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NORM OF THE BERGMAN PROJECTION ONTO THE BLOCH SPACE WITH $\mathcal{M}$-INVARIANT GRADIENT NORM

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Abstract. The operator norm of Bergman projections P_α from $L^\infty(\mathbf{B}^n)$ to the Bloch space was found in [4]. In the same paper the authors made a conjecture on the norms of P_α with respect to $\mathcal{M}$-invariant gradient norm. In this paper we prove their conjecture.

1. Introduction

1.1. Bergman projection. Let $\mathbf{B}^n$ denote the unit ball in $\mathbf{C}^n$, $n \geq 1$ and let be dv_α the measure given by

$$dv_\alpha(z) = c_\alpha \left(1 - |z|^2\right)^\alpha dv(z),$$

where $dv(z)$ is the Lebesgue measure on $\mathbf{B}^n$ and

$$c_\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\pi^n} \tag{1}$$

is a normalizing constant i.e. $v_\alpha(\mathbf{B}^n) = 1$. We will also use the symbol v_n for the $n-$dimensional Lebesgue measure at places where dimensions must be distinguished.

For $\alpha > -1$ the Bergman projection operator is given by

$$P_\alpha f(z) = c_\alpha \int_{\mathbf{B}^n} K_\alpha(z, w) f(w)\, dv_\alpha(w), \quad f \in L^p(\mathbf{B}^n), \quad 1 < p \leq \infty,$$

where

$$K_\alpha(z, w) = \frac{1}{(1 - \langle z, w\rangle)^{n+\alpha+1}}, \quad z, w \in \mathbf{B}^n.$$

Here $\langle z, w\rangle$ stands for scalar product given by $z_1\overline{w}_1 + z_2\overline{w}_2 + \cdots + z_n\overline{w}_n$. These projections are among the most important operators in theory of analytic function spaces. In [3], Forelli and Rudin proved that P_α is bounded as operator from $L^p(\mathbf{B}^n)$ to Bergman space of all p-integrable analytic functions on $\mathbf{B}^n$ if and only if $\alpha > \frac{1}{p} - 1$. They also found the exact operator norm in cases $p = 1$ and $p = 2$. Mateljević and Pavlović extended these result for $0 < p < 1$, see [11].

The problem of finding the exact value of the operator norm of P_α on L^p spaces turned out to be quite difficult, even for $P = P_0$. In [18], Zhu obtained asymptotically sharp two sided norm estimates , while Dostanić in [2] gave the following estimate:

$$\frac{1}{2} \csc \frac{\pi}{p} \leq \|P\|_p \leq \pi \csc \frac{\pi}{p},$$

for $1 < p < \infty$. Liu improved these estimates in [7], for the unit ball $\mathbf{B}^n$. Also, papers [8] and [9] give the estimates for the Bergman projection in the Siegel upper-half space

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and for the weighted Bergman projections in the unit disk. In recent years, there has been increasing interest in studying projections of this type in various spaces. See also [6, 16, 10].

Here, we investigate the operator norm of $P_\alpha\colon L^\infty \to \mathcal{B}$ with $\mathcal{M}$-invariant gradient. In [1] or [17], for $n = 1$, the reader can find proof of boundedness and surjectivity of P_α from L^∞ to $\mathcal{B}$. In [14] and [15], Peralla found the exact value of the norm $\|P\|$ in $\mathbf{D}$, while [4] contains a generalization of this result to P_α and $\mathbf{B}^n$.

In [4], the authors (Kalaj and Marković) also have settled the problem of finding the exact value of $\|P_\alpha\|_{L^\infty(\mathbf{B}^n)\to\mathcal{B}(\mathbf{B}^n)}$ with a different norm on $\mathcal{B}$. They obtained the two-sided estimate and conjectured that the norm is equal to the estimate from above. Using a new technique, we will obtain the appropriate series expansion of certain elliptic integral considered in [4] and the exact norm as the maximum of that series. We hope that technique can be used for a variety of similar extremal problems.

