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THE RICKMAN–PICARD THEOREM

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Dedicated to the memory of Juha Heinonen and Seppo Rickman

Abstract. We give a new and conceptually simple proof of the Rickman–Picard theorem for quasiregular maps based on potential-theoretic methods.

1. Introduction

The classical Picard theorem in complex analysis states that a non-constant holomorphic map $f\colon \mathbf{C} \to \widehat{\mathbf{C}}$ defined on the complex plane $\mathbf{C}$ omits at most two values in the Riemann sphere $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. There are a dozen or so proofs of this theorem using surprisingly diverse and unexpected approaches. In the 1970s, efforts were made to generalize this theorem to quasiregular maps defined on real Euclidean spaces. Although quasiregular maps (defined below) provide a generalization of holomorphic maps, many of the proofs that work in the holomorphic case fail in the higher-dimensional setting. In 1980 Rickman [Ri1] was the first to establish an analog of Picard's theorem for quasiregular maps in higher dimensions. His proof was based on the concept of modulus for path families. Later non-linear potential theory was used to give alternative proofs. To formulate the Rickman–Picard theorem, we first review some basic definitions.

Let M and N be connected and oriented Riemannian n-manifolds, where $n \geq 2$. A non-constant continuous map $f\colon M \to N$ is called K-*quasiregular*, where $K \geq 1$, if f has distributional derivatives that are locally L^n-integrable (with respect to the Riemannian measure on M) and if the formal differential $Df(p)\colon T_pM \to T_{f(p)}N$ satisfies

$$\|Df(p)\|^n \leq K \det(Df(p))$$

for almost every $p \in M$. Here $\|Df(p)\|$ denotes operator norm of $Df(p)$ with respect to the norms on the tangent spaces T_pM and $T_{f(p)}N$ induced by the Riemannian structures on M and N, respectively. We call $f\colon M \to N$ *quasiregular* if it is K-quasiregular for some $K \geq 1$.

We included continuity in the definition of a quasiregular map, because one can show that even without this assumption, a quasiregular map admits a continuous representative in its Sobolev class. It is known that a (continuous) quasiregular map is open and discrete. These regularity results for quasiregular maps go back to work by Reshetnyak (see [Re] for a systematic exposition).

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For $n \in \mathbf{N}$, let $\mathbf{R}^n$ be Euclidean n-space, and $\mathbf{S}^n$ be the unit sphere in $\mathbf{R}^{n+1}$ equipped with the induced Riemannian metric. Then the Rickman–Picard theorem can be formulated as follows (see [Ri3, Chapter 4]).

Theorem 1.1. *Let $f\colon \mathbf{R}^n \to \mathbf{S}^n$ be a K-quasiregular map, where $n \geq 2$ and $K \geq 1$. Then f can omit at most $q_0 = q_0(n, K) < \infty$ points in $\mathbf{S}^n$.*

Here the maximal number $q_0(n, K)$ of omitted points only depends on n and K, and not on the specific map f. In dimension $n = 2$ we actually have $q_0(2, K) = 2$ independently of K. This follows from the classical Picard theorem and the well-known fact that every quasiregular map $f\colon \mathbf{R}^2 \to \mathbf{S}^2$ can be written in the form $f = g \circ \varphi$, where $g\colon \mathbf{R}^2 \cong \mathbf{C} \to \mathbf{S}^2 \cong \widehat{\mathbf{C}}$ is holomorphic and $\varphi\colon \mathbf{R}^2 \to \mathbf{R}^2$ is a quasiconformal homeomorphism. In dimensions $n \geq 3$ we have $q_0(n, K) \to \infty$ as $K \to \infty$, and so Theorem 1.1 is qualitatively best possible. This was shown by Rickman for $n = 3$ [Ri2] and more recently for arbitrary $n \geq 3$ by Drasin and Pankka [DP].

In this paper we give a new and streamlined proof for the Rickman–Picard theorem for quasiregular maps. Our proof is based on ideas from non-linear potential theory. In contrast to earlier proofs, notably by Eremenko–Lewis [EL] and by Lewis [Le], we will not use any deeper results from non-linear potential theory established in the literature such as the Harnack inequality. We will rely on simple integral inequalities that are fairly easy to establish from first principles.

This paper is essentially self-contained except that we take the regularity theory of quasiregular maps and some of their basic topological properties for granted.

Acknowledgments. Our work would not have been possible without Seppo Rickman's deep insights into the geometry of quasiregular maps. Many of the basic ideas in this paper originated in discussions with our late friend Juha Heinonen. He would be a co-author if he were still alive. We dedicate this paper to Juha's and Seppo's memory.

2. Synopsis

In this introductory section we will give an outline of our proof of the Rickman–Picard theorem.

The starting point is, as usual, a (non-constant) K-quasiregular map $f\colon \mathbf{R}^n \to \mathbf{S}^n$, $n \geq 2$, that omits q distinct values $a_1, \ldots, a_q$ in $\mathbf{S}^n$. The goal is to find a bound on q depending only on the dimension n and the distortion $K \geq 1$.

We consider the spherical measure σ_n on $\mathbf{S}^n \cong \widehat{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$ given by the explicit expression

$$d\sigma_n(x) = \frac{2^n}{(1 + |x|^2)^n}\, d\lambda_n(x), \tag{2.1}$$

where λ_n denotes Lebesgue measure on $\mathbf{R}^n$. The growth behavior of f is controlled by the measure A on $\mathbf{R}^n$ obtained by pulling σ_n back by f; so

$$dA(x) = \frac{2^n \det(Df(x))}{(1 + |f(x)|^2)^n}\, d\lambda_n(x). \tag{2.2}$$

Part of our argument implies that if $q \geq 3$, then $A(\mathbf{R}^n) = \infty$. Actually, one can show that if $f\colon \mathbf{R}^n \to \mathbf{S}^n$ is a quasiregular map with $A(\mathbf{R}^n) < \infty$, then f has an extension to ∞ and can therefore omit at most one value, but we will not use this fact.

The basic idea now is to show that there are constants $C_0 > 0$ and $C_1 = C_1(q, n, K) > 0$ such that

$$A(8B) > C_1 A(B), \tag{2.3}$$

whenever $B \subseteq \mathbf{R}^n$ is a (Euclidean) ball with $A(B) > C_0$. Here one has no good quantitative control for C_0, but we have $C_1 = C_1(q, n, K) \to \infty$ as $q \to \infty$.

On the other hand, the following elementary fact is true.

Lemma 2.1. (Rickman's Hunting Lemma)[1] *Let μ be a Borel measure on $\mathbf{R}^n$ without atoms such that $\mu(\mathbf{R}^n) = \infty$ and $\mu(B) < \infty$ for each ball $B \subseteq \mathbf{R}^n$. Then for some constant $D = D(n) > 1$ the following statement is true: for every $C > 0$ there exists a ball $B = B(a, r) \subseteq \mathbf{R}^n$ such that*

$$\mu(B) > C \text{ and } \mu(8B) \leq D\mu(B).$$

See Section 5 for the proof. A version of this statement was formulated in [Ri1, Lemma 5.1]. If one applies Lemma 2.1 to $\mu = A$, then one can derive a contradiction with (2.3) if q is large enough depending on n and K, because then $C_1(q, n, K) > D(n)$.

Inequality (2.3) should be thought of as an "inflation mechanism" for the measure A if q is large: if the mass of a ball B exceeds a certain critical threshold C_0, then the mass "explodes" and increases by a large multiplicative constant C_1 if we pass to the eight times larger ball $8B$. Lemma 2.1 says that such inflationary behavior is impossible for measures on $\mathbf{R}^n$ if the measure satisfies some mild conditions and if $C_1 > D(n)$.

The main part of the proof is now to set up this inflation mechanism. For this one constructs certain auxiliary functions $v_1, \ldots, v_q$ on $\mathbf{S}^n$ so that v_k becomes large only near the omitted value a_k, and one pulls these functions back by f. This idea is standard and the common choice is to use functions of the form

$$v_k(y) = \log^+ \frac{\delta}{|y - a_k|} \tag{2.4}$$

with suitable $\delta > 0$ (where $a_k \neq \infty$). These functions were employed in the proof of the Rickman–Picard theorem by Eremenko and Lewis [EL], for example. The basic function $\log^+ |y|$ is well-known in this context and can be traced back to Nevanlinna's theory of value distribution for analytic functions.

