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CHARACTERIZATIONS OF HARDY-TYPE, BERGMAN-TYPE AND DIRICHLET-TYPE SPACES ON CERTAIN CLASSES OF COMPLEX-VALUED FUNCTIONS

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Abstract. In this paper, we continue our investigation of function spaces on certain classes of complex-valued functions. In particular, we give characterizations on Hardy-type, Bergman-type and Dirichlet-type spaces. Furthermore, we present applications of our results to certain nonlinear PDEs.

1. Introduction and main results

For a positive integer $n \geq 1$, let $\mathbf{C}^n$ denote the complex *Euclidean n-space*. For $z := (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbf{C}^n$, we let $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_n)$, and $\langle z, w\rangle := \sum_{k=1}^{n} z_k \overline{w}_k$ with the *Euclidean norm* $\|z\| := \langle z, z\rangle^{1/2}$ which makes $\mathbf{C}^n$ into an n-dimensional complex *Hilbert* space. For $a \in \mathbf{C}^n$ and $r > 0$, $\mathbf{B}^n(a, r)$ denotes the (open) ball of radius r with center a. Also, we let $\mathbf{B}^n(r) := \mathbf{B}^n(0, r)$ and denote the unit ball by $\mathbf{B}^n := \mathbf{B}^n(1)$. In particular, let $\mathbf{B}^1(r) = \mathbf{D}(r)$ and $\mathbf{D} = \mathbf{B}^1$. For a domain $\Omega \subset \mathbf{C}^n$ with non-empty boundary, let $d_\Omega(z)$ be the Euclidean distance from z to the boundary $\partial\Omega$ of Ω. Moreover, we always use $d(z)$ to denote the Euclidean distance from z to the boundary of $\mathbf{B}^n$. We denote by $\mathcal{C}^m(\mathbf{B}^n)$ the set of all m-time continuously differentiable complex-valued functions f of $\mathbf{B}^n$ into $\mathbf{C}$, where $m \in \{0, 1, \ldots\}$.

For $k \in \{1, \ldots, n\}$, $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ and $f \in \mathcal{C}^1(\mathbf{B}^n)$, we introduce the following notations:

$$\nabla f = (f_{z_1}, \ldots, f_{z_n}), \quad \overline{\nabla} f = (f_{\overline{z}_1}, \ldots, f_{\overline{z}_n}) \quad \text{and} \quad D_f = (\nabla f, \overline{\nabla} f),$$

where $f_{z_k} = \partial f/\partial z_k = 1/2\big(\partial f/\partial x_k - i\partial f/\partial y_k\big)$, $f_{\overline{z}_k} = \partial f/\partial \overline{z}_k = 1/2\big(\partial f/\partial x_k + i\partial f/\partial y_k\big)$ and $z_k = x_k + iy_k$, with x_k and y_k real. Let $\|D_f\|$ be the *Hilbert–Schmidt semi-norm* given by

$$\|D_f\| = (\|\nabla f\|^2 + \|\overline{\nabla} f\|^2)^{1/2}.$$

Let $f = u + iv \in \mathcal{C}^1(\mathbf{B}^n)$, where u and v are real-valued functions. Then for $z = (z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n) \in \mathbf{B}^n$,

$$\|\nabla f(z)\| + \|\overline{\nabla} f(z)\| \leq \|\nabla u(z)\| + \|\nabla v(z)\|, \tag{1.1}$$

where

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial y_n}\right) \quad \text{and} \quad \nabla v = \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial y_1}, \ldots, \frac{\partial v}{\partial x_n}, \frac{\partial v}{\partial y_n}\right).$$

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Note that the converse of (1.1) is not always true (see [5]).

Generalized Hardy spaces. For $p \in (0,\infty]$, the *generalized Hardy space* $\mathcal{H}^p_g(\mathbf{B}^n)$ consists of measurable functions $f\colon \mathbf{B}^n \to \mathbf{C}$ such that $M_p(r,f)$ exists for all $r \in (0,1)$ and $\|f\|_p < \infty$, where

$$\|f\|_p = \begin{cases} \sup\limits_{0<r<1} M_p(r,f), & \text{if } p \in (0,\infty), \\ \sup\limits_{z\in\mathbf{B}^n} |f(z)|, & \text{if } p = \infty, \end{cases} \qquad M_p(r,f) = \left(\int_{\partial\mathbf{B}^n} |f(r\zeta)|^p \, d\sigma(\zeta)\right)^{1/p}$$

and $d\sigma$ denotes the normalized Lebesgue surface measure in $\partial\mathbf{B}^n$.

There are numerous characterizations of the classical analytic Hardy spaces in the literature, see for example [12, 17, 18, 21, 22, 27]. But, to our knowledge, there are few analogous results for general complex-valued functions. In this paper, we give the following characterization of a class of complex-valued functions f in Hardy-type spaces.

Theorem 1. *For $p \geq 2$, let $f \in \mathcal{C}^2(\mathbf{B}^n)$ with $\mathrm{Re}(f\overline{\Delta f}) \geq 0$. Then,*

$$\int_{\mathbf{B}^n} d(z)\Delta\left(|f(z)|^p\right) dV_N(z) < \infty$$

if and only if $f \in \mathcal{H}^p_g(\mathbf{B}^n)$, where dV_N denotes the normalized Lebesgue volume measure in $\mathbf{B}^n$ and Δ is the usual complex Laplacian operator

$$\Delta := 4\sum_{k=1}^n \frac{\partial^2}{\partial z_k \partial \overline{z}_k} = \sum_{k=1}^n \left(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2}\right)$$

for $z = (z_1,\ldots,z_n) = (x_1+iy_1,\ldots,x_n+iy_n) \in \mathbf{B}^n$.

Yukawa PDE. Let $\tau,\eta\colon \mathbf{B}^n \to [0,\infty)$ be continuous and $f = u+iv \in \mathcal{C}^2(\mathbf{B}^n)$, where u and v are real-valued functions in $\mathbf{B}^n$. The nonlinear elliptic partial differential equation (PDE) of the form

$$\Delta f(z) = \tau(z)f(z) + \eta(z)\mathrm{Re}\big(f(z)\big) \tag{1.2}$$

is called the *non-homogeneous Yukawa PDE*, where $z \in \mathbf{B}^n$. If τ in (1.2) is a positive constant function and $\eta \equiv 0$, then we have the usual Yukawa PDE. This equation arose from the work of the Japanese Nobel physicist Hideki Yukawa, who used it to describe the nuclear potential of a point charge as $e^{-\sqrt{\tau}r}/r$ (cf. [1, 3, 7, 9, 10, 11, 16, 30, 34]).

As an application of Theorem 1, we obtain the following result.

Corollary 1.1. *For $p \geq 2$, let $f \in \mathcal{C}^2(\mathbf{B}^n)$ satisfying* (1.2). *Then,*

$$\int_{\mathbf{B}^n} d(z)\Delta\left(|f(z)|^p\right) dV_N(z) < \infty$$

if and only if $f \in \mathcal{H}^p_g(\mathbf{B}^n)$.

A continuous increasing function $\omega\colon [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ is called a *majorant* if $\omega(t)/t$ is non-increasing for $t > 0$ (cf. [14, 15, 25, 26]). Given a subset Ω of $\mathbf{C}^n$, a function $f\colon \Omega \to \mathbf{C}$ is said to belong to the *Lipschitz space* $L_\omega(\Omega)$ if there is a positive constant C such that

$$|f(z) - f(w)| \leq C\omega(|z-w|) \quad \text{for all } z,w \in \Omega.$$

A classical result of Hardy and Littlewood asserts that if $p \in (0, \infty]$, $\alpha \in (1, \infty)$ and f is an analytic function in $\mathbf{D}$, then (cf. [12, 21, 22])

$$M_p(r, f') = O\left(\left(\frac{1}{1-r}\right)^{\alpha}\right) \quad \text{as } r \to 1,$$

if and only if

$$M_p(r, f) = O\left(\left(\log \frac{1}{1-r}\right)^{\alpha-1}\right) \quad \text{as } r \to 1.$$

In [17], via the closed graph theorem, Girela, Pavlović and Peláez refined the above result for the case $\alpha = 1$ as follows.

Theorem A. [17, Theorem 1.1] *Let $p \in (2, \infty)$. For $r \in (0, 1)$, if f is analytic in $\mathbf{D}$ such that*

$$M_p(r, f') = O\left(\frac{1}{1-r}\right) \quad \text{as } r \to 1,$$

then

$$M_p(r, f) = O\left(\left(\log \frac{1}{1-r}\right)^{\frac{1}{2}}\right) \quad \text{as } r \to 1$$

and the exponent $1/2$ is sharp.

Theorem A gives an affirmative answer to the open problem in [18, p. 464, Equation (26)]. For related investigations on this topic, we refer to [3, 4, 7, 32].

Next we study the relationship between the integral means of solutions to the equation (1.2) and those of their second order partial derivative. Our result is given as follows.

Theorem 2. *Let ω be a majorant and $f \in \mathcal{C}^2(\mathbf{B}^n)$ satisfying (1.2) with $\eta + \tau < 4n/p$, where τ and η are nonnegative constant functions. For $p \geq 2$ and $r \in (0, 1)$, if*

$$M_p(r, D_f^*) \leq M^* \omega\left(\frac{1}{1-r}\right),$$

then

$$M_p(r, D_f) \leq \sqrt{M_2^*}\left[\|D_f(0)\|^2 + M_1^* \int_0^1 \omega\left(\frac{1}{1-rt}\right) dt\right]^{\frac{1}{2}},$$

and $f \in \mathcal{H}_g^p(\mathbf{B}^n)$, where M^* is a positive constant,

$$D_f^* = \left[\sum_{j=1}^{n}\sum_{k=1}^{n}\left(|f_{z_k z_j}|^2 + |f_{z_k \overline{z}_j}|^2 + |f_{\overline{z}_k z_j}|^2 + |f_{\overline{z}_k \overline{z}_j}|^2\right)\right]^{\frac{1}{2}},$$

$M_1^* = 2p(2p-3)(M^*)^2\omega(1)$ and $M_2^* = 1/\left[1 - p(\eta + \tau)/(4n)\right]$.

In particular, by taking $\omega(t) = t$ in Theorem 2, we obtain the following result.

Corollary 1.2. *Let $p \geq 2$ and $f \in \mathcal{C}^2(\mathbf{B}^n)$ satisfying (1.2) with $\eta + \tau < 4n/p$, where τ and η are nonnegative constant functions. For $r \in (0, 1)$, if*

$$M_p(r, D_f^*) = O\left(\frac{1}{1-r}\right) \quad \text{as } r \to 1,$$

then

$$M_p(r, D_f) = O\left(\left(\log \frac{1}{1-r}\right)^{\frac{1}{2}}\right) \quad \text{as } r \to 1,$$

and $f \in \mathcal{H}_g^p(\mathbf{B}^n)$.