Also, let us say that the paper [5] consider Bergman projections on Bloch space with the family of seminorms and norms inhereted from Besov spaces. This is a generalization of [4] and can be of some good motivation for some future work on this topic.

Let us, first, recall that the Bloch space consists of functions f analytic in $\mathbf{B}^n$ for which the following semi-norm is finite:

$$\|f\|_\beta := \sup_{|z|<1}(1-|z|^2)|\nabla f(z)|,$$

where

$$\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z)\right).$$

But, we can also define the semi-norm invariant with respect to the group $\mathrm{Aut}(\mathbf{B}^n)$. For analytic f, the invariant gradient $\widetilde{\nabla} f(z)$ is defined by:

$$\widetilde{\nabla} f(z) = \nabla(f\circ\varphi_z)(0),$$

where φ_z is an automorphism of the unit ball for which $\varphi_z(0) = z$. We have

$$|\widetilde{\nabla}(f\circ\varphi)| = |(\widetilde{\nabla} f)\circ\varphi)|,$$

exactly what we want. Then the Bloch space can be described also as the space of all holomorphic functions f for which

$$\|f\|_{\widetilde{\beta}} := \sup_{|z|<1}|\widetilde{\nabla} f(z)| < \infty.$$

Now, we can equip $\mathcal{B}$ with the norm $\|f\|_{\widetilde{\mathcal{B}}} := |f(0)| + \|f\|_{\widetilde{\beta}}$.

1.2. Statement of the problem. In order to formulate the problem and the known result, we define the following function of one real variable $t \in [0, \frac{\pi}{2}]$:

$$l(t) = (n+\alpha+1)\int_{\mathbf{B}^n} \frac{|(1-w_1)\cos t + w_2 \sin t|}{|w_1-1|^{n+\alpha+1}}\, dv_\alpha(w).$$

Kalaj and Marković in [4] proved:

Theorem 1. *For $\alpha > -1$, $n > 1$, we have*

$$l\left(\frac{\pi}{2}\right) = \frac{\pi}{2} l(0) = \frac{\pi}{2} C_\alpha,$$

where $C_\alpha = \frac{\Gamma(n+\alpha+2)}{\Gamma^2(\frac{n+\alpha}{2}+1)}$. *For the* $\widetilde{\beta}$*-semi-norm of the Bergman projection* P_α *we have*

$$\widetilde{C}_\alpha := \|P_\alpha\|_{\widetilde{\beta}} = \max_{0\le t\le \frac{\pi}{2}} l(t),$$

and

$$\frac{\pi}{2}C_\alpha \le \|P_\alpha\| \le \frac{\sqrt{\pi^2+4}}{2}C_\alpha.$$

They also conjectured that

$$\|P_\alpha\|_{\widetilde{\beta}} = \frac{\pi}{2}C_\alpha.$$

From these facts we can conclude that it is enough to prove that $l(t)$ attains its maximum in $t = \frac{\pi}{2}$. We will prove that this conjecture is true. This is contained in the following theorem.

Theorem 2. *For* $\alpha > -1$ *and* $n \ge 2$, *we have*

$$\|P_\alpha\|_{\widetilde{\beta}} = \frac{\pi}{2}\frac{\Gamma(n+\alpha+2)}{\Gamma^2(\frac{n+\alpha}{2}+1)}$$

and

$$\frac{\pi}{2}\frac{\Gamma(n+\alpha+2)}{\Gamma^2(\frac{n+\alpha}{2}+1)} \le \|P_\alpha\|_{\mathcal{B}} \le 1 + \frac{\pi}{2}\frac{\Gamma(n+\alpha+2)}{\Gamma^2(\frac{n+\alpha}{2}+1)}.$$

Moreover, we have the following representation for function l *in terms of hypergeometric series:*

$$l(t) = \frac{\pi\Gamma(n+\alpha+2)}{2\Gamma^2(\frac{n+\alpha}{2}+1)}\cdot {}_2F_1\left(\frac{1}{2},-\frac{1}{2};1;\cos^2 t\right)$$

and $l(t)$ *is increasing in* $t \in [0, \frac{\pi}{2}]$.

