Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 44, 2019, 91–101

THE CLOSURE OF DIRICHLET SPACES IN THE BLOCH SPACE

Petros Galanopoulos and Daniel Girela

Aristotle University of Thessaloniki, Department of Mathematics
54124, Thessaloniki, Greece; petrosgala@math.auth.gr

Universidad de Málaga, Análisis Matemático
Campus de Teatinos, 29071 Málaga, Spain; girela@uma.es

Abstract. If $0 < p < \infty$ and $\alpha > -1$, the space of Dirichlet type $\mathcal{D}^p_\alpha$ consists of those functions f which are analytic in the unit disc $\mathbf{D}$ and have the property that f' belongs to the weighted Bergman space A^p_α. Of special interest are the spaces $\mathcal{D}^p_{p-1}$ $(0 < p < \infty)$ and the analytic Besov spaces $B^p = \mathcal{D}^p_{p-2}$ $(1 < p < \infty)$. Let $\mathcal{B}$ denote the Bloch space. It is known that the closure of B^p $(1 < p < \infty)$ in the Bloch norm is the little Bloch space $\mathcal{B}_0$. A description of the closure in the Bloch norm of the spaces $H^p \cap \mathcal{B}$ has been given recently. Such closures depend on p. In this paper we obtain a characterization of the closure in the Bloch norm of the spaces $\mathcal{D}^p_\alpha \cap \mathcal{B}$ $(1 \le p < \infty$, $\alpha > -1)$. In particular, we prove that for all $p \ge 1$ the closure of the space $\mathcal{D}^p_{p-1} \cap \mathcal{B}$ coincides with that of $H^2 \cap \mathcal{B}$. Hence, contrary with what happens with Hardy spaces, these closures are independent of p. We apply these results to study the membership of Blaschke products in the closure in the Bloch norm of the spaces $\mathcal{D}^p_\alpha \cap \mathcal{B}$.

1. Introduction and main results

Let $\mathbf{D} = \{z \in \mathbf{C} \colon |z| < 1\}$ denote the open unit disc in the complex plane $\mathbf{C}$, $\partial\mathbf{D}$ will be the unit circle. Also, dA will denote the area measure on $\mathbf{D}$, normalized so that the area of $\mathbf{D}$ is 1. Thus $dA(z) = \frac{1}{\pi}\,dx\,dy = \frac{1}{\pi}\,r\,dr\,d\theta$. The space of all analytic functions in $\mathbf{D}$ will be denoted by $\mathcal{H}ol(\mathbf{D})$. We also let H^p $(0 < p \le \infty)$ be the classical Hardy spaces. We refer to [9] for the notation and results regarding Hardy spaces. The space $BMOA$ consists of those functions $f \in H^1$ whose boundary values have bounded mean oscillation on $\partial\mathbf{D}$. The "little oh" version of $BMOA$ is the space $VMOA$. We refer to [15] for the theory of $BMOA$-functions.

For $0 < p < \infty$ and $\alpha > -1$ the weighted Bergman space A^p_α consists of those $f \in \mathcal{H}ol(\mathbf{D})$ such that

$$\|f\|_{A^p_\alpha} \overset{\text{def}}{=} \left((\alpha+1)\int_{\mathbf{D}} (1-|z|^2)^\alpha |f(z)|^p\,dA(z)\right)^{1/p} < \infty.$$

The unweighted Bergman space A^p_0 is simply denoted by A^p. We refer to [10, 19, 31] for the notation and results about Bergman spaces. The space of Dirichlet type $\mathcal{D}^p_\alpha$ $(0 < p < \infty$ and $\alpha > -1)$ consists of those $f \in \mathcal{H}ol(\mathbf{D})$ such that $f' \in A^p_\alpha$. In other

https://doi.org/10.5186/aasfm.2019.4402
2010 Mathematics Subject Classification: Primary 30H30; Secondary 46E15.
Key words: Bloch space, Dirichlet spaces, Besov spaces, weighted Bergman spaces, closure in the Bloch norm, Blaschke product.
This research is supported by grants from "El Ministerio de Economía y Competitividad, Spain" MTM2014-52865-P; and from "La Junta de Andalucía" (FQM210).

words, a function $f \in \mathcal{H}ol(\mathbf{D})$ belongs to $\mathcal{D}^p_\alpha$ if and only if

$$\|f\|_{\mathcal{D}^p_\alpha} \overset{\text{def}}{=} |f(0)| + \left((\alpha+1)\int_{\mathbf{D}} (1-|z|^2)^\alpha |f'(z)|^p \, dA(z)\right)^{1/p} < \infty.$$

If $\alpha > p-1$ then it is well known that $\mathcal{D}^p_\alpha = A^p_{\alpha-p}$ (see, e.g., [11, Theorem 6]). For $1 < p < \infty$, the space $\mathcal{D}^p_{p-2}$ is the analytic Besov space B^p. The space B^1 requires a special definition: it is the space of all functions $f \in \mathcal{H}ol(\mathbf{D})$ such that $f'' \in A^1$. Although the corresponding semi-norm is not conformally invariant, the space itself is. Another possible definition (with a conformally invariant semi-norm) is given in the fundamental article [3], where B^1 was denoted by $\mathcal{M}$. The spaces B^p, $1 \le p < \infty$, form a nested scale of conformally invariant spaces which are contained in $VMOA$ and show up naturally in different settings (see [3], [8] and [30]). In particular, $\mathcal{D}^2_0 = B^2$ is the classical Dirichlet space $\mathcal{D}$.

