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EXAMPLES OF DE BRANGES–ROVNYAK SPACES GENERATED BY NONEXTREME FUNCTIONS

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Abstract. We describe de Branges–Rovnyak spaces $\mathcal{H}(b_\alpha)$, $\alpha > 0$, where the function b_α is not extreme in the unit ball of H^∞ on the unit disk $\mathbf{D}$, defined by the equality $b_\alpha(z)/a_\alpha(z) = (1-z)^{-\alpha}$, $z \in \mathbf{D}$, where a_α is the outer function such that $a_\alpha(0) > 0$ and $|a_\alpha|^2 + |b_\alpha|^2 = 1$ a.e. on $\partial\mathbf{D}$.

1. Introduction

Let H^2 denote the standard Hardy space in the open unit disk $\mathbf{D}$ and let $\mathbf{T} = \partial\mathbf{D}$. For $\chi \in L^\infty(\mathbf{T})$ let T_χ denote the bounded Toeplitz operator on H^2, that is, $T_\chi f = P_+(\chi f)$, where P_+ is the orthogonal projection of $L^2(\mathbf{T})$ onto H^2. In particular, $S = T_z$ is called the shift operator. We will denote by $\mathcal{M}(\chi)$ the range of T_χ equipped with the range norm, that is, the norm that makes the operator T_χ a coisometry of H^2 onto $\mathcal{M}(\chi)$.

Let $\mathcal{S}$ denote the closed unit ball of H^∞, that is, $\mathcal{S} = \{f \in H^\infty \colon \|f\|_\infty \le 1\}$. Let us recall that $f \in \mathcal{S}$ is an extreme point of $\mathcal{S}$ if it is not a proper convex combination of two different elements of $\mathcal{S}$. It is known that $f \in \mathcal{S}$ is an extreme point of $\mathcal{S}$ if and only if

$$\int_0^{2\pi} \log(1 - |f(e^{it})|)\, dt = -\infty$$

(see [3, pp. 125–127] and [6, Thm. 6.7]).

Given a function b in $\mathcal{S}$, the *de Branges–Rovnyak space* $\mathcal{H}(b)$ is the image of H^2 under the operator $(I - T_b T_{\overline{b}})^{1/2}$ with the corresponding range norm $\|\cdot\|_b$. The space $\mathcal{H}(b)$ is a Hilbert space with reproducing kernel

$$k_w^b(z) = \frac{1 - \overline{b(w)}b(z)}{1 - \overline{w}z} \quad (z, w \in \mathbf{D}).$$

Here we are interested in the case when the function b is not an extreme point of $\mathcal{S}$, that is, when $\log(1 - |b|) \in L^1(\mathbf{T})$. Then there exists an outer function $a \in H^\infty$ for which $|a|^2 + |b|^2 = 1$ a.e. on $\mathbf{T}$. Moreover, if we suppose that $a(0) > 0$, then a is uniquely determined, and, following Sarason, we say that (b, a) is a *pair*. The function a is sometimes called the *Pythagorean mate* associated with b (see [6, Vol. 2, p. 274]).

It is known that both $\mathcal{M}(a)$ and $\mathcal{M}(\overline{a})$ are contained contractively in $\mathcal{H}(b)$ (see [12, p. 25]). Moreover, if (b, a) is a corona pair, that is, $|a| + |b|$ is bounded away from 0 in $\mathbf{D}$, then $\mathcal{H}(b) = \mathcal{M}(\overline{a})$ (see e.g. [12, p. 62]).

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Let us recall that the Smirnov class $\mathcal{N}^+$ consists of those holomorphic functions in $\mathbf{D}$ that are quotients of functions in H^∞ in which the denominators are outer functions. If (b,a) is a pair, then the quotient $\varphi = b/a$ is in $\mathcal{N}^+$, and conversely, for every nonzero function $\varphi \in \mathcal{N}^+$ there exists a unique pair (b,a) such that $\varphi = b/a$ ([14]).

Many properties of $\mathcal{H}(b)$ can be expressed in terms of the function $\varphi = b/a$ in the Smirnov class $\mathcal{N}^+$. It is worth noting here that if φ is rational, then the functions a and b in the representation of φ are also rational (see [14]) and in such a case (b,a) is called a rational pair. Spaces $\mathcal{H}(b)$ for rational pairs have been studied in [13], [1], [2], [4] and [8] where, among other results, the connection between $\mathcal{H}(b)$ and the local Dirichlet spaces has been discussed. Recently, in [4], the authors studied also the spaces $\mathcal{H}(b^r)$, where b is a rational outer function in the closed unit ball of H^∞ and r is a positive number.

