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SINGULAR INTEGRAL OPERATORS WITH ROUGH KERNELS ON CENTRAL MORREY SPACES WITH VARIABLE EXPONENT

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Abstract. In this paper we define the λ-central BMO spaces and the central Morrey spaces with variable exponent. We obtain the boundedness of the singular integral operator $T_{\Omega,\alpha}$ and its commutator $[b, T_{\Omega,\alpha}]$ on central Morrey spaces with variable exponent, where $\Omega \in L^s(\mathrm{S}^{n-1})$ for $s \geq 1$ be homogeneous function of degree zero, $0 \leq \alpha < n$ and b be λ-central BMO functions. As applications, we consider the regularity in the central Morrey spaces with variable exponent of strong solutions to nondivergence elliptic equations with VMO coefficients.

1. Introduction

The study of Morrey spaces can be traced to the work of Morrey [16] on the regularity of solutions of some partial differential equations. In [1], Alvarez, Lakey and Guzmán-Partida introduced λ-central bounded mean oscillation spaces and central Morrey spaces, which are generalizations of spaces of bounded central mean oscillations. These λ-central bounded mean oscillation spaces, Morrey type spaces and related functional spaces have interesting applications in studying boundedness of operators including singular integrals and Hausdorff operators; see, for example [5, 6, 11, 21, 22, 23, 29, 30]. On the other hand, due to their wide applications in electrorheological fluids [19], image processing [2, 3] and partial dierential equations with non-standard growth [12, 31], the theory of function spaces with variable exponents have attracted a lot of attentions in recent years. In particular, such theory have achieved great progresses after the notable work of Kováčik and Rákosník [14] in 1991; see [6, 8] and the references therein. In 2015, Mizuta, Ohno and Shimomura introduced the non-homogeneous central Morrey spaces of variable exponent in [15].

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Recently, Wang et al. introduced the central BMO spaces with variable exponent and gave the boundedness of some operators in [25].

Motivated by [1,11,15,25], we aim to develop the λ-central BMO spaces and central Morrey spaces in the setting of variable exponents. We will study the boundedness of the singular integral operator with rough kernel $T_{\Omega,\alpha}$ and its commutator $[b, T_{\Omega,\alpha}]$ on central Morrey spaces with variable exponent. We will also consider the regularity in the central Morrey spaces with variable exponent of strong solutions to nondivergence elliptic equations with VMO coefficients.

Let us explain the outline of this article. In Section 2, we first briefly recall some standard notations and lemmas in variable Lebesgue spaces. Then we will define the λ-central BMO spaces and central Morrey space with variable exponent, and give some properties of these spaces. In Section 3, we will establish the boundedness for Calderón-Zygmund operators and fractional integral operators on central Morrey spaces with variable exponent. Subsequently the boundedness of singular integral operator with rough kernel $T_{\Omega,\sigma}$ and its commutator $[b, T_{\Omega,\sigma}]$ on central Morrey spaces with variable exponent will also be obtained in Section 3. In Section 4, we will consider the regularity in the central Morrey spaces with variable exponent of strong solutions to nondivergence elliptic equations with VMO coefficients.

In addition, we denote the Lebesgue measure and the characteristic function of a measurable set $A \subset \mathbf{R}^n$ by $|A|$ and χ_A respectively. The notation $f \approx g$ means that there exist constants $C_1, C_2 > 0$ such that $C_1 g \le f \le C_2 g$.

2. $\boldsymbol{\lambda}$-central BMO spaces and central Morrey spaces with variable exponent

In this section we first recall some basic properties of variable Lebesgue spaces, and then introduce λ-central BMO spaces and central Morrey spaces with variable exponent.

Given an open set $E \subset \mathbf{R}^n$, and a measurable function $p(\cdot)\colon E \longrightarrow [1,\infty)$, $L^{p(\cdot)}(E)$ denotes the set of measurable functions f on E such that for some $\eta > 0$,

$$\int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} dx < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg–Nakano norm

$$\|f\|_{L^{p(\cdot)}(E)} = \inf\left\{\eta > 0\colon \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} dx \le 1\right\}.$$

These spaces are referred as variable L^p spaces, since they generalized the standard L^p spaces: if $p(x) = p$ is constant, then $L^{p(\cdot)}(E)$ is isometrically isomorphic to $L^p(E)$.

The space $L^{p(\cdot)}_{\mathrm{loc}}(E)$ is defined by

$$L^{p(\cdot)}_{\mathrm{loc}}(E) := \{f\colon f \in L^{p(\cdot)}(F) \text{ for all compact subsets } F \subset E\}.$$

Define $\mathcal{P}(E)$ to be set of measurable functions $p(\cdot)\colon E \longrightarrow [1,\infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x)\colon x \in E\} > 1, \quad p^+ = \operatorname{ess\,sup}\{p(x)\colon x \in E\} < \infty.$$

Denote $p'(x) = p(x)/(p(x) - 1)$.

Let f be a locally integrable function. The Hardy–Littlewood maximal operator is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B \cap E} |f(y)|\, dy,$$

where the supremum is taken over all balls B containing x. Let $\mathcal{B}(E)$ be the set of $p(\cdot) \in \mathcal{P}(E)$ such that M is bounded on $L^{p(\cdot)}(E)$.

In variable L^p spaces there are some important lemmas as follows.

Lemma 2.1. [3] *Given an open set $E \subset \mathbf{R}^n$. If $p(\cdot) \in \mathcal{P}(E)$ and satisfies*

$$|p(x) - p(y)| \le \frac{C}{-\log(|x-y|)}, \quad |x-y| \le 1/2, \tag{2.1}$$

and

$$|p(x) - p(y)| \le \frac{C}{\log(|x|+e)}, \quad |y| \ge |x|, \tag{2.2}$$

then $p(\cdot) \in \mathcal{B}(E)$, that is the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(E)$.

Lemma 2.2. [14] *Given an open set $E \subset \mathbf{R}^n$ and let $p(\cdot) \in \mathcal{P}(E)$. If $f \in L^{p(\cdot)}(E)$ and $g \in L^{p'(\cdot)}(E)$, then fg is integrable on E and*

$$\int_E |f(x)g(x)|\, dx \le r_p \|f\|_{L^{p(\cdot)}(E)} \|g\|_{L^{p'(\cdot)}(E)},$$

where

$$r_p = 1 + 1/p^- - 1/p^+.$$

This inequality is named the generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 2.3. [13] *Let $p(\cdot) \in \mathcal{B}(\mathbf{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbf{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \le C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbf{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbf{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2},$$

where $0 < \delta_1, \delta_2 < 1$ depend on $p(\cdot)$.

Lemma 2.4. [13] *Suppose $p(\cdot) \in \mathcal{B}(\mathbf{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbf{R}^n$,*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbf{R}^n)} \le C.$$

Now we recall that the λ-central bounded mean oscillation space and the central Morrey space in [1] are defined as follows.