In contrast, the elegant Ahlfors–Shimizu variant of value distribution theory (see [Ne, Section VI.3]) uses the function $\log(1 + |y|^2)$ here instead (this is essentially the Kähler potential of the spherical metric on $\widehat{\mathbf{C}}$). Our main new observation is that by using a higher-dimensional analog of this function, the potential-theoretic approach to the Rickman–Picard theorem becomes substantially simpler on a conceptual level.

Namely, for each dimension $n \geq 2$, Lemma 3.1 below guarantees the existence of a radially symmetric function $v\colon \mathbf{R}^n \to [0, \infty)$ with a logarithmic singularity at ∞ such that $\Delta_n v = \sigma_n$, where Δ_n is the n-Laplacian and the equation has to be interpreted in the distributional sense. So v is an n-subharmonic function with Riesz measure σ_n. We then define $v_k = v \circ R_k$, where R_k is a rotation of $\mathbf{S}^n$ that moves a_k to the point $\infty \in \mathbf{S}^n \cong \widehat{\mathbf{R}}^n$. Then v_k is still an n-subharmonic function with Riesz measure σ_n, which implies that $u_k = v_k \circ f$ is a non-negative $\mathcal{A}$-subharmonic function

[1] In discussions with Juha we jokingly referred to the lemma under this name, because Seppo Rickman used it to establish his Picard theorem and in the proof of the lemma a suitable point is "hunted down". Our proof presented in Section 5 is very similar to an argument that can be found in [Ri3, p. 85].

with Riesz measure A (see Lemma 5.1). The main point here is that each of these functions has the *same* Riesz measure A giving a direct link to the map f.

Now it is well-known that the growth of a non-negative subharmonic function is related to the growth of its Riesz measure (see Lemmas 4.2 and 5.4). Due to the construction of the functions $u_1, \dots, u_q$, the superlevel sets $\{u_k > L_0\}$, $k = 1, \dots, q$, are pairwise disjoint if L_0 is large enough. On the other hand, one can show that if a ball $B \subseteq \mathbf{R}^n$ has sufficiently large A-mass, then B meets all these superlevel sets (Corollary 5.2). So these sets are crowded together near such a ball B; if q is large and so there are many such sets, one of them has to be fairly narrow (a precise quantitative version of this crowding phenomenon is given in Lemma 5.3 based on a notion of n-capacity). On the other hand, if a non-negative subharmonic function is supported on a narrow set, then it has to grow fast which in turn drives up its Riesz measure A (see Lemma 5.4). In this way, we obtain the desired inflation mechanism (2.3), which leads to the proof of Theorem 1.1 as we discussed. For the full details of this argument see Section 5.

In our proof of Theorem 1.1, we will not rely on any auxiliary results from non-linear potential theory, but will give proofs of all relevant facts. An advantage of our approach is that we do not use any Harnack-type inequality for $\mathcal{A}$-harmonic functions, but we will only use simple integral estimates of Caccioppoli-type that are fairly easy to establish.

As we remarked in the introduction, we will take the basic regularity theory of quasiregular maps for granted though; namely, a quasiregular map $f\colon \mathbf{R}^n \to \mathbf{S}^n$ is open and discrete, and its derivative $Df(x)$ exists and is non-singular for almost every $x \in \mathbf{R}^n$.

3. n-harmonic and n-subharmonic functions

In this and the following section we will develop the necessary tools from non-linear potential theory. We will prove all the relevant statements, but we assume that the reader is familiar with some basic facts from the theory of Sobolev spaces (see [Zi] for background).

We use fairly standard notation. We write $X \lesssim Y$ for two quantities X and Y if $X \le CY$ for some constant $C \ge 0$ only depending on some ambient parameters specified in the given setting.

We denote by $B(a, r) = \{x \in \mathbf{R}^n\colon |x - a| < r\}$ the open Euclidean ball of radius $r > 0$ centered at $a \in \mathbf{R}^n$. If $B = B(a, r)$ is such a ball and $\lambda > 0$, then we define $\lambda B = B(a, \lambda r)$. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are points in $\mathbf{R}^n$, then

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n$$

stands for the standard Euclidean scalar product (the "dot product").

If $K \subseteq \mathbf{R}^n$ is a measurable set and $1 \le p < \infty$, then $L^p(K)$ denotes the space of all (equivalence classes of) L^p-integrable functions on K. Unless the measure is explicitly specified, all integrals are taken against Lebesgue measure λ_n. We also leave out the integration domain, if it is identical with $\mathbf{R}^n$. So, for example,

$$\int u = \int_{\mathbf{R}^n} u \, d\lambda_n$$

denotes its Lebesgue integral of a function $u \in L^1(\mathbf{R}^n)$.

If $\Omega \subseteq \mathbf{R}^n$ is some open set, then $C_c(\Omega)$ is the space of all continuous and $C_c^\infty(\Omega)$ the space of all C^∞-smooth (real-valued) functions on Ω with compact support.

The space $W^{1,n}_{\mathrm{loc}}(\Omega)$ is the Sobolev space of all functions f on Ω that are locally L^n-integrable with first-order weak partial derivatives that are also locally L^n-integrable. We also use this notation for $\mathbf{R}^n$-valued maps $f\colon \Omega \to \mathbf{R}^n$ with the understanding that then each component function of f lies in $W^{1,n}_{\mathrm{loc}}(\Omega)$.

Let $\Omega \subseteq \mathbf{R}^n$ be an open set. A function $u\colon \Omega \to \mathbf{R}$ is called *n-harmonic* if its *n-Laplacian* vanishes, i.e., if

$$\Delta_n u := \operatorname{div}(|\nabla u|^{n-2}\nabla u) = 0. \tag{3.1}$$

Here $\nabla u\colon \Omega \to \mathbf{R}^n$ denotes the gradient of u and $\operatorname{div} V$ the divergence of a vector field V. This equation arises as the Euler–Lagrange equation for the minimization of the *n-energy* of u given as

$$\int_\Omega |\nabla u|^n.$$

Recall that by our convention, integration here is against Lebesgue measure λ_n on $\mathbf{R}^n$.

For (3.1) to be meaningful, one has to assume that the function u is sufficiently smooth, say C^2-smooth. To allow more general functions, one can formulate an equivalent definition based on integration against smooth test functions. Accordingly, we say that a function $u \in W^{1,n}_{\mathrm{loc}}(\Omega)$ is *n-harmonic* if

$$\int_\Omega |\nabla u|^{n-2}\nabla u \cdot \nabla\varphi = 0 \tag{3.2}$$

for all $\varphi \in C^\infty_c(\Omega)$. For C^2-smooth functions this definition of an n-harmonic function is equivalent with the one given in (3.1).

We say that $u \in W^{1,n}_{\mathrm{loc}}(\Omega)$ is *n-subharmonic* if there exists a positive Borel measure μ on Ω such that

$$-\int_\Omega |\nabla u|^{n-2}\nabla u \cdot \nabla\varphi = \int_\Omega \varphi\, d\mu \tag{3.3}$$

for all $\varphi \in C^\infty_c(\Omega)$. The measure μ is uniquely determined by u and is called the *Riesz measure* of u. We will write the relation of u and μ in the symbolic (or rather distributional) form

$$\Delta_n u = \mu$$

with the understanding that this is interpreted to mean that (3.3) holds. This property is *local*, i.e., in order to verify it, it is enough to show that for each $p \in \Omega$ the identity (3.3) is true for all functions in $C^\infty_c(\Omega)$ with support in a small neighborhood of p. An n-subharmonic function is n-harmonic precisely if its Riesz measure vanishes identically.

Under conformal maps, n-harmonic or n-subharmonic functions pull-back to functions of the same type. More precisely, let $\Omega, \widetilde{\Omega} \subseteq \mathbf{R}^n$ be open sets, and $f\colon \Omega \to \widetilde{\Omega}$ be a (smooth) conformal map. If $\widetilde{u}\colon \widetilde{\Omega} \to \mathbf{R}$ is n-subharmonic, then $u := \widetilde{u} \circ f\colon \Omega \to \mathbf{R}$ is also n-subharmonic. Moreover, if $\widetilde{\mu}$ and μ are the Riesz measures of $\widetilde{u}$ and u, respectively, then $f_*\mu = \widetilde{\mu}$. Here $f_*\mu$ denotes the push-forward of μ by f. This immediately follows from (3.3) and the transformation formula for integrals. If $\widetilde{u}$ is n-harmonic, then $\widetilde{\mu}$ and μ vanish identically, and so u is also n-harmonic. This can also be deduced from the conformal invariance of the n-energy.