Here we describe the Branges–Rovnyak spaces $\mathcal{H}(b_\alpha)$, $\alpha > 0$, where (b_α, a_α) is such a pair that

$$\varphi_\alpha(z) = \frac{b_\alpha(z)}{a_\alpha(z)} = \frac{1}{(1-z)^\alpha}$$

(principal branch).

Following Sarason [14], for a function φ that is holomorphic on $\mathbf{D}$ we define T_φ to be the operator of multiplication by φ on the domain $\mathcal{D}(T_\varphi) = \{f \in H^2 \colon\ \varphi f \in H^2\}$. It is easy to verify that T_φ is a closed operator (see [6, Vol. 2, p. 309]). It was proved in [14] that the domain $\mathcal{D}(T_\varphi)$ is dense in H^2 if and only if $\varphi \in \mathcal{N}^+$. More precisely, if φ is a nonzero function in $\mathcal{N}^+$ with canonical representation $\varphi = b/a$, then $\mathcal{D}(T_\varphi) = aH^2$. In this case T_φ has a unique, densely defined adjoint T_φ^* that is also closed. In what follows we denote $T_{\overline{\varphi}} = T_\varphi^*$. The reason for such a notation for T_φ^* is explained in [14, pp. 286–288]. The next theorem says that the domain of $T_{\overline{\varphi}}$ coincides with the de Branges–Rovnyak space $\mathcal{H}(b)$.

Theorem 1.1. [14], [6, Thm. 23.31] *Let (b,a) be a pair and let $\varphi = b/a$. Then the domain of $T_{\overline{\varphi}}$ is $\mathcal{H}(b)$ and for $f \in \mathcal{H}(b)$,*

$$\|f\|_b^2 = \|f\|_2^2 + \|T_{\overline{\varphi}} f\|_2^2.$$

The next proposition was also proved in [14].

Proposition 1.2. [14] *If φ is in $\mathcal{N}^+$, ψ is in H^∞, and f is in $\mathcal{D}(T_{\overline{\varphi}})$, then*

$$T_{\overline{\varphi}} T_{\overline{\psi}} f = T_{\overline{\varphi\psi}} f = T_{\overline{\psi}} T_{\overline{\varphi}} f.$$

Corollary 1.3. *Let $\varphi_1, \varphi_2 \in \mathcal{N}^+$ have canonical representations $\varphi_i = b_i/a_i$, $i = 1,2$. If $\varphi_2/\varphi_1 \in H^\infty$, then $\mathcal{H}(b_1) \subset \mathcal{H}(b_2)$.*

Proof. Put $\psi = \varphi_2/\varphi_1$. It follows from Proposition 1.2 that $\mathcal{D}(T_{\overline{\varphi_1}}) \subset \mathcal{D}(T_{\overline{\varphi_1\psi}})$, and so

$$\mathcal{H}(b_1) = \mathcal{D}(T_{\overline{\varphi_1}}) \subset \mathcal{D}(T_{\overline{\varphi_1\psi}}) = \mathcal{D}(T_{\overline{\varphi_2}}) = \mathcal{H}(b_2). \tag{1.1}$$ □

In the proof of our main theorem we will use the following description of invertible Toeplitz operators with unimodular symbols.

Devinatz–Widom Theorem. [9, p. 250] *Let $\psi \in L^\infty(\partial\mathbf{D})$ be such that $|\psi| = 1$ a.e. on $\partial\mathbf{D}$. The following are equivalent.*

(a) *T_ψ is invertible.*
(b) $\operatorname{dist}(\psi, H^\infty) < 1$ *and* $\operatorname{dist}(\overline{\psi}, H^\infty) < 1$.
(c) *There exists an outer function $h \in H^\infty$ such that $\|\psi - h\|_\infty < 1$.*

(d) *There exist real valued bounded functions u, v and a constant $c \in \mathbf{R}$ such that $\psi = e^{i(u+\tilde{v}+c)}$ and $\|u\|_\infty < \frac{\pi}{2}$, where $\tilde{v}$ denotes the conjugate function of v.*

We will need also the notion of a rigid function in H^1. A function in H^1 is called rigid if no other functions in H^1, except for positive scalar multiples of itself, have the same argument as it almost everywhere on $\partial\mathbf{D}$. As observed in [11], every rigid function is outer. It is known that the function $(1-z)^\alpha$ is rigid if $0 < \alpha \le 1$ and is not rigid if $\alpha > 1$ (see e.g. [6, Section 6.8]).