Finally, we recall that a function $f \in \mathcal{H}ol(\mathbf{D})$ is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} \overset{\text{def}}{=} |f(0)| + \sup_{z \in \mathbf{D}} (1-|z|^2)|f'(z)| < \infty.$$

The space of all Bloch functions will be denoted by $\mathcal{B}$. It is a non-separable Banach space with the norm $\|\cdot\|_{\mathcal{B}}$ just defined. A classical source for the theory of Bloch functions is [1]. The closure of the polynomials in the Bloch norm is the *little Bloch space* $\mathcal{B}_0$ which consists of those $f \in \mathcal{H}ol(\mathbf{D})$ with the property that

$$\lim_{|z|\to 1} (1-|z|^2)|f'(z)| = 0.$$

It is well known that

$$H^\infty \subsetneq BMOA \subsetneq \cap_{0<p<\infty} H^p, \quad H^\infty \subsetneq BMOA \subsetneq \mathcal{B}, \quad VMOA \subsetneq \mathcal{B}_0 \subsetneq \mathcal{B}.$$

Anderson, Clunie and Pommerenke [1, p. 36] raised the question of determining the closure of H^∞ in $\mathcal{B}$. They remarked that this closure strictly contains $\mathcal{B}_0$ but is not identical with $\mathcal{B}$. The problem is still open. However, Jones gave an unpublished description of the closure of $BMOA$ in $\mathcal{B}$ (see [2, Theorem 9]). Given $f \in \mathcal{B}$ and $\varepsilon > 0$, we define

$$\Omega_\varepsilon(f) = \{z \in \mathbf{D} \colon (1-|z|^2)|f'(z)| \ge \varepsilon\}.$$

Then a Bloch function f is in the closure of $BMOA$ in the Bloch norm if and only if for every $\varepsilon > 0$ the Borel measure $(1-|z|^2)^{-1}\chi_{\Omega_\varepsilon(f)}(z)\, dA(z)$ is a Carleson measure in $\mathbf{D}$. As usual, for a Borel subset E of $\mathbf{D}$, χ_E denotes the characteristic function of E. A proof of Jones' description is provided by Ghatage and Zheng [14].

This study has been broaden to determine the closure in the Bloch norm of other subspaces of $\mathcal{B}$. For simplicity, if X is a subspace of the Bloch space we shall let $\mathcal{C}_{\mathcal{B}}(X)$ denote the closure in the Bloch norm of the space X.

Tjani [26] proved that if $f \in \mathcal{B}$, then $f \in \mathcal{B}_0$ if and only if $\int_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^2} < \infty$ for every $\varepsilon > 0$. Since all Besov spaces contain the polynomials and are contained in $\mathcal{B}_0$, we have

$$\mathcal{C}_{\mathcal{B}}(B^p) = \mathcal{B}_0, \quad 1 \le p < \infty. \tag{1.1}$$

This was observed in [29] where the closures in the Bloch norm of other conformally invariant spaces were also described. Bao and Göğüş [5] have recently characterized the closure in the Bloch norm of the space $\mathcal{D}^2_\alpha \cap \mathcal{B}$ $(-1 < \alpha \le 1)$.

Monreal Galán and Nicolau [22] described the closure in the Bloch norm of $\mathcal{B}\cap H^p$, for $1<p<\infty$. Galanopoulos, Monreal Galán and Pau [13] have extended this result to the whole range $0<p<\infty$.

Let us fix some notation. Given a Lebesgue measurable subset Ω of $\mathbf{D}$, we let $A_h(\Omega)$ be the hyperbolic area of Ω, that is,

$$A_h(\Omega)=\int_\Omega \frac{dA(z)}{(1-|z|^2)^2}.$$

Also, for fixed $a>1$ and for $\xi\in\partial\mathbf{D}$, we let $\Gamma_a(\xi)=\{z\in\mathbf{D}\colon |z-\xi|<a(1-|z|)\}$ be the Stolz angle with vertex at ξ. Putting [22, Theorem 1] and [13, Theorem 1] together yields the following result.