Definition 2.1. Let $1 < q < \infty$ and $\lambda < 1/n$. The λ-central bounded mean oscillation space $\mathrm{CBMO}^{q,\lambda}(\mathbf{R}^n)$ is defined by

$$\mathrm{CBMO}^{q,\lambda}(\mathbf{R}^n) = \left\{ f \in L^q_{\mathrm{loc}}(\mathbf{R}^n) \colon \|f\|_{\mathrm{CBMO}^{q,\lambda}(\mathbf{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\mathrm{CBMO}^{q,\lambda}(\mathbf{R}^n)} = \sup_{R>0} \left(\frac{1}{|B(0,R)|^{1+\lambda q}} \int_{B(0,R)} |f(x) - f_{B(0,R)}|^q dx \right)^{1/q}.$$

Definition 2.2. Let $1 < q < \infty$ and $\lambda \in \mathbf{R}$. The central Morrey space $\dot{\mathcal{B}}^{q,\lambda}(\mathbf{R}^n)$ is defined by

$$\dot{\mathcal{B}}^{q,\lambda}(\mathbf{R}^n) = \left\{ f \in L^q_{\mathrm{loc}}(\mathbf{R}^n) \colon \|f\|_{\dot{\mathcal{B}}^{q,\lambda}(\mathbf{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\dot{\mathcal{B}}^{q,\lambda}(\mathbf{R}^n)} = \sup_{R>0} \left(\frac{1}{|B(0,R)|^{1+\lambda q}} \int_{B(0,R)} |f(x)|^q \, dx \right)^{1/q}.$$

Next we extend the above definitions to the case of function spaces with variable exponent.

Definition 2.3. Let $q(\cdot) \in \mathcal{P}(\mathbf{R}^n)$ and $\lambda < 1/n$. The λ-central BMO space with variable exponent $\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbf{R}^n)$ is defined by

$$\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbf{R}^n) = \left\{ f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbf{R}^n) \colon \|f\|_{\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbf{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbf{R}^n)} = \sup_{R>0} \frac{\|(f - f_{B(0,R)})\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}{|B(0,R)|^{\lambda} \|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}.$$

Definition 2.4. Let $q(\cdot) \in \mathcal{P}(\mathbf{R}^n)$ and $\lambda \in \mathbf{R}$. The central Morrey space with variable exponent $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)$ is defined by

$$\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n) = \left\{ f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbf{R}^n) \colon \|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)} = \sup_{R>0} \frac{\|f\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}{|B(0,R)|^{\lambda} \|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}.$$

Remark 2.1 If $q(\cdot) = q$ is constant, then we can easily get the Definition 2.1 and Definition 2.2 respectively.

Remark 2.2. The space $\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbf{R}^n)$ when $\lambda = 0$ is just the space $\mathrm{CBMO}^{q(\cdot)}$ $(\mathbf{R}^n)$ defined in [25].

Remark 2.3. Denote by $\mathrm{CMO}^{q(\cdot),\lambda}(\mathbf{R}^n)$ and $\mathcal{B}^{q(\cdot),\lambda}(\mathbf{R}^n)$ the inhomogeneous versions of the λ-central BMO space and the central Morrey space with variable exponent, which are defined, respectively, by taking the supremum over $R \geq 1$ in Definition 2.3 and Definition 2.4 instead of $R > 0$ there. Obviously, $\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbf{R}^n) \subset \mathrm{CMO}^{q(\cdot),\lambda}(\mathbf{R}^n)$ for $q(\cdot) \in \mathcal{P}(\mathbf{R}^n)$ and $\lambda < 1/n$, and $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n) \subset \mathcal{B}^{q(\cdot),\lambda}(\mathbf{R}^n)$ for $q(\cdot) \in \mathcal{P}(\mathbf{R}^n)$ and $\lambda \in \mathbf{R}$. Recall that the inhomogeneous central Morrey space of variable exponent was introduced by Mizuta, Ohno and Shimomura in [15].

Now we give the relationship between the spaces $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)$ and $\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbf{R}^n)$.

Theorem 2.1. *Let $q(\cdot) \in \mathcal{B}(\mathbf{R}^n)$ and $\lambda < 1/n$. Then $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)$ is a Banach space continuously included in* $\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbf{R}^n)$.

Proof. Firstly we will prove $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)$ is a Banach space. Let $\{f_k\}$ be a Cauchy sequence in $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)$. Then for $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbf{N}$ such that $\|f_i - f_j\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)} < \varepsilon$ for $i, j > N(\varepsilon)$. Let $n_k = N(\frac{1}{2^k})$. Then there is a subsequence, denoted by $\{f_{n_k}\}$, such that

$$\|f_{n_{k+1}} - f_{n_k}\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)} < \frac{1}{2^k}.$$

We set

$$f(x)=f_{n_1}(x)+\sum_{k=1}^{\infty}(f_{n_{k+1}}(x)-f_{n_k}(x)),\quad x\in\mathbf{R}^n,$$

and

$$g(x)=|f_{n_1}(x)|+\sum_{k=1}^{\infty}|f_{n_{k+1}}(x)-f_{n_k}(x)|,\quad x\in\mathbf{R}^n.$$

Let $S_N(f)=f_{n_1}+\sum_{k=1}^{N-1}(f_{n_{k+1}}-f_{n_k})=f_{n_N}$ and $S_N(g)=|f_{n_1}|+\sum_{k=1}^{N-1}|f_{n_{k+1}}-f_{n_k}|$. Then by the Minkowski inequality, we have

$$\begin{aligned}
&\|S_N(g)\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)}=\sup_{R>0}\frac{\|S_N(g)\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}\\
&\le\sup_{R>0}\frac{\|f_{n_1}\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}+\sum_{k=1}^{N-1}\sup_{R>0}\frac{\|(f_{n_{k+1}}-f_{n_k})\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}\\
&\le\|f_{n_1}\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)}+\sum_{k=1}^{N-1}\frac{1}{2^k}\le\|f_{n_1}\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)}+1.
\end{aligned}$$

So we have $\|g\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)}<\infty$. Noting that $|f|\le g$, then $\|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)}\le\|g\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)}<\infty$. Thus we have $f\in\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)$.

Since

$$\begin{aligned}
\lim_{N\to\infty}\|f-f_{n_N}\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)}&\le\lim_{N\to\infty}\sum_{k=N}^{\infty}\|f_{n_{k+1}}-f_{n_k}\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)}\le\lim_{N\to\infty}\sum_{k=N}^{\infty}2^{-k}\\
&=\lim_{N\to\infty}2^{-N+1}=0,
\end{aligned}$$

we know that the sequence $\{f_{n_k}\}$ converges to f in $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)$. Thus the Cauchy sequence $\{f_k\}$ converges to f in $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)$.

By Lemma 2.4 and the generalized Hölder inequality we have

$$\begin{aligned}
&\|f\|_{\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbf{R}^n)}=\sup_{R>0}\frac{\|(f-f_{B(0,R)})\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}\\
&\le C\left(\sup_{R>0}\frac{\|f\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}+\sup_{R>0}\frac{\|f_{B(0,R)}\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}\right)\\
&\le C\left(\sup_{R>0}\frac{\|f\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}+\sup_{R>0}\frac{\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}|f_{B(0,R)}|}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}\right)\\
&\le C\Bigg(\sup_{R>0}\frac{\|f\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}\\
&\qquad+\sup_{R>0}\frac{\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}\left(\frac{1}{|B(0,R)|}\|f\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}\|\chi_{B(0,R)}\|_{L^{q'(\cdot)}(\mathbf{R}^n)}\right)}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}\Bigg)\\
&\le C\sup_{R>0}\frac{\|f\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}\\
&=C\|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)},
\end{aligned}$$

so we obtain $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)$ is continuously included in $\mathrm{CBMO}^{q(\cdot),\lambda}(\mathbf{R}^n)$. Thus we complete the proof of Theorem 2.1.

Similarly, we can obtain the following theorem for the inhomogeneous versions.

Theorem 2.2. *$\mathcal{B}^{q(\cdot),\lambda}(\mathbf{R}^n)$ is a Banach space continuously included in* $\mathrm{CMO}^{q(\cdot),\lambda}$ $(\mathbf{R}^n)$.