Dirichlet-type spaces and Bergman-type spaces. For $\nu,\ \mu,\ t\in\mathbf{R}$,

$$\mathcal{D}_f(\nu,\mu,t)=\int_{\mathbf{B}^n} d^{\nu}(z)|f(z)|^{\mu}\|D_f(z)\|^t\,dV_N(z)<\infty$$

is called *Dirichlet-type energy integral* of the complex-valued function f (cf. [1, 2, 7, 16, 17, 19, 31, 32, 33, 34]). In particular, for $\nu\geq 0$, $\mu=0$ and $0<t<\infty$, we use $\mathcal{D}_{\nu,t}(\mathbf{B}^n)$ to denote the *Dirichlet-type space* consisting of all $f\in\mathcal{C}^1(\mathbf{B}^n)$ with the norm

$$\|f\|_{\mathcal{D}_{\nu,t}}=|f(0)|+\big(\mathcal{D}_f(\nu,0,t)\big)^{1/t}<\infty.$$

Moreover, for $\nu>-1$, $0<\mu<\infty$ and $t=0$, we denote by $b_{\nu,\mu}(\mathbf{B}^n)$ the *Bergman-type space* consisting of all $f\in\mathcal{C}^0(\mathbf{B}^n)$ with the norm

$$\|f\|_{b_{\nu,\mu}}=|f(0)|+\big(\mathcal{D}_f(\nu,\mu,0)\big)^{1/\mu}<\infty.$$

We refer to [13, 17, 19, 20, 26, 28, 35] for basic characterizations of analytic (or harmonic) Bergman-type spaces and Dirichlet-type spaces. Again, for general complex-valued functions, very little related research can be found from the literature. The following is a characterization of a class of complex-valued functions f in Bergman-type spaces.

Theorem 3. *Let $f\in\mathcal{C}^2(\mathbf{B}^n)$ with $\mathrm{Re}(f\overline{\Delta f})\geq 0$. Then, for $p\geq 2$ and $\alpha\geq 2$,*

$$\int_{\mathbf{B}^n}(1-|z|^2)^{\alpha}\Delta\left(|f(z)|^p\right)dV_N(z)<\infty,$$

if and only if $f\in b_{\alpha-2,p}(\mathbf{B}^n)$.

The following result easily follows from Theorem 3.

Corollary 1.3. *Let $f\in\mathcal{C}^2(\mathbf{B}^n)$ satisfy (1.2). Then, for $p\geq 2$ and $\alpha\geq 2$,*

$$\int_{\mathbf{B}^n}(1-|z|^2)^{\alpha}\Delta\left(|f(z)|^p\right)dV_N(z)<\infty$$

if and only if $f\in b_{\alpha-2,p}(\mathbf{B}^n)$.

Definition 1. For $m\in\{2,3,\ldots\}$, we denote by $\mathcal{HZ}_m(\mathbf{B}^n)$ the class of all functions $f\in\mathcal{C}^m(\mathbf{B}^n)$ satisfying *Heinz's* nonlinear differential inequality (cf. [23])

$$|\Delta f(z)|\leq a(z)\|D_f(z)\|+b(z)|f(z)|+c(z),$$

where $a(z)$, $b(z)$ and $c(z)$ are real-valued nonnegative continuous functions in $\mathbf{B}^n$.

Theorem 4. *Let M be a nonnegative constant and $f\in\mathcal{HZ}_3(\mathbf{B}^n)\cap\mathcal{D}_{\gamma,\alpha}(\mathbf{B}^n)$ with $\mathrm{Re}(f\overline{\Delta f})\geq 0$ and $\mathrm{Re}\left\{\sum_{k=1}^n\left[\overline{f_{z_k}}(\Delta f)_{z_k}+\overline{f_{\overline{z}_k}}(\Delta f)_{\overline{z}_k}\right]\right\}\geq 0$, where $2\leq\alpha\leq 2n$, $\gamma>0$, $\sup_{z\in\mathbf{B}^n}a(z)<\infty$, $\sup_{z\in\mathbf{B}^n}b(z)<\infty$ and $c(z)\leq M\big(d(z)\big)^{-q}$. Then for $p\geq 2$,*

$$\int_{\mathbf{B}^n}\big(d(z)\big)^{pq}\Delta\left(|f(z)|^p\right)dV_N(z)<\infty,$$

where $q=(2n+\gamma)/\alpha-1$.

The result given below is a consequence of Theorem 4.

Corollary 1.4. *For $2\leq\alpha\leq 2n$ and $\gamma>0$, let $f\in\mathcal{HZ}_3(\mathbf{B}^n)\cap\mathcal{D}_{\gamma,\alpha}(\mathbf{B}^n)$ satisfying (1.2), where τ and η are nonnegative constant functions. Then for $p\geq 2$,*

$$\int_{\mathbf{B}^n}\big(d(z)\big)^{pq}\Delta\left(|f(z)|^p\right)dV_N(z)<\infty,$$

where $q=(2n+\gamma)/\alpha-1$.

Proof. By elementary calculations, we see that if f is a solution to (1.2), then f satisfies Heinz's nonlinear differential inequality. Hence Corollary 1.4 follows from (2.8), (2.9) and Theorem 4. □

By Corollaries 1.1, 1.3 and 1.4, we get

Corollary 1.5. *For $2 \leq \alpha \leq 2n$ and $\gamma > 0$, let $q = (2n+\gamma)/\alpha - 1$ and let $f \in \mathcal{HZ}_3(\mathbf{B}^n) \cap \mathcal{D}_{\gamma,\alpha}(\mathbf{B}^n)$ satisfy (1.2), where τ and η are nonnegative constant functions.*

(1) *If $p = \frac{1}{q} \geq 2$, then $f \in \mathcal{H}^p_g(\mathbf{B}^n)$;*
(2) *If $p \geq 2$ and $pq \geq 2$, then $f \in b_{pq-2,p}(\mathbf{B}^n)$.*

Definition 2. For $p \geq 2$, $t_1 > 0$, $t_2 > 0$ and $m \in \{2, 3, \ldots\}$, we denote by $\mathcal{IHZ}^{t_1,t_2}_m(\mathbf{B}^n)$ the class of all functions $f \in \mathcal{C}^m(\mathbf{B}^n)$ satisfying the inverse *Heinz's* nonlinear differential inequality

$$\Delta(|f(z)|^p) \geq a_1(z)\|D_f(z)\|^{t_1} + b_1(z)|f(z)|^{t_2} + c_1(z),$$

where $a_1(z)$, $b_1(z)$ and $c_1(z)$ are real-valued nonnegative continuous functions in $\mathbf{B}^n$.

Theorem 5. *Let $f \in \mathcal{IHZ}^{t_1,t_2}_2(\mathbf{B}^n) \cap \mathcal{H}^p_g(\mathbf{B}^n)$, where $\inf_{z\in\mathbf{B}^n} a_1(z) + \inf_{z\in\mathbf{B}^n} b_1(z) > 0$ and $\inf_{z\in\mathbf{B}^n} c_1(z) \geq 0$.*

(1) *If $\inf_{z\in\mathbf{B}^n} a_1(z) > 0$, then $f \in \mathcal{D}_{1,t_1}(\mathbf{B}^n)$;*
(2) *If $\inf_{z\in\mathbf{B}^n} b_1(z) > 0$, then $f \in b_{1,t_2}(\mathbf{B}^n)$.*

For $k \in \{1, \ldots, n\}$, let $\lambda_k \in \mathbf{R}$ be a constant and let $f \in \mathcal{C}^1(\mathbf{B}^n)$ satisfy the following nonlinear PDE,

$$\frac{\partial f}{\partial \overline{z}_k} = \lambda_k |f|^\alpha, \tag{1.3}$$

where $\alpha \geq 0$. If, for each $k \in \{1, \ldots, n\}$, $\lambda_k = 0$, then f is holomorphic. Moreover, if $\alpha = 0$, then f is pluriharmonic (cf. [6, 29]). It has attracted the attention of many authors when $n = \lambda_1 = 1$ and $\alpha \in (0, 1)$ (cf. [2, 8, 24]).

Corollary 1.6. *For $\sum_{k=1}^n \lambda_k^2 \neq 0$, $\alpha \geq 0$ and $p > \max\{2, (\alpha-2)^2/4\}$, if $f \in \mathcal{H}^p_g(\mathbf{B}^n) \cap \mathcal{C}^2(\mathbf{B}^n)$ satisfies (1.3), then $f \in b_{1,\vartheta}(\mathbf{B}^n)$, where $\vartheta = p + 2\alpha - 2$.*

The proofs of Theorems 1 and 2 will be presented in Section 2, and the proofs of Theorems 3, 4, 5 and Corollary 1.6 will be given in Section 3.

2. Hardy-type spaces and applications to pdes

We start this section by recalling the following result.

Theorem B. [27] *Let g be a function of class $\mathcal{C}^2(\mathbf{B}^n)$. Then, for $r \in (0, 1)$,*

$$\int_{\partial\mathbf{B}^n} g(r\zeta)\, d\sigma(\zeta) = g(0) + \int_{\mathbf{B}^n(r)} \Delta g(z) G_{2n}(z, r)\, dV_N(z),$$

where

$$G_{2n}(z, r) = \begin{cases} \dfrac{|z|^{2(1-n)} - r^{2(1-n)}}{4n(n-1)}, & \text{if } n \geq 2, \\ \dfrac{1}{2} \log \dfrac{r}{|z|}, & \text{if } n = 1. \end{cases}$$

Lemma 1. *Let $p \geq 2$ and $f \in \mathcal{C}^2(\mathbf{B}^n)$ with $\mathrm{Re}(f\overline{\Delta f}) \geq 0$. Then $M^p_p(r, f)$ is increasing with respect to $r \in (0, 1)$.*

Proof. Case 1. Let $p \in [4, \infty)$. By elementary calculations, we get

$$\Delta\big(|f|^p\big) = p(p-2)|f|^{p-4}\sum_{k=1}^{n}|f_{z_k}\overline{f} + \overline{f_{\overline{z}_k}}f|^2 + 2p|f|^{p-2}\|D_f\|^2 + p|f|^{p-2}\mathrm{Re}(f\overline{\Delta f}) \geq 0,$$

which implies that, for $p \in [4, \infty)$, $M_p^p(r, f)$ is increasing in $(0, 1)$.

Case 2. Let $p \in [2, 4)$. For $m \in \{1, 2, \ldots\}$, let $T_m^p = \big(|f|^2 + \frac{1}{m}\big)^{\frac{p}{2}}$. By computations, we have

$$\begin{aligned}\Delta\big(T_m^p\big) &= 4\sum_{k=1}^{n}\frac{\partial^2}{\partial z_k \partial \overline{z}_k}\big(T_m^p\big) = 4\sum_{k=1}^{n}\big(T_m^p\big)_{z_k\overline{z}_k}\\ &= p(p-2)\left(|f|^2 + \frac{1}{m}\right)^{\frac{p}{2}-2}\sum_{k=1}^{n}|f_{z_k}\overline{f} + \overline{f_{\overline{z}_k}}f|^2\\ &\quad + 2p\left(|f|^2 + \frac{1}{m}\right)^{\frac{p}{2}-1}\|D_f\|^2 + p\left(|f|^2 + \frac{1}{m}\right)^{\frac{p}{2}-1}\mathrm{Re}(f\overline{\Delta f}).\end{aligned}$$

Let $Q_m = \Delta\big(T_m^p\big)$. It is not difficult to show that, for $r \in (0, 1)$, Q_m is integrable in $\mathbf{B}^n(r)$ and $0 < Q_m \leq \Lambda_f$, where

$$\begin{aligned}\Lambda_f &= p(p-2)|f|^{p-2}\sum_{k=1}^{n}\big(|f_{z_k}| + |f_{\overline{z}_k}|\big)^2 + 2p\left(|f|^2 + 1\right)^{\frac{p}{2}-1}\|D_f\|^2\\ &\quad + p\left(|f|^2 + 1\right)^{\frac{p}{2}-1}\mathrm{Re}(f\overline{\Delta f})\end{aligned}$$

and Λ_f is integrable in $\mathbf{B}^n(r)$.