Theorem A. *Let $0<p<\infty$ and $a>1$. A Bloch function f is in the closure in the Bloch norm of $\mathcal{B}\cap H^p$ if and only if for every $\varepsilon>0$ the function $F_\varepsilon(f)$ defined by*

$$F_\varepsilon(f)(\xi)=A_h^{1/2}\left(\Gamma_a(\xi)\cap\Omega_\varepsilon(f)\right),\quad \xi\in\partial\mathbf{D},$$

belongs to $L^p(\partial\mathbf{D})$, that is,

$$\int_{\partial\mathbf{D}}\left(\int_{\Gamma_a(\xi)\cap\Omega_\varepsilon(f)}\frac{dA(z)}{(1-|z|^2)^2}\right)^{p/2}|d\xi|<\infty.$$

It is well known that there exists a positive constant C such that

$$|f(z)|\le C\|f\|_{\mathcal{B}}\log\frac{2}{1-|z|},\quad (z\in\mathbf{D}),\ \text{ for every } f\in\mathcal{B},$$

(see [1, p. 13]). Then it follows trivially that $\mathcal{B}\subset A^p_\alpha$ whenever $0<p<\infty$ and $\alpha>-1$. So the question of characterizing $\mathcal{C}_{\mathcal{B}}(A^p_\alpha\cap\mathcal{B})$ is trivial:

$$\mathcal{C}_{\mathcal{B}}(A^p_\alpha\cap\mathcal{B})=\mathcal{C}_{\mathcal{B}}(\mathcal{B})=\mathcal{B},\quad 0<p<\infty,\quad \alpha>-1. \tag{1.2}$$

The main object of this paper is to characterize the closure in the Bloch norm of the spaces $\mathcal{D}^p_\alpha\cap\mathcal{B}$. As we mentioned above, if $p-1<\alpha$ then $\mathcal{D}^p_\alpha=A^p_{\alpha-p}$. Thus, using (1.2) we obtain

$$\mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_\alpha\cap\mathcal{B})=\mathcal{B},\quad 0<p<\infty,\quad p-1<\alpha. \tag{1.3}$$

If $-1<\alpha\le p-2$ then we have that $\mathcal{D}^p_\alpha\subset\mathcal{D}^p_{p-2}=B^p\subset\mathcal{B}$, and then (1.1) implies that

$$\mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_\alpha\cap\mathcal{B})=\mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_\alpha)\subset\mathcal{C}_{\mathcal{B}}(B^p)=\mathcal{B}_0.$$

Now it is clear that the polynomials lie in $\mathcal{D}^p_\alpha$ and then it follows that $\mathcal{B}_0\subset\mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_\alpha)$. Consequently, we have

$$\mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_\alpha\cap\mathcal{B})=\mathcal{B}_0,\quad 0<p<\infty,\quad \alpha\le p-2. \tag{1.4}$$

If remains to consider the case where $p-2<\alpha\le p-1$ and we shall pay a special attention to the case $\alpha=p-1$ because the spaces $\mathcal{D}^p_{p-1}$ are the closest ones to Hardy spaces among all the $\mathcal{D}^p_\alpha$-spaces. By the Littlewood–Paley identity, we have $\mathcal{D}^2_1=H^2$. We have also [21]

$$H^p\subsetneq\mathcal{D}^p_{p-1},\quad \text{for } 2<p<\infty,$$

and [11, 27]

$$\mathcal{D}^p_{p-1}\subsetneq H^p,\quad \text{for } 0<p<2.$$

A number of similarities and differences between the spaces H^p and $\mathcal{D}^p_{p-1}$ are presented in [4, 16, 17, 23, 27]. As in the case of Hardy spaces, there is no inclusion relation between the spaces $\mathcal{D}^p_{p-1}$ and the Bloch space. Despite the fact that there is no relation of inclusion between $\mathcal{D}^p_{p-1}$ and $\mathcal{D}^q_{q-1}$ for $p \neq q$ (see [4, 17, 12]), it was observed in [7] that

$$\mathcal{D}^p_{p-1} \cap \mathcal{B} \subset \mathcal{D}^q_{q-1} \cap \mathcal{B}, \quad 0 < p < q < \infty.$$

In the next theorem we give a characterization of the closures in the Bloch norm of the spaces $\mathcal{D}^p_{p-1} \cap \mathcal{B}$ $(1 \leq p < \infty)$. We remark that, contrary to what happens with Hardy spaces, these closures are independent of p.

Theorem 1. *Let $p \in [1, \infty)$ and $f \in \mathcal{B}$. Then the following conditions are equivalent.*

(i) $f \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_{p-1} \cap \mathcal{B})$.

(ii) *For every $\varepsilon > 0$*

$$\int_{\Omega_\varepsilon(f)} \frac{dA(z)}{1 - |z|^2} < \infty.$$

(iii) $f \in \mathcal{C}_{\mathcal{B}}(H^2 \cap \mathcal{B})$.