The locality and conformal invariance properties of n-harmonic and n-subharmonic functions make it possible to extend these concepts to functions defined on open sets $\Omega \subseteq \widehat{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$, possibly containing the point $\infty \in \widehat{\mathbf{R}}^n$. One then verifies (3.3) near ∞ after the conformal coordinate change $x \mapsto x/|x|^2$ that sends ∞ to 0.

We denote by σ_n the *spherical measure* on $\widehat{\mathbf{R}}^n$; so

$$d\sigma_n(x) = \frac{2^n}{(1+|x|^2)^n}\, d\lambda_n(x).$$

We set

$$\omega_n = \sigma_n(\mathbf{S}^n) = \sigma_n(\widehat{\mathbf{R}}^n) = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}.$$

The following lemma provides an auxiliary n-subharmonic function on $\mathbf{R}^n$ whose Riesz measure is equal to σ_n.

Lemma 3.1. *For fixed $n \geq 2$ we define $h\colon [0,\infty) \to [0,\infty)$ by setting*

$$h(r) = \frac{1}{\omega_{n-1}^{1/(n-1)}} \int_0^r \sigma_n(B(0,t))^{1/(n-1)}\, \frac{dt}{t} \qquad \text{for } r \geq 0.$$

Let $v\colon \mathbf{R}^n \to \mathbf{R}$ be the radially symmetric function given by $v(x) := h(|x|)$ for $x \in \mathbf{R}^n$. Then v and the vector field $V := |\nabla v|^{n-2}\nabla v$ are C^1-smooth on $\mathbf{R}^n$, and we have

$$(\operatorname{div} V)(x) = \frac{2^n}{(1+|x|^2)^n}, \quad x \in \mathbf{R}^n. \tag{3.4}$$

The last identity implies that v is n-subharmonic and satisfies

$$\Delta_n v = \sigma_n \tag{3.5}$$

in the distributional sense. It follows from the definition of v that v is bounded on compact subsets of $\mathbf{R}^n$ and that $v(x) \to +\infty$ as $x \to \infty \in \widehat{\mathbf{R}}^n$.

Proof. For $r > 0$ we have

$$\sigma_n(B(0,r)) = 2^n\omega_{n-1} \int_0^r \frac{t^{n-1}}{(1+t^2)^n}\, dt. \tag{3.6}$$

From this it easily follows that the function h in the statement is C^∞-smooth on $(0,\infty)$. This in turn implies that v is C^∞-smooth on $\mathbf{R}^n\backslash\{0\}$. In order to investigate the behavior of v near 0, we expand the integral in (3.6) in a power series near 0. Then for $r > 0$ sufficiently close to 0 we have

$$\sigma_n(B(0,r)) = \frac{2^n}{n}\omega_{n-1} r^n (1 + O(r^2)),$$

where here and below $1 + O(r^2)$ indicates a power series in r^2 that converges near 0.

This and the definition of h in turn imply that for $r \geq 0$ close to 0 we have

$$h(r) = a_n r^{n/(n-1)}(1+O(r^2)), \text{ and } h'(r) = b_n r^{1/(n-1)}(1+O(r^2))$$

with some constants $a_n, b_n > 0$. Hence, h' exists and is continuous on $[0,\infty)$ with $h'(0) = 0$. Since $v(x) = h(r)$ with $r = |x|$ for $x \in \mathbf{R}^n$, it easily follows that v is differentiable at $x = 0$ with $\nabla v(0) = 0$ and that

$$\nabla v(x) = \frac{h'(r)}{r} x$$

is a continuous function of $x \in \mathbf{R}^n$. Hence v is indeed C^1-smooth on $\mathbf{R}^n$.

In order to verify the statements about V, first note that

$$V(x) = |\nabla v(x)|^{n-2}\nabla v(x) = \frac{h'(r)^{n-1}}{r} x = \frac{\sigma_n(B(0,r))}{\omega_{n-1} r^n} x \tag{3.7}$$

for $x \in \mathbf{R}^n \backslash \{0\}$. This implies that V is C^∞-smooth on $\mathbf{R}^n \backslash \{0\}$. To investigate the behavior of V near 0, we define

$$\phi(r) := \frac{\sigma_n(B(0,r))}{\omega_{n-1} r^n}.$$

Then by (3.6), for $r \geq 0$ near 0 we have

$$\phi(r) = c_n(1 + O(r^2))$$

with some constant $c_n > 0$. In particular, the function $x \mapsto \phi(|x|)$ is C^1-smooth (actually C^∞-smooth) near 0. Since $V(x) = \phi(r)x$, this implies that V is C^1-smooth near 0, and so C^1-smooth on $\mathbf{R}^n$.

In order to show (3.4), it is enough to establish the distributional version of this identity, namely that

$$-\int V \cdot \nabla \varphi = \int \varphi \, d\sigma_n$$

for each $\varphi \in C_c^\infty(\mathbf{R}^n)$ (recall that by our convention, here the integral on the left-hand side is against Lebesgue measure λ_n, and both integrals are extended over $\mathbf{R}^n$). Indeed, if $\varphi \in C_c^\infty(\mathbf{R}^n)$ is arbitrary, then a computation in polar coordinates (see [Fo, Theorem 2.49]) based on (3.7) and (3.6) shows that

$$\begin{aligned}
-\int V \cdot \nabla \varphi &= -\int_{\mathbf{S}^{n-1}} \left(\int_0^\infty \frac{\sigma_n(B(0,r))}{\omega_{n-1}} \frac{\partial \varphi(r\xi)}{\partial r} \, dr \right) d\sigma_{n-1}(\xi) \\
&= \frac{1}{\omega_{n-1}} \int_{\mathbf{S}^{n-1}} \left(\int_0^\infty \frac{d\sigma_n(B(0,r))}{dr} \varphi(r\xi) \, dr \right) d\sigma_{n-1}(\xi) \\
&= \int_{\mathbf{S}^{n-1}} \left(\int_0^\infty \varphi(r\xi) \frac{2^n r^{n-1}}{(1+r^2)^n} \, dr \right) d\sigma_{n-1}(\xi) = \int \varphi \, d\sigma_n.
\end{aligned}$$

The identity (3.4) follows. □

If A is a (real) $n \times n$-matrix, we denote the *adjunct matrix* of A by $\operatorname{adj} A$. This is an $n \times n$-matrix, whose entries are given by

$$(\operatorname{adj} A)_{ij} = (-1)^{i+j} \det A_{ji}$$

for $i, j = 1, \ldots, n$, where A_{ji} is the minor of A obtained by deleting the jth row and the ith column of A. Note that

$$A(\operatorname{adj} A) = (\det A) I_n,$$

where I_n is the $n \times n$-unit matrix.

Lemma 3.2. *Let $\Omega, \Omega' \subseteq \mathbf{R}^n$ be open sets in $\mathbf{R}^n$, $f \colon \Omega \to \Omega'$ be a continuous map in $W^{1,n}_{\mathrm{loc}}(\Omega)$, and $V \colon \Omega' \to \mathbf{R}^n$ be a C^1-smooth vector field on Ω'. Then*

$$\int_\Omega [(\operatorname{adj} Df)(V \circ f)] \cdot \nabla \varphi = -\int_\Omega \varphi [(\operatorname{div} V) \circ f] J_f \tag{3.8}$$

for all $\varphi \in C_c^\infty(\Omega)$.

Here $J_f = \det(Df)$ denotes the Jacobian of f and $\operatorname{adj} Df$ the matrix-valued function $x \mapsto \operatorname{adj} Df(x)$. Both are defined almost everywhere on Ω. Moreover, $(\operatorname{adj} Df)(V \circ f)$ is the $\mathbf{R}^n$-valued function obtained from pointwise multiplication of the matrix $(\operatorname{adj} Df)(x)$ with the column vector $(V \circ f)(x)$.