By using Theorem B and Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}\lim_{m\to\infty} r^{2n-1}\frac{d}{dr}M_p^p(r, T_m) &= \frac{1}{2n}\lim_{m\to\infty}\int_{\mathbf{B}^n(r)} Q_m\, dV_N = \frac{1}{2n}\int_{\mathbf{B}^n(r)}\lim_{m\to\infty} Q_m\, dV_N\\ &= \frac{1}{2n}\int_{\mathbf{B}^n(r)}\Big[p(p-2)|f|^{p-4}\sum_{k=1}^{n}|f\overline{f_{z_k}} + \overline{f}f_{\overline{z}_k}|^2\\ &\quad + 2p|f|^{p-2}\|D_f\|^2 + p|f|^{p-2}\mathrm{Re}\big(f\overline{\Delta f}\big)\Big]\, dV_N\\ &= r^{2n-1}\frac{d}{dr}M_p^p(r, f) \geq 0,\end{aligned}$$

which implies that $M_p^p(r, f)$ is increasing in r on $(0, 1)$ for $p \in [2, 4)$. □

By using Theorem B and a similar argument as in the proof of Lemma 1, we obtain the following result.

Lemma 2. *Let $p \geq 2$ and $f \in \mathcal{C}^2(\mathbf{B}^n)$ with $\mathrm{Re}(f\overline{\Delta f}) \geq 0$. Then, for $r \in (0, 1)$,*

$$M_p^p(r, f) = |f(0)|^p + \int_{\mathbf{B}^n(r)}\Delta(|f(z)|^p)G_{2n}(z, r)\, dV_N(z),$$

where G_{2n} is the function defined in Theorem B.

Proof of Theorem 1. Case 1. Let $n \geq 2$. We first prove the necessity. For a fixed positive constant $r_0 \in (0,1)$, let $r \in (r_0, 1)$. Then, by Lemma 2, we have

$$\begin{aligned} M_p^p(r,f) &= |f(0)|^p + \int_{\mathbf{B}^n(r)} \Delta\big(|f(z)|^p\big) G_{2n}(z,r)\, dV_N(z) \\ &= |f(0)|^p + \frac{1}{4n(n-1)} \int_{\mathbf{B}^n(r)\setminus \mathbf{B}^n(r_0)} \big(|z|^{2(1-n)} - r^{2(1-n)}\big)\Delta\big(|f(z)|^p\big)\, dV_N(z) \\ &\quad + \frac{1}{4n(n-1)} \int_{\mathbf{B}^n(r_0)} \big(|z|^{2(1-n)} - r^{2(1-n)}\big)\Delta\big(|f(z)|^p\big)\, dV_N(z). \end{aligned} \tag{2.1}$$

Since $\Delta\big(|f|^p\big) \geq 0$,

$$\begin{aligned} \infty &> 2n \int_{\partial \mathbf{B}^n} \int_0^{r_0} \big(\rho - \rho^{2n-1}\big)\Delta\big(|f(\rho\zeta)|^p\big)\, d\rho\, d\sigma(\zeta) \\ &\geq 2n \int_{\partial \mathbf{B}^n} \int_0^{r_0} \big(\rho - r^{2(1-n)}\rho^{2n-1}\big)\Delta\big(|f(\rho\zeta)|^p\big)\, d\rho\, d\sigma(\zeta) \\ &= \int_{\mathbf{B}^n(r_0)} \big(|z|^{2(1-n)} - r^{2(1-n)}\big)\Delta\big(|f(z)|^p\big)\, dV_N(z) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\int_{\mathbf{B}^n(r)\setminus \mathbf{B}^n(r_0)} \big(|z|^{2(1-n)} - r^{2(1-n)}\big)\Delta\big(|f(z)|^p\big)\, dV_N(z) \\ &= \int_{\mathbf{B}^n(r)\setminus \mathbf{B}^n(r_0)} \frac{(r-|z|)\big(\sum_{k=0}^{2n-3} r^{2n-3-k}|z|^k\big)}{|z|^{2n-2}r^{2n-2}} \Delta\big(|f(z)|^p\big)\, dV_N(z) \\ &\leq \frac{(2n-2)}{r_0^{4n-4}} \int_{\mathbf{B}^n(r)\setminus \mathbf{B}^n(r_0)} (r-|z|)\Delta\big(|f(z)|^p\big)\, dV_N(z) \\ &\leq \frac{(2n-2)}{r_0^{4n-4}} \int_{\mathbf{B}^n} d(z)\Delta\big(|f(z)|^p\big)\, dV_N(z) < \infty. \end{aligned}$$

By (2.1) and Lemma 1, we see that the limit

$$\lim_{r\to 1-} M_p(r,f)$$

exists. Hence $f \in \mathcal{H}_g^p(\mathbf{B}^n)$.

Next we prove the sufficiency. Applying (2.1), (2.2) and $f \in \mathcal{H}_g^p(\mathbf{B}^n)$, we observe that

$$\begin{aligned} \infty &> \int_{\mathbf{B}^n(r)\setminus \mathbf{B}^n(r_0)} \big(|z|^{2(1-n)} - r^{2(1-n)}\big)\Delta\big(|f(z)|^p\big)\, dV_N(z) \\ &= \int_{\mathbf{B}^n(r)\setminus \mathbf{B}^n(r_0)} \frac{(r-|z|)\big(\sum_{k=0}^{2n-3} r^{2n-3-k}|z|^k\big)}{|z|^{2n-2}r^{2n-2}} \Delta\big(|f(z)|^p\big)\, dV_N(z) \geq I(r), \end{aligned}$$

which, together with the monotonicity of $I(r)$ on $r \in [r_0, 1)$, yields that

$$\lim_{r\to 1-} \int_{\mathbf{B}^n(r)\setminus \mathbf{B}^n(r_0)} (r-|z|)\Delta\left(|f(z)|^p\right) dV_N(z)$$

exists, where

$$I(r) = (2n-2)r_0^{2n-3} \int_{\mathbf{B}^n(r)\setminus \mathbf{B}^n(r_0)} (r-|z|)\Delta\left(|f(z)|^p\right) dV_N(z).$$

Therefore,

$$\int_{\mathbf{B}^n} d(z)\Delta\left(|f(z)|^p\right) dV_N(z) < \infty.$$

Case 2. Let $n = 1$. In this case, we also first prove the necessity. Fix $r \in (0,1)$. Since

$$\lim_{|z|\to r} \frac{\log r - \log|z|}{r - |z|} = \frac{1}{r},$$

we see that there exists $r_0 \in (0, r)$ such that

$$\frac{1}{2r} \le \frac{\log r - \log|z|}{r - |z|} \le \frac{3}{2r} \tag{2.3}$$

for $r_0 \le |z| < r$. It is not difficult to see that, for $|z| \le r < 1$,

$$\frac{r - |z|}{r} \le 1 - |z|. \tag{2.4}$$

Because

$$\lim_{\rho\to 0+} \rho \log\frac{1}{\rho} = 0,$$

it follows that

$$\begin{aligned} \int_{\mathbf{D}(r_0)} \Delta\big(|f(z)|^p\big) \log\frac{r}{|z|}\, dA(z) &= \int_{\mathbf{D}(r_0)} \Delta\big(|f(z)|^p\big) \log\frac{1}{|z|}\, dA(z) \\ &= \frac{1}{\pi}\int_0^{2\pi}\int_0^{r_0} \Delta\big(|f(\rho e^{i\theta})|^p\big)\rho \log\frac{1}{\rho}\, d\theta < \infty, \end{aligned} \tag{2.5}$$

where dA denotes the normalized area measure in $\mathbf{D}$.

By (2.3), (2.4), (2.5), Lemmas 1 and 2, we see that

$$\begin{aligned} M_p^p(r,f) &= |f(0)|^p + \frac{1}{2}\int_{\mathbf{D}(r)} \Delta\big(|f(z)|^p\big) \log\frac{r}{|z|}\, dA(z) \\ &= |f(0)|^p + \frac{1}{2}\int_{\mathbf{D}(r_0)} \Delta\big(|f(z)|^p\big) \log\frac{r}{|z|}\, dA(z) + \frac{1}{2}\int_{\mathbf{D}(r)\backslash\mathbf{D}(r_0)} \Delta\big(|f(z)|^p\big) \log\frac{r}{|z|}\, dA(z) \\ &\le |f(0)|^p + \frac{1}{2}\int_{\mathbf{D}(r_0)} \Delta\big(|f(z)|^p\big) \log\frac{r}{|z|}\, dA(z) + \frac{3}{4}\int_{\mathbf{D}(r)\backslash\mathbf{D}(r_0)} \Delta\big(|f(z)|^p\big) \frac{(r-|z|)}{r}\, dA(z) \\ &\le |f(0)|^p + \frac{1}{2}\int_{\mathbf{D}(r_0)} \Delta\big(|f(z)|^p\big) \log\frac{r}{|z|}\, dA(z) + \frac{3}{4}\int_{\mathbf{D}\backslash\mathbf{D}(r_0)} \Delta\big(|f(z)|^p\big) d(z)\, dA(z) < \infty, \end{aligned}$$

which implies that the limit

$$\lim_{r\to 1-} M_p(r,f)$$

exists. Hence $f \in \mathcal{H}_g^p(\mathbf{B}^n)$.