As remarked in [22], the equivalence (ii) $\Leftrightarrow$ (iii) follows immediately from the case where $p = 2$ in Theorem A by using Fubini's theorem. Indeed, using Fubini's theorem, for $f \in \mathcal{B}$, $\varepsilon > 0$, and $a > 1$, we have

$$\begin{aligned}
&\int_{\partial \mathbf{D}} \int_{\Gamma_a(\xi) \cap \Omega_\varepsilon(f)} \frac{1}{(1 - |z|^2)^2} \, dA(z) \, |d\xi| \\
&= \int_{\partial \mathbf{D}} \int_{\Omega_\varepsilon(f)} \chi_{\Gamma_a(\xi)}(z) \frac{1}{(1 - |z|^2)^2} \, dA(z) \, |d\xi| \\
&= \int_{\Omega_\varepsilon(f)} \left(\int_{\partial \mathbf{D}} \chi_{\Gamma_a(\xi)}(z) \, |d\xi| \right) \frac{1}{(1 - |z|^2)^2} \, dA(z) \\
&\asymp \int_{\Omega_\varepsilon(f)} (1 - |z|^2) \frac{dA(z)}{(1 - |z|^2)^2} = \int_{\Omega_\varepsilon(f)} \frac{dA(z)}{1 - |z|^2}.
\end{aligned}$$

Bearing in mind that (ii) $\iff$ (iii), Theorem 1 follows from the following one where we give a characterization of $\mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_\alpha \cap \mathcal{B})$ whenever $1 \leq p < \infty$ and $p - 2 < \alpha \leq p - 1$.

Theorem 2. *Suppose that $1 \leq p < \infty$, $p - 2 < \alpha \leq p - 1$, and let f be a Bloch function. Then the following conditions are equivalent.*

(i) $f \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_\alpha \cap \mathcal{B})$.

(ii) *For every $\varepsilon > 0$ we have that*

$$\int_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1 - |z|^2)^{p-\alpha}} < \infty.$$

The proof of Theorem 2 will be presented in Section 2. In Section 3 we discuss the case $0 < p < 1$ and we study also the membership of Blaschke products in the spaces $\mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_\alpha \cap \mathcal{B})$.

We close this section noticing that, as usual, we shall be using the convention that $C = C(p, \alpha, q, \beta, \dots)$ will denote a positive constant which depends only upon the displayed parameters $p, \alpha, q, \beta \dots$ (which sometimes will be omitted) but not necessarily the same at different occurrences. Moreover, for two real-valued functions E_1, E_2 we write $E_1 \lesssim E_2$, or $E_1 \gtrsim E_2$, if there exists a positive constant C independent of the arguments such that $E_1 \leq C E_2$, respectively $E_1 \geq C E_2$. If we have $E_1 \lesssim E_2$ and $E_1 \gtrsim E_2$ simultaneously then we say that E_1 and E_2 are equivalent and we write $E_1 \asymp E_2$.

2. Proof of Theorem 2

We start recalling a well known lemma (see [31, Lemma 3. 10, p. 55]).

Lemma A. *Suppose that c is real and $t > -1$, and set*

$$F(z) = \int_{\mathbf{D}} \frac{(1-|w|^2)^t}{|1-\overline{w}z|^{2+t+c}}\, dA(w), \quad z \in \mathbf{D}.$$

(i) *If $c < 0$, then $F(z)$ is a bounded function of z*
(ii) *If $c > 0$, then $F(z) \asymp (1-|z|^2)^{-c}$, $|z| \to 1^-$.*
(iii) *If $c = 0$, then $F(z) \asymp \log \frac{1}{(1-|z|^2)}$, $|z| \to 1^-$.*

We shall also need the following representation formula for Bloch functions (see [31, Proposition 4. 27 and p. 112]).

Proposition A. *Let f be a Bloch function with $f(0) = f'(0) = 0$, then*

$$f(z) = \int_{\mathbf{D}} \frac{(1-|w|^2)f'(w)}{(1-z\overline{w})^2\,\overline{w}}\, dA(w), \quad z \in \mathbf{D}.$$

Proof of the implication (i) $\implies$ (ii) *in Theorem 2.* Take a function f in the closure in the Bloch norm of $\mathcal{D}^p_\alpha \cap \mathcal{B}$ and $\varepsilon > 0$. Then there exists a function $g \in \mathcal{D}^p_\alpha \cap \mathcal{B}$ such that $\|f-g\|_{\mathcal{B}} < \frac{\varepsilon}{2}$. Clearly, this implies that $\Omega_\varepsilon(f) \subseteq \Omega_{\frac{\varepsilon}{2}}(g)$. Then it follows that

$$\begin{aligned}\int_{\mathbf{D}} |g'(z)|^p(1-|z|^2)^\alpha\, dA(z) &\geq \int_{\Omega_{\frac{\varepsilon}{2}}(g)} |g'(z)|^p(1-|z|^2)^\alpha\, dA(z)\\ &= \int_{\Omega_{\frac{\varepsilon}{2}}(g)} \frac{|g'(z)|^p(1-|z|^2)^p}{(1-|z|^2)^{p-\alpha}}\, dA(z)\\ &\geq \left(\frac{\varepsilon}{2}\right)^p \int_{\Omega_{\frac{\varepsilon}{2}}(g)} \frac{dA(z)}{(1-|z|^2)^{p-\alpha}}\\ &\geq \left(\frac{\varepsilon}{2}\right)^p \int_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^{p-\alpha}}.\end{aligned}$$