The next theorem shows a close connection between kernels of Toeplitz operators and rigid functions in H^1 ([12, p. 70]).

Theorem 1.4. *If f is an outer function in H^2, then f^2 is rigid if and only if the operator $T_{\overline{f}/f}$ has a trivial kernel.*

Moreover, for a pair (b, a) the following sufficient condition for density of $\mathcal{M}(a)$ in $\mathcal{H}(b)$ is known ([12, p. 72], [6, vol. 2, p. 496]).

Theorem 1.5. *If the function a^2 is rigid, then $\mathcal{M}(a)$ is dense in $\mathcal{H}(b)$.*

2. The spaces $\mathcal{H}(b_\alpha)$, $\alpha > 0$

Recall that for $\alpha > 0$ we define the pair (b_α, a_α) by

$$\varphi_\alpha(z) = \frac{b_\alpha(z)}{a_\alpha(z)} = \frac{1}{(1-z)^\alpha}.$$

Consequently, the outer function a_α is given by

$$a_\alpha(z) = \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log\frac{|1-e^{it}|^{2\alpha}}{1+|1-e^{it}|^{2\alpha}}\, dt\right\}. \tag{2.1}$$

Since both a_α and $(1-z)^\alpha$ are outer functions, by the uniqueness of inner–outer factorization, the equality $(1-z)^\alpha b_\alpha(z) = a_\alpha(z)$ implies that b_α is also outer. Hence

$$b_\alpha(z) = a_\alpha(z)\varphi_\alpha(z) = \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log\frac{1}{1+|1-e^{it}|^{2\alpha}}\, dt\right\}. \tag{2.2}$$

This formula shows that $\log|b_\alpha(z)|$ is a function harmonic in $\mathbf{D}$ and continuous in $\overline{\mathbf{D}}$ with $\log|b_\alpha(e^{it})|^2 = \log\frac{1}{1+|1-e^{it}|^{2\alpha}}$. In particular, $\log|b_\alpha(1)| = \log 1 = 0$. We now prove that actually $b_\alpha(1) = 1$. To this end, it is enough to note that $\arg b_\alpha(r) = 0$ for all $0 < r < 1$. Indeed,

$$\begin{aligned}\arg b_\alpha(r) &= \frac{1}{4\pi}\int_0^{2\pi} \operatorname{Im}\left(\frac{e^{it}+r}{e^{it}-r}\right) \log\frac{1}{1+|1-e^{it}|^{2\alpha}}\, dt \\ &= -\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{2r\sin t}{|e^{it}-r|^2} \log\frac{1}{1+|1-e^{it}|^{2\alpha}}\, dt = 0,\end{aligned}$$

because the integrand is an odd function.

The following proposition says for which α a nontangential limit at 1 of each function (and its derivatives up to a given order) from $\mathcal{H}(b_\alpha)$ exists.

Proposition 2.1. *Every $f \in \mathcal{H}(b_\alpha)$ along with its derivatives up to order $n-1$ has a nontangential limit at the point 1 if and only if $n < \alpha + 1/2$.*

This is a consequence of Theorem 3.2 from [7] (see also [12] and [4]), which states that the following two conditions are equivalent:

(i) for every $f \in \mathcal{H}(b_\alpha)$ the functions $f(z), f'(z), \dots, f^{(n-1)}(z)$ have finite limits as z tends nontangentially to 1;

(ii)
$$\int_0^{2\pi} \frac{|\log|b_\alpha(e^{it})||}{|1-e^{it}|^{2n}}\,dt < +\infty.$$

Since
$$\log|b_\alpha(e^{it})|^2 = \log\frac{1}{1+|1-e^{it}|^{2\alpha}} = \log\left(1-\frac{|1-e^{it}|^{2\alpha}}{1+|1-e^{it}|^{2\alpha}}\right)$$
and $|\log(1-x)| \approx |x|$ for x sufficiently close to zero, we have
$$\log|b_\alpha(e^{it})| \approx \frac{|1-e^{it}|^{2\alpha}}{1+|1-e^{it}|^{2\alpha}} \approx |1-e^{it}|^{2\alpha}$$
whenever t is sufficiently close to 0 or 2π. This implies that
$$\int_0^{2\pi} \frac{|\log|b_\alpha(e^{it})||}{|1-e^{it}|^{2n}}\,dt < \infty$$
if and only if
$$\int_0^{2\pi} \frac{1}{|1-e^{it}|^{2n-2\alpha}}\,dt < \infty,$$
which holds exactly when $\alpha > n-1/2$.