Proof. Suppose first that f is C^∞-smooth. If we define

$$W := \varphi[(\operatorname{adj} Df)(V \circ f)],$$

then W is a C^1-smooth vector field on Ω with compact support in Ω, and so

$$\int_\Omega \operatorname{div} W = 0.$$

Now

$$\operatorname{div}[(\operatorname{adj} Df)(V \circ f)] = [(\operatorname{div} V) \circ f] J_f. \tag{3.9}$$

This is straightforward to establish by direct computation if one uses the easily verified (and well-known) fact that each column of the matrix function $\operatorname{adj} Df$, considered as a vector field, is divergence free. We obtain

$$\operatorname{div} W = [(\operatorname{adj} Df)(V \circ f)] \cdot \nabla\varphi + \varphi[(\operatorname{div} V) \circ f] J_f,$$

and (3.8) follows if f is C^∞-smooth.

The general case can be derived from the smooth case by an approximation argument. Namely, if $f \in W^{1,n}_{\mathrm{loc}}(\Omega)$ we can find C^∞-smooth maps $f_k\colon \Omega \to \mathbf{R}^n$ such that on the support K of φ we have uniform convergence $f_k \to f$, and convergence $Df_k \to Df$ in $L^n(K)$. Then $J_{f_k} \to J_f$ and $\operatorname{adj} Df_k \to \operatorname{adj} Df$ in $L^1(K)$. By the first part of the proof, (3.8) is true for f_k with k large; by taking the limit as $k \to \infty$ on both sides, we obtain (3.8) for the map f. □

Using the language of differential forms, one can outline a more conceptual way of verifying the crucial identity (3.9) as follows. We can identify the vector field V with a C^1-smooth $(n-1)$-form α on Ω' with the same n components up to sign in standard coordinates on $\mathbf{R}^n$. If d denotes exterior differentiation of forms and Vol the standard volume form on $\mathbf{R}^n$, then we can choose these signs so that

$$d\alpha = (\operatorname{div} V)\mathrm{Vol}.$$

Moreover, if f^* denotes the pull-back operation on forms by f, then $f^*\alpha$ is an $(n-1)$-form whose coefficients in standard coordinates correspond to the components of the vector field $(\operatorname{adj} Df)(V \circ f)$. Since $f^* \circ d = d \circ f^*$ for smooth f, we obtain

$$\operatorname{div}[(\operatorname{adj} Df)(V \circ f)]\mathrm{Vol} = d(f^*\alpha) = f^*(d\alpha) = f^*((\operatorname{div} V)\mathrm{Vol}) = [(\operatorname{div} V) \circ f] J_f \mathrm{Vol}.$$

Equation (3.9) follows.

We conclude this section with a brief discussion of capacity based on the notion of n-energy and we will establish a corresponding estimate. To set this up, we fix $a \in \mathbf{R}^n$, $r > 0$, $t > 1$, and let

$$R = \{x \in \mathbf{R}^n : r < |x - a| < tr\}.$$

For two given sets $E, F \subseteq \mathbf{R}^n$ that are closed and disjoint, we define

$$S = R \setminus (E \cup F).$$

We consider the open set S as a condenser in the ring domain R with the complementary sets

$$E' := R \cap E \text{ and } F' := R \cap F.$$

We define the capacity of S (in the given setup) as

$$\operatorname{Cap}(S) := \inf\left\{ \int_S |\nabla\psi|^n : \psi \in C^\infty_c(\mathbf{R}^n), \psi|E' \geq 1,\ \psi|F' \leq 0 \right\}. \tag{3.10}$$

If we set

$$\Sigma(\rho) = \{x \in \mathbf{R}^n : |x - a| = \rho\}$$

for $\rho > 0$, then the following statement is true.

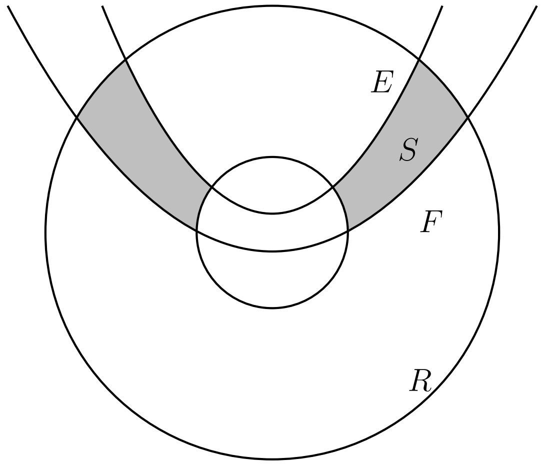

Figure 1. The condenser S.

Lemma 3.3. (First capacity estimate) *With the setup as above, suppose* $E \cap \Sigma(\rho) \neq \emptyset$ *and* $F \cap \Sigma(\rho) \neq \emptyset$ *for each* $r < \rho < tr$*. Then we have*

$$\int_r^{tr} \frac{d\rho}{\sigma_{n-1}(S \cap \Sigma(\rho))^{1/(n-1)}} \leq c \operatorname{Cap}(S), \tag{3.11}$$

where $c = c(n) > 0$*.*

For a geometric illustration see Figure 1.

Proof. In this proof all implicit multiplicative constants in inequalities of the form $X \lesssim Y$ only depend on n. By our hypotheses, for each fixed $\rho \in (r, tr)$ we can choose $x_\rho \in E \cap \Sigma(\rho)$ and $y_\rho \in F \cap \Sigma(\rho)$. We may assume that $|x_\rho - y_\rho|$ is minimal among all such points. Then $S \cap \Sigma(\rho)$ contains a spherical cap of radius comparable to $|x_\rho - y_\rho|$. Hence

$$|x_\rho - y_\rho|^{n-1} \lesssim \sigma_{n-1}(S \cap \Sigma(\rho)). \tag{3.12}$$

Let $\psi \in C_c^\infty(\mathbf{R}^n)$ be a test function for the condenser S as in (3.10), and define $\widetilde{\psi} := \min\{1, \max\{\psi, 0\}\}$. Then $\widetilde{\psi}(x_\rho) = 1$ and $\widetilde{\psi}(y_\rho) = 0$. Moreover, $\nabla\widetilde{\psi}(x) = \nabla\psi(x)$ for a.e. (= almost every) $x \in S$ and $\nabla\widetilde{\psi}(x) = 0$ for a.e. $x \in R \backslash S$. Hence by Fubini we have

$$\int_{\Sigma(\rho)} |\nabla\widetilde{\psi}|^n \, d\sigma_{n-1} = \int_{S \cap \Sigma(\rho)} |\nabla\psi|^n \, d\sigma_{n-1}$$

for a.e. $\rho \in (r, tr)$.

If we apply the Sobolev embedding theorem for the supercitical exponent $p = n$ on the $(n-1)$-dimensional sphere $\Sigma(\rho)$ to the function $\widetilde{\psi}$ (this is essentially [GT, Theorem 7.17]), then we conclude that

$$1 = |\widetilde{\psi}(y_\rho) - \widetilde{\psi}(x_\rho)| \lesssim |x_\rho - y_\rho|^{1-(n-1)/n} \left(\int_{S \cap \Sigma(\rho)} |\nabla\psi|^n \, d\sigma_{n-1} \right)^{1/n},$$

and so by (3.12) we obtain that

$$\frac{1}{\sigma_{n-1}(S \cap \Sigma(\rho))^{1/(n-1)}} \lesssim \int_{S \cap \Sigma(\rho)} |\nabla\psi|^n \, d\sigma_{n-1}$$

for a.e. $\rho \in (r, tr)$. Integrating over (r, tr) we arrive at

$$\int_r^{tr} \frac{d\rho}{\sigma_{n-1}(S \cap \Sigma(\rho))^{1/(n-1)}} \lesssim \int_S |\nabla\psi|^n.$$

If we take the infimum over all test functions ψ here, then we obtain the desired inequality (3.11). □

4. $\mathcal{A}$-harmonic and $\mathcal{A}$-subharmonic functions

Pull-backs of n-harmonic or n-subharmonic functions by quasiregular maps are not of the same type in general, but one obtains functions that still satisfy a non-linear degenerate elliptic equation of divergence type. This is well-known and the basis of the potential-theoretic method to investigate quasiregular maps. Here we only discuss some basic facts relevant for our approach.

Let $\mathcal{A}\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be a measurable map such that for almost every $p \in \mathbf{R}^n$ the map $\xi \mapsto \mathcal{A}(p, \xi)$ is defined on $\mathbf{R}^n$ and satisfies

(i) $\mathcal{A}(p, \lambda\xi) = |\lambda|^{n-2}\lambda\mathcal{A}(p, \xi)$,
(ii) $|\mathcal{A}(p, \xi)| \le c_1|\xi|^{n-1}$,
(iii) $\mathcal{A}(p, \xi) \cdot \xi \ge c_2|\xi|^n$

for all $\xi \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$. Here c_1 and c_2 are positive constants independent of p and ξ. These requirements are modeled on properties of the basic example $\mathcal{A}(p, \xi) = |\xi|^{n-2}\xi$.