3. Singular integral operators with rough kernels

The standard Calderón–Zygmund operator T is defined by

$$Tf(x) = \text{p.v.} \int_{\mathbf{R}^n} K(x-y)f(y)\,dy,$$

with the kernel K satisfying the following size condition:

$$|K(x)| \le C|x|^{-n}, \quad x \ne 0,$$

and some smoothness assumption. The Calderón-Zygmund operator is a direct generalization of the Hilbert transform and the Riesz transform. The former is originated from researches of boundary value of conjugate harmonic functions on the upper half-plane, and the latter is tightly associated to the regularity of solution of second order elliptic equation.

Given $0 < \alpha < n$, the fractional integral operator (also known as the Riesz potential) T_α is defined by

$$T_\alpha f(x) = \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

It is closely related to the Laplacian operator of fractional degree. When $n > 2$ and $\alpha = 2$, $T_\alpha f$ is a solution of Poisson equation $-\triangle u = f$. The importance of fractional integral operators is owing to the fact that they are smooth operators and have been extensively used in various areas such as potential analysis, harmonic analysis and partial differential equations.

For simplicity, we denote by $T_\alpha(0 \le \alpha < n)$ the fractional integral operator when $0 < \alpha < n$, and the standard Calderón–Zygmund operator when $\alpha = 0$ (i.e. $T = T_0$). In [7], Cruz-Uribe et al. obtained the $(L^{p(\cdot)}, L^{q(\cdot)})$-boundedness of $T_\alpha(0 < \alpha < n)$ and the $(L^{p(\cdot)}, L^{p(\cdot)})$-boundedness of $T_\alpha(\alpha = 0)$. We can give the corresponding result about the operator $T_\alpha(0 \le \alpha < n)$ on central Morrey space with variable exponent.

Proposition 3.1. *Let $0 \le \alpha < n$, $p(\cdot)$, $q(\cdot) \in \mathcal{P}(\mathbf{R}^n)$ satisfy conditions (2.1) and (2.2) in Lemma 2.1 with $p(\cdot) < n/\alpha$ and $1/q(\cdot) = 1/p(\cdot) - \alpha/n$. If $\lambda_1 < -\alpha/n$ and $\lambda_2 = \lambda_1 + \alpha/n$, then T_α is bounded from $\dot{\mathcal{B}}^{p(\cdot),\lambda_1}(\mathbf{R}^n)$ to $\dot{\mathcal{B}}^{q(\cdot),\lambda_2}(\mathbf{R}^n)$.*

On the other hand, the commutator $[b, T_\alpha]$ generated by a locally integrable function b and the operator $T_\alpha(0 \le \alpha < n)$ is defined by

$$[b, T_\alpha]f(x) = b(x)T_\alpha f(x) - T_\alpha(bf)(x)$$

for suitable functions f. It is well known that the commutator plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order.

Let us recall that the space $\mathrm{BMO}(\mathbf{R}^n)$ consists of all locally integrable functions f such that

$$\|f\|_* = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q|\,dx < \infty,$$

where $f_Q = |Q|^{-1} \int_Q f(y)\,dy$, the supremum is taken over all cubes $Q \subset \mathbf{R}^n$ with sides parallel to the coordinate axes and $|Q|$ denotes the Lebesgue measure of Q.

For $0 < \beta \leq 1$, the Lipschitz function class $\mathrm{Lip}_\beta(\mathbf{R}^n)$ is defined as

$$\mathrm{Lip}_\beta(\mathbf{R}^n) = \left\{ f \colon \|f\|_{\mathrm{Lip}_\beta} = \sup_{x,y\in\mathbf{R}^n; x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty \right\}.$$

Cruz-Uribe et al. obtained the $(L^{p(\cdot)}, L^{p(\cdot)})$-boundedness of $[b, T_\alpha](\alpha = 0)$ in [7] when b is a BMO function. In [28], Wang, Fu and Liu obtained the $(L^{p(\cdot)}, L^{q(\cdot)})$-boundedness of $[b, T_\alpha](0 < \alpha < n)$ when b is a BMO function or Lipschitz funtion. We can also give the corresponding result about the operator $[b, T_\alpha](0 \leq \alpha < n)$ on central Morrey space with variable exponent.

Proposition 3.2. *Suppose that $0 \leq \alpha < n$, $p_1(\cdot)$, $p_2(\cdot)$, $q(\cdot) \in \mathcal{P}(\mathbf{R}^n)$ satisfies conditions (2.1) and (2.2) in Lemma 2.1 with $p_1(\cdot) < n/\alpha$, $p_1'(\cdot) < p_2(\cdot)$ and $1/q(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot) - \alpha/n$. Let $0 \leq \lambda_2 < 1/n$, $\lambda_1 < -\lambda_2 - \alpha/n$ and $\lambda = \lambda_1 + \lambda_2 + \alpha/n$. If $b \in \mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)$, then the commutator $[b, T_\alpha]$ is bounded from $\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)$ to $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)$ and satisfies the following inequality:*

$$\|[b, T_\alpha]f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)} \leq C\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}.$$

Remark 3.1. We will not give the proofs of Propositions 3.1 and 3.2 since they are direct consequences of Theorems 3.1 and 3.2 below.

Remark 3.2. Our results in this paper remain true for the inhomogeneous versions of λ-central BMO spaces and central Morrey spaces with variable exponent.

Furthermore, we will consider the more general singular integral operators with rough kernels on central Morrey space with variable exponent.

Suppose that S^{n-1} denote the unit sphere in $\mathbf{R}^n(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let T_Ω be a Calderón–Zygmund singular integral operator, defined as

$$T_\Omega f(x) = \mathrm{p.v.} \int_{\mathbf{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy,$$

where $\Omega \in L^s(\mathrm{S}^{n-1})$ for $s \geq 1$ is homogeneous of degree zero and has mean value zero on the unit sphere. It is well known that T_Ω is of type $(p,p)(1 < p < \infty)$, see [9] for details.

Let $\Omega \in L^s(\mathrm{S}^{n-1})$ for $s \geq 1$ be a homogeneous function of degree zero. For $0 < \alpha < n$, the homogeneous fractional integral operator with rough kernel $T_{\Omega,\alpha}$ is defined by

$$T_{\Omega,\alpha} f(x) = \int_{\mathbf{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)\,dy.$$

In 1971, Muckenhoupt and Wheeden [17] gave the weighted (L^p, L^q)-boundedness of $T_{\Omega,\alpha}$ with power weights. Recently, Tan and Liu [24] gave the $(L^{p(\cdot)}, L^{q(\cdot)})$-boundedness of $T_{\Omega,\alpha}(0 < \alpha < n)$, and Wang [26] gave the $(L^{p(\cdot)}, L^{p(\cdot)})$-boundedness of $T_{\Omega,\alpha}(\alpha = 0)$.

For simplicity, we denote by $T_{\Omega,\alpha}(0 \leq \alpha < n)$ the fractional integral operator with rough kernel when $0 < \alpha < n$, and the Calderón–Zygmund singular integral operator with rough kernel when $\alpha = 0$ (i.e. $T_\Omega = T_{\Omega,0}$).

For $0 \leq \alpha < n$, let b be a locally integrable function, the commutator of singular integral operator with rough kernel $[b, T_{\Omega,\alpha}]$ is defined by

$$[b, T_{\Omega,\alpha}]f(x) = b(x)T_{\Omega,\alpha}f(x) - T_{\Omega,\alpha}(bf)(x) = \int_{\mathbf{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\sigma}}(b(x)-b(y))f(y)\,dy.$$

Fu, Lin and Lu established λ-central BMO estimates for commutators of singular integral operators with rough kernels on central Morrey spaces in [11]. In [26] and [27], the third author of this paper gave some boundedness of $[b, T_{\Omega,\alpha}]$ on function spaces with variable exponent.