In particular, we see that every $f \in \mathcal{H}(b_\alpha)$ has a nontangential limit at 1 if and only if $\alpha > 1/2$.

The next proposition is an immediate consequence of Corollary 1.3.

Proposition 2.2. *For every* $0 < \alpha \le \beta < \infty$,
$$\mathcal{H}(b_\beta) \subset \mathcal{H}(b_\alpha).$$

Observe now that b_α is bounded below. Indeed, by (2.2),
$$\begin{aligned} |b_\alpha(z)| &= \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \mathrm{Re}\frac{e^{it}+z}{e^{it}-z}\log\frac{1}{1+|1-e^{it}|^{2\alpha}}\,dt\right\} \\ &\ge \exp\left\{\log\sqrt{\frac{1}{1+4^\alpha}}\cdot\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\frac{e^{it}+z}{e^{it}-z}\,dt\right\} = \sqrt{\frac{1}{1+4^\alpha}}. \end{aligned} \tag{2.3}$$
Clearly, this implies that (b_α, a_α) is a corona pair for $\alpha > 0$.

Corollary 2.3. *For* $\alpha > 0$,
$$\mathcal{M}(a_\alpha) = \mathcal{M}((1-z)^\alpha) \quad \text{and} \quad \mathcal{H}(b_\alpha) = \mathcal{M}(\overline{a}_\alpha) = \mathcal{M}(\overline{(1-z)^\alpha})$$
with equivalence of norms.

Proof. The equality of $\mathcal{H}(b_\alpha)$ and $\mathcal{M}(\overline{a}_\alpha)$ follows from the fact that (b_α, a_α) is a corona pair, which as noted above is a consequence of the fact that b_α is bounded below. The latter implies that $1/b_\alpha \in H^\infty$ and so T_{b_α} and $T_{\overline{b}_\alpha}$ are invertible. Hence
$$\mathcal{M}((1-z)^\alpha) = T_{\frac{a_\alpha}{b_\alpha}}H^2 = T_{a_\alpha}H^2$$
and
$$\mathcal{M}(\overline{(1-z)^\alpha}) = T_{\frac{\overline{a}_\alpha}{\overline{b}_\alpha}}H^2 = T_{\overline{a}_\alpha}H^2.$$
Both $\mathcal{M}(a_\alpha)$ and $\mathcal{M}((1-z)^\alpha)$ are boundedly contained in H^2. Hence, the Closed Graph Theorem implies equivalence of their norms. Similarly, one obtains the equivalence of norms in $\mathcal{M}(\overline{a}_\alpha)$ and $\mathcal{M}(\overline{(1-z)^\alpha})$. ∎

It is worth mentioning here that many results on the space $\mathcal{M}(\overline{a})$ where $a \in H^\infty$ is an outer function have been recently obtained in [5]. In this paper the authors study, in particular, boundary behavior of the functions from $\mathcal{M}(\overline{a})$ and describe a natural orthogonal decomposition of this space.

3. Main results

We start with the following.

Theorem 3.1. *For any $n \in \mathbf{N}$ and $n - 1/2 < \alpha < n + 1/2$ we have*

$$\mathcal{M}(\overline{(1-z)^\alpha}) = \mathcal{M}((1-z)^\alpha) + \operatorname{span}\{S^*(1-z)^\alpha, \dots, S^{*n}(1-z)^\alpha\}.$$

Proof. Let

$$Q(z) = \frac{1-z}{1-\overline{z}}, \quad z \in \mathbf{D}.$$

Then Q has a continuous extension to $\overline{\mathbf{D}} \setminus \{1\}$ and

$$Q(z) = -z \quad \text{for } z \in \mathbf{T} \setminus \{1\},$$

which implies that

$$T_{Q^n} = (-1)^n S^n \quad \text{for } n \geq 1. \tag{3.1}$$

Moreover, we observe that for $n - 1/2 < \alpha < n + 1/2$, $n \geq 1$, we have

$$T_{Q^\alpha} = T_{Q^{\alpha-n}Q^n} = (-1)^n T_{Q^{\alpha-n}} S^n.$$

Consequently,

$$T_{(1-z)^\alpha} = T_{\overline{(1-z)^\alpha} Q^\alpha} = (-1)^n T_{\overline{(1-z)^\alpha}} T_{Q^{\alpha-n}} S^n. \tag{3.2}$$

Observe now that the operator $T_{Q^{\alpha-n}}$ is invertible. Indeed, we have

$$Q^{\alpha-n}(e^{it}) = e^{i(\alpha-n)(t-\pi)}, \quad t \in (0, 2\pi),$$

where $|\alpha - n| < 1/2$. So invertibility of $T_{Q^{\alpha-n}}$ follows from part (d) of the Devinatz–Widom Theorem.