Let $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be a K-quasiregular map. Then $Df(p)$ exists for almost $p \in \mathbf{R}^n$ and is an invertible linear map on $\mathbf{R}^n$. We identify $Df(p)$ with the Jacobi matrix (the matrix representation of $Df(p)$ with respect to the standard basis on $\mathbf{R}^n$). Then

$$G(p) := \det(Df(p))^{2/n} Df(p)^{-1}(Df(p)^{-1})^t \tag{4.1}$$

is defined for almost every $p \in \mathbf{R}^n$. Here A^{-1} and A^t indicate the inverse and the transpose of a matrix A, respectively. Since f is a K-quasiregular map, we have

$$\frac{1}{C}|\xi|^2 \le G(p)\xi \cdot \xi \le C|\xi|^2$$

for almost every $p \in \mathbf{R}^n$ and all $\xi \in \mathbf{R}^n$, where $C = C(n, K) > 0$ only depends on n and K.

If we define

$$\mathcal{A}(p, \xi) = (G(p)\xi \cdot \xi)^{(n-2)/2} G(p)\xi, \tag{4.2}$$

then $\mathcal{A}$ has the above properties (i)–(iii) with constants $c_1 = c_1(n, K)$ and $c_2 = c_2(n, K)$.

Suppose $\mathcal{A}$ satisfying (i)–(iii) is given, and $u \in W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n)$. For ease of notation we write $\mathcal{A}_u$ for the almost everywhere defined measurable function $\mathcal{A}_u\colon \mathbf{R}^n \to \mathbf{R}^n$, $x \in \mathbf{R}^n \mapsto \mathcal{A}(x, \nabla u(x))$. We say that $u\colon \mathbf{R}^n \to \mathbf{R}$ is $\mathcal{A}$-*subharmonic* if $u \in W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n)$ and if there exists a positive measure μ on $\mathbf{R}^n$ (the *Riesz measure* of u) such that

$$-\int \mathcal{A}_u \cdot \nabla\varphi = \int \varphi\, d\mu \tag{4.3}$$

for all $\varphi \in C^\infty_c(\mathbf{R}^n)$. In other words, u satisfies the equation

$$\operatorname{div} \mathcal{A}_u = \mu$$

in the distributional sense. For our purposes it is actually enough to only consider continuous $\mathcal{A}$-subharmonic functions. A standard approximation argument shows that (4.3) remains valid for all functions $\varphi \in C_c(\mathbf{R}^n) \cap W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n)$.

Lemma 4.1. (Caccioppoli inequality) *Let $u\colon \mathbf{R}^n \to [0,\infty)$ be a non-negative continuous $\mathcal{A}$-subharmonic function. Then*

$$\int \varphi^n |\nabla u|^n \le C \int u^n |\nabla \varphi|^n$$

for all non-negative $\varphi \in C_c^\infty(\mathbf{R}^n)$, where $C = C(n, c_1, c_2) > 0$.

Proof. In the following proof all implicit multiplicative constants only depend on n, c_1, and c_2. If μ is the Riesz measure of u, then for each non-negative function $\psi \in C_c(\mathbf{R}^n) \cap W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n)$ we have

$$\int \mathcal{A}_u \cdot \nabla \psi = -\int \psi \, d\mu \le 0.$$

Now let $\varphi \in C_c^\infty(\mathbf{R}^n)$ with $\varphi \ge 0$ be arbitrary, and choose $\psi = \varphi^n u$ in the previous inequality. This is possible, because $u \ge 0$ and $u \in C_c(\mathbf{R}^n) \cap W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n)$. Then

$$\nabla \psi = n u \varphi^{n-1} \nabla \varphi + \varphi^n \nabla u,$$

and so

$$\begin{aligned}\int \varphi^n \mathcal{A}_u \cdot \nabla u &= \int \mathcal{A}_u \cdot \nabla \psi - n \int u \varphi^{n-1} \mathcal{A}_u \cdot \nabla \varphi \\ &\le -n \int u \varphi^{n-1} \mathcal{A}_u \cdot \nabla \varphi \le n \int u \varphi^{n-1} |\mathcal{A}_u| \cdot |\nabla \varphi|.\end{aligned}$$

Using this and the properties (ii) and (iii) of $\mathcal{A}$, we obtain

$$\begin{aligned}\int \varphi^n |\nabla u|^n &\lesssim \int \varphi^n \mathcal{A}_u \cdot \nabla u \lesssim \int u \varphi^{n-1} |\mathcal{A}_u| \cdot |\nabla \varphi| \lesssim \int u \varphi^{n-1} |\nabla u|^{n-1} |\nabla \varphi| \\ &\le \left(\int \varphi^n |\nabla u|^n \right)^{(n-1)/n} \left(\int u^n |\nabla \varphi|^n \right)^{1/n}.\end{aligned}$$

The desired inequality follows. □

Lemma 4.2. (Growth controls Riesz measure) *Let $u\colon \mathbf{R}^n \to [0,\infty)$ be a non-negative continuous $\mathcal{A}$-subharmonic function with Riesz measure μ, and $B = B(a, r)$, where $a \in \mathbf{R}^n$ and $r > 0$. Then*

$$\mu(B) \le C \sup_{x \in 2B} u(x)^{n-1}, \tag{4.4}$$

where $C = C(n, c_1, c_2) > 0$.

Proof. In the following, all implicit multiplicative constants again only depend on n, c_1, and c_2. We can pick a function $\varphi \in C_c^\infty(\mathbf{R}^n)$ such that

$$0 \le \varphi \le 1, \quad \varphi | B = 1, \quad \varphi | \mathbf{R}^n \setminus 2B = 0, \quad |\nabla \varphi| \lesssim 1/r.$$

Then $\int |\nabla\varphi|^n \lesssim 1$. Applying (4.3) and the Caccioppoli inequality, we conclude that

$$\begin{aligned}\mu(B) &\le \int \varphi^n\, d\mu = -n\int \varphi^{n-1}\mathcal{A}_u\cdot\nabla\varphi \lesssim \int \varphi^{n-1}|\nabla u|^{n-1}|\nabla\varphi|\\ &\le \Big(\int \varphi^n|\nabla u|^n\Big)^{(n-1)/n}\Big(\int|\nabla\varphi|^n\Big)^{1/n}\\ &\lesssim \Big(\int \varphi^n|\nabla u|^n\Big)^{(n-1)/n} \lesssim \Big(\int u^n|\nabla\varphi|^n\Big)^{(n-1)/n}\\ &\le \Big(\sup_{x\in 2B} u(x)^{n-1}\Big)\Big(\int|\nabla\varphi|^n\Big)^{(n-1)/n} \lesssim \sup_{x\in 2B} u(x)^{n-1}. \qquad\square\end{aligned}$$

Lemma 4.3. *Let $u\colon \mathbf{R}^n\to[0,\infty)$ be a non-negative continuous $\mathcal{A}$-subharmonic function. If u is bounded, then u is a constant function.*

Proof. If u is bounded, then by Lemma 4.1 we can find $C\ge 0$ such that

$$\int |\nabla u|^n\varphi^n \le C\int|\nabla\varphi|^n$$

for all $\varphi\in C_c^\infty(\mathbf{R}^n)$. Now it is a well-known fact that for each $r\ge 0$ there exist functions $\varphi\in C_c^\infty(\mathbf{R}^n)$ with $\varphi\ge 0$, $\varphi|B(0,r)=1$, and $\int|\nabla\varphi|^n$ arbitrarily small (because the n-capacity of ∞ vanishes in $\mathbf{R}^n$). This implies that

$$\int_{B(0,r)}|\nabla u|^n = 0$$

for each $r\ge 0$, and so $\nabla u = 0$ almost everywhere on $\mathbf{R}^n$. It follows that u is equal to a constant function (see [Zi, Corollary 2.1.9]). $\square$

If $g\colon\mathbf{R}^n\to\mathbf{R}$ is any function, we use some obvious notation for sets related to sub- and superlevels of g. For example, if $a,b\in\mathbf{R}$ and $a\le b$, then

$$\{a\le g\le b\} := \{x\in\mathbf{R}^n\colon a\le g(x)\le b\},\quad \{g>b\} := \{x\in\mathbf{R}^n\colon u(x)>b\},\quad \text{etc.}$$

We require the following version of the maximum principle for $\mathcal{A}$-subharmonic functions, expressed as a statement about their superlevel sets.