Next we will prove the boundedness of the singular integral operators with rough kernels $T_{\Omega,\alpha}$ and its commutator $[b, T_{\Omega,\alpha}]$ on central Morrey spaces with variable exponent.

Theorem 3.1. *Suppose that $0 \le \alpha < n$, $\Omega \in L^s(\mathrm{S}^{n-1})$, $s > n/(n-\alpha)$. Let $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbf{R}^n)$ satisfy conditions (2.1) and (2.2) in Lemma 2.1 with $p(\cdot) < n/\alpha$ and $1/q(\cdot) = 1/p(\cdot) - \alpha/n$. If λ_1 satisfies either of the following two conditions:*

(i) *when $s' = p(\cdot)$, there is $\lambda_1 < -\alpha/n$;*

(ii) *when $q(\cdot) = s$, there is $\lambda_1 < -\alpha/n - 1/s$,*

and $\lambda_2 = \lambda_1 + \alpha/n$, then $T_{\Omega,\alpha}$ is bounded from $\dot{\mathcal{B}}^{p(\cdot),\lambda_1}(\mathbf{R}^n)$ to $\dot{\mathcal{B}}^{q(\cdot),\lambda_2}(\mathbf{R}^n)$.

Theorem 3.2. *Suppose that $0 \le \alpha < n$, $\Omega \in L^s(\mathrm{S}^{n-1})$, $s > n/(n-\alpha)$, $p_1(\cdot)$, $p_2(\cdot)$, $q(\cdot) \in \mathcal{P}(\mathbf{R}^n)$ satisfies conditions (2.1) and (2.2) in Lemma 2.1 with $p_1(\cdot) < n/\alpha$, $p_1'(\cdot) < p_2(\cdot)$ and $1/q(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot) - \alpha/n$. Let $0 \le \lambda_2 < 1/n$, λ_1 satisfy either of the following two conditions:*

(i) *when $1/p_2(\cdot) + 1/s = 1/p_1'(\cdot)$, there is $\lambda_1 < -\lambda_2 - \alpha/n$;*

(ii) *when $1/p_1(\cdot) = \alpha/n + 1/s$, there is $\lambda_1 < -\lambda_2 - \alpha/n - 1/s$,*

and $\lambda = \lambda_1 + \lambda_2 + \alpha/n$. If $b \in \mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)$, then the commutator $[b, T_{\Omega,\alpha}]$ is bounded from $\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)$ to $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)$ and the following inequality holds:

$$\|[b, T_{\Omega,\alpha}]f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)} \le C\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}.$$

Here, we give only the proof of Theorem 3.2 and omit the proof of Theorem 3.1 due to their similarity. To prove the Theorem 3.2, we need the following lemmas.

Lemma 3.1. [18] *Define a variable exponent $\tilde{q}(\cdot)$ by $\frac{1}{p(x)} = \frac{1}{\tilde{q}(x)} + \frac{1}{q}$ $(x \in \mathbf{R}^n)$. Then we have*

$$\|fg\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le C\|f\|_{L^{\tilde{q}(\cdot)}(\mathbf{R}^n)}\|g\|_{L^q(\mathbf{R}^n)}$$

for all measurable functions f and g.

Lemma 3.2. [8] *Let $p(\cdot) \in \mathcal{P}(\mathbf{R}^n)$ satisfy conditions (2.1) and (2.2) in Lemma 2.1. Then*

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)} \approx \begin{cases} |Q|^{\frac{1}{p(x)}} & \text{if } |Q| \le 2^n \text{ and } x \in Q, \\ |Q|^{\frac{1}{p(\infty)}} & \text{if } |Q| \ge 1, \end{cases}$$

for every cube (or ball) $Q \subset \mathbf{R}^n$, where $p(\infty) = \lim_{x\to\infty} p(x)$.

Lemma 3.3. [8] *Let $p(\cdot), q(\cdot), s(\cdot) \in \mathcal{P}(\mathbf{R}^n)$ be such that*

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}$$

for almost every $x \in \mathbf{R}^n$. Then

$$\|fg\|_{L^{s(\cdot)}(\mathbf{R}^n)} \le 2\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)}\|g\|_{L^{q(\cdot)}(\mathbf{R}^n)}$$

for all $f \in L^{p(\cdot)}(\mathbf{R}^n)$ and $g \in L^{q(\cdot)}(\mathbf{R}^n)$.

Lemma 3.4. [6] *Given a measurable set E and $p(\cdot) \in \mathcal{P}(E)$, let $f\colon E \times E \to \mathbf{R}$ be a measurable function (with respect to the product measure) such that for almost* *every* $y \in E$, $f(\cdot, y) \in L^{p(\cdot)}(E)$. *Then*

$$\left\| \int_E f(\cdot, y)\, dy \right\|_{L^{p(\cdot)}(E)} \le C \int_E \|f(\cdot, y)\|_{L^{p(\cdot)}(E)}\, dy.$$

Proof Theorem 3.2. Let f be a function in $\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)$. For fixed $R > 0$, denote $B(0, R)$ by B. We need to prove

$$\|[b, T_{\Omega,\alpha}]f\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)} \le C|B|^{\lambda}\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)},$$

where C is a constant independent of R.

Using the Minkowski inequality we write

$$\begin{aligned} \|[b, T_{\Omega,\alpha}]f\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)} &\le \|(b - b_B)(T_{\Omega,\alpha}(f\chi_{2B}))\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)} \\ &\quad + \|(b - b_B)(T_{\Omega,\alpha}(f\chi_{(2B)^c}))\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)} \\ &\quad + \|T_{\Omega,\alpha}((b - b_B)(f\chi_{2B}))\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)} \\ &\quad + \|T_{\Omega,\alpha}((b - b_B)(f\chi_{(2B)^c}))\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)} \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.1}$$

We first estimate I_1. Set $1/t(\cdot) = 1/p_1(\cdot) - \alpha/n$, then $1/q(\cdot) = 1/p_2(\cdot) + 1/t(\cdot)$. Noticing that $1 < p_1(\cdot) < n/\alpha$, by Lemma 2.3, Lemma 3.2, Lemma 3.3 and the $(L^{p_1(\cdot)}(\mathbf{R}^n), L^{t(\cdot)}(\mathbf{R}^n))$-boundedness of $T_{\Omega,\alpha}$ in [24], we have

$$\begin{aligned} I_1 &\le \|(b - b_B)\chi_B\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\|T_{\Omega,\alpha}(f\chi_{2B})\|_{L^{t(\cdot)}(\mathbf{R}^n)} \\ &\le C\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}|B|^{\lambda_2}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\|f\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)} \\ &\le C\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}|B|^{\lambda_2}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}|2B|^{\lambda_1}\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)} \\ &\le C\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}|B|^{\lambda_1+\lambda_2}\|\chi_B\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbf{R}^n)} \\ &\le C|B|^{\lambda}\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)}, \end{aligned} \tag{3.2}$$

where

$$\|\chi_B\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbf{R}^n)} \approx |B|^{1/p_1(\cdot)+1/p_2(\cdot)} = |B|^{1/q(\cdot)+\alpha/n} \approx |B|^{\alpha/n}\|\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)}.$$