In the next section we give some preliminary facts which we need for the proof.

1.3. Beta and hypergeometric functions. Here we recall some properties of hypergeometric functions. They are defined by

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{+\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}.$$

It converges for all $|z| < 1$, and, for $\mathrm{Re}(c-a-b) > 0$ also for $z = 1$. Here $(a)_k$ stands for Pochhammer symbol $a(a+1)\dots(a+k-1)$, and a is not negative integer.

We will use the next theorem due to Gauss:

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Also, we use the following Euler transformation

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\cdot {}_2F_1(c-a,c-b;c;z).$$

Beta function is defined as

$$\mathrm{B}(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt.$$

The identity

$$\mathrm{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

connects Beta function with Gamma function and we will exploit this relation later.

2. Proof of the Theorem 2

Let us recall the integral representation of the constant C_α. We start from the expression

$$L(\xi_t) := c_\alpha \int_{\mathbf{B}^n} \frac{|\langle w - e_1, \xi_t\rangle|(1-|w|^2)^\alpha}{|1-\langle w, e_1\rangle|^{n+\alpha+1}}\, dv_n(w)$$
$$= c_\alpha \int_{\mathbf{B}^n} \frac{|(1-w_1)\cos t + w_2 \sin t|}{|1-w_1|^{n+\alpha+1}}(1-|w|^2)^\alpha\, dv_n(w)$$

where $\xi_t = e_1 \cos t + e_2 \sin t$, $t \in [0, \frac{\pi}{2}]$ and c_α is given in (1).

Let us fix $t \in [0, \frac{\pi}{2}]$. Changing coordinates with $A_t w = z$, where A_t is a real $n \times n$ orthogonal matrix

$$\begin{pmatrix} \cos t & \sin t & 0 & \cdots & 0 \\ -\sin t & \cos t & 0 & \cdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

such that $A_t \xi_t = e_1$, we obtain

$$L(\xi_t) = c_\alpha \int_{\mathbf{B}^n} \frac{|\langle A_t w - A_t e_1, e_1\rangle|(1-|w|^2)^\alpha}{|1-\langle A_t w, A_t e_1\rangle|^{n+\alpha+1}}\, dv_n(w)$$
$$= c_\alpha \int_{\mathbf{B}^n} \frac{|\langle z - A_t e_1, e_1\rangle|(1-|z|^2)^\alpha}{|1-\langle z, A_t e_1\rangle|^{n+\alpha+1}}\, dv_n(z).$$

Since $A_t e_1 = \big(\cos t, -\sin t, 0, \ldots, 0\big)$, we have:

$$L(\xi_t) = c_\alpha \int_{\mathbf{B}^n} \frac{|z_1 - \cos t|(1-|z|^2)^\alpha}{|1 - z_1 \cos t + z_2 \sin t|^{n+\alpha+1}}\, dv_n(z)$$
$$= c_\alpha \int_{\mathbf{B}^n} \frac{|z_1 - \cos t|(1-|z|^2)^\alpha}{|1 - z_1 \cos t - z_2 \sin t|^{n+\alpha+1}}\, dv_n(z).$$

Now, as in [12], we use Fubini's theorem:

$$L(\xi_t) = c_\alpha \int_{\mathbf{B}^n} \frac{|z_1 - \cos t|(1-|z_1|^2-|z_2|^2-|z'|^2)^\alpha}{|1 - z_1 \cos t - z_2 \sin t|^{n+\alpha+1}}\, dv_n(z)$$
$$= c_\alpha \int_{\mathbf{B}^2} \frac{|z_1 - \cos t| J(z_1, z_2)\, dv_2(z_1, z_2)}{|1 - z_1 \cos t - z_2 \sin t|^{n+\alpha+1}}$$

where

$$J(z_1, z_2) = \int_{\sqrt{1-|z_1|^2-|z_2|^2}\mathbf{B}^{n-2}} (1-|z_1|^2-|z_2|^2-|z'|^2)^\alpha\, dv_{n-2}(z');$$

here $z = (z_1, z_2, z')$, $z' \in \mathbf{C}^{n-2}$.