Lemma 4.4. *Let $u\colon\mathbf{R}^n\to[0,\infty)$ be a non-constant, non-negative, and continuous $\mathcal{A}$-subharmonic function. Then for each $L\ge 0$ the open set $\{u>L\}$ has no bounded components.*

Proof. Let $L\ge 0$ be arbitrary. Then the set $\{u>L\}$ is open, and it is non-empty by Lemma 4.3. We argue by contradiction and assume that $\{u>L\}$ has a (non-empty) bounded component Ω. Then $u|\partial\Omega = L$. Define $\varphi=\max\{(u-L),0\}$ on Ω and $\varphi=0$ on $\mathbf{R}^n\backslash\Omega$. Then φ is a non-negative function in $C_c(\mathbf{R}^n)\cap W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n)$ with $\nabla\varphi=\nabla u$ a.e. on Ω and $\nabla\varphi=0$ a.e. on $\mathbf{R}^n\backslash\Omega$ (since u is absolutely continuous on almost every line, one easily checks that $\nabla\varphi$ defined in this way is indeed a distributional gradient of φ; see [Zi, Corollary 2.1.8]).

If we denote by μ the Riesz measure of u, then we can apply (4.3) for this function φ. Hence

$$\int_\Omega \mathcal{A}_u\cdot\nabla u = \int\mathcal{A}_u\cdot\nabla\varphi = -\int\varphi\, d\mu\le 0.$$

On the other hand,

$$\mathcal{A}_u\cdot\nabla u\ge c_2|\nabla u|^n\ge 0$$

almost everywhere on $\mathbf{R}^n$. This is only possible if $\nabla u(x) = 0$ for a.e. $x \in \Omega$. Therefore, u is locally constant on Ω, and hence constant on Ω, because Ω is open and connected. Since $u = L$ on $\partial\Omega$, this implies that $u = L$ on Ω; but we know that $u > L$ on Ω. This is a contradiction. □

5. Proof of the Rickman–Picard theorem

We first provide a proof of Rickman's Hunting Lemma.

Proof of Lemma 2.1. We can find a number $D = D(n) \in \mathbf{N}$ so that every ball $B \subseteq \mathbf{R}^n$ of radius $\rho > 0$ can be covered by D balls of radius $\rho/16$ centered in B (this is true, because $\mathbf{R}^n$ is a "doubling" metric space).

We argue by contradiction and assume that there is a constant $C > 0$ such that for every ball $B \subseteq \mathbf{R}^n$ with $\mu(B) > C$ we have $\mu(8B) > D\mu(B)$.

Pick $x_0 = 0$. Since $\mu(\mathbf{R}^n) = \infty$ there exists $r > 0$ such that for $B_0 := B(x_0, r)$ we have $\mu(B_0) > C$. Then by our hypotheses $\mu(8B_0) > D\mu(B_0)$. By choice of D, the ball $8B_0$, which has radius $8r$, can be covered by D balls of radius $r/2$ centered at points in $8B_0$. Hence there exists $x_1 \in \mathbf{R}^n$ with $|x_1 - x_0| < 8r$ such that

$$\mu(B(x_1, r/2)) \geq \frac{1}{D}\mu(B(x_0, 8r)) \geq \mu(B(x_0, r)) > C.$$

Now we repeat the argument for $B_1 := B(x_1, r/2)$, and so on, decreasing the radii of the balls by the factor 2 in each step. In this way, we obtain a sequence of points x_k, $k \in \mathbf{N}_0$, such that for all $k \in \mathbf{N}_0$ we have

$$|x_{k+1} - x_k| < 2^{3-k}r$$

and $\mu(B_k) > C$, where $B_k := B(x_k, 2^{-k}r)$.

The points x_k form a Cauchy sequence and so there exists $x_\infty \in \mathbf{R}^n$ such that $x_k \to x_\infty$ as $k \to \infty$. If $\delta > 0$ is arbitrary, then $B_k \subseteq B(x_\infty, \delta)$ for some $k \in \mathbf{N}$, and so $\mu(B(x_\infty, \delta)) \geq \mu(B_k) > C$. Hence

$$\mu(\{x_\infty\}) = \lim_{\delta \to 0} \mu(B(x_\infty, \delta)) \geq C > 0.$$

Here we used that $\mu(B(x_\infty, \delta)) < \infty$ as follows from our hypotheses. We obtain a contradiction, because $\mu(\{x_\infty\}) > 0$, but μ has no atoms. □

To set up the proof of the Rickman–Picard theorem, we consider a fixed K-quasiregular map $f\colon \mathbf{R}^n \to \mathbf{S}^n = \widehat{\mathbf{R}}^n$, where $n \geq 2$ and $K \geq 1$. We want to show that f cannot omit a set of points if their number is sufficiently large depending on n and K. By considering f followed by a rotation if necessary, we may assume that $\infty \in \widehat{\mathbf{R}}^n$ is among the omitted values. So suppose $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is a K-quasiregular map omitting the distinct values $a_1, \dots, a_q \in \mathbf{R}^n$. We want to derive a contradiction if q is sufficiently large only depending on n and K. Note that f is K-quasiregular with the same K independently of whether we equip the target $\mathbf{R}^n$ with the Euclidean metric or the restriction of the spherical metric on $\widehat{\mathbf{R}}^n$ to $\mathbf{R}^n$. This follows from the conformal equivalence of these metrics.

As already outlined in Section 2, we now consider the measure A on $\mathbf{R}^n$ obtained by pulling back the spherical measure σ_n on $\mathbf{R}^n \subseteq \widehat{\mathbf{R}}^n$ to $\mathbf{R}^n$ by f. Recall that (see (2.2))

$$dA(x) = \frac{2^n \det(Df(x))}{(1 + |f(x)|^2)^n}\, d\lambda_n(x),$$

where as before $Df(x)$ is the almost everywhere defined Jacobi matrix of f.

In the following, $\mathcal{A}$ will be as in (4.2) with G as in (4.1). We want to use the omitted values to construct certain non-negative continuous $\mathcal{A}$-subharmonic functions whose Riesz measure is equal to A.

For each $k = 1, \ldots, q$ we pick a rotation R_k on $\mathbf{S}^n \cong \widehat{\mathbf{R}}^n$ such that $R_k(a_k) = \infty$. Let v be the function from Lemma 3.1 and define $v_k := v \circ R_k$. Then v_k is a non-negative C^1-smooth function on $\mathbf{R}^n \backslash \{a_k\} \subseteq \mathbf{S}^n \backslash \{a_k\}$. Since R_k is a conformal map and the spherical measure σ_n is rotation-invariant, it follows that v_k is n-subharmonic on $\mathbf{R}^n \backslash \{a_k\}$ with Riesz measure σ_n (restricted to $\mathbf{R}^n \backslash \{a_k\}$ to be precise). Actually, $|\nabla v_k|^{n-2} \nabla v_k$ is C^1-smooth and

$$\operatorname{div}(|\nabla v_k(y)|^{n-2} \nabla v_k(y)) = \frac{2^n}{(1+|y|^2)^n}$$

for $y \in \mathbf{R}^n \backslash \{a_k\}$. Moreover, v_k is bounded outside each neighborhood of a_k with $v_k(x) \to +\infty$ as $x \to a_k$ for $k = 1, \ldots, q$.

We fix $\delta > 0$ such that the Euclidean balls $B(a_1, \delta), \ldots, B(a_k, \delta)$ are pairwise disjoint. The behavior of the functions v_k near the singularities a_k implies that if we choose δ small enough, then we can find a constant $L_0 > 0$ with the following property: if $v_k(y) > L_0$ for some $k = 1, \ldots, q$, then $y \in B(a_k, \delta)$. In particular, the sets $\{v_k > L_0\}$, $k = 1, \ldots, q$, are pairwise disjoint.

Now define $u_k := v_k \circ f$. This is meaningful, because f omits the value a_k. It follows from the definition of L_0 that the sets

$$\{u_k > L_0\}, \quad k = 1, \ldots, q, \tag{5.1}$$

are pairwise disjoint.