Next we estimate I_3. Denote $1/l(\cdot) = 1/p_1(\cdot)+1/p_2(\cdot)$, then $1/q(\cdot) = 1/l(\cdot)-\alpha/n$. Since $1 < p_1(\cdot) < n/\alpha$ and $p_1'(\cdot) < p_2(\cdot) < \infty$, there is $1 < l(\cdot) < n/\alpha$. Using Lemma 2.4, Lemma 3.2, Lemma 3.3 and the $(L^{l(\cdot)}(\mathbf{R}^n), L^{q(\cdot)}(\mathbf{R}^n))$-boundedness of $T_{\Omega,\alpha}$, we have

$$\begin{aligned} I_3 &\le C\|(b - b_B)f\chi_{2B}\|_{L^{l(\cdot)}(\mathbf{R}^n)} \\ &\le C\|(b - b_B)\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\|f\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)} \\ &\le C\left[\|(b - b_{2B})\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)} + |b_{2B} - b_B|\|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\right]\|f\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)} \\ &\le C\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}|2B|^{\lambda_2}\|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\|f\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)} \\ &\le C\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}|2B|^{\lambda_2}\|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}|2B|^{\lambda_1}\|\chi_{2B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)} \\ &\le C\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}|B|^{\lambda_1+\lambda_2}\|\chi_B\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbf{R}^n)} \\ &\le C|B|^{\lambda}\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned}|b_{2B}-b_B| &\le \frac{1}{|B|}\|(b-b_{2B})\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\|\chi_{2B}\|_{L^{p_2'(\cdot)}(\mathbf{R}^n)}\\ &\le C\|(b-b_{2B})\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\frac{1}{\|\chi_{2B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}}.\end{aligned}$$

To estimate I_2 and I_4, we will consider two cases, respectively.

(i) $1/p_2(\cdot)+1/s=1/p_1'(\cdot)$. On the one hand, $1/p_2(\cdot)+1/s=1/p_1'(\cdot)$ implies that $1/s<1/p_1'(\cdot)$, thus $1-1/s-1/p_1(\cdot)>0$. Given $x\in B$, by Lemma 3.1 and the generalized Hölder inequality we have

$$\begin{aligned}&|T_{\Omega,\alpha}(f\chi_{(2B)^c})(x)|\\ &\le \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\frac{|\Omega(x-y)||f(y)|}{|x-y|^{n-\alpha}}dy\\ &\le C\sum_{k=1}^{\infty}|2^kB|^{-1+\alpha/n}\int_{2^{k+1}B\setminus 2^kB}|\Omega(x-y)||f(y)|dy\\ &\le C\sum_{k=1}^{\infty}|2^kB|^{-1+\alpha/n}\|\Omega(x-\cdot)\chi_{2^{k+1}B}(\cdot)\|_{L^{p_1'(\cdot)}(\mathbf{R}^n)}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\\ &\le C\sum_{k=1}^{\infty}|2^kB|^{-1+\alpha/n}\|\Omega(x-\cdot)\chi_{2^{k+1}B}(\cdot)\|_{L^s(\mathbf{R}^n)}\|\chi_{2^{k+1}B}(\cdot)\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}.\end{aligned}$$

For $x\in B$ and $y\in 2^{k+1}B$, $x-y\in 2^{k+2}B$. Noticing that Ω is homogeneous of degree zero and $\Omega\in L^s(\mathrm{S}^{n-1})$, we obtain

$$\begin{aligned}\|\Omega(x-\cdot)\chi_{2^{k+1}B}(\cdot)\|_{L^s(\mathbf{R}^n)} &= \left(\int_{2^{k+1}B}|\Omega(x-y)|^s dy\right)^{1/s}\\ &\le \left(\int_{2^{k+2}B}|\Omega(z)|^s dz\right)^{1/s}\\ &= \left(\int_0^{2^{k+2}R}\int_{\mathrm{S}^{n-1}}|\Omega(z')|^s d\sigma(z')r^{n-1}dr\right)^{1/s}\\ &= C\|\Omega\|_{L^s(\mathrm{S}^{n-1})}|2^kB|^{1/s}.\end{aligned}\tag{3.4}$$

When $|2^{k+1}B|\le 2^n$ and $x\in 2^{k+1}B$, by Lemma 3.2 and $1/p_2(\cdot)+1/s=1/p_1'(\cdot)$ we have

$$\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\approx|2^{k+1}B|^{\frac{1}{p_2(x)}}\approx\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbf{R}^n)}|2^{k+1}B|^{-\frac{1}{s}}.$$

When $|2^{k+1}B|\ge 1$ we have

$$\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\approx|2^{k+1}B|^{\frac{1}{p_2(\infty)}}\approx\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbf{R}^n)}|2^{k+1}B|^{-\frac{1}{s}}.$$

So we obtain

$$\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\approx\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbf{R}^n)}|2^{k+1}B|^{-\frac{1}{s}}.$$

Thus, it follows from Lemma 2.4 and $\lambda_1 < -\lambda_2 - \alpha/n \le -\alpha/n$ that

$$
\begin{aligned}
&|T_{\Omega,\alpha}(f\chi_{(2B)^c})(x)| \\
&\le C\sum_{k=1}^{\infty} |2^kB|^{-1+\alpha/n}|2^kB|^{1/s}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)} \\
&\le C\sum_{k=1}^{\infty} |2^kB|^{-1+\alpha/n+1/s}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}|2^kB|^{\lambda_1}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)} \\
&\le C\sum_{k=1}^{\infty} |2^kB|^{-1+\alpha/n+1/s}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}|2^kB|^{\lambda_1}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)} \\
&\qquad \cdot \|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbf{R}^n)}|2^{k+1}B|^{-\frac{1}{s}} \\
&\le C\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}\sum_{k=1}^{\infty}|2^kB|^{-1+\alpha/n+1/s+\lambda_1+1-1/s} \\
&\le C\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}|B|^{\alpha/n+\lambda_1}.
\end{aligned}
$$

Since $1/q(\cdot) = 1/p_2(\cdot) + 1/t(\cdot)$, by Lemma 3.2 and Lemma 3.3 we have

$$
\begin{aligned}
I_2 &= \|(b-b_B)(T_{\Omega,\alpha}(f\chi_{(2B)^c}))\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)} \\
&\le C\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}|B|^{\alpha/n+\lambda_1}\|(b-b_B)\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)} \\
&\le C\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}|B|^{\alpha/n+\lambda_1}\|(b-b_B)\chi_B\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\|\chi_B\|_{L^{t(\cdot)}(\mathbf{R}^n)} \qquad (3.5)\\
&\le C\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}|B|^{\alpha/n+\lambda_1+\lambda_2}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\|\chi_B\|_{L^{t(\cdot)}(\mathbf{R}^n)} \\
&\le C|B|^{\lambda}\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)}
\end{aligned}
$$

On the other hand, $1/p_2(\cdot)+1/s = 1/p_1'(\cdot)$ implies that $1-1/s-1/p_1(\cdot)-1/p_2(\cdot) = 0$. Given $x \in B$, noticing that $\lambda_2 \ge 0$ and $\lambda_1 < -\lambda_2 - \alpha/n$, by (3.4), Lemma 3.1, the Minkowski inequality and the generalized Hölder inequality we have

$$
\begin{aligned}
&|T_{\Omega,\alpha}((b-b_B)f\chi_{(2B)^c})(x)| \\
&\le \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\frac{|b(y)-b_B||\Omega(x-y)||f(y)|}{|x-y|^{n-\alpha}}dy \\
&\le C\sum_{k=1}^{\infty}|2^kB|^{-1+\alpha/n}\int_{2^{k+1}B\setminus 2^kB}|b(y)-b_B||\Omega(x-y)||f(y)|dy \\
&\le C\sum_{k=1}^{\infty}|2^kB|^{-1+\alpha/n}\|(b(\cdot)-b_B)\Omega(x-\cdot)\chi_{2^{k+1}B}(\cdot)\|_{L^{p_1'(\cdot)}(\mathbf{R}^n)}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)} \\
&\le C\sum_{k=1}^{\infty}|2^kB|^{-1+\alpha/n}\|(b(\cdot)-b_B)\chi_{2^{k+1}B}(\cdot)\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\|\Omega(x-\cdot)\chi_{2^{k+1}B}(\cdot)\|_{L^s(\mathbf{R}^n)} \\
&\qquad \cdot \|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)} \\
&\le C\sum_{k=1}^{\infty}|2^kB|^{-1+\alpha/n+1/s}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}|2^kB|^{\lambda_1}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)} \\
&\qquad \cdot \Big[\|(b-b_{2^{k+1}B})\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)} + |b_{2^{k+1}B}-b_B|\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\Big].
\end{aligned}
$$