We make a substitution $z' = \lambda w$, $\lambda = \sqrt{1-|z_1|^2-|z_2|^2}$, in the expression for $J(z_1, z_2)$, which gives

$$\int_{\lambda\mathbf{B}^{n-2}} (\lambda^2 - |z'|^2)^\alpha\, dv_{n-2}(z') = \lambda^{2\alpha+2n-4} \int_{\mathbf{B}^{n-2}} (1-|w|^2)^\alpha\, dv_{n-2}(w).$$

We easily find $\int_{\mathbf{B}^{n-2}}(1-|w|^2)^\alpha\, dv_{n-2}(w) = k_\alpha = \pi^{n-2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n-1)}$, so

$$L(\xi_t) = c_\alpha k_\alpha I(\cos t, \sin t),$$

where

$$I(\cos t, \sin t) = \int_{\mathbf{B}^2} \frac{|z_1-\cos t|(1-|z_1|^2-|z_2|^2)^{n+\alpha-2}}{|1-z_1\cos t - z_2\sin t|^{n+\alpha+1}}\, dv_2(z_1,z_2).$$

Now, the proof of Theorem 2 is reduced to proving monotonicity of $I(\cos t, \sin t)$ as a function of $0 \le t \le \frac{\pi}{2}$.

Again, Fubini's theorem gives us

$$I(\cos t, \sin t) = \int_{\mathbf{D}} |z_1 - \cos t|\, dv(z_1) \int_{\sqrt{1-|z_1|^2}\mathbf{D}} \frac{(1-|z_1|^2-|z_2|^2)^{n+\alpha-2}}{|1-z_1\cos t - z_2\sin t|^{n+\alpha+1}}\, dv(z_2).$$

Next, we make substitution $z_2 = \sqrt{1-|z_1|^2}\rho e^{i\theta}$, $0 \le \rho < 1$, $\theta \in [0, 2\pi]$:

$$\begin{aligned}
&\int_{\sqrt{1-|z_1|^2}\mathbf{D}} \frac{(1-|z_1|^2-|z_2|^2)^{n+\alpha-2}}{|1-z_1\cos t - z_2\sin t|^{n+\alpha+1}}\, dv(z_2)\\
&= \int_0^1 d\rho \int_0^{2\pi} \frac{(1-|z_1|^2)^{n+\alpha-2}(1-\rho^2)^{n+\alpha-2}\rho(1-|z_1|^2)}{|1-z_1\cos t - \sqrt{1-|z_1|^2}\rho\sin t e^{i\theta}|^{n+\alpha+1}}\, d\theta\\
&= (1-|z_1|^2)^{n+\alpha-1}\int_0^1 \rho(1-\rho^2)^{n+\alpha-2}\Phi(z_1,\rho,t)\, d\rho,
\end{aligned}$$

where $\Phi(z_1,\rho,t) = \int_0^{2\pi} \frac{d\theta}{|1-z_1\cos t-\sqrt{1-|z_1|^2}\rho\sin t e^{i\theta}|^{n+\alpha+1}}$.

Next, we use Parseval's identity and Taylor's expansion of $(1-z)^{-\frac{n+\alpha+1}{2}}$:

$$\begin{aligned}
\Phi(z_1,\rho,t) &= \int_0^{2\pi} \frac{d\theta}{|1-z_1\cos t - \sqrt{1-|z_1|^2}\rho\sin t e^{i\theta}|^{n+\alpha+1}}\\
&= \frac{1}{|1-z_1\cos t|^{n+\alpha+1}}\int_0^{2\pi} \frac{d\theta}{\left|1-\frac{\sqrt{1-|z_1|^2}\rho\sin t}{1-z_1\cos t}e^{i\theta}\right|^{n+\alpha+1}}\\
&= \frac{2\pi}{|1-z_1\cos t|^{n+\alpha+1}}\sum_{k=0}^{+\infty}\binom{\frac{n+\alpha+1}{2}+k-1}{k}^2 \frac{(1-|z_1|^2)^k\rho^{2k}\sin^{2k}t}{|1-z_1\cos t|^{2k}}.
\end{aligned}$$