Lemma 5.1. *Each function u_k is an unbounded, non-negative, and continuous $\mathcal{A}$-subharmonic function with Riesz measure A.*

Proof. Essentially, this follows from the fact that $u = u_k$ is the pull-back of the non-negative and continuous n-subharmonic function $v = v_k$ which has Riesz measure $\sigma = \sigma_n$, and σ pulls back to A. A more general version of the statement is true for arbitrary n-subharmonic functions. For the proof one has to struggle with regularity issues. Here the argument is easier, because we have sufficient smoothness.

Consider $u = v \circ f \geq 0$. It is clear that u is continuous. Since v is C^1-smooth and f is in $W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n)$, the function u has a weak derivative ∇u that is locally L^n-integrable. We can apply the chain rule and obtain

$$\nabla u(x) = Df(x)^t \nabla v(f(x)).$$

Here and in similar equations below this is understood to hold for a.e. $x \in \mathbf{R}^n$. Using (4.1) we find that

$$G(x)\nabla u(x) \cdot \nabla u(x) = \det(Df(x))^{2/n} |\nabla v(f(x))|^2.$$

Inserting this into (4.2), we obtain

$$\mathcal{A}(x, \nabla u(x)) = |\nabla v(f(x))|^{n-2} \det(Df(x)) Df(x)^{-1} \nabla v(f(x)).$$

This can be rewritten as

$$\mathcal{A}(x, \nabla u(x)) = \operatorname{adj}(Df(x)) \left(|\nabla v(f(x))|^{n-2} \nabla v(f(x))\right) = \operatorname{adj}(Df(x)) \left((V \circ f)(x)\right),$$

where

$$V := |\nabla v|^{n-2} \nabla v.$$

It follows from Lemma 3.1 that V is a C^1-smooth vector field, and that

$$(\operatorname{div} V)(x) = \frac{2^n}{(1+|x|^2)^n}. \tag{5.2}$$

Now let $\varphi \in C_c^\infty(\mathbf{R}^n)$. Then based on Lemma 3.2 we conclude from (5.2) and (2.2) that

$$-\int \mathcal{A}_u \cdot \nabla\varphi = -\int [\operatorname{adj}(Df)(V\circ f)]\cdot\nabla\varphi = \int \varphi((\operatorname{div} V)\circ f)J_f = \int \varphi\, dA.$$

So equation (4.3) is satisfied with $\mu = A$. Therefore, u is a non-negative continuous $\mathcal{A}$-subharmonic function with Riesz measure $\mu = A$ as claimed. Obviously, the function u is non-constant (otherwise $\mu = A = 0$), and so it is unbounded by Lemma 4.3. ☐

We are now ready to set up the "inflation mechanism" discussed in Section 2. The starting point is the following statement.

Corollary 5.2. (Meeting superlevel sets) *There exists a constant $C_0 > 0$ with the following property: if $B = B(a,r)$ with $a \in \mathbf{R}^n$ and $r > 0$ is any ball satisfying $A(\frac{1}{2}B) \geq C_0$, then*

$$B \cap \{u_k \geq 3L_0\} \neq \emptyset$$

for all $k = 1,\dots,q$.

In other words, if a ball has sufficiently large A-mass, then the twice larger ball with the same center meets each superlevel set $\{u_k \geq 3L_0\}$. In general, the constant C_0 will not only depend on n and K, but on other data (such as L_0 and the points $a_1,\dots,a_q$).

Proof. By Lemma 5.1 an inequality as in Lemma 4.2 holds for each function u_k with $\mu = A$ and a constant C only depending on n and K. So for each ball $B \subseteq \mathbf{R}^n$ we have

$$A(\tfrac{1}{2}B) \lesssim \sup_{x\in B} u_k(x)^{n-1}$$

with an implicit constant only depending on n and K. This shows that if the A-mass of $\frac{1}{2}B$ exceeds a large enough constant C_0 (independent of k and B), then for each of the functions u_k there exists a point $x \in B$ with $u_k(x) \geq 3L_0$. The claim follows. ☐

The inequality in Lemma 4.2 says that the growth of an $\mathcal{A}$-subharmonic function controls its Riesz measure. One can also prove a similar inequality in the other direction. There seems to be no simple proof of this fact for general $\mathcal{A}$-subharmonic functions. We will only need this for our functions $u_1,\dots,u_q$, where one can prove a related inequality by a rather simple argument. For this we will use the notion of capacity as introduced towards the end of section Section 3. We use a similar setup as was discussed there.

So we fix $a \in \mathbf{R}^n$, $r > 0$, $t > 1$, and let $B = B(a,r)$, $R = \{x \in \mathbf{R}^n : r < |x-a| < tr\}$. For $k = 1,\dots,q$ we define

$$M_k := \sup_{x\in B} u_k(x), \quad E_k := \{u_k \geq 2M_k/3\}, \quad F_k := \{u_k \leq M_k/3\}. \tag{5.3}$$

For fixed k the sets E_k and F_k are closed and disjoint, and give the condenser

$$S_k := R \setminus (E_k \cup F_k) = R \cap \{M_k/3 < u_k < 2M_k/3\}. \tag{5.4}$$

in R with the complementary sets

$$E_k' = R \cap E_k = R \cap \{u_k \geq 2M_k/3\} \quad \text{and} \quad F_k' = R \cap F_k = R \cap \{u_k \leq M_k/3\}.$$

For the capacity of S_k as defined in (3.10) we have

$$\operatorname{Cap}(S_k) := \inf\left\{\int_{S_k} |\nabla\psi|^n \colon \psi \in C_c^\infty(\mathbf{R}^n), \psi|E_k' \geq 1,\ \psi|F_k' \leq 0\right\}.$$

Based on an approximation argument, one can here replace the class $C_c^\infty(\mathbf{R}^n)$ of test functions by the larger class of all functions in $W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n)$ with compact support. If $\varphi \in C_c^\infty(\mathbf{R}^n)$ and $\varphi|tB = 1$, then $\psi = (3u_k/M_k - 1)\varphi$ lies in the latter class. It follows that

$$\frac{M_k^n}{3^n}\operatorname{Cap}(S_k) \leq \int_{S_k} |\nabla u_k|^n. \tag{5.5}$$

If $A(\frac{1}{2}B) \geq C_0$, where C_0 is the constant in Corollary 5.2, then $M_k/3 \geq L_0$ for each $k = 1, \dots, q$. This implies that $S_k \subseteq \{u_k > L_0\}$ and so by (5.1) the sets S_k, $k = 1, \dots, q$, are pairwise disjoint. Then the condensers $S_1, \dots, S_q$ are crowded together in the ring R. It is not hard to see that if q is large, then there exists $k \in \{1, \dots, q\}$ such that $\operatorname{Cap}(S_k)$ is large. Actually, we will record a corresponding capacity estimate that gives an essentially optimal bound.

Lemma 5.3. (Second capacity estimate) *With the setup as above, there exists a constant $C_0 > 0$ such that if $q \geq 2$ and $A(\frac{1}{2}B) > C_0$, then*

$$\sum_{k=1}^{q} \operatorname{Cap}(S_k) \geq c\log(t) q^{n/(n-1)}, \tag{5.6}$$

where $c = c(n) > 0$.

The proof will show that (5.6) is also true if $q \geq 2$ and r is large enough (without the assumption $A(\frac{1}{2}B) > C_0$).

Proof. In this proof all implicit multiplicative constants in inequalities of the form $X \lesssim Y$ only depend on n. By Corollary 5.2 we can find a constant $C_0 > 0$ independent of the ball $B = B(a,r)$ such that if $A(\frac{1}{2}B) > C_0$, then $M_k \geq 3L_0 > 0$. In this case, by (5.1) the superlevel sets $\{u_k > M_k/3\} \subseteq \{u_k > L_0\}$ are pairwise disjoint for $k = 1, \dots q$. This last statement (and hence the rest of the argument) is also true without the assumption $A(\frac{1}{2}B) > C_0$ if r is large enough, because the functions u_k are unbounded and so $M_k/3 \geq L_0$ for all $k = 1, \dots, q$, if $r > 0$ is large enough.