For $\lambda_2 \geq 0$,

$$\begin{aligned}
|b_{2^{k+1}B} - b_B| &\leq \sum_{j=0}^{k} |b_{2^{j+1}B} - b_{2^jB}| \\
&\leq \sum_{j=0}^{k} \frac{1}{|2^jB|} \int_{2^jB} |b(y) - b_{2^{j+1}B})| dy \\
&\leq C \sum_{j=0}^{k} \frac{1}{|2^jB|} \|(b(\cdot) - b_{2^{j+1}B})\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)} \|\chi_{2^{j+1}B}\|_{L^{p_2'(\cdot)}(\mathbf{R}^n)} \\
&\leq C \sum_{j=0}^{k} \frac{1}{|2^jB|} \|(b(\cdot) - b_{2^{j+1}B})\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)} \frac{|2^{j+1}B|}{\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}} \\
&\leq C\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)} \sum_{j=0}^{k} |2^{j+1}B|^{\lambda_2} \|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)} \frac{1}{\|\chi_{2^{j+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}} \\
&\leq C\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)} (k+1)|2^{k+1}B|^{\lambda_2}.
\end{aligned} \tag{3.6}$$

So by Lemma 3.2 we have

$$\begin{aligned}
&|T_{\Omega,\alpha}((b - b_B)f\chi_{(2B)^c})(x)| \\
&\leq C \sum_{k=1}^{\infty} |2^kB|^{-1+\alpha/n+1/s+\lambda_1} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)} \|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)} \\
&\qquad \cdot \|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)} k|2^{k+1}B|^{\lambda_2} \|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)} \\
&\leq C\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)} \sum_{k=1}^{\infty} k|2^kB|^{-1+\alpha/n+1/s+\lambda_1+\lambda_2+1-1/s} \\
&\leq C\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)} |B|^{\alpha/n+\lambda_1+\lambda_2} \\
&= C|B|^{\lambda} \|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
I_4 &= \|T_{\Omega,\alpha}((b - b_B)f\chi_{(2B)^c})\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)} \\
&\leq C|B|^{\lambda} \|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)} \|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)} \|\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)}.
\end{aligned} \tag{3.7}$$

(ii) $1/p_1(\cdot) = \alpha/n + 1/s$. Similarly to (3.4), when $y \in 2^{k+1}B$, it is true that

$$\|\Omega(\cdot - y)\chi_B(\cdot)\|_{L^s(\mathbf{R}^n)} \leq C\|\Omega\|_{L^s(\mathrm{S}^{n-1})} |2^kB|^{1/s}. \tag{3.8}$$

On the one hand, $1/p_1(\cdot) = \alpha/n + 1/s$ implies that $q(\cdot) < s$ and $1/q(\cdot) - 1/p_2(\cdot) - 1/s = 0$. By (3.8), Lemma 3.1, Lemma 3.2, Lemma 3.4, the generalized Hölder inequality and the fact that $\lambda_1 < -\lambda_2 - \alpha/n - 1/s \leq -\alpha/n - 1/s$, we have

$$\begin{aligned}
I_2 &\leq C \sum_{k=1}^{\infty} \int_{2^{k+1}B \backslash 2^kB} \left\| \frac{(b(\cdot) - b_B)\Omega(\cdot - y)\chi_B(\cdot)}{|\cdot - y|^{n-\alpha}} \right\|_{L^{q(\cdot)}(\mathbf{R}^n)} |f(y)|\, dy \\
&\leq C \sum_{k=1}^{\infty} |2^kB|^{-1+\alpha/n} \|(b(\cdot) - b_B)\Omega(\cdot - y)\chi_B(\cdot)\|_{L^{q(\cdot)}(\mathbf{R}^n)} \int_{2^{k+1}B} |f(y)|\, dy \\
&\leq C \sum_{k=1}^{\infty} |2^kB|^{-1+\alpha/n} \|(b - b_B)\chi_B\|_{L^{p_2(\cdot)}(\mathbf{R}^n)} \|\Omega(\cdot - y)\chi_B(\cdot)\|_{L^s(\mathbf{R}^n)} \int_{2^{k+1}B} |f(y)|\, dy
\end{aligned}$$

$$
\begin{aligned}
(3.9)\quad &\le C\sum_{k=1}^{\infty}|2^kB|^{-1+\alpha/n}\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}|B|^{\lambda_2}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}\\
&\quad\cdot|2^kB|^{1/s}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbf{R}^n)}\\
&\le C\sum_{k=1}^{\infty}|2^kB|^{-1+\alpha/n+1/s}|B|^{\lambda_2}\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}\\
&\quad\cdot\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}|2^{k+1}B|^{\lambda_1}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\|\chi_B\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\|\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbf{R}^n)}\\
&\le C\sum_{k=1}^{\infty}|2^kB|^{-1+\alpha/n+1/s+\lambda_1+1}|B|^{\lambda_2}\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}\\
&\quad\cdot\|\chi_B\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\\
&\le C\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}|B|^{\alpha/n+1/s+\lambda_1+\lambda_2+1/p_2(\cdot)}\\
&\le C|B|^{\lambda}\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)}.
\end{aligned}
$$

On the other hand, $p_1'(\cdot) < p_2(\cdot) < \infty$ implies that $1 - 1/p_1(\cdot) - 1/p_2(\cdot) > 0$. Let $1/p_1'(\cdot) = 1/p_2(\cdot) + 1/m$, by (3.6), (3.8), Lemma 3.2, Lemma 3.4, the Minkowski inequality, the generalized Hölder inequality and the fact that $\lambda_1 < -\lambda_2 - \alpha/n - 1/s$, we have

$$
\begin{aligned}
I_4 &= \|T_{\Omega,\alpha}((b-b_B)f\chi_{(2B)^c})\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)}\\
&\le C\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\left\|\frac{\Omega(\cdot-y)\chi_B(\cdot)}{|\cdot-y|^{n-\alpha}}\right\|_{L^{q(\cdot)}(\mathbf{R}^n)}|b(y)-b_B||f(y)|\,dy\\
&\le C\sum_{k=1}^{\infty}|2^kB|^{-1+\alpha/n}\|\Omega(\cdot-y)\chi_B(\cdot)\|_{L^s(\mathbf{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)}|B|^{-1/s}\\
&\quad\cdot\int_{2^{k+1}B}|b(y)-b_B||f(y)|\,dy\\
&\le C\sum_{k=1}^{\infty}|2^kB|^{-1+\alpha/n}\|\Omega\|_{L^s(\mathrm{S}^{n-1})}|2^kB|^{1/s}\|\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)}|B|^{-1/s}\\
&\quad\cdot\|(b-b_B)\chi_{2^{k+1}B}\|_{L^{p_1'(\cdot)}(\mathbf{R}^n)}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\\
&\le C\sum_{k=1}^{\infty}|2^kB|^{-1+\alpha/n+1/s}\|\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)}|B|^{-1/s}\|f\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\\
&\quad\cdot\|(b-b_B)\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\|\chi_{2^{k+1}B}\|_{L^m(\mathbf{R}^n)}\\
&\le C\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}\sum_{k=1}^{\infty}|2^kB|^{\lambda_1-1+\alpha/n+1/s+1/m}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)}|B|^{-1/s}\\
&\quad\cdot\|(b-b_B)\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\\
&\le C\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}\sum_{k=1}^{\infty}|2^kB|^{\lambda_1-1+\alpha/n+1/s+1/m}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)}|B|^{-1/s}\\
&\quad\cdot\Big[\|(b-b_{2^{k+1}B})\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}+|b_{2^{k+1}B}-b_B|\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)}|B|^{-1/s}\sum_{k=1}^{\infty}|2^kB|^{\lambda_1-1+\alpha/n+1/s+1/m}\|\chi_{2^{k+1}B}\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}\\
&\qquad\cdot\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}k|2^{k+1}B|^{\lambda_2}\|\chi_{2^{k+1}B}\|_{L^{p_2(\cdot)}(\mathbf{R}^n)}\\
&\leq C\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)}\\
&\qquad\cdot\sum_{k=1}^{\infty}|2^kB|^{\lambda_1-1+\alpha/n+1/s+1/m+1/p_1(\cdot)+\lambda_2+1/p_2(\cdot)}k|B|^{-1/s}\\
&\leq C|B|^{\lambda}\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)}.
\end{aligned}
$$