Now we prove the sufficiency. By (2.3), we have

$$\begin{aligned} \int_{\mathbf{D}(r)\backslash\mathbf{D}(r_0)} \Delta\big(|f(z)|^p\big) \log\frac{r}{|z|}\, dA(z) &\ge \frac{1}{2r}\int_{\mathbf{D}(r)\backslash\mathbf{D}(r_0)} \Delta\big(|f(z)|^p\big)(r-|z|)\, dA(z) \\ &\ge \frac{I^*(r)}{2}, \end{aligned} \tag{2.6}$$

where

$$I^*(r) = \int_{\mathbf{D}(r)\backslash\mathbf{D}(r_0)} \Delta\big(|f(z)|^p\big)(r-|z|)\, dA(z).$$

By (2.6), Lemmas 1 and 2, we have

$$
\begin{aligned}
M_p^p(r,f) &= |f(0)|^p + \frac{1}{2}\int_{\mathbf{D}(r)} \Delta\big(|f(z)|^p\big)\log\frac{r}{|z|}\,dA(z)\\
&= |f(0)|^p + \frac{1}{2}\int_{\mathbf{D}(r_0)} \Delta\big(|f(z)|^p\big)\log\frac{r}{|z|}\,dA(z) + \frac{1}{2}\int_{\mathbf{D}(r)\backslash\mathbf{D}(r_0)} \Delta\big(|f(z)|^p\big)\log\frac{r}{|z|}\,dA(z)\\
&\geq |f(0)|^p + \frac{1}{2}\int_{\mathbf{D}(r_0)} \Delta\big(|f(z)|^p\big)\log\frac{r}{|z|}\,dA(z) + \frac{1}{4}I^*(r),
\end{aligned}
$$

which yields that $I^*(r)<\infty$. Since $I^*(r)$ is increasing on r, we see that

$$\lim_{r\to 1-} I^*(r)$$

exists. Then

$$\int_{\mathbf{D}} d(z)\Delta\left(|f(z)|^p\right)dA(z) < \infty$$

concluding the proof of the theorem. □

Lemma 3. *Let* $f\in\mathcal{C}^3(\mathbf{B}^n)$ *and* $\mathrm{Re}\left\{\sum_{k=1}^n\left[\overline{f_{z_k}}(\Delta f)_{z_k}+\overline{f_{\overline{z}_k}}(\Delta f)_{\overline{z}_k}\right]\right\}\geq 0$. *Then, for* $\alpha\geq 2$, $\|D_f\|^\alpha$ *is subharmonic in* $\mathbf{B}^n$.

Proof. First we consider the case $\alpha\in[4,\infty)$. Since

$$
\begin{aligned}
\Delta\big(\|D_f\|^\alpha\big) &= \alpha(\alpha-2)\|D_f\|^{\alpha-4}\left|\sum_{j=1}^n\sum_{k=1}^n\big(f_{z_kz_j}\overline{f_{z_k}}+\overline{f_{z_k\overline{z}_j}}f_{z_k}+f_{\overline{z}_kz_j}\overline{f_{\overline{z}_k}}+\overline{f_{\overline{z}_k\overline{z}_j}}f_{\overline{z}_k}\big)\right|^2\\
&\quad+2\alpha\|D_f\|^{\alpha-2}\sum_{j=1}^n\sum_{k=1}^n\big(|f_{z_jz_k}|^2+|f_{z_j\overline{z}_k}|^2+|f_{\overline{z}_jz_k}|^2+|f_{\overline{z}_j\overline{z}_k}|^2\big)\\
&\quad+\alpha\|D_f\|^{\alpha-2}\mathrm{Re}\left\{\sum_{k=1}^n\left[\overline{f_{z_k}}(\Delta f)_{z_k}+\overline{f_{\overline{z}_k}}(\Delta f)_{\overline{z}_k}\right]\right\}\geq 0,
\end{aligned}
$$

we see that, for $\alpha\in[4,\infty)$, $\|D_f\|^\alpha$ is subharmonic in $\mathbf{B}^n$.

Next we deal with the case $\alpha\in[2,4)$. In this case, for $m\in\{1,2,\ldots\}$, we let $F_m^\alpha=(\|D_f\|^2+\frac{1}{m})^{\frac{\alpha}{2}}$. Then, by elementary computations, we have

$$
\begin{aligned}
\Delta(F_m^\alpha) &= 4\sum_{j=1}^n (F_m^\alpha)_{z_k\overline{z}_k} = 4\sum_{j=1}^n\frac{\partial^2}{\partial z_j\partial\overline{z}_j}\left\{\left[\frac{1}{m}+\sum_{k=1}^n\big(f_{z_k}\overline{f_{z_k}}+f_{\overline{z}_k}\overline{f_{\overline{z}_k}}\big)\right]^{\frac{\alpha}{2}}\right\}\\
&= \alpha(\alpha-2)\left(\|D_f\|^2+\frac{1}{m}\right)^{\frac{\alpha}{2}-2}\left[\sum_{j=1}^n\frac{\partial}{\partial z_j}(\|D_f\|^2)\right]\left[\sum_{j=1}^n\frac{\partial}{\partial\overline{z}_j}(\|D_f\|^2)\right]\\
&\quad+2\alpha\left(\|D_f\|^2+\frac{1}{m}\right)^{\frac{\alpha}{2}-1}\sum_{j=1}^n\sum_{k=1}^n\big(|f_{z_jz_k}|^2+|f_{z_j\overline{z}_k}|^2+|f_{\overline{z}_jz_k}|^2+|f_{\overline{z}_j\overline{z}_k}|^2\big)\\
&\quad+\alpha\left(\|D_f\|^2+\frac{1}{m}\right)^{\frac{\alpha}{2}-1}\mathrm{Re}\left\{\sum_{k=1}^n\left[\overline{f_{z_k}}(\Delta f)_{z_k}+\overline{f_{\overline{z}_k}}(\Delta f)_{\overline{z}_k}\right]\right\}
\end{aligned}
$$

$$= \alpha(\alpha-2)\left(\|D_f\|^2+\frac{1}{m}\right)^{\frac{\alpha}{2}-2}\left|\sum_{j=1}^{n}\sum_{k=1}^{n}\left(f_{z_kz_j}\overline{f_{z_k}}+\overline{f_{z_k\overline{z}_j}}f_{z_k}+f_{\overline{z}_kz_j}\overline{f_{\overline{z}_k}}+\overline{f_{\overline{z}_k\overline{z}_j}}f_{\overline{z}_k}\right)\right|^2$$
$$+2\alpha\left(\|D_f\|^2+\frac{1}{m}\right)^{\frac{\alpha}{2}-1}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(|f_{z_jz_k}|^2+|f_{z_j\overline{z}_k}|^2+|f_{\overline{z}_jz_k}|^2+|f_{\overline{z}_j\overline{z}_k}|^2\right)$$
$$+\alpha\left(\|D_f\|^2+\frac{1}{m}\right)^{\frac{\alpha}{2}-1}\mathrm{Re}\left\{\sum_{k=1}^{n}\left[\overline{f_{z_k}}(\Delta f)_{z_k}+\overline{f_{\overline{z}_k}}(\Delta f)_{\overline{z}_k}\right]\right\}.$$

By the Cauchy–Schwarz inequality, we have

$$\left|\sum_{j=1}^{n}\sum_{k=1}^{n}\left(f_{z_kz_j}\overline{f_{z_k}}+\overline{f_{z_k\overline{z}_j}}f_{z_k}+f_{\overline{z}_kz_j}\overline{f_{\overline{z}_k}}+\overline{f_{\overline{z}_k\overline{z}_j}}f_{\overline{z}_k}\right)\right|^2 \tag{2.7}$$
$$\leq\left[\sum_{j=1}^{n}\sum_{k=1}^{n}\left(|f_{z_kz_j}\overline{f_{z_k}}|+|\overline{f_{z_k\overline{z}_j}}f_{z_k}|+|f_{\overline{z}_kz_j}\overline{f_{\overline{z}_k}}|+|\overline{f_{\overline{z}_k\overline{z}_j}}f_{\overline{z}_k}|\right)\right]^2$$
$$\leq\left\{\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\left(2|f_{z_k}|^2+2|f_{\overline{z}_k}|^2\right)^{\frac{1}{2}}\left(|f_{z_kz_j}|^2+|f_{z_k\overline{z}_j}|^2+|f_{\overline{z}_kz_j}|^2+|f_{\overline{z}_k\overline{z}_j}|^2\right)^{\frac{1}{2}}\right]\right\}^2$$
$$\leq 2\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\left(2|f_{z_k}|^2+2|f_{\overline{z}_k}|^2\right)\left(|f_{z_kz_j}|^2+|f_{z_k\overline{z}_j}|^2+|f_{\overline{z}_kz_j}|^2+|f_{\overline{z}_k\overline{z}_j}|^2\right)\right]$$
$$\leq 4\|D_f\|^2\sum_{j=1}^{n}\sum_{k=1}^{n}\left(|f_{z_kz_j}|^2+|f_{z_k\overline{z}_j}|^2+|f_{\overline{z}_kz_j}|^2+|f_{\overline{z}_k\overline{z}_j}|^2\right).$$

Hence, by (2.7) and Lebesgue's dominated convergence theorem, we obtain

$$\lim_{m\to\infty}\Delta(F_m^{\alpha})=\alpha(\alpha-2)\|D_f\|^{\alpha-4}\left|\sum_{j=1}^{n}\sum_{k=1}^{n}\left(f_{z_kz_j}\overline{f_{z_k}}+\overline{f_{z_k\overline{z}_j}}f_{z_k}+f_{\overline{z}_kz_j}\overline{f_{\overline{z}_k}}+\overline{f_{\overline{z}_k\overline{z}_j}}f_{\overline{z}_k}\right)\right|^2$$
$$+2\alpha\|D_f\|^{\alpha-2}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(|f_{z_jz_k}|^2+|f_{z_j\overline{z}_k}|^2+|f_{\overline{z}_jz_k}|^2+|f_{\overline{z}_j\overline{z}_k}|^2\right)$$
$$+\alpha\|D_f\|^{\alpha-2}\mathrm{Re}\left\{\sum_{k=1}^{n}\left[\overline{f_{z_k}}(\Delta f)_{z_k}+\overline{f_{\overline{z}_k}}(\Delta f)_{\overline{z}_k}\right]\right\}\geq 0.$$

Then, for $\alpha\in[2,4)$, $\|D_f\|^{\alpha}$ is subharmonic in $\mathbf{B}^n$. □

Proof of Theorem 2 It is not difficult to see that if τ and η are constant functions, then each solution f to (1.2) belongs to $C^{\infty}(\mathbf{B}^n)$, i.e., they are infinitely differentiable in $\mathbf{B}^n$.

By elementary calculations, we get

$$\sum_{k=1}^{n}\mathrm{Re}\left[\overline{f_{z_k}}(\Delta f)_{z_k}+\overline{f_{\overline{z}_k}}(\Delta f)_{\overline{z}_k}\right]=\sum_{k=1}^{n}\left[\tau\left(|f_{z_k}|^2+|f_{\overline{z}_k}|^2\right)+\frac{\eta}{2}\left|f_{z_k}+\overline{f_{\overline{z}_k}}\right|^2\right] \tag{2.8}$$
$$\leq(\eta+\tau)\|D_f\|^2,$$

and

$$\mathrm{Re}\left(f\overline{\Delta f}\right)=\tau|f|^2+\eta\left(\mathrm{Re}(f)\right)^2\leq(\eta+\tau)|f|^2. \tag{2.9}$$

By using Hölder's inequality, for $\rho \in (0,1)$, we see that

$$\int_{\partial \mathbf{B}^n} \|D_f(\rho\zeta)\|^{p-2}\big(D_f^*(\rho\zeta)\big)^2\, d\sigma(\zeta) \leq M_p^2(\rho, D_f^*)M_p^{p-2}(\rho, D_f) \tag{2.10}$$

and

$$\int_{\partial \mathbf{B}^n} \|f(\rho\zeta)\|^{p-2}\|D_f(\rho\zeta)\|^2\, d\sigma(\zeta) \leq M_p^2(\rho, D_f)M_p^{p-2}(\rho, f). \tag{2.11}$$

For $t \in [0,1]$, $r \in (0,1)$ and $\rho \in (0,r]$, we obtain

$$\frac{t(1-t^{2n-2})}{2(n-1)} \leq 1-t \tag{2.12}$$

and

$$\rho \log \frac{r}{\rho} \leq r - \rho, \tag{2.13}$$

where $n \geq 2$.