By Lemma 4.4 no component of $\{u_k > 2M_k/3\}$ is bounded. Since $\{u_k > 2M_k/3\} \supseteq \{u_k \geq M_k\}$ meets $\overline{B}$, it follows that $E_k = \{u_k \geq 2M_k/3\}$ has non-empty intersection with each sphere

$$\Sigma(\rho) = \{x \in \mathbf{R}^n \colon |x - a| = \rho\}$$

for $r < \rho < tr$. Now for $l \in \{1, \dots q\}$ with $l \neq k$ we have

$$E_l = \{u_l \geq 2M_l/3\} \subseteq \{u_l > M_l/3\} \subseteq \mathbf{R}^n \setminus \{u_k > M_k/3\} = \{u_k \leq M_k/3\} = F_k,$$

and so F_k also meets each sphere $\Sigma(\rho)$ with $r < \rho < tr$ (here it is important that $q \geq 2$ which implies that there exists $l \in \{1, \dots, k\}$ with $l \neq k$).

This shows that the hypotheses of Lemma 3.3 are true for each condensers S_k, and so estimate (3.11) holds for all condensers $S_1, \dots, S_q$. Since they are pairwise disjoint, we have

$$\sum_{k=1}^{q} \sigma_{n-1}(S_k \cap \Sigma(\rho)) \leq \sigma_{n-1}(\Sigma(\rho)) \lesssim \rho^{n-1}$$

for all $\rho \in (r, tr)$. Now by Hölder's inequality,

$$q \leq \left(\sum_{k=1}^{q} \frac{1}{\sigma_{n-1}(\Sigma(\rho) \cap S_k)^{1/(n-1)}} \right)^{(n-1)/n} \left(\sum_{k=1}^{q} \sigma_{n-1}(\Sigma(\rho) \cap S_k) \right)^{1/n},$$

and so

$$q^{n/(n-1)}/\rho \lesssim \sum_{k=1}^{q} \frac{1}{\sigma_{n-1}(\Sigma(\rho) \cap S_k)^{1/(n-1)}}.$$

If we integrate over (r, tr) with respect to ρ, then the claim follows from (3.11). $\square$

The next result will be used in two ways, either when the value of the parameter t is suitably large or when $t = 2$.

Lemma 5.4. (Riesz measure controls growth) *Let $a \in \mathbf{R}^n$, $r > 0$, $t \geq 2$, $B = B(a, r)$, and $R = \{x \in \mathbf{R}^n \colon r < |x - a| < tr\}$. For some $k \in \{1, \ldots, q\}$, set $u = u_k$, $M = \sup_{x \in B} u(x)$, and*

$$S = R \cap \{M/3 < u < 2M/3\}.$$

Then we have

$$(\operatorname{Cap}(S) - C_1) \sup_{x \in B} u(x)^{n-1} \leq C_2 A(2tB), \tag{5.7}$$

where $C_1 = C_1(n, K) > 0$ and $C_2 = C_2(n, K) > 0$.

The inequality is trivial if $\operatorname{Cap}(S) \leq C_1$. The main point is that if t is large or q is large, then necessarily $\operatorname{Cap}(S_k) >> C_1$ for some k and so we get a non-trivial estimate.

Before we turn to the proof of the lemma, we want to apply it to verify that the measure A satisfies the hypotheses of Lemma 2.1 if $q \geq 2$ (which we may assume). To see this, it is enough to show that $A(\mathbf{R}^n) = \infty$. We use the remark after the statement of Lemma 5.3. Namely, (5.6) is true for $q \geq 2$ and the ball $B = B(0, r)$ (without the assumption $A(\frac{1}{2}B) > C_0$) if $r > 0$ is large enough, say $r \geq r_0 > 0$.

By (5.6) we can choose $t \geq 2$ so large (independently of $r \geq r_0$) that for one of the condensers S_k we have $\operatorname{Cap}(S_k) > 2C_1$, where C_1 is the constant in Lemma 5.4. Inequality (5.7) then shows that we must have $A(\mathbf{R}^n) = \infty$, because the function u_k is unbounded and so the right hand side in (5.7) can be made arbitrarily large if we choose r large enough.

Proof of Lemma 5.4. In this proof all implicit multiplicative constants will only depend on n and K. We have

$$-\int \mathcal{A}_u \cdot \nabla \varphi = \int \varphi \, dA$$

for all $\varphi \in C_c(\mathbf{R}^n) \cap W^{1,n}_{\mathrm{loc}}(\mathbf{R}^n)$. We can use this identity for the test function

$$\varphi = (M - u)_+ \psi^n,$$

where we choose $\psi \in C_c^\infty(\mathbf{R}^n)$ so that it satisfies

$$0 \leq \psi \leq 1, \quad \psi | tB = 1, \quad \psi | \mathbf{R}^n \backslash 2tB = 0, \quad |\nabla \psi| \lesssim \frac{1}{tr}.$$

Then $0 \leq \varphi \leq M$,

$$\nabla \varphi(x) = -\nabla u(x) \psi(x)^n + n(M - u(x))_+ \nabla \psi(x) \psi^{n-1}(x)$$

for a.e. $x \in U$, and $\nabla \varphi(x) = 0$ for a.e. $x \in \mathbf{R}^n \backslash U$, where

$$U := 2tB \cap \{u < M\} \supseteq S.$$

Hence

$$\begin{aligned}\int_U \psi^n |\nabla u|^n &\lesssim \int_U \psi^n \mathcal{A}_u \cdot \nabla u = \int (M-u)_+ \psi^n \, dA + n \int_U (M-u)_+ \psi^{n-1} \mathcal{A}_u \cdot \nabla \psi \\ &\lesssim MA(2tB) + M \int_U \psi^{n-1} |\mathcal{A}_u| \cdot |\nabla \psi| \\ &\lesssim MA(2tB) + M \int_U \psi^{n-1} \, |\nabla u|^{n-1} |\nabla \psi| \\ &\lesssim MA(2tB) + M \Big(\int_U \psi^n |\nabla u|^n \Big)^{(n-1)/n} \Big(\int |\nabla \psi|^n \Big)^{1/n} \\ &\lesssim MA(2tB) + M \Big(\int_U \psi^n |\nabla u|^n \Big)^{(n-1)/n}.\end{aligned}$$

In the last step we used that $\int |\nabla \psi|^n \lesssim 1$.

Now if $\alpha \leq \beta + \gamma \alpha^{(n-1)/n}$ for $\alpha, \beta, \gamma \geq 0$, then $\alpha \leq \max\{2\beta, 2^n \gamma^n\}$. It follows that

$$\int_U \psi^n |\nabla u|^n \lesssim \max\{MA(2tB), M^n\}.$$

By (5.5) we have

$$M^n \operatorname{Cap}(S) \lesssim \int_S |\nabla u|^n \leq \int_U \psi^n |\nabla u|^n$$

and so we conclude that

$$M^n \operatorname{Cap}(S) \lesssim \max\{MA(2tB), M^n\} \leq MA(2tB) + M^n.$$

The claim follows. □

We are now ready to wrap things up.

Proof of Theorem 1.1. We use the setup and the notation introduced in the beginning of this section. Let $C_0 > 0$ be the constant from Lemma 5.3, and $D = D(n) > 0$ be the constant from Lemma 2.1. As we have seen, we can apply this lemma to the measure A if $q \geq 2$ as we may assume. It follows that there exists a ball $B = B(a, r)$ such that

$$A(\tfrac{1}{2}B) > C_0 \text{ and } A(4B) \leq DA(\tfrac{1}{2}B).$$

On the other hand, by Lemma 5.3 there exists $k \in \{1, \dots, q\}$ such that for the condenser $S = S_k$ as defined in (5.4) with $t = 2$, we have

$$\operatorname{Cap}(S) \gtrsim q^{1/(n-1)}.$$

This implies that if q is larger than a constant only depending on n and K (as we may assume), then $\operatorname{Cap}(S) \geq 2C_1$, where $C_1 = C_1(n, K)$ is the constant in Lemma 5.4. Combining Lemma 5.4 for $t = 2$ and $u = u_k$ with Lemma 4.2, we obtain

$$\begin{aligned}DA(\tfrac{1}{2}B) \geq A(4B) &\gtrsim \operatorname{Cap}(S) \sup_{x \in B} u(x)^{n-1} \\ &\gtrsim q^{1/(n-1)} \sup_{x \in B} u(x)^{n-1} \gtrsim q^{1/(n-1)} A(\tfrac{1}{2}B).\end{aligned}$$

In the previous inequalities all implicit multiplicative constants only depend on n and K. It follows that the previous inequality is impossible if q exceeds a constant only depending on n and K. □

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