Combining the estimates of (3.5) and (3.9), (3.7) and (3.10), respectively, in both cases, there are

$$
\begin{aligned}
I_2 &\leq C|B|^{\lambda}\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)},\\
I_4 &\leq C|B|^{\lambda}\|b\|_{\mathrm{CBMO}^{p_2(\cdot),\lambda_2}(\mathbf{R}^n)}\|f\|_{\dot{\mathcal{B}}^{p_1(\cdot),\lambda_1}(\mathbf{R}^n)}\|\chi_B\|_{L^{q(\cdot)}(\mathbf{R}^n)}.
\end{aligned}
$$

This completes the proof of Theorem 3.2. □

Remark 3.3. It should be pointed out that the results of Theorem 3.1 and Theorem 3.2 still hold for a more general sublinear operator T with rough kernel Ω, which has $(L^{p(\cdot)}, L^{q(\cdot)})$-boundedness and satisfies the following condition:

$$|Tf(x)| \leq C\int_{\mathbf{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|\,dy,$$

for $f \in L^1$ with compact support and $x \notin \mathrm{supp} f$.

Moreover, by Remark 3.3 and the idea of the proof of Theorem 3.2 we can also obtain the boundedness for commutators of Marcinkiewicz integrals, multipliers and oscillatory singular integrals with rough kernel on central Morrey spaces with variable exponent, respectively.

4. Further results and some applications

In this section, we will give further results and some applications of our main results to nondivergence elliptic equations. Firstly we recall the following definitions for Calderón–Zygmund operators.

Definition 4.1. Let $k\colon \mathbf{R}^n\setminus\{0\} \to \mathbf{R}$. We say that $k(x)$ is a constant Calderón–Zygmund kernel (constant C–Z kernel) if

(i) $k \in C^{\infty}(\mathbf{R}^n)$;
(ii) k is homogeneous of degree $-n$;
(iii) $\int_{\Sigma} k(x)\,d\sigma = 0$, where $\Sigma = \{x \in \mathbf{R}^n\colon |x| = 1\}$.

Definition 4.2. Let E be an open subset of $\mathbf{R}^n$ and $k\colon E \times \{\mathbf{R}^n \setminus \{0\}\} \to \mathbf{R}$. We say that k is a variable C–Z kernel on E if

(i) $k(x,\cdot)$ is a constant C–Z kernel for a.e. $x \in E$;
(ii) $\max\limits_{|j|\leq 2n}\|(\partial^j/\partial z^j)k(x,z)\|_{L^{\infty}(E\times\Sigma)} < \infty$.

Let k be a constant or a variable C–Z kernel on E. We define the corresponding C–Z operator by

$$Tf(x) = \mathrm{p.v.}\int_{\mathbf{R}^n} k(x-y)f(y)\,dy \quad \text{or} \quad Tf(x) = \mathrm{p.v.}\int_{E} k(x,x-y)f(y)\,dy.$$

Similar to Definition 2.4, we give the following definition.

Definition 4.3. Let E be an open subset of $\mathbf{R}^n$, $\lambda \in \mathbf{R}$, and $q(\cdot) \in \mathcal{P}(E)$. The central Morrey space with variable exponent $\dot{\mathcal{B}}^{q(\cdot),\lambda}(E)$ is defined by

$$\dot{\mathcal{B}}^{q(\cdot),\lambda}(E) = \left\{ f \in L^{q(\cdot)}_{\mathrm{loc}}(E) \colon \|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(E)} < \infty \right\},$$

where

$$\|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(E)} = \sup_{R>0} \frac{\|f\chi_{B(0,R)\cap E}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}{|B(0,R)\cap E|^{\lambda}\|\chi_{B(0,R)\cap E}\|_{L^{q(\cdot)}(\mathbf{R}^n)}}.$$

Subsequently, we have the following definition.

Definition 4.4. Let E be an open subset of $\mathbf{R}^n$, $\lambda \in \mathbf{R}$, and $q(\cdot) \in \mathcal{P}(E)$. $f \in L^1_{\mathrm{loc}}(E)$ is said to belong to the central Sobolev–Morrey space with variable exponent $W^2\dot{\mathcal{B}}^{q(\cdot),\lambda}(E)$ if and only if u and its distributional derivatives, u_{x_i}, $u_{x_ix_j} (i,j = 1,\dots,n)$ are in $\dot{\mathcal{B}}^{q(\cdot),\lambda}(E)$. Moreover, let

$$\|u\|_{W^2\dot{\mathcal{B}}^{q(\cdot),\lambda}(E)} \equiv \|u\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(E)} + \sum_{i=1}^{n} \|u_{x_i}\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(E)} + \sum_{i,j=1}^{n} \|u_{x_ix_j}\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(E)}.$$

The Dirichlet problem on the second-order elliptic equation in nondivergence form is

$$\begin{cases} Lu \equiv \displaystyle\sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} = f & \text{a.e. in } F, \\ u = 0 & \text{on} \partial F, \end{cases} \tag{4.1}$$

where $x = (x_1,\dots,x_n) \in \mathbf{R}^n$, F is a bounded domain of $\mathbf{R}^n$. The coefficients $\{a_{ij}\}_{i,j=1}^n$ of L are symmetric and uniformly elliptic, that is, for some $\nu \geq 1$ and every $\xi \in \mathbf{R}^n$,

$$a_{ij}(x) = a_{ji}(x) \quad \text{and} \quad \nu^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \nu|\xi|^2 \tag{4.2}$$

with a.e. $x \in F$. In [10], Fan, Lu and Yang studied the regularity in classic Morrey space of strong solution to (4.1) with $a_{ij} \in VMO(F)$. The space of the functions of vanishing mean oscillation VMO was introduced by Sarason in [20]. The main methods of [10] are based on integral representation formulas established in [4] for the second derivatives of the solution u to (4.1), a priori estimate of the solution to (4.1) and about the theories of singular integrals and sublinear commutators in corresponding Morrey spaces.

By generalizing some theorems of [10] to function spaces with variable exponent, we can obtain the regularity in central Morrey spaces with variable exponent of strong solutions to problem (4.1).