Case 1. Let $n \geq 2$.

Step 1. By (2.7), (2.8), (2.10), Lemma 3, Theorem B and Lebesgue's Dominated Convergence theorem, we see that

$$\begin{aligned}
M_p^p(r, D_f) &= \|D_f(0)\|^p + \int_{\mathbf{B}^n(r)} \Delta\big(\|D_f(z)\|^p\big) G_{2n}(z,r)\, dV_N(z) = \|D_f(0)\|^p \\
&\quad + \int_{\mathbf{B}^n(r)} \Bigg\{ p\|D_f(z)\|^{p-2} \sum_{k=1}^n \mathrm{Re}\Big[\overline{f_{z_k}(z)}(\Delta f(z))_{z_k} + \overline{f_{\overline{z}_k}(z)}(\Delta f(z))_{\overline{z}_k}\Big] \\
&\quad + p(p-2)\|D_f(z)\|^{p-4} \Bigg| \sum_{j=1}^n \sum_{k=1}^n \Big(f_{z_k z_j}(z)\overline{f_{z_k}(z)} \\
&\quad + \overline{f_{z_k \overline{z}_j}(z)} f_{z_k}(z) + f_{\overline{z}_k z_j}(z)\overline{f_{\overline{z}_k}(z)} + \overline{f_{\overline{z}_k \overline{z}_j}(z)} f_{\overline{z}_k}(z)\Big)\Bigg|^2 \\
&\quad + 2p\|D_f(z)\|^{p-2}\big(D_f^*(z)\big)^2 \Bigg\} G_{2n}(z,r)\, dV_N(z) \\
&\leq \|D_f(0)\|^p + p\int_{\mathbf{B}^n(r)} \Big[(\eta+\tau)\|D_f(z)\|^p \\
&\quad + 2(2p-3)\|D_f(z)\|^{p-2}\big(D_f^*(z)\big)^2\Big] G_{2n}(z,r)\, dV_N(z) \\
&= \|D_f(0)\|^p + \frac{p(\eta+\tau)}{2(n-1)} \int_0^r \big(\rho - \rho^{2n-1}r^{2(1-n)}\big) M_p^p(\rho, D_f)\, d\rho \\
&\quad + \frac{p(2p-3)}{(n-1)} \int_0^r \big(\rho - \rho^{2n-1}r^{2(1-n)}\big) \int_{\partial \mathbf{B}^n} \|D_f(\rho\zeta)\|^{p-2}\big(D_f^*(\rho\zeta)\big)^2\, d\sigma(\zeta)\, d\rho \\
&\leq \|D_f(0)\|^p + \frac{p(\eta+\tau)}{2(n-1)} \int_0^r \big(\rho - \rho^{2n-1}r^{2(1-n)}\big) M_p^p(\rho, D_f)\, d\rho \\
&\quad + \frac{p(2p-3)}{(n-1)} \int_0^r \big(\rho - \rho^{2n-1}r^{2(1-n)}\big) M_p^2(\rho, D_f^*)M_p^{p-2}(\rho, D_f)\, d\rho.
\end{aligned}$$

The above together with (2.12) and subharmonicity of $\|D_f\|^p$, shows that

$$\left[1-\frac{p(\eta+\tau)}{2(n-1)}\int_0^r(\rho-\rho^{2n-1}r^{2(1-n)})\,d\rho\right]M_p^2(r,D_f)=\left[1-\frac{pr^2(\eta+\tau)}{4n}\right]M_p^2(r,D_f)$$
$$\le\|D_f(0)\|^2+\frac{p(2p-3)}{(n-1)}\int_0^r\big(\rho-\rho^{2n-1}r^{2(1-n)}\big)M_p^2(\rho,D_f^*)\,d\rho$$
$$=\|D_f(0)\|^2+2p(2p-3)r^2\int_0^1\frac{t(1-t^{2n-2})}{2(n-1)}M_p^2\left(rt,D_f^*\right)dt$$
$$\le\|D_f(0)\|^2+2p(2p-3)r^2(M^*)^2\int_0^1\left[\omega\Big(\frac{1}{1-rt}\Big)\right]^2(1-t)\,dt$$
$$\le\|D_f(0)\|^2+2p(2p-3)r^2(M^*)^2\int_0^1\left[\omega\Big(\frac{1}{1-rt}\Big)\right]^2(1-rt)\,dt$$
$$\le\|D_f(0)\|^2+2p(2p-3)r^2(M^*)^2\omega(1)\int_0^1\omega\left(\frac{1}{1-rt}\right)dt.$$

Then

$$M_p^2(r,D_f)\le M_2^*\left[\|D_f(0)\|^2+M_1^*\int_0^1\omega\left(\frac{1}{1-rt}\right)dt\right],\tag{2.14}$$

where $M_1^*=2p(2p-3)(M^*)^2\omega(1)$ and $M_2^*=1/\left[1-p(\eta+\tau)/(4n)\right]$.

Step 2. By (2.9), (2.11), Lemmas 1 and 2, we obtain

$$M_p^p(r,f)=|f(0)|^p+\int_{\mathbf{B}^n(r)}\Delta(|f(z)|^p)G_{2n}(z,r)\,dV_N(z)$$
$$\le|f(0)|^p+\int_0^r\int_{\partial\mathbf{B}^n}4np(p-1)\rho^{2n-1}|f(\rho\zeta)|^{p-2}|D_f(\rho\zeta)|^2G_{2n}(\rho\zeta,r)\,d\sigma(\zeta)\,d\rho$$
$$+p(\eta+\tau)\int_{\mathbf{B}^n(r)}|f(z)|^pG_{2n}(z,r)\,dV_N(z)$$
$$\le|f(0)|^p+4p(p-1)\int_0^r n\rho^{2n-1}G_{2n}(\rho\zeta,r)M_p^2(\rho,D_f)M_p^{p-2}(\rho,f)\,d\rho$$
$$+\frac{p(\eta+\tau)r^2}{4n}M_p^p(r,f).$$

By the above estimates, (2.12), (2.14) and the monotonicity of $M_p(r,f)$ on r,

$$\frac{M_p^2(r,f)}{M_2^*}\le\left[1-\frac{p(\eta+\tau)r^2}{4n}\right]M_p^2(r,f)$$
$$\le|f(0)|^2+4p(p-1)\int_0^r n\rho^{2n-1}G_{2n}(\rho\zeta,r)M_p^2(\rho,D_f)\,d\rho$$
$$=|f(0)|^2+2p(p-1)\int_0^1 r^2M_p^2(r\rho,D_f)\cdot\frac{\rho(1-\rho^{2n-2})}{2(n-1)}\,d\rho$$
$$\le|f(0)|^2+2p(p-1)\int_0^1 M_p^2(r\rho,D_f)(1-\rho)\,d\rho$$
$$\le|f(0)|^2+2p(p-1)M_2^*\|D_f(0)\|^2$$
$$+2p(p-1)M_1^*M_2^*\int_0^1\left[\int_0^1\omega\Big(\frac{1}{1-r\rho t}\Big)(1-\rho)\,dt\right]d\rho$$

$$\begin{aligned}&\leq |f(0)|^2 + 2p(p-1)M_2^*\|D_f(0)\|^2\\&\quad + 2p(p-1)M_1^*M_2^*\int_0^1\left[\int_0^1 \omega\Big(\frac{1}{1-r\rho t}\Big)(1-rt\rho)\,dt\right]d\rho\\&\leq |f(0)|^2 + 2p(p-1)M_2^*\|D_f(0)\|^2 + 2p(p-1)M_1^*M_2^*\omega(1) < \infty.\end{aligned}$$

Hence $f \in \mathcal{H}_g^p(\mathbf{B}^n)$.

Case 2. Let $n = 1$.

Step 3. By (2.7), (2.8), (2.10), Lemma 3, Theorem B and Lebesgue's dominated convergence theorem, we see that

$$\begin{aligned}M_p^p(r, D_f) &= \|D_f(0)\|^p + \frac{1}{2}\int_{\mathbf{D}(r)} \Delta\big(\|D_f(z)\|^p\big)\log\frac{r}{|z|}\,dA(z)\\&= \|D_f(0)\|^p + \frac{1}{2}\int_{\mathbf{D}(r)}\Big\{p\|D_f(z)\|^{p-2}\mathrm{Re}\Big[\overline{f_z(z)}(\Delta f(z))_z + \overline{f_{\overline{z}}(z)}(\Delta f(z))_{\overline{z}}\Big]\\&\quad + p(p-2)\|D_f(z)\|^{p-4}\Big|\Big(f_{zz}(z)\overline{f_z(z)} + \overline{f_{z\overline{z}}(z)}f_z(z) + f_{\overline{z}z}(z)\overline{f_{\overline{z}}(z)} + \overline{f_{\overline{z}\overline{z}}(z)}f_{\overline{z}}(z)\Big)\Big|^2\\&\quad + 2p\|D_f(z)\|^{p-2}\big(D_f^*(z)\big)^2\Big\}\log\frac{r}{|z|}\,dA(z)\\&\leq \|D_f(0)\|^p + \frac{p}{2}\int_{\mathbf{D}(r)}\Big[(\eta+\tau)\|D_f(z)\|^p + 2(2p-3)\|D_f(z)\|^{p-2}\big(D_f^*(z)\big)^2\Big]\log\frac{r}{|z|}\,dA(z)\\&= \|D_f(0)\|^p + p(\eta+\tau)\int_0^r M_p^p(\rho, D_f)\rho\log\frac{r}{\rho}d\rho\\&\quad + 2p(2p-3)\int_0^r \rho\log\frac{r}{\rho}\left(\frac{1}{2\pi}\int_0^{2\pi}\|D_f(\rho e^{i\theta})\|^{p-2}\big(D_f^*(\rho e^{i\theta})\big)^2 d\theta\right)d\rho\\&\leq \|D_f(0)\|^p + p(\eta+\tau)\int_0^r M_p^p(\rho, D_f)\rho\log\frac{r}{\rho}d\rho\\&\quad + 2p(2p-3)\int_0^r M_p^2(\rho, D_f^*)M_p^{p-2}(\rho, D_f)\rho\log\frac{r}{\rho}\,d\rho\\&\leq \|D_f(0)\|^p + p(\eta+\tau)M_p^p(r, D_f)\int_0^r \rho\log\frac{r}{\rho}d\rho\\&\quad + 2p(2p-3)\int_0^r M_p^2(\rho, D_f^*)M_p^{p-2}(\rho, D_f)\rho\log\frac{r}{\rho}\,d\rho,\end{aligned}$$