To use the above series expansion, we have to explain why

$$\left|\frac{\sqrt{1-|z_1|^2}\rho\sin t}{1-z_1\cos t}\right| \le \rho < 1.$$

In fact, from the Cauchy–Schwarz inequality, we have

$$\sqrt{1-|z_1|^2}\sin t + |z_1|\cos t \le \sqrt{(\sqrt{1-|z_1|^2})^2+|z_1|^2}\sqrt{\sin^2 t+\cos^2 t} = 1$$

and hence

$$\sqrt{1-|z_1|^2}\sin t \le 1-|z_1|\cos t.$$

Triangle inequality gives

$$1-|z_1|\cos t \le |1-z_1\cos t|.$$

The last two inequalities imply

$$\left|\frac{\sqrt{1-|z_1|^2}\sin t}{1-z_1\cos t}\right|\le 1,$$

therefore, for $0\le\rho<1$,

$$\left|\frac{\sqrt{1-|z_1|^2}\rho\sin t}{1-z_1\cos t}\right|\le\rho<1.$$

Therefore, from the last expansion we have

$$\begin{aligned}&\int_{\sqrt{1-|z_1|^2}\mathbf{D}}\frac{(1-|z_1|^2-|z_2|^2)^{n+\alpha-2}}{|1-z_1\cos t-z_2\sin t|^{n+\alpha+1}}\,dv(z_2)\\ &=2\pi\sum_{k=0}^{+\infty}\binom{\frac{n+\alpha+1}{2}+k-1}{k}^2\frac{(1-|z_1|^2)^{k+n+\alpha-1}\sin^{2k}t}{|1-z_1\cos t|^{2k+n+\alpha+1}}\cdot\int_0^1\rho^{2k+1}(1-\rho^2)^{n+\alpha-2}\,d\rho\\ &=\pi\sum_{k=0}^{+\infty}\binom{\frac{n+\alpha+1}{2}+k-1}{k}^2\frac{(1-|z_1|^2)^{k+n+\alpha-1}\sin^{2k}t}{|1-z_1\cos t|^{2k+n+\alpha+1}}\mathrm{B}(k+1,n+\alpha-1),\end{aligned}$$

and hence

$$\begin{aligned}I(\cos t,\sin t)=&\pi\sum_{k=0}^{+\infty}\mathrm{B}(k+1,n+\alpha-1)\binom{\frac{n+\alpha+1}{2}+k-1}{k}^2\sin^{2k}t\\ &\cdot\int_{\mathbf{D}}\frac{|z_1-\cos t|(1-|z_1|^2)^{k+n+\alpha-1}}{|1-z_1\cos t|^{2k+n+\alpha+1}}\,dv(z_1).\end{aligned}$$

We calculate these integrals by changing coordinates with $z_1=\frac{\cos t-\zeta}{1-\zeta\overline{\cos t}}=\frac{\cos t-\zeta}{1-\zeta\cos t}$ (since $\cos t\in\mathbf{R}$). Here, we assume $t>0$. Then, we have

$$\zeta=\frac{\cos t-z_1}{1-z_1\cos t},\qquad J_{\mathbf{R}}=\frac{(1-\cos^2 t)^2}{|1-\zeta\cos t|^4}.$$