Theorem 4.1. *Let F is a bounded domain of $\mathbf{R}^n$, $q(\cdot) \in \mathcal{P}(F)$ satisfy conditions (2.1) and (2.2) in Lemma 2.1 and $f \in \dot{\mathcal{B}}^{q(\cdot),\lambda}(F)$ with $\lambda < 0$. Then (4.1) has a unique solution $u \in W^2\dot{\mathcal{B}}^{q(\cdot),\lambda}(F)$ satisfying*

$$\|u\|_{W^2\dot{\mathcal{B}}^{q(\cdot),\lambda}(F)} \leq C\|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(F)}.$$

Similar to the method of [10], we can easily prove Theorem 4.1, so we omit the details here. We only give two main results to explain this similarity. All other proofs of the corresponding theorems are straightforward. By Theorem 3.1 and Remark 3.3

in Section 3 we can obtain similar results to Theorem 2.1 in [10] on $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n)$. Next, we will prove the similar results to Theorem 2.3 in [10].

Let $\mathbf{R}^n_+ = \{x = (x', x_n) \colon x' = (x_1, \ldots, x_{n-1}) \in \mathbf{R}^{n-1},\ x_n > 0\}$. To give the boundary estimates of the solutions to (4.1), we need to prove the following general theorem for sublinear operators.

Lemma 4.1. *Let $\lambda < 0$, $q(\cdot) \in \mathcal{P}(\mathbf{R}^n)$ satisfy conditions (2.1) and (2.2) in Lemma 2.1 and $\tilde{x} = (x', -x_n)$ for $x = (x', x_n) \in \mathbf{R}^n_+$. If a sublinear operator $\mathcal{T}$ is bounded on $L^{q(\cdot)}(\mathbf{R}^n_+)$ and for any $f \in L^1(\mathbf{R}^n_+)$ with compact support and $x \in \mathbf{R}^n_+$,*

$$|\mathcal{T} f(x)| \le C \int_{\mathbf{R}^n_+} \frac{|f(y)|}{|\tilde{x} - y|^n} dy, \tag{4.3}$$

then $\mathcal{T}$ is bounded on $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n_+)$.

Proof. Let f be a function in $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n_+)$. For fixed $R > 0$, set $B^+_R = B(0,R) \cap \mathbf{R}^n_+$. We need to prove

$$\left\| \mathcal{T} f \chi_{B^+_R} \right\|_{L^{q(\cdot)}(\mathbf{R}^n_+)} \le C |B^+_R|^{\lambda} \|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n_+)} \left\| \chi_{B^+_R} \right\|_{L^{q(\cdot)}(\mathbf{R}^n_+)}.$$

Denote

$$f(y) = f(y) \chi_{B^+_{2^4 R}}(y) + \sum_{l=4}^{\infty} f(y) \chi_{B^+_{2^{l+1}R} \setminus B^+_{2^l R}}(y) \equiv \sum_{l=3}^{\infty} f_l(y).$$

Then by Lemma 2.4, $L^{q(\cdot)}(\mathbf{R}^n_+)$-boundedness of $\mathcal{T}$ and (4.3), we have

$$\begin{aligned}
&\left\| \mathcal{T} f \chi_{B^+_R} \right\|_{L^{q(\cdot)}(\mathbf{R}^n_+)} \le \sum_{l=3}^{\infty} \left\| \mathcal{T} f_l \chi_{B^+_R} \right\|_{L^{q(\cdot)}(\mathbf{R}^n_+)} \\
&\le C \|f_3\|_{L^{q(\cdot)}(\mathbf{R}^n_+)} + C \sum_{l=4}^{\infty} \left\| \int_{B^+_{2^{l+1}R} \setminus B^+_{2^l R}} \frac{|f(y)|}{|\tilde{x} - y|^n} dy \right\|_{L^{q(\cdot)}(B^+_R)} \\
&\le C |B^+_R|^{\lambda} \|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n_+)} \left\| \chi_{B^+_R} \right\|_{L^{q(\cdot)}(\mathbf{R}^n_+)} + C \sum_{l=4}^{\infty} \frac{1}{(2^l R)^n} \left\| \int_{B^+_{2^{l+1}R} \setminus B^+_{2^l R}} |f(y)| dy \right\|_{L^{q(\cdot)}(B^+_R)} \\
&\le C |B^+_R|^{\lambda} \|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n_+)} \left\| \chi_{B^+_R} \right\|_{L^{q(\cdot)}(\mathbf{R}^n_+)} \\
&\quad + C \sum_{l=4}^{\infty} \frac{1}{(2^l R)^n} \left\| f \chi_{B^+_{2^{l+1}R}} \right\|_{L^{q(\cdot)}(\mathbf{R}^n_+)} \left\| \chi_{B^+_{2^{l+1}R}} \right\|_{L^{q'(\cdot)}(\mathbf{R}^n_+)} \left\| \chi_{B^+_R} \right\|_{L^{q(\cdot)}(\mathbf{R}^n_+)} \\
&\le C |B^+_R|^{\lambda} \|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n_+)} \left\| \chi_{B^+_R} \right\|_{L^{q(\cdot)}(\mathbf{R}^n_+)} + C \sum_{l=4}^{\infty} \frac{1}{(2^l R)^n} |B^+_{2^{l+1}R}|^{\lambda} \|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n_+)} \\
&\quad \cdot \left\| \chi_{B^+_{2^{l+1}R}} \right\|_{L^{q(\cdot)}(\mathbf{R}^n_+)} \left\| \chi_{B^+_{2^{l+1}R}} \right\|_{L^{q'(\cdot)}(\mathbf{R}^n_+)} \left\| \chi_{B^+_R} \right\|_{L^{q(\cdot)}(\mathbf{R}^n_+)} \\
&\le C |B^+_R|^{\lambda} \|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n_+)} \left\| \chi_{B^+_R} \right\|_{L^{q(\cdot)}(\mathbf{R}^n_+)} \\
&\quad + C \sum_{l=4}^{\infty} \frac{1}{(2^l R)^n} |B^+_{2^{l+1}R}|^{\lambda} \|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n_+)} |B^+_{2^{l+1}R}| \left\| \chi_{B^+_R} \right\|_{L^{q(\cdot)}(\mathbf{R}^n_+)}
\end{aligned}$$

$$
\begin{aligned}
&\le C|B_R^+|^\lambda \|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n_+)} \left\|\chi_{B_R^+}\right\|_{L^{q(\cdot)}(\mathbf{R}^n_+)} \left(1+\sum_{l=4}^{\infty} 2^{ln\lambda}\right)\\
&\le C|B_R^+|^\lambda \|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbf{R}^n_+)} \left\|\chi_{B_R^+}\right\|_{L^{q(\cdot)}(\mathbf{R}^n_+)}.
\end{aligned}
$$

This completes the proof of Lemma 4.1. □

To state the following lemma, we need more notation. Let $a(x) = \{a_{in}(x)\}_{i=1}^n$ be as in (4.2) and define

$$
T(x,y) \equiv x - \frac{2x_n}{a_{nn}(y)} a(y).
$$

Then, by the above Lemma 4.1, Lemma 3.1 in [4] and a simple computation, we have

Lemma 4.2. *Let E be an open subset of $\mathbf{R}^n_+$, $\lambda < 0$, and $q(\cdot) \in \mathcal{P}(E)$ satisfy conditions (2.1) and (2.2) in Lemma 2.1. If k is a variable $C-Z$ kernel on E, and*

$$
\tilde{\mathcal{T}} f(x) = \int_E k(x, T(x)-y) f(y)\, dy, \ \forall x \in E
$$

with $T(x) = T(x,x)$, then there exists a constant C such that for all $f \in \dot{\mathcal{B}}^{q(\cdot),\lambda}(E)$,

$$
\|\tilde{\mathcal{T}} f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(E)} \le C \|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(E)}.
$$

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