which, together with (2.13), gives that

$$\begin{aligned}&\left[1 - p(\eta+\tau)\int_0^r \rho\log\frac{r}{\rho}\,d\rho\right]M_p^2(r, D_f) = \left[1 - \frac{pr^2(\eta+\tau)}{4}\right]M_p^2(r, D_f)\\&\leq \|D_f(0)\|^2 + 2p(2p-3)\int_0^r \rho\log\frac{r}{\rho}M_p^2(\rho, D_f^*)\,d\rho\\&\leq \|D_f(0)\|^2 + 2p(2p-3)\int_0^r (r-\rho)M_p^2(\rho, D_f^*)\,d\rho\\&= \|D_f(0)\|^2 + 2p(2p-3)r^2\int_0^1 (1-t)M_p^2\left(rt, D_f^*\right)dt\\&\leq \|D_f(0)\|^2 + 2p(2p-3)r^2(M^*)^2\int_0^1\left[\omega\Big(\frac{1}{1-rt}\Big)\right]^2(1-t)\,dt\end{aligned}$$

$$\leq \|D_f(0)\|^2 + 2p(2p-3)r^2(M^*)^2 \int_0^1 \left[\omega\Big(\frac{1}{1-rt}\Big)\right]^2 (1-rt)\, dt$$
$$\leq \|D_f(0)\|^2 + 2p(2p-3)r^2(M^*)^2\omega(1) \int_0^1 \omega\left(\frac{1}{1-rt}\right) dt,$$

where dA denotes the normalized area measure in $\mathbf{D}$. Then

$$M_p^2(r, D_f) \leq M_2^{**}\left[\|D_f(0)\|^2 + M_1^* \int_0^1 \omega\Big(\frac{1}{1-rt}\Big) dt\right], \tag{2.15}$$

where $M_2^{**} = 1/\left[1 - p(\eta+\tau)/4\right]$.

Step 4. By (2.9), (2.11), Lemmas 1 and 2, we obtain

$$M_p^p(r,f) = |f(0)|^p + \frac{1}{2}\int_{\mathbf{D}(r)} \Delta(|f(z)|^p)\log\frac{r}{|z|}\, dV_N(z)$$
$$\leq |f(0)|^p + 2p(p-1)\int_0^r \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|^{p-2}|D_f(\rho e^{i\theta})|^2\, d\theta\right)\rho\log\frac{r}{\rho}\, d\rho$$
$$+ p(\eta+\tau)\int_0^r \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|^p\, d\theta\right)\rho\log\frac{r}{\rho}\, d\rho$$
$$\leq |f(0)|^p + 2p(p-1)\int_0^r \rho\log\frac{r}{\rho}M_p^2(\rho, D_f)M_p^{p-2}(\rho, f)\, d\rho$$
$$+ p(\eta+\tau)M_p^p(r,f)\int_0^r \rho\log\frac{r}{\rho}\, d\rho$$
$$\leq |f(0)|^p + 2p(p-1)\int_0^r \rho\log\frac{r}{\rho}M_p^2(\rho, D_f)M_p^{p-2}(\rho, f)\, d\rho$$
$$+ \frac{pr^2(\eta+\tau)}{4}M_p^p(r,f).$$

The above, (2.13), (2.15) and the monotonicity of $M_p(r,f)$ on r, imply that

$$\frac{M_p^2(r,f)}{M_2^{**}} \leq \left[1 - \frac{p(\eta+\tau)r^2}{4}\right] M_p^2(r,f)$$
$$\leq |f(0)|^2 + 2p(p-1)\int_0^r M_p^2(\rho, D_f)\rho\log\frac{r}{\rho}\, d\rho$$
$$\leq |f(0)|^2 + 2p(p-1)\int_0^r M_p^2(\rho, D_f)(r-\rho)\, d\rho$$
$$\leq |f(0)|^2 + 2p(p-1)\int_0^1 M_p^2(r\rho, D_f)(1-\rho)\, d\rho$$
$$\leq |f(0)|^2 + 2p(p-1)M_2^{**}\|D_f(0)\|^2$$
$$+ 2p(p-1)M_1^*M_2^{**}\int_0^1 \left[\int_0^1 \omega\Big(\frac{1}{1-r\rho t}\Big)(1-\rho)\, dt\right] d\rho$$
$$\leq |f(0)|^2 + 2p(p-1)M_2^{**}\|D_f(0)\|^2$$
$$+ 2p(p-1)M_1^*M_2^{**}\int_0^1 \left[\int_0^1 \omega\Big(\frac{1}{1-r\rho t}\Big)(1-rt\rho)\, dt\right] d\rho$$
$$\leq |f(0)|^2 + 2p(p-1)M_2^{**}\|D_f(0)\|^2 + 2p(p-1)M_1^*M_2^{**}\omega(1) < \infty.$$

Hence $f \in \mathcal{H}_g^p(\mathbf{D})$. The proof of the theorem is complete. □

3. Dirichlet-type spaces, Bergman-type spaces and applications to PDEs

Proof of Theorem 3. We first prove the necessity. Since $\mathrm{Re}(f\overline{\Delta f}) \geq 0$, we observe that $\Delta\big(|f|^p\big) \geq 0$ and

$$0 \leq \int_{\mathbf{B}^n} (1-|z|^2)^\alpha \Delta\left(|f(z)|^p\right) dV_N(z) < \infty. \tag{3.1}$$

Let $r \in (0,1)$. For $\alpha \geq 2$, it is not difficult to see that

$$(r^2-|z|^2)^\alpha|_{\partial\mathbf{B}^n(r)} = 0 \quad \text{and} \quad \frac{\partial}{\partial\varepsilon}\big[(r^2-|z|^2)^\alpha\big]|_{\partial\mathbf{B}^n(r)} = 0,$$

where $\partial/\partial\varepsilon$ denotes an outer normal derivative. Then, by Green's theorem, we get

$$\int_{\mathbf{B}^n(r)} (r^2-|z|^2)^\alpha \Delta\left(|f(z)|^p\right) dV_N(z) = \int_{\mathbf{B}^n(r)} |f(z)|^p \Delta\big[(r^2-|z|^2)^\alpha\big]\, dV_N(z)$$

$$= 4\alpha \int_{\mathbf{B}^n(r)} |f(z)|^p (r^2-|z|^2)^{\alpha-2}\left[|z|^2(n+\alpha-1)-nr^2\right] dV_N(z), \tag{3.2}$$

which, together with (3.1), gives that

$$\begin{aligned}
\infty &> 4\alpha \int_{\mathbf{B}^n(R_1)} |f(z)|^p (1-|z|^2)^{\alpha-2}\left[n-|z|^2(n+\alpha-1)\right] dV_N(z) \\
&\quad + \int_{\mathbf{B}^n} (1-|z|^2)^\alpha \Delta\left(|f(z)|^p\right) dV_N(z) \\
&\geq 4\alpha \int_{\mathbf{B}^n(rR_1)} |f(z)|^p (1-|z|^2)^{\alpha-2}\left[n-|z|^2(n+\alpha-1)\right] dV_N(z) \\
&\quad + \int_{\mathbf{B}^n(r)} (1-|z|^2)^\alpha \Delta\left(|f(z)|^p\right) dV_N(z) \\
&\geq 4\alpha \int_{\mathbf{B}^n(rR_1)} |f(z)|^p (r^2-|z|^2)^{\alpha-2}\left[nr^2-|z|^2(n+\alpha-1)\right] dV_N(z) \\
&\quad + \int_{\mathbf{B}^n(r)} (r^2-|z|^2)^\alpha \Delta\left(|f(z)|^p\right) dV_N(z) \\
&= 4\alpha \int_{\mathbf{B}^n(r)\backslash\mathbf{B}^n(rR_1)} |f(z)|^p (r^2-|z|^2)^{\alpha-2}\left[|z|^2(n+\alpha-1)-nr^2\right] dV_N(z) \\
&\geq 2r^2\alpha \int_{\mathbf{B}^n(r)\backslash\mathbf{B}^n(rR_2)} |f(z)|^p (r^2-|z|^2)^{\alpha-2}\, dV_N(z),
\end{aligned}$$

where $R_1 = \sqrt{\frac{n}{n+\alpha-1}}$ and $R_2 = \sqrt{\frac{n+\frac{1}{2}}{n+\alpha-1}}$.

For $R_2 < r < 1$, we conclude that

$$\infty > 2r^2\alpha \int_{\mathbf{B}^n(r)\backslash\mathbf{B}^n(rR_2)} |f(z)|^p (r^2-|z|^2)^{\alpha-2}\, dV_N(z) \geq 2\alpha R_2^2 U(r), \tag{3.3}$$

where

$$U(r) = \int_{\mathbf{B}^n(r)\backslash\mathbf{B}^n(R_2)} |f(z)|^p (r^2-|z|^2)^{\alpha-2}\, dV_N(z).$$

Then, for $R_2 < r < 1$, $U(r)$ is increasing and bounded, from which we conclude that

$$\lim_{r\to 1-} U(r)$$

exists. Hence for $p \geq 2$, $f \in b_{\alpha-2,p}(\mathbf{B}^n)$.

Next we prove the sufficiency. For $\alpha \geq 2$, by (3.2), we have

$$
\begin{aligned}
(3.4)\qquad & 4\alpha\int_{\mathbf{B}^n(rR_1)} |f(z)|^p(r^2-|z|^2)^{\alpha-2}\big[nr^2-|z|^2(n+\alpha-1)\big]\,dV_N(z)\\
&+\int_{\mathbf{B}^n(r)}(r^2-|z|^2)^{\alpha}\Delta\big(|f(z)|^p\big)\,dV_N(z)\\
&=4\alpha\int_{\mathbf{B}^n(r)\backslash\mathbf{B}^n(rR_1)}|f(z)|^p(r^2-|z|^2)^{\alpha-2}\big[|z|^2(n+\alpha-1)-nr^2\big]\,dV_N(z)\\
&\leq 4\alpha(n+\alpha-1)\int_{\mathbf{B}^n(r)\backslash\mathbf{B}^n(rR_1)}|f(z)|^p(r^2-|z|^2)^{\alpha-2}\,dV_N(z)\\
&\leq 4\alpha(n+\alpha-1)\int_{\mathbf{B}^n}|f(z)|^p(1-|z|^2)^{\alpha-2}\,dV_N(z)<\infty.
\end{aligned}
$$

Since

$$
\begin{aligned}
\infty &> \int_{\mathbf{B}^n(R_1)}|f(z)|^p(1-|z|^2)^{\alpha-2}\big[n-|z|^2(n+\alpha-1)\big]\,dV_N(z)\\
(3.5)\qquad &\geq \int_{\mathbf{B}^n(rR_1)}|f(z)|^p(r^2-|z|^2)^{\alpha-2}\big[nr^2-|z|^2(n+\alpha-1)\big]\,dV_N(z),
\end{aligned}
$$

which, together with (3.4) and $\Delta\big(|f|^p\big)\geq 0$, implies that

$$\lim_{r\to 1-}\int_{\mathbf{B}^n(r)}(r^2-|z|^2)^{\alpha}\Delta\big(|f(z)|^p\big)\,dV_N(z)$$

does exist. Therefore,

$$\int_{\mathbf{B}^n}(1-|z|^2)^{\alpha}\Delta\big(|f(z)|^p\big)\,dV_N(z)<\infty,$$

and thus the theorem is proved. □

The following result is well-known.