Also, we need the following identities

$$1-z_1\cos t=1-\frac{\cos-\zeta}{1-\zeta\cos t}\cos t=\frac{1-\cos^2 t}{1-\zeta\cos t}$$

and

$$1-|z_1|^2=\frac{(1-\cos^2 t)(1-|\zeta|^2)}{|1-\zeta\cos t|^2}.$$

Using the above substitution, we get

$$\begin{aligned}&\int_{\mathbf{D}}\frac{|z_1-\cos t|(1-|z_1|^2)^{k+n+\alpha-1}}{|1-z_1\cos t|^{2k+n+\alpha+1}}\,dv(z_1)\\ &=\int_{\mathbf{D}}|\zeta|\frac{(1-\cos^2 t)^{k+n+\alpha-1}(1-|\zeta|^2)^{k+n+\alpha-1}}{|1-\zeta\cos t|^{2k+2n+2\alpha-2}}\frac{|1-\zeta\cos t|^{2k+n+\alpha}}{(1-\cos^2 t)^{2k+n+\alpha}}\frac{(1-\cos^2 t)^2}{|1-\zeta\cos t|^4}\,dv(\zeta)\\ &=(1-\cos^2 t)^{1-k}\int_{\mathbf{D}}|\zeta|\frac{(1-|\zeta|^2)^{k+n+\alpha-1}}{|1-\zeta\cos t|^{n+\alpha+2}}\,dv(\zeta).\end{aligned}$$

Passing to the polar coordinates, we have

$$\int_{\mathbf{D}}|\zeta|\frac{(1-|\zeta|^2)^{k+n+\alpha-1}}{|1-\zeta\cos t|^{n+\alpha+2}}\,dv(\zeta)=\int_0^1 r^2(1-r^2)^{k+n+\alpha-1}\,dr\int_0^{2\pi}\frac{d\varphi}{|1-r\cos te^{i\varphi}|^{n+\alpha+2}},$$

and then, again, by Parseval's identity

$$\int_0^{2\pi} \frac{d\varphi}{|1 - re^{i\varphi}\cos t|^{n+\alpha+2}} = 2\pi \sum_{m=0}^{+\infty} \binom{\frac{n+\alpha+2}{2} + m - 1}{m}^2 r^{2m} \cos^{2m} t,$$

thus

$$\begin{aligned}
&\int_{\mathbf{D}} |\zeta| \frac{(1 - |\zeta|^2)^{k+n+\alpha-1}}{|1 - \zeta \cos t|^{n+\alpha+2}}\, dv(\zeta) \\
&= 2\pi \sum_{m=0}^{\infty} \binom{\frac{n+\alpha+2}{2} + m - 1}{m}^2 \cos^{2m} t \int_0^1 r^{2m+2} (1 - r^2)^{k+n+\alpha-1}\, dr \\
&= \pi \sum_{m=0}^{\infty} \binom{\frac{n+\alpha+2}{2} + m - 1}{m}^2 \cos^{2m} t\, \mathrm{B}(m + \tfrac{3}{2}, k + n + \alpha).
\end{aligned}$$

This gives

$$\begin{aligned}
I(\cos t, \sin t) = \pi^2 &\sum_{k=0}^{+\infty} \binom{\frac{n+\alpha+1}{2} + k - 1}{k}^2 \mathrm{B}(k+1, n+\alpha-1) \sin^{2k} t \\
&\cdot \sum_{m=0}^{\infty} \binom{\frac{n+\alpha+2}{2} + m - 1}{m}^2 \mathrm{B}(m + \tfrac{3}{2}, k + n + \alpha) \cos^{2m} t (1 - \cos^2 t)^{1-k}.
\end{aligned}$$

Since $(1 - \cos^2 t)^{1-k} \sin^{2k} t = 1 - \cos^2 t$, we conclude

$$\begin{aligned}
I(\cos t, \sin t) = \pi^2 (1 - \cos^2 t) &\sum_{k,m=0}^{+\infty} \binom{\frac{n+\alpha+1}{2} + k - 1}{k}^2 \binom{\frac{n+\alpha+2}{2} + m - 1}{m}^2 \\
&\cdot \mathrm{B}(k+1, n+\alpha-1)\, \mathrm{B}(m + \tfrac{3}{2}, k + n + \alpha) \cos^{2m} t.
\end{aligned} \tag{2}$$