Since $g \in \mathcal{D}^p_\alpha$, (ii) follows. □

Proof of the implication (ii) $\implies$ (i) *in Theorem 2.* Suppose that $1 \leq p < \infty$, $p-2 < \alpha \leq p-1$, and let f be a Bloch function which satisfies (ii). Assume without loss of generality that $f(0) = f'(0) = 0$. Using Proposition A we can write f as follows

$$f(z) = \int_{\mathbf{D}} \frac{(1-|w|^2)f'(w)}{(1-z\overline{w})^2\,\overline{w}}\, dA(w), \quad z \in \mathbf{D}.$$

Take $\varepsilon > 0$. We decompose f in the following way

$$\begin{aligned}f(z) &= \int_{\Omega_\varepsilon(f)} \frac{(1-|w|^2)f'(w)}{(1-\overline{w}z)^2\overline{w}}\, dA(w) + \int_{\mathbf{D}\setminus\Omega_\varepsilon(f)} \frac{(1-|w|^2)f'(w)}{(1-\overline{w}z)^2\overline{w}}\, dA(w)\\ &= f_1(z) + f_2(z).\end{aligned}$$

For any $z \in \mathbf{D}$, we have

$$\begin{aligned}(1-|z|^2)|f_2'(z)| &\le 2(1-|z|^2)\int_{\mathbf{D}\setminus\Omega_\varepsilon(f)} \frac{(1-|w|^2)|f'(w)|}{|1-\overline{w}z|^3}\,dA(w)\\ &\le 2\varepsilon(1-|z|^2)\int_{\mathbf{D}\setminus\Omega_\varepsilon(f)} \frac{dA(w)}{|1-\overline{w}z|^3}\\ &\le 2\varepsilon(1-|z|^2)\int_{\mathbf{D}} \frac{dA(w)}{|1-\overline{w}z|^3}.\end{aligned}$$

Using Lemma A with $t=0$ and $c=1$, we obtain that $(1-|z|^2)|f_2'(z)| \le C\,\varepsilon$ where C is a positive constant. Hence, $\|f_2\|_{\mathcal{B}} \le C\varepsilon$. Equivalently, f_1 is a Bloch function with

$$\|f-f_1\|_{\mathcal{B}} \le C\varepsilon.$$

The proof will be finished if we prove that $f_1 \in \mathcal{D}^p_\alpha$ or, equivalently, that $f_1' \in A^p_\alpha$. We have

$$\begin{aligned}&\int_{\mathbf{D}} (1-|z|^2)^\alpha |f_1'(z)|^p\,dA(z) = \int_{\mathbf{D}} (1-|z|^2)^\alpha |f_1'(z)|^{p-1}|f_1'(z)|\,dA(z)\\ &= \int_{\mathbf{D}} (1-|z|^2)^{p-1}|f_1'(z)|^{p-1}(1-|z|^2)^{\alpha-p+1}|f_1'(z)|\,dA(z)\\ &\le \|f_1\|_{\mathcal{B}}^{p-1}\int_{\mathbf{D}} (1-|z|^2)^{\alpha-p+1}|f_1'(z)|\,dA(z)\\ &\le \|f_1\|_{\mathcal{B}}^{p-1}\int_{\mathbf{D}} (1-|z|^2)^{\alpha-p+1}\left(\int_{\Omega_\varepsilon(f)} \frac{(1-|w|^2)|f'(w)|}{|1-\overline{w}\,z|^3}\,dA(w)\right)dA(z)\\ &\le \|f_1\|_{\mathcal{B}}^{p-1}\|f\|_{\mathcal{B}}\int_{\Omega_\varepsilon(f)}\left(\int_{\mathbf{D}} \frac{(1-|z|^2)^{\alpha-p+1}}{|1-\overline{w}\,z|^3}\,dA(z)\right)dA(w).\end{aligned}$$

Observe that $\alpha-p+1>-1$ and $p-\alpha>0$. Then, using Lemma A with $t=\alpha-p+1$ and $c=p-\alpha$ and (ii), we obtain

$$\int_{\mathbf{D}} (1-|z|^2)^\alpha |f_1'(z)|^p\,dA(z) \lesssim \|f_1\|_{\mathcal{B}}^{p-1}\|f\|_{\mathcal{B}}\int_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^{p-\alpha}} < \infty,$$

that is, $f_1' \in A^p_\alpha$ as desired. □

3. The case $0<p<1$ and some further results

Putting together (1.3), (1.4) and Theorem 2 we have the following result.