Let $f \in \mathcal{M}(\overline{(1-z)^\alpha})$ and $f = T_{\overline{(1-z)^\alpha}} g$ for a function $g \in H^2$. Since $T_{Q^{\alpha-n}}$ is invertible, there exists $g_0 \in H^2$ such that $(-1)^n g = T_{Q^{\alpha-n}} g_0$. Hence, using (3.2), we obtain

$$\begin{aligned} f = T_{\overline{(1-z)^\alpha}} g &= (-1)^n T_{\overline{(1-z)^\alpha}} T_{Q^{\alpha-n}} g_0 \\ &= (-1)^n T_{\overline{(1-z)^\alpha}} T_{Q^{\alpha-n}} \left(S^n S^{*n} g_0 + \sum_{k=0}^{n-1} \langle g_0, z^k \rangle z^k \right) \\ &= T_{(1-z)^\alpha} S^{*n} g_0 + (-1)^n \sum_{k=0}^{n-1} \langle g_0, z^k \rangle T_{\overline{(1-z)^\alpha}} T_{Q^{\alpha-n}} z^k. \end{aligned}$$

Since for $0 \leq k \leq n-1$,

$$\begin{aligned} (-1)^n T_{\overline{(1-z)^\alpha}} T_{Q^{\alpha-n}} z^k &= (-1)^n T_{\overline{Q}^n (1-z)^\alpha} S^k 1 \\ &= S^{*(n-k)} T_{(1-z)^\alpha} 1 = S^{*(n-k)} (1-z)^\alpha \quad \text{by (3.1)}, \end{aligned}$$

we get

$$\begin{aligned} f &= (1-z)^\alpha S^{*n} g_0 + \sum_{k=0}^{n-1} \langle g_0, z^k \rangle S^{*(n-k)} (1-z)^\alpha \\ &\in \mathcal{M}((1-z)^\alpha) + \operatorname{span}\{S^*(1-z)^\alpha, \dots, S^{*n}(1-z)^\alpha\}. \end{aligned}$$

To show that

$$\mathcal{M}((1-z)^\alpha) + \operatorname{span}\{S^*(1-z)^\alpha, \dots, S^{*n}(1-z)^\alpha\} \subset \mathcal{M}(\overline{(1-z)^\alpha}),$$

it is enough to observe that $\mathcal{M}((1-z)^\alpha) \subset \mathcal{M}(\overline{(1-z)^\alpha})$ and $\mathcal{M}(\overline{(1-z)^\alpha})$ is S^*-invariant. □

Now we prove our main result.

Theorem 3.2. *Let $0 < \alpha < \infty$ and let (b_α, a_α) be a pair, with the functions b_α and a_α given by (2.2) and (2.1), respectively. Then*

(i) *for $0 < \alpha < 1/2$,*

$$\mathcal{H}(b_\alpha) = \mathcal{M}(a_\alpha) = (1-z)^\alpha H^2,$$

(ii) *for $n - 1/2 < \alpha < n + 1/2$, $n = 1, 2, \dots$,*

$$\mathcal{H}(b_\alpha) = \mathcal{M}(a_\alpha) + \mathcal{P}_n = (1-z)^\alpha H^2 + \mathcal{P}_n,$$

where $\mathcal{P}_n$ is the set of all polynomials of degree at most $n-1$,

(iii)

$$\mathcal{H}(b_{1/2}) = \overline{\mathcal{M}(a_{1/2})} = \overline{(1-z)^{1/2} H^2},$$

where the closure is taken with respect to the $\mathcal{H}(b_{1/2})$-norm,

(iv) *for $\alpha = n + 1/2$, $n = 1, 2, \dots$,*

$$\mathcal{H}(b_\alpha) = \overline{\mathcal{M}(a_\alpha)} + \mathcal{A}_n,$$

where the closure is taken with respect to the $\mathcal{H}(b_\alpha)$-norm and $\mathcal{A}_n$ is the n-dimensional subspace of $\mathcal{H}(b_\alpha)$ defined by

$$\mathcal{A}_n = \left\{ p_n \cdot P_+\left(\overline{(1-z)^\alpha}(1-z)^{1/2}\right) + P_+\left(p_n P_-\left(\overline{(1-z)^\alpha}(1-z)^{1/2}\right)\right) : p_n \in \mathcal{P}_n \right\},$$

where $P_- = I - P_+$.