Lemma 4. *Suppose that $a,b\in[0,\infty)$ and $q\in(0,\infty)$. Then*

$$(a+b)^q\leq 2^{\max\{q-1,0\}}(a^q+b^q).$$

Proof of Theorem 4. By Lemma 3, for $\rho\in[0,d(z))$, we get

$$\|D_f(z)\|^{\alpha}\leq\int_{\partial\mathbf{B}^n}\|D_f(z+\rho\zeta)\|^{\alpha}\,d\sigma(\zeta). \tag{3.6}$$

Multiplying both sides of the inequality (3.6) by $2n\rho^{2n-1}$ and integrating from 0 to $d(z)/2$, we have

$$
\begin{aligned}
\frac{d(z)^{2n}\|D_f(z)\|^{\alpha}}{2^{2n}} &\leq \int_{\partial\mathbf{B}^n}\int_0^{\frac{d(z)}{2}}2n\rho^{2n-1}\|D_f(z+\rho\zeta)\|^{\alpha}\,d\rho\,d\sigma(\zeta)\\
&=\int_{\mathbf{B}^n\left(z,\frac{d(z)}{2}\right)}\|D_f(\xi)\|^{\alpha}\,dV_N(\xi)\\
&\leq 2^{\gamma}(d(z))^{-\gamma}\int_{\mathbf{B}^n\left(z,\frac{d(z)}{2}\right)}(1-|\xi|)^{\gamma}\|D_f(\xi)\|^{\alpha}\,dV_N(\xi)\\
&\leq\frac{2^{\gamma}\|f\|^{\alpha}_{\mathcal{D}_{\gamma,\alpha}}}{\big(d(z)\big)^{\gamma}},
\end{aligned}
$$

which implies that

$$\|D_f(z)\| \le \frac{M_1}{(d(z))^{q+1}}, \tag{3.7}$$

where $M_1 = 2^{1+q}\|f\|_{\mathcal{D}_{\gamma,\alpha}}$ and $q = \frac{\gamma+2n}{\alpha} - 1$. By (3.7), we know that

$$\begin{aligned} |f(z)| &\le |f(0)| + \left|\int_{[0,z]} df(\varsigma)\right| \le |f(0)| + \sqrt{2}\int_{[0,z]} \|D_f(\varsigma)\|\|d\varsigma\| \\ &\le |f(0)| + \frac{M_2}{\big(d(z)\big)^q}, \end{aligned} \tag{3.8}$$

where $M_2 = M_1\sqrt{2}/q$ and $[0,z]$ denotes the line segment from 0 to z.

By (3.8) and Lemma 4, we see that for $z \in \mathbf{B}^n$,

$$|f(z)|^{p-2} \le \left[|f(0)| + \frac{M_2}{\big(d(z)\big)^q}\right]^{p-2} \le 2^{p-2}\left[|f(0)|^{p-2} + \frac{M_2^{p-2}}{\big(d(z)\big)^{q(p-2)}}\right], \tag{3.9}$$

$$|f(z)|^{p-1} \le \left[|f(0)| + \frac{M_2}{\big(d(z)\big)^q}\right]^{p-1} \le 2^{p-1}\left[|f(0)|^{p-1} + \frac{M_2^{p-1}}{\big(d(z)\big)^{q(p-1)}}\right] \tag{3.10}$$

and

$$|f(z)|^{p} \le \left[|f(0)| + \frac{M_2}{\big(d(z)\big)^q}\right]^{p} \le 2^{p}\left[|f(0)|^{p} + \frac{M_2^{p}}{\big(d(z)\big)^{qp}}\right]. \tag{3.11}$$

Case 1. Let $p \in [4,\infty)$. By direct calculations, we get

$$\begin{aligned} \Delta\big(|f|^p\big) &= p(p-2)|f|^{p-4}\sum_{k=1}^n |f_{z_k}\overline{f} + \overline{f_{\overline{z}_k}}f|^2 + 2p|f|^{p-2}\|D_f\|^2 + p|f|^{p-2}\mathrm{Re}(f\overline{\Delta f}) \\ &\le p(p-2)|f|^{p-4}\sum_{k=1}^n |f_{z_k}\overline{f} + \overline{f_{\overline{z}_k}}f|^2 + 2p|f|^{p-2}\|D_f\|^2 + p|f|^{p-1}|\Delta f| \\ &\le 2p(p-1)|f|^{p-2}\|D_f\|^2 + pa|f|^{p-1}\|D_f\| + pb|f|^p + pc|f|^{p-1}. \end{aligned} \tag{3.12}$$

It follows from (3.9), (3.10), (3.11) and (3.12) that

$$\begin{aligned} \big(d(z)\big)^{pq}\Delta\big(|f|^p\big) &\le 2p(p-1)\big(d(z)\big)^{pq}|f|^{p-2}\|D_f\|^2 \\ &\quad + pa\big(d(z)\big)^{pq}|f|^{p-1}\|D_f\| + pb\big(d(z)\big)^{pq}|f|^p \\ &\quad + pc\big(d(z)\big)^{pq}|f|^{p-1} \\ &= 2p(p-1)\big(d(z)\big)^{pq-\frac{2\gamma}{\alpha}}|f|^{p-2}\|D_f\|^2\big(d(z)\big)^{\frac{2\gamma}{\alpha}} \\ &\quad + pa\big(d(z)\big)^{pq-\frac{\gamma}{\alpha}}|f|^{p-1}\|D_f\|\big(d(z)\big)^{\frac{\gamma}{\alpha}} \\ &\quad + pb\big(d(z)\big)^{pq}|f|^p + pc\big(d(z)\big)^{pq}|f|^{p-1} \\ &\le M_3\|D_f\|^2\big(d(z)\big)^{\frac{2\gamma}{\alpha}} + M_4\|D_f\|\big(d(z)\big)^{\frac{\gamma}{\alpha}} + M_5, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} M_3 &= 2^{p-1}p(p-1)\left(|f(0)|^{p-2} + M_2^{p-2}\right), \\ M_4 &= p2^{p-1}\left(|f(0)|^{p-1} + M_2^{p-1}\right)\sup_{z\in\mathbf{B}^n} a(z) \end{aligned}$$

and

$$M_5 = p2^p \left(|f(0)|^p + M_2^p\right) \sup_{z\in\mathbf{B}^n} b(z) + pM2^{p-1}\left(|f(0)|^{p-1} + M_2^{p-1}\right).$$

By Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{\mathbf{B}^n} \big(d(z)\big)^{\frac{2\gamma}{\alpha}} \|D_f(z)\|^2\, dV_N(z) \\
&\le \left(\int_{\mathbf{B}^n} \big(d(z)\big)^{\gamma} \|D_f(z)\|^{\alpha}\, dV_N(z)\right)^{\frac{2}{\alpha}} \left(\int_{\mathbf{B}^n} dV_N(z)\right)^{1-\frac{2}{\alpha}} \le \|f\|^2_{\mathcal{D}_{\gamma,\alpha}},
\end{aligned} \tag{3.14}$$

which gives

$$\begin{aligned}
&\int_{\mathbf{B}^n} \big(d(z)\big)^{\frac{\gamma}{\alpha}} \|D_f(z)\|\, dV_N(z) \\
&\le \left(\int_{\mathbf{B}^n} \big(d(z)\big)^{\frac{2\gamma}{\alpha}} \|D_f(z)\|^2\, dV_N(z)\right)^{\frac{1}{2}} \left(\int_{\mathbf{B}^n} dV_N(z)\right)^{\frac{1}{2}} \le \|f\|_{\mathcal{D}_{\gamma,\alpha}}.
\end{aligned} \tag{3.15}$$

It follows from (3.13), (3.14) and (3.15) that

$$\begin{aligned}
&\int_{\mathbf{B}^n} \big(d(z)\big)^{pq} \Delta\big(|f(z)|^p\big)\, dV_N(z) \\
&= \int_{\mathbf{B}^n} \Big[M_3\|D_f\|^2\big(d(z)\big)^{\frac{2\gamma}{\alpha}} + M_4\|D_f\|\big(d(z)\big)^{\frac{\gamma}{\alpha}} + M_5\Big]\, dV_N(z) \\
&\le M_3\|f\|^2_{\mathcal{D}_{\gamma,\alpha}} + M_4\|f\|_{\mathcal{D}_{\gamma,\alpha}} + M_5 < \infty.
\end{aligned}$$

Case 2. Let $p \in [2,4)$. For $p \in [2,4)$, $m \in \{1,2,\ldots\}$ and $r \in (0,1)$, let $T_m^p = \big(|f|^2 + \frac{1}{m}\big)^{\frac{p}{2}}$. Then, by (3.13), (3.14), (3.15) and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{r\to 1-}\left\{\lim_{m\to\infty} \int_{\mathbf{B}^n(r)} \big(d(z)\big)^{pq} \Delta\big(T_m^p(z)\big)\, dV_N(z)\right\} \\
&= \lim_{r\to 1-} \int_{\mathbf{B}^n(r)} \big(d(z)\big)^{pq} \lim_{m\to\infty} \Delta\big(T_m^p(z)\big)\, dV_N(z) \\
&= \lim_{r\to 1-} \int_{\mathbf{B}^n(r)} \big(d(z)\big)^{pq} \Bigg[p(p-2)|f(z)|^{p-4} \sum_{k=1}^{n} \Big|f_{z_k}(z)\overline{f(z)} \\
&\quad + \overline{f_{\overline{z}_k}(z)}f(z)\Big|^2 + 2p|f(z)|^{p-2}\|D_f(z)\|^2 + p|f(z)|^{p-2}\mathrm{Re}\big(f(z)\overline{\Delta f(z)}\big)\Bigg]\, dV_N(z) \\
&\le \int_{\mathbf{B}^n} \Big(M_3\|D_f\|^2\big(d(z)\big)^{\frac{2\gamma}{\alpha}} + M_4\|D_f\|\big(d(z)\big)^{\frac{\gamma}{\alpha}} + M_5\Big)\, dV_N(z) < \infty.
\end{aligned}$$

This concludes the proof of the theorem. □

Proof of Theorem 5. Case 1. Let $n \ge 2$. Without loss of generality, we may assume that

$$\inf_{z\in\mathbf{B}^n} a_1(z) > 0 \quad \text{and} \quad \inf_{z\in\mathbf{B}^n} b_1(z) > 0.$$