Theorem 3. *Suppose that $0<p<\infty$ and $\alpha>-1$.*

(i) *If $\alpha \le p-2$, then $\mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_\alpha \cap \mathcal{B}) = \mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_\alpha) = \mathcal{B}_0$.*

(ii) *If $\alpha > p-1$, then $\mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_\alpha \cap \mathcal{B}) = \mathcal{B}$.*

(iii) *If $p \ge 1$ and $p-2<\alpha\le p-1$, then*

$$\mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_\alpha \cap \mathcal{B}) = \left\{ f \in \mathcal{B}\colon \int_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^{p-\alpha}} < \infty \text{ for all } \varepsilon>0 \right\}.$$

We do not know whether (iii) remains true for $0<p<1$. In particular, we do not know whether $\mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_{p-1} \cap \mathcal{B})$ coincides with $\mathcal{C}_{\mathcal{B}}(H^2 \cap \mathcal{B})$ when $0<p<1$.

We can prove the following result.

Theorem 4. *Suppose that $0<p<1$, $-1<\alpha\le p-1$, and let f be a Bloch function.*

(a) *If $f \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_\alpha \cap \mathcal{B})$, then $\int_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^{p-\alpha}} < \infty$ for every $\varepsilon>0$.*

(b) *If there exists* $\gamma > 2 - \frac{1+\alpha}{p}$ *such that* $\int_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^\gamma} < \infty$ *for every* $\varepsilon > 0$*, then* $f \in \mathcal{C}_\mathcal{B}(\mathcal{D}^p_\alpha \cap \mathcal{B})$.

Proof. For $f \in \mathcal{B}$, we have

$$\begin{aligned}\int_\mathbf{D} (1-|z|^2)^{\alpha+1-p} |f'(z)|\, dA(z) &= \int_\mathbf{D} (1-|z|^2)^\alpha |f'(z)|^p \left[(1-|z|^2)|f'(z)|\right]^{1-p} dA(z)\\ &\le \|f\|_\mathcal{B}^{1-p} \int_\mathbf{D} (1-|z|^2)^\alpha |f'(z)|^p\, dA(z).\end{aligned}$$

Hence, it follows that $\mathcal{D}^p_\alpha \cap \mathcal{B} \subset \mathcal{D}^1_{\alpha+1-p} \cap \mathcal{B}$. Using this, the fact that $-1 < \alpha+1-p \le 0$, and Theorem 2, (a) follows.

We turn to prove (b). Observe that

$$1 \le 2 - \frac{1+\alpha}{p} < 2.$$

Suppose that $\gamma > 2 - \frac{1+\alpha}{p}$ and that $\int_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^\gamma} < \infty$ for every $\varepsilon > 0$. Clearly, we may assume without loss of generality that $\gamma < 2$. Arguing as is the proof of the implication (ii) $\Longrightarrow$ (i) in Theorem 2, the fact $f \in \mathcal{C}_\mathcal{B}(\mathcal{D}^p_{p-1} \cap \mathcal{B})$ will follow if we prove that the Bloch function f_1 defined by

$$f_1(z) = \int_{\Omega_\varepsilon(f)} \frac{(1-|w|^2) f'(w)}{(1-\overline{w}z)^2 \overline{w}}\, dA(w), \quad z \in \mathbf{D},$$

belongs to the space $\mathcal{D}^p_\alpha$ or, equivalently, that

$$f_1' \in A^p_\alpha. \tag{3.1}$$

We are going to present two proofs of (3.1), the second one has been suggested to us by one of the referees. Observe that $0 < 2-\gamma < \frac{\alpha+1}{p}$ and $1-\gamma > -1$. Then it follows that $A^1_{1-\gamma} \subset A^p_\alpha$ (see [20, p. 703] or [6, Lemma 1.2])). Hence it suffices to show that

$$f_1' \in A^1_{1-\gamma}. \tag{3.2}$$

We have

$$\begin{aligned}&\int_\mathbf{D} (1-|z|^2)^{1-\gamma} |f_1'(z)|\, dA(z)\\ &\le \int_\mathbf{D} (1-|z|^2)^{1-\gamma} \int_{\Omega_\varepsilon(f)} \frac{(1-|w|^2)|f'(w)|}{|1-\overline{w}\,z|^3}\, dA(w)\, dA(z)\\ &\le \|f\|_\mathcal{B} \int_{\Omega_\varepsilon(f)} \left(\int_\mathbf{D} \frac{(1-|z|^2)^{1-\gamma}}{|1-\overline{w}\,z|^3}\, dA(z)\right) dA(w)\\ &\le \|f\|_\mathcal{B} \int_{\Omega_\varepsilon(f)} \frac{dA(w)}{(1-|w|^2)^\gamma}.\end{aligned}$$

To obtain the last inequality we have used Lemma A with $t = 1-\gamma$ and $c = \gamma$. Then (3.2) follows.