Proof. (i) We know from Corollary 2.3 that for $\alpha > 0$,

$$\mathcal{H}(b_\alpha) = \mathcal{M}(\overline{a}_\alpha) = \mathcal{M}(\overline{(1-z)^\alpha}).$$

As in the proof of Theorem 3.1 it follows from the Devinatz–Widom Theorem that for $0 < \alpha < 1/2$ the operator T_{Q^α} is invertible. Consequently,

$$\mathcal{M}(\overline{(1-z)^\alpha}) = T_{\overline{(1-z)^\alpha}} H^2 = T_{\overline{(1-z)^\alpha}} T_{Q^\alpha} H^2 = (1-z)^\alpha H^2.$$

(ii) Since $\mathcal{H}(b_\alpha)$ contains $\mathcal{M}(a_\alpha) = \mathcal{M}((1-z)^\alpha)$ and all polynomials (see e.g. [12, p. 25]), to prove (ii) it is enough to show that

$$\mathcal{H}(b_\alpha) \subset \mathcal{P}_n + \mathcal{M}((1-z)^\alpha).$$

By Theorem 3.1 we have

$$\mathcal{H}(b_\alpha) = \mathcal{M}(\overline{(1-z)^\alpha}) = \mathcal{M}((1-z)^\alpha) + \operatorname{span}\{S^*(1-z)^\alpha, \dots, S^{*n}(1-z)^\alpha\}.$$

Therefore, we only need to show that

$$\operatorname{span}\{S^*(1-z)^\alpha, \dots, S^{*n}(1-z)^\alpha\} \subset \mathcal{P}_n + \mathcal{M}((1-z)^\alpha).$$

Clearly,

$$\begin{aligned} S^*(1-z)^\alpha &= \frac{(1-z)^\alpha - 1}{z} = \frac{(1-z)^\alpha - (1-z)^n + (1-z)^n - 1}{z} \\ &= S^*(1-z)^n - (1-z)^\alpha S^*(1-z)^{n-\alpha} \in \mathcal{P}_n + \mathcal{M}((1-z)^\alpha) \end{aligned}$$

($(1-z)^{n-\alpha} \in H^2$ since $n-\alpha > -1/2$). Now assume that for any $1 \le k < n$,

$$S^{*k}(1-z)^\alpha \in \mathcal{P}_n + \mathcal{M}((1-z)^\alpha),$$

or, in other words,

$$S^{*k}(1-z)^\alpha = p_n + (1-z)^\alpha h_k \quad \text{for some } p_n \in \mathcal{P}_n \text{ and } h_k \in H^2.$$

Then

$$\begin{aligned} S^{*(k+1)}(1-z)^\alpha &= S^*(S^{*k}(1-z)^\alpha) = \frac{p_n + (1-z)^\alpha h_k - p_n(0) - h_k(0)}{z} \\ &= \frac{p_n + (1-z)^\alpha h_k - (1-z)^\alpha h_k(0) + (1-z)^\alpha h_k(0) - p_n(0) - h_k(0)}{z} \\ &= S^* p_n + h_k(0) S^*(1-z)^\alpha + (1-z)^\alpha S^* h_k \in \mathcal{P}_n + \mathcal{M}((1-z)^\alpha). \end{aligned}$$

This completes the proof of (ii).

(iii) In view of Theorem 1.5, to prove (iii) it is enough to show that $a_{1/2}^2$ is a rigid function. We actually prove that a_α^2 is rigid for every $0 < \alpha \le 1/2$. To this end, we observe that by (2.3), for $\alpha > 0$,

$$\frac{1}{\sqrt{1+4^\alpha}}|1-z|^\alpha \le |a_\alpha(z)| \le |1-z|^\alpha, \quad z \in \mathbf{D}.$$

and so $(1-z)^\alpha / a_\alpha \in H^\infty$.

