1] Allard, W.: On the first variation of a varifold. - Ann. of Math. (2) 95:3, 1972, 417–491.

[2] David, G., K. Kenig, and T. Toro: Asymptotically optimally doubling measures and Reifenberg flat sets with vanishing constant. - Comm. Pure Appl. Math. 54:4, 2001, 385–449.

[3] Guillemin, V., and A. Pollack: Differential topology. - Mathematics Series, Prentice-Hall, 1974.

[4] Hirsch, M. W.: Differential topology. - Grad. Texts in Math., Springer, 3rd edition, 1976.

[5] Hong, G., and L. Wang: A new proof of Reifenberg's topological disc theorem. - Pacific J. Math. 246:2, 2010, 325–332.

[6] Jones, P. W.: The traveling salesman problem and harmonic analysis. - Publ. Mat. 35:1, 1991, 259–267.

[7] Käfer, B.: Geometric curvature functionals in the scale-invariant case, and characterization of Lipschitz- and C^1-submanifolds. - PhD thesis, RWTH Aachen University (in progress).

[8] Kolasinski, S., P. Strzelecki, and H. von der Mosel: Characterizing $W^{2,p}$ submanifolds by p-integrability of global curvatures. - Geom. Funct. Anal. 23:3, 2013, 937–984.

[9] Kolasinski, S., P. Strzelecki, and H. von der Mosel: Compactness and isotopy finiteness for submanifolds with uniformly bounded geometric curvature energies. - arXiv:1504.04538 [math.DG], 2015.

[10] Morrey, C. B.: Multiple integrals in the calculus of variations. - Springer, 1966.

[11] Ranjbar-Motlagh, A.: Almost flat subsets of Euclidean spaces and Lipschitz graphs. - Preprint, 1998.

[12] Reifenberg, E. R.: Solution of the Plateau problem for m-dimensional surfaces of varying topological type. - Acta Math. 104, 1960, 1–92.

[13] Toro, T.: Geometric conditions and existence of bi-Lipschitz parameterizations. - Duke Math. J. 77:1, 1995, 193–227.

[14] Toro, T.: Doubling and flatness: Geometry of measures. - Notices Amer. Math. Soc. 44:9, 1997, 1087–1094.

Received 6 April 2018 • Accepted 19 November 2018