Let $r_0 \in (0,1)$ be a constant. Then, by Lemma 2, for $0 < r_0 \le r < 1$, we have

$$
\begin{aligned}
(3.16)\quad M_p^p(r,f) &= |f(0)|^p + \int_{\mathbf{B}^n(r)} \Delta\big(|f(z)|^p\big) G_{2n}(z,r)\, dV_N(z) \\
&\ge |f(0)|^p + \int_{\mathbf{B}^n(r)} \Big(a_1(z)\|D_f(z)\|^{t_1} + b_1(z)|f(z)|^{t_2} + c_1(z)\Big) G_{2n}(z,r)\, dV_N(z) \\
&\ge |f(0)|^p + \inf_{z\in\mathbf{B}^n} a_1(z) \int_{\mathbf{B}^n(r)} \|D_f(z)\|^{t_1} G_{2n}(z,r)\, dV_N(z) \\
&\quad + \inf_{z\in\mathbf{B}^n} b_1(z) \int_{\mathbf{B}^n(r)} |f(z)|^{t_2} G_{2n}(z,r)\, dV_N(z) \\
&\quad + \inf_{z\in\mathbf{B}^n} c_1(z) \int_{\mathbf{B}^n(r)} G_{2n}(z,r)\, dV_N(z) \\
&= |f(0)|^p + \inf_{z\in\mathbf{B}^n} a_1(z) \int_{\mathbf{B}^n(r_0)} \|D_f(z)\|^{t_1} G_{2n}(z,r)\, dV_N(z) \\
&\quad + \inf_{z\in\mathbf{B}^n} a_1(z) \int_{\mathbf{B}^n(r)\backslash\mathbf{B}^n(r_0)} \|D_f(z)\|^{t_1} G_{2n}(z,r)\, dV_N(z) \\
&\quad + \inf_{z\in\mathbf{B}^n} b_1(z) \int_{\mathbf{B}^n(r_0)} |f(z)|^{t_2} G_{2n}(z,r)\, dV_N(z) \\
&\quad + \inf_{z\in\mathbf{B}^n} b_1(z) \int_{\mathbf{B}^n(r)\backslash\mathbf{B}^n(r_0)} |f(z)|^{t_2} G_{2n}(z,r)\, dV_N(z) \\
&\quad + \inf_{z\in\mathbf{B}^n} c_1(z) \int_{\mathbf{B}^n(r)} G_{2n}(z,r)\, dV_N(z).
\end{aligned}
$$

It is easy to see that, for all $r \in (0,1)$,

$$
(3.17)\qquad 0 < \int_{\mathbf{B}^n(r)} G_{2n}(z,r)\, dV_N(z) < \infty.
$$

Since

$$
\int_{\mathbf{B}^n(r_0)} \|D_f(z)\|^{t_1} G_{2n}(z,r)\, dV_N(z) \le \int_{\mathbf{B}^n(r_0)} \|D_f(z)\|^{t_1} G_{2n}(z,1)\, dV_N(z) < \infty
$$

and

$$
\begin{aligned}
\infty &> \int_{\mathbf{B}^n(r)\backslash\mathbf{B}^n(r_0)} \|D_f(z)\|^{t_1} G_{2n}(z,r)\big)\, dV_N(z) \\
&= \frac{1}{4n(n-1)} \int_{\mathbf{B}^n(r)\backslash\mathbf{B}^n(r_0)} \frac{(r-|z|)\big(\sum_{k=0}^{2n-3} r^{2n-3-k}|z|^k\big)}{|z|^{2n-2}r^{2n-2}} \|D_f(z)\|^{t_1}\, dV_N(z) \\
&\ge \delta(r),
\end{aligned}
$$

which, together with $f \in \mathcal{H}_g^p(\mathbf{B}^n)$, (3.16), (3.17) and the monotonicity of $\delta(r)$, yield that the limit

$$
(3.18)\qquad \lim_{r\to 1-} \int_{\mathbf{B}^n(r)\backslash\mathbf{B}^n(r_0)} (r-|z|)\|D_f(z)\|^{t_1}\, dV_N(z)
$$

exists, where

$$
\delta(r) = \frac{r_0^{2n-3}}{2n} \int_{\mathbf{B}^n(r)\backslash\mathbf{B}^n(r_0)} (r-|z|)\|D_f(z)\|^{t_1}\, dV_N(z).
$$

Then $f\in\mathcal{D}_{1,t_1}(\mathbf{B}^n)$.

By using a similar argument as in the proof of (3.18), we see that $f\in b_{1,t_2}(\mathbf{B}^n)$.

Case 2. Let $n=1$. In this case, the proof is similar to the proof of the case 2 in Theorem 1. Therefore, proof of the theorem is complete. $\square$

Proof of Corollary 1.6. Without loss of generality, we assume that $\prod_{k=1}^{n}\lambda_k\neq 0$.

Case 1. Let $p\in[4,\infty)$. By computations, for $k\in\{1,\ldots,n\}$, we have

$$\begin{aligned}
(|f|^p)_{z_k\overline{z}_k} &= [(f^{\frac{p}{2}}\overline{f}^{\frac{p}{2}})_{\overline{z}_k}]_{z_k} = \frac{p}{2}\left(f^{\frac{p}{2}-1}f_{\overline{z}_k}\overline{f}^{\frac{p}{2}} + f^{\frac{p}{2}}\overline{f}^{\frac{p}{2}-1}\overline{f_{z_k}}\right)_{z_k}\\
&= \frac{p}{2}\left(\lambda_k f^{\frac{p}{2}+\frac{\alpha}{2}-1}\overline{f}^{\frac{p}{2}+\frac{\alpha}{2}} + f^{\frac{p}{2}}\overline{f}^{\frac{p}{2}-1}\overline{f_{z_k}}\right)_{z_k}\\
&= \frac{p}{2}\Big[\lambda_k\big(\frac{p}{2}+\frac{\alpha}{2}-1\big)f^{\frac{p}{2}+\frac{\alpha}{2}-2}\overline{f}^{\frac{p}{2}+\frac{\alpha}{2}}f_{z_k} + \lambda_k\big(\frac{p}{2}+\frac{\alpha}{2}\big)|f|^{p+\alpha-2}\overline{f_{\overline{z}_k}}\\
&\quad + \frac{p}{2}|f|^{p-2}|f_{z_k}|^2 + \big(\frac{p}{2}-1\big)\overline{f_{z_k}}f^{\frac{p}{2}}\overline{f}^{\frac{p}{2}-2}\overline{f_{\overline{z}_k}} + f^{\frac{p}{2}}\overline{f}^{\frac{p}{2}-1}\overline{f_{z_k\overline{z}_k}}\Big]\\
&= \frac{p}{2}\Big[\lambda_k\big(\frac{p}{2}+\frac{\alpha}{2}-1\big)|f|^{p+\alpha-4}f_{z_k}\overline{f}^2 + \lambda_k^2\big(\frac{p+\alpha}{2}\big)|f|^{p+\alpha-2}|f|^{\alpha}\\
&\quad + \frac{p}{2}|f|^{p-2}|f_{z_k}|^2 + \lambda_k\big(\frac{\alpha}{2}+\frac{p}{2}-1\big)|f|^{p+\alpha-4}\overline{f_{z_k}}f^2 + \frac{\alpha\lambda_k^2}{2}|f|^{p+2\alpha-2}\Big]\\
&= \frac{p}{2}\Big[\lambda_k\big(\frac{p}{2}+\frac{\alpha}{2}-1\big)|f|^{p+\alpha-4}f_{z_k}\overline{f}^2 + \lambda_k\big(\frac{p}{2}+\frac{\alpha}{2}-1\big)|f|^{p+\alpha-4}\overline{f_{z_k}}f^2\\
&\quad + \lambda_k^2\big(\alpha+\frac{p}{2}\big)|f|^{p+2\alpha-2} + \frac{p}{2}|f_{z_k}|^2|f|^{p-2}\Big]\\
&= \frac{p}{2}\Big\{\mathrm{Re}\big[\lambda_k(p+\alpha-2)|f|^{p+\alpha-4}f_{z_k}\overline{f}^2\big] + \lambda_k^2\big(\alpha+\frac{p}{2}\big)|f|^{p+2\alpha-2}\\
&\quad + \frac{p}{2}|f_{z_k}|^2|f|^{p-2}\Big\},
\end{aligned}$$

which implies

$$\begin{aligned}
\Delta(|f|^p) = 4\sum_{k=1}^{n}(|f|^p)_{z_k\overline{z}_k} &= 2p\Big\{\sum_{k=1}^{n}\mathrm{Re}\big[\lambda_k(p+\alpha-2)|f|^{p+\alpha-4}f_{z_k}\overline{f}^2\big]\\
&\quad + \sum_{k=1}^{n}\lambda_k^2\big(\alpha+\frac{p}{2}\big)|f|^{p+2\alpha-2} + \frac{p}{2}\sum_{k=1}^{n}|f_{z_k}|^2|f|^{p-2}\Big\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Delta(|f|^p) - [4p-(2-\alpha)^2]|f|^{p+2\alpha-2}\sum_{k=1}^{n}\lambda_k^2\\
&= \sum_{k=1}^{n}\Big\{\lambda_k^2\big[p^2+2p(\alpha-2)+(\alpha-2)^2\big]|f|^{p+2\alpha-2}\\
&\quad + \mathrm{Re}\big[2p\lambda_k(p+\alpha-2)|f|^{p+\alpha-4}f_{z_k}\overline{f}^2\big] + p^2|f_{z_k}|^2|f|^{p-2}\Big\}\\
&\geq \sum_{k=1}^{n}\Big\{\lambda_k^2(p+\alpha-2)^2|f|^{p+2\alpha-2} + p^2|f_{z_k}|^2|f|^{p-2}\\
&\quad - |2p\lambda_k(p+\alpha-2)||f|^{p+\alpha-2}|f_{z_k}|\Big\}\\
&= |f|^{p-2}\sum_{k=1}^{n}\big(|\lambda_k(p+\alpha-2)||f|^{\alpha} - |p||f_{z_k}|\big)^2 \geq 0,
\end{aligned}$$

which yields

$$\Delta(|f|^p) \geq \left[4p - (2-\alpha)^2\right]|f|^{p+2\alpha-2}\sum_{k=1}^{n}\lambda_k^2. \tag{3.19}$$

Case 2. Let $p \in [2,4)$. For $m \in \{1,2,\ldots\}$, let $G_m^p = \left(|f|^2 + \frac{1}{m}\right)^{\frac{p}{2}}$. Then by Lebesgue's dominated convergence theorem and (3.19), we have

$$\begin{aligned}\lim_{m\to\infty}\Delta\left(G_m^p\right) &= \lim_{m\to\infty} 4\sum_{k=1}^{n}(G_m^p)_{z_k\overline{z}_k}\\ &= 2p\Bigg\{\sum_{k=1}^{n}\mathrm{Re}\left[\lambda_k(p+\alpha-2)|f|^{p+\alpha-4}f_{z_k}\overline{f}^2\right]\\ &\quad + \sum_{k=1}^{n}\lambda_k^2\Big(\alpha+\frac{p}{2}\Big)|f|^{p+2\alpha-2} + \frac{p}{2}\sum_{k=1}^{n}|f_{z_k}|^2|f|^{p-2}\Bigg\}\\ &\geq \left[4p-(2-\alpha)^2\right]|f|^{p+2\alpha-2}\sum_{k=1}^{n}\lambda_k^2.\end{aligned}$$

Applying Theorem 5, we conclude that $f \in b_{1,\vartheta}(\mathbf{B}^n)$, where $\vartheta = p + 2\alpha - 2$. The proof of the Corollary is complete. □

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