Now we use a reasoning analogous to that in [12, (X–5)]. If a_α^2 is not rigid for some $0 < \alpha \le 1/2$, then by Theorem 1.4 there is a nonzero function g in the kernel of $T_{\overline{a}_\alpha / a_\alpha}$. Then

$$T_{\frac{\overline{(1-z)^\alpha}}{(1-z)^\alpha}} \left(\tfrac{(1-z)^\alpha g}{a_\alpha} \right) = P_+ \left(\tfrac{\overline{(1-z)^\alpha} g}{a_\alpha} \right) = P_+ \left(\tfrac{\overline{(1-z)^\alpha} g}{\overline{a}_\alpha} \cdot \tfrac{\overline{a}_\alpha}{a_\alpha} \right) = T_{\frac{\overline{(1-z)^\alpha}}{\overline{a}_\alpha}} T_{\frac{\overline{a}_\alpha}{a_\alpha}} g = 0,$$

which means that $(1-z)^\alpha g / a_\alpha \in H^2$ is a nonzero function in the kernel of $T_{\overline{(1-z)^\alpha}/(1-z)^\alpha}$, contrary to the fact that $(1-z)^{2\alpha}$ is rigid for $0 < \alpha \le 1/2$ (see, e.g., [6, Section 6.8]).

(iv) We know that for every $\alpha > 0$,

$$\mathcal{H}(b_\alpha) = \mathcal{M}(\overline{a}_\alpha) = \mathcal{M}(\overline{(1-z)^\alpha}) = T_{\overline{(1-z)^\alpha}} H^2$$

and $\mathcal{M}(a_\alpha) = \mathcal{M}((1-z)^\alpha)$ is the image under $T_{\overline{(1-z)^\alpha}}$ of the range of $T_{(1-z)^\alpha / \overline{(1-z)^\alpha}}$, that is,

$$\mathcal{M}((1-z)^\alpha) = T_{\overline{(1-z)^\alpha}} T_{\frac{(1-z)^\alpha}{\overline{(1-z)^\alpha}}} H^2.$$

It follows that the orthogonal complement of $\mathcal{M}((1-z)^\alpha)$ in the space $\mathcal{M}(\overline{(1-z)^\alpha})$ is the image under $T_{\overline{(1-z)^\alpha}}$ of $\ker T_{\overline{(1-z)^\alpha}/(1-z)^\alpha}$.

Clearly, for $\alpha = n + 1/2$ we have

$$\ker T_{\frac{\overline{(1-z)^\alpha}}{(1-z)^\alpha}} = \ker T_{\overline{z^n}} T_{\frac{\overline{(1-z)^{1/2}}}{(1-z)^{1/2}}}.$$

Since $1-z$ is a rigid function, we get

$$\ker T_{\overline{z^n}} T_{\frac{\overline{(1-z)^{1/2}}}{(1-z)^{1/2}}} = (1-z)^{1/2} \mathcal{P}_n,$$

where $\mathcal{P}_n$ is the set of all polynomials of degree at most $n-1$. Finally, note that if p_n is in $\mathcal{P}_n$, then

$$\begin{aligned} &T_{\overline{(1-z)^\alpha}} \left((1-z)^{1/2} p_n \right) = P_+ \left(\overline{(1-z)^\alpha} (1-z)^{1/2} p_n \right) \\ &= P_+ \left(\overline{(1-z)^\alpha} (1-z)^{1/2} \right) p_n + P_+ \left(P_- \left(\overline{(1-z)^\alpha} (1-z)^{1/2} \right) p_n \right). \end{aligned}$$

Our claim follows. □

The following corollary is just another statement of (ii) in Theorem 3.2.

Corollary 3.3. *For any $n \in \mathbf{N}$ and $n - 1/2 < \alpha < n + 1/2$ we have*

$$\mathcal{H}(b_\alpha) = \mathcal{M}(a_\alpha) + \mathcal{P}_n = \mathcal{M}(a_\alpha) + \operatorname{span}\{T_{\overline{a}_\alpha}1, \ldots, T_{\overline{a}_\alpha}z^{n-1}\}.$$

Remark 3.4. We observe that since a_α^2 is rigid for all $0 < \alpha \leq 1/2$, Theorem 1.5 implies that the space $\mathcal{M}(a_\alpha)$ is dense in $\mathcal{H}(b_\alpha)$ for all such α. However, for $0 < \alpha < 1/2$ we have $\mathcal{M}(a_\alpha) = \mathcal{H}(b_\alpha)$, while $\mathcal{M}(a_{1/2}) \subsetneq \mathcal{H}(b_{1/2})$. The latter follows from the fact that every $h \in H^2$ satisfies $|h(z)| = o((1-|z|)^{-1/2})$ as $|z| \to 1^-$. Thus if $f \in \mathcal{M}(a_{1/2})$, then $f(z) = (1-z)^{1/2}h(z)$, $h \in H^2$, and

$$|f(z)| = |1-z|^{\frac{1}{2}}|h(z)| = \left(\frac{|1-z|}{1-|z|}\right)^{\frac{1}{2}} |h(z)|(1-|z|)^{\frac{1}{2}}.$$

This shows that the nontangential limit of f at 1 is 0. On the other hand, $\mathcal{H}(b_{1/2})$ contains nonzero constants, so $\mathcal{M}(a_{1/2})$ cannot be equal to $\mathcal{H}(b_{1/2})$.