Now, we consider the function ϕ defined as

$$\begin{aligned}
\phi(x) = (1 - x) &\sum_{k,m=0}^{+\infty} \binom{\frac{n+\alpha+1}{2} + k - 1}{k}^2 \binom{\frac{n+\alpha+2}{2} + m - 1}{m}^2 \mathrm{B}(k+1, n+\alpha-1) \\
&\cdot \mathrm{B}(m + \tfrac{3}{2}, k + n + \alpha) x^m,
\end{aligned}$$

for $0 \le x < 1$ (because of condition $0 \le \cos t < 1$!). Using $\mathrm{B}(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ we have

$$\begin{aligned}
\phi(x) = \Gamma(n + \alpha - 1)(1 - x) &\sum_{k,m=0}^{+\infty} \binom{\frac{n+\alpha+1}{2} + k - 1}{k}^2 \binom{\frac{n+\alpha+2}{2} + m - 1}{m}^2 \\
&\cdot \frac{k!\, \Gamma(m + \frac{3}{2})}{\Gamma(k + n + \alpha + m + \frac{3}{2})} x^m.
\end{aligned}$$

Let us sum, over k, the terms which depend on k,

$$\begin{aligned}
&\sum_{k=0}^{+\infty}\binom{\frac{n+\alpha+1}{2}+k-1}{k}^2\frac{k!}{\Gamma(k+n+\alpha+m+\frac{3}{2})}\\
&=\sum_{k=0}^{+\infty}\frac{\left(\frac{n+\alpha+1}{2}+k-1\right)^2\cdot\dots\cdot\left(\frac{n+\alpha+1}{2}+1\right)^2\left(\frac{n+\alpha+1}{2}\right)^2}{k!\,\Gamma(k+m+n+\alpha+\frac{3}{2})}\\
&=\sum_{k=0}^{+\infty}\frac{\left(\frac{n+\alpha+1}{2}\right)_k\left(\frac{n+\alpha+1}{2}\right)_k}{k!\left(n+k+\alpha+m+\frac{1}{2}\right)\cdot\dots\cdot\left(n+\alpha+m+\frac{3}{2}\right)\Gamma(n+\alpha+m+\frac{3}{2})}\\
&=\frac{1}{\Gamma(n+\alpha+m+\frac{3}{2})}\sum_{k=0}^{+\infty}\frac{1}{k!}\frac{\left(\frac{n+\alpha+1}{2}\right)_k\left(\frac{n+\alpha+1}{2}\right)_k}{\left(n+\alpha+m+\frac{3}{2}\right)_k}.
\end{aligned}$$

We recognize that the last sum is ${}_2F_1(\frac{n+\alpha+1}{2},\frac{n+\alpha+1}{2};n+\alpha+m+\frac{3}{2};1)$, and by Gauss's theorem this is equal to

$$\frac{\Gamma(n+\alpha+m+\frac{3}{2})\Gamma(m+\frac{1}{2})}{\Gamma^2(m+1+\frac{n+\alpha}{2})}.$$

Hence, the double sum in (2) is equal to

$$\sum_{m=0}^{+\infty}\binom{\frac{n+\alpha+2}{2}+m-1}{m}^2\frac{\Gamma(m+\frac{3}{2})\Gamma(m+\frac{1}{2})}{\Gamma^2(m+1+\frac{n+\alpha}{2})}x^m.$$

Note that

$$\binom{\frac{n+\alpha+2}{2}+m-1}{m}^2=\frac{1}{(m!)^2}\left(\frac{n+\alpha+2}{2}+m-1\right)^2\cdot\dots\cdot\left(\frac{n+\alpha+2}{2}\right)^2=\frac{1}{(m!)^2}\frac{\Gamma^2(\frac{n+\alpha+2}{2}+m)}{\Gamma^2(\frac{n+\alpha+2}{2})},$$

and hence

$$\phi(x)=\frac{\Gamma(n+\alpha-1)}{\Gamma^2(\frac{n+\alpha}{2}+1)}(1-x)\sum_{m=0}^{+\infty}\frac{\Gamma(m+\frac{1}{2})\Gamma(m+\frac{3}{2})}{(m!)^2}x^m,\quad 0\le x<1.$$