Let us turn to the other promised proof of (3.1). Notice that

$$0 < (1-p)(\alpha+1) < p(1-\alpha). \tag{3.3}$$

Pick δ with

$$0 < \delta < (1-p)(\alpha+1) \tag{3.4}$$

and define $h(z) = (1-|z|^2)^\delta$ $(z \in \mathbf{D})$. Using Hölder's inequality, Fubini's theorem, the facts that $\frac{\delta}{1-p} - \alpha < 1$, $\alpha + \frac{\delta}{p} > -1$ and $1 - \alpha - \frac{\delta}{p} > 0$, and Lemma A, we obtain

$$\int_{\mathbf{D}} |f_1'(z)|^p (1-|z|^2)^\alpha dA(z) \lesssim \int_{\mathbf{D}} \left(\int_{\Omega_\varepsilon(f)} \frac{|f'(w)|(1-|w|^2)}{|1-\overline{w}\,z|^3} dA(w) \right)^p (1-|z|^2)^\alpha dA(z)$$

$$\leq \|f\|_{\mathcal{B}}^p \int_{\mathbf{D}} \left(\int_{\Omega_\varepsilon(f)} \frac{dA(w)}{|1-\overline{w}\,z|^3} \right)^p h(z)\, h(z)^{-1} (1-|z|^2)^{\alpha p} (1-|z|^2)^{\alpha(1-p)} dA(z)$$

$$\lesssim \|f\|_{\mathcal{B}}^p \left(\int_{\mathbf{D}} (1-|z|^2)^{\alpha + \frac{\delta}{p}} \int_{\Omega_\varepsilon(f)} \frac{dA(w)}{|1-\overline{w}\,z|^3}\, dA(z) \right)^p \left(\int_{\mathbf{D}} (1-|z|^2)^{\alpha - \frac{\delta}{1-p}} dA(z) \right)^{1-p}$$

$$\lesssim \|f\|_{\mathcal{B}}^p \left(\int_{\Omega_\varepsilon(f)} \int_{\mathbf{D}} \frac{(1-|z|^2)^{\alpha+\frac{\delta}{p}}}{|1-\overline{w}\,z|^3} dA(z)\, dA(w) \right)^p \lesssim \|f\|_{\mathcal{B}}^p \left(\int_{\Omega_\varepsilon(f)} \frac{dA(w)}{(1-|w|^2)^{1-\alpha-\frac{\delta}{p}}} \right)^p .$$

Since $1 - \alpha - \frac{(1-p)(\alpha+1)}{p} = 2 - \frac{\alpha+1}{p}$, (3.1) follows choosing δ sufficiently close to $(1-p)(\alpha+1)$. □

Our next aim is to give applications of the results that we have obtained so far to study the membership of a Blaschke product in $\mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_\alpha \cap \mathcal{B})$ for distinct values of p and α. We refer to [9] for the definition, notation, and results about Blaschke products. Since $H^\infty \subset H^2 \cap \mathcal{B}$, Theorem 1 trivially implies that

$$H^\infty \subset \mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_{p-1} \cap \mathcal{B}), \quad 1 \leq p < \infty.$$

In particular any Blaschke product lies in $\mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_{p-1} \cap \mathcal{B})$ whenever $1 \leq p < \infty$.

For $0 < p < 2$ the space H^∞ is not included in $\mathcal{D}^p_{p-1}$. Rudin [25, Theorem III] proved that there exists a Blaschke product B with $B \notin \mathcal{D}^1_0$. Later on, Vinogradov [27] gave examples of Blaschke products B such that $B \notin \mathcal{D}^p_{p-1}$ for all $p \in (0,2)$.

On the other hand, Rudin also proved in [25] that if a sequence $\{a_n\} \subset \mathbf{D}$ satisfies the condition

$$\sum (1-|a_n|) \log \frac{1}{1-|a_n|} < \infty \tag{3.5}$$

then the Blaschke product whose sequence of zeros is $\{a_n\}$ belongs to $\mathcal{D}^1_0$ (and, consequently to $\mathcal{D}^p_{p-1}$ for all $p \geq 1$). The converse of this is not true. Indeed, a result of Vinogradov [27, Theorem 2. 9, p. 3814] implies that a Blaschke product with zeros in a Stolz angle lies in $\mathcal{D}^p_{p-1}$ for all p.

Protas proved in [24, Theorem 1] that if $0 < s < 1$ and the sequence $\{a_n\}$ of the zeros of the Blaschke product B satisfies the condition $\sum (1-|a_n|^2)^s < \infty$, then $B' \in A^1_{s-1}$. Using again [6, Lemma 1. 2] we see that $A^1_{s-1} \subset A^p_{p-1}$ for all $p \in (0,1)$, whenever $0 < s < 1$. Then we deduce the following:

If the sequence $\{a_n\}$ of the zeros of the Blaschke product B satisfies the condition $\sum (1-|a_n|^2)^s < \infty$ for some $s < 1$, then $B \in \cap_{0<p<\infty} \mathcal{D}^p_{p-1}$.