Corollary 3.5. *If $n - 1/2 < \alpha < n + 1/2$, $n \in \mathbf{N}$, and $f \in \mathcal{H}(b_\alpha)$, then there is a function h in H^2 such that*

$$f(z) = f(1) + f'(1)(z-1) + \ldots + \frac{f^{(n-1)}(1)}{(n-1)!}(z-1)^{n-1} + (1-z)^\alpha h(z).$$

Proof. It follows from Proposition 2.1 that f and its derivatives of order up to $n-1$ have nontangential limits at 1, say $f(1), f'(1), \ldots, f^{(n-1)}(1)$. By Theorem 3.2(ii), f can be written as

$$f(z) = p_n(z) + (1-z)^\alpha h(z) = \sum_{k=0}^{n-1} a_k(z-1)^k + (1-z)^\alpha h(z), \quad h \in H^2.$$

Since every h in H^2 satisfies

$$|h^{(k)}(z)| \leq \frac{c_k}{(1-|z|)^{k+\frac{1}{2}}},$$

we find that

$$a_k = \frac{p_n^{(k)}(1)}{k!} = \frac{f^{(k)}(1)}{k!} \quad \text{for } k = 0, 1, \ldots, n-1. \tag{3.3}$$

□

The next theorem describes the space $\mathcal{H}(\tilde{b}_\alpha)$ where $\tilde{b}_\alpha$ is an outer function from the unit ball of H^∞ whose Pythagorean mate is $\left(\frac{1-z}{2}\right)^\alpha$, $\alpha > 0$.

Theorem 3.6. *For $\alpha > 0$ let $\tilde{a}_\alpha(z) = \left(\frac{1-z}{2}\right)^\alpha$ and let $\tilde{b}_\alpha$ be the outer function such that $(\tilde{b}_\alpha, \tilde{a}_\alpha)$ is a pair. Then*

$$\mathcal{H}(\tilde{b}_\alpha) = \mathcal{H}(b_\alpha).$$

Proof. It is enough to show that $(\tilde{b}_\alpha, \tilde{a}_\alpha)$ is a corona pair. We will use the reasoning similar to that in the proof of Lemma 3.3 in [4]. The function $\tilde{a}_\alpha$ is continuous on $\overline{\mathbf{D}}$ and vanishes only at 1. Since $|\tilde{b}_\alpha(1)| = \tilde{a}_\alpha(-1) = 1$, there exist $\delta > 0$ such that $|\tilde{b}_\alpha(z)| > 1/2$ on $D_1 = \overline{\mathbf{D}} \cap \{z \colon |z-1| < \delta\}$ and $|\tilde{a}_\alpha(z)| > 1/2$ on $D_2 = \overline{\mathbf{D}} \cap \{z \colon |z+1| < \delta\}$. Then the continuous function $|\tilde{b}_\alpha|^2 + |\tilde{a}_\alpha|^2$ is positive on the compact set $\overline{\mathbf{D}} \setminus (D_1 \cup D_2)$, so it is bounded from below by a strictly positive number $\varepsilon > 0$. □

Remark 3.7. Since $\frac{1-z}{2}$ is the Pythagorean mate for $\frac{1+z}{2}$, we remark that it follows from [4] that for $\alpha > 0$,

$$\mathcal{H}\left(\left(\tfrac{1+z}{2}\right)^{\alpha}\right) = \mathcal{H}\left(\tfrac{1+z}{2}\right) = c + (1-z)H^2$$

as sets.

Finally, we remark that if u is a finite Blaschke product and b_α is given by (2.2), then

$$\mathcal{H}(ub_\alpha) = \mathcal{H}(b_\alpha). \tag{3.4}$$

Since every function in $\mathcal{H}(u)$ is holomorphic in $\overline{\mathbf{D}}$ (see, e.g. [6, Sec. 14.2]) and $\mathcal{H}(b_\alpha)$ is invariant under multiplication by functions holomorphic in $\overline{\mathbf{D}}$ (see, e.g. [12, (IV-6)]), (3.4) follows from the equality

$$\mathcal{H}(ub_\alpha) = \mathcal{H}(u) + u\mathcal{H}(b_\alpha).$$

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