Let us denote

$$a_m=\frac{\Gamma(m+\frac{1}{2})\Gamma(m+\frac{3}{2})}{(m!)^2}.$$

It is easily verified that a_m is strictly decreasing in $m\ge 0$:

$$\frac{a_{m+1}}{a_m}=\frac{\Gamma(m+\frac{3}{2})\Gamma(m+\frac{5}{2})(m!)^2}{\Gamma(m+\frac{1}{2})\Gamma(m+\frac{3}{2})\left((m+1)!\right)^2}=\frac{(m+\frac{1}{2})(m+\frac{3}{2})}{(m+1)^2}<1.$$

In particular, $a_m\le a_0$. Then, we may conclude

$$(1-x)\sum_{m=0}^{+\infty}a_mx^m\le(1-x)a_0\sum_{m=0}^{+\infty}x^m=a_0,$$

that is $\phi(x)\le\phi(0)$. Moreover, $\phi(x)$ is decreasing, since we can write it in the following form

$$\phi(x)=\frac{\Gamma(n+\alpha+1)}{\Gamma^2(\frac{n+\alpha}{2}+1)}\left(a_0+\sum_{m=1}^{+\infty}(a_m-a_{m-1})x^m\right).$$

This is the crux of the proof of our Theorem.

So, the best constant is $\widetilde{C}_\alpha = (1+n+\alpha)c_\alpha k_\alpha \pi^2 \frac{\Gamma(\alpha+n-1)}{\Gamma^2(\frac{n+\alpha}{2}+1)}\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})$, i.e.

$$\widetilde{C}_\alpha = \frac{\pi}{2}\frac{\Gamma(\alpha+n+2)}{\Gamma^2(\frac{n+\alpha}{2}+1)}.$$

According to [4], for $\xi = (1,0,0\dots,0)$ we have $l(0) = \frac{2}{\pi}\widetilde{C}_\alpha$. This can be also obtained from the above series by letting x tends to 1.

Finally, all these computations give us

$$\begin{aligned} l(t) &= (1+n+\alpha)c_\alpha k_\alpha \pi^2 \frac{\Gamma(\alpha+n-1)}{\Gamma^2(\frac{\alpha+n}{2}+1)}\sin^2 t \sum_{m=0}^{+\infty} \frac{\Gamma(m+\frac{1}{2})\Gamma(m+\frac{3}{2})}{(m!)^2}\cos^{2m} t \\ &= \frac{\Gamma(n+\alpha+2)}{\Gamma^2(\frac{n+\alpha}{2}+1)}\sin^2 t \sum_{m=0}^{+\infty} \frac{\Gamma(m+\frac{1}{2})\Gamma(m+\frac{3}{2})}{(m!)^2}\cos^{2m} t, \quad 0 < t \le \frac{\pi}{2}, \end{aligned}$$

and $l(t)$ is increasing in $t \in [0, \frac{\pi}{2}]$. (Because $\phi(x)$ is decreasing and $l(t) = \phi(\cos^2 t)$.)

Using the definition od Pochhammer symbol $(a)_k$, hypergeometric functions and Euler transformation from subsection 1.3 we get

$$l(t) = \frac{\pi\Gamma(n+\alpha+2)}{2\Gamma^2(\frac{n+\alpha}{2}+1)} \cdot {}_2F_1(\frac{1}{2}, -\frac{1}{2}; 1; \cos^2 t).$$

The proof of the second part of our Theorem easily follows from its first part and the inequality

$$\|P_\alpha\|_{\widetilde{\beta}} \le \|P_\alpha\|_{\widetilde{\mathcal{B}}} \le 1 + \|P_\alpha\|_{\widetilde{\beta}}.$$

This concludes the proof of Theorem 2.

Let us say that the function $l(t)$ also can be expressed as $\frac{\Gamma(n+\alpha+2)}{\Gamma^2(\frac{n+\alpha}{2}+1)}E(\cos t)$, where E is the complete elliptic integral of the second kind.

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