Let us summarize these facts in the following theorem.

Theorem 5. *Let B be a Blaschke product and let $\{a_n\}$ be its sequence of zeros.*

(i) *$B \in \mathcal{C}_{\mathcal{B}}(\mathcal{D}^p_{p-1} \cap \mathcal{B})$ whenever $1 \leq p < \infty$.*
(ii) *If $\sum (1-|a_n|) \log \frac{1}{1-|a_n|} < \infty$, then $B \in \cap_{1 \leq p < \infty} \mathcal{D}^p_{p-1}$.*
(iii) *If $\sum (1-|a_n|^2)^s < \infty$ for some $s < 1$, then $B \in \cap_{0<p<\infty} \mathcal{D}^p_{p-1}$.*

Suppose that $1 \le \gamma < 2$ and let B be the Blaschke product whose sequence of zeros is $\{a_n\}$. Take $\varepsilon > 0$. We have

$$|B'(z)| \le \sum \frac{1-|a_n|^2}{|1-\overline{a_n}z|^2}, \quad z \in \mathbf{D},$$

and hence

$$z \in \Omega_\varepsilon(B) \quad \Longrightarrow \quad 1 \le \frac{1}{\varepsilon}(1-|z|^2)\sum \frac{1-|a_n|^2}{|1-\overline{a_n}z|^2}.$$

Then it follows that

$$\begin{aligned}\int_{\Omega_\varepsilon(B)} \frac{dA(z)}{(1-|z|^2)^\gamma} &\le \frac{1}{\varepsilon}\sum(1-|a_n|^2)\int_{\Omega_\varepsilon(B)} \frac{(1-|z|^2)^{1-\gamma}}{|1-\overline{a_n}z|^2}\,dA(z)\\ &\le \frac{1}{\varepsilon}\sum(1-|a_n|^2)\int_{\mathbf{D}} \frac{(1-|z|^2)^{1-\gamma}}{|1-\overline{a_n}z|^2}\,dA(z).\end{aligned}$$

Now, using Lemma A with $t = 1-\gamma$ and $c = \gamma - 1$, we obtain

$$\int_{\Omega_\varepsilon(B)} \frac{dA(z)}{1-|z|^2} \lesssim \frac{1}{\varepsilon}\sum(1-|a_n|^2)\,\log\frac{1}{1-|a_n|^2} \tag{3.6}$$

and

$$\int_{\Omega_\varepsilon(B)} \frac{dA(z)}{(1-|z|^2)^\gamma} \lesssim \frac{1}{\varepsilon}\sum(1-|a_n|^2)^{2-\gamma}, \quad \text{if } 1<\gamma<2. \tag{3.7}$$

Using these inequalities and Theorem 1 and Theorem 4 with $\alpha = p-1$, we obtain results which are weaker than those stated in Theorem 5. However, using (3.7) and Theorem 4 in the case $\alpha < p-1$, we obtain the following result.

Theorem 6. *Let B be the Blaschke product whose sequence of zeros is $\{a_n\}$. If $1 \le p < \infty$, $p-2 < \alpha < p-1$, and $\sum(1-|a_n|^2)^{2-(p-\alpha)} < \infty$, then $B \in \mathcal{C}_\mathcal{B}(\mathcal{D}^p_\alpha \cap \mathcal{B})$.*

Restricting ourselves to interpolating Blaschke products (that is, Blaschke products whose sequences of zeros are universal interpolation sequences [9, Chapter 9]), we have the following result.

Theorem 7. *Let B be an interpolating Blaschke product whose sequence of zeros is $\{a_n\}_{n=1}^\infty$. Suppose that $1 \le p < \infty$ and $p-2 < \alpha < p-1$. Then the following conditions are equivalent.*

(i) $\sum(1-|a_n|^2)^{2-(p-\alpha)} < \infty$.
(ii) $B \in \mathcal{C}_\mathcal{B}(\mathcal{D}^p_\alpha \cap \mathcal{B})$.

We remark that this was proved in [5] for the case where $p = 2$ and $0 < \alpha < 1$.

Proof of Theorem 7. The implication (i) $\Longrightarrow$ (ii) follows trivially from Theorem 6. To prove the other implication, suppose that $B \in \mathcal{C}_\mathcal{B}(\mathcal{D}^p_\alpha \cap \mathcal{B})$. By Theorem 3 we have

$$\int_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^{p-\alpha}} < \infty. \tag{3.8}$$

Since B is an interpolating Blaschke product, the sequence $\{a_n\}$ is uniformly separated, that is, there exists $\delta > 0$ such that

$$\inf_{m\ge 1} \prod_{n=1, n\ne m}^{\infty} \varrho(a_n, a_m) \ge \delta.$$

Here ϱ denotes the pseudo-hyperbolic distance:

$$\varrho(z,w) = \left|\frac{z-w}{1-\overline{w}z}\right|, \quad z, w \in \mathbf{D}.$$

Also, for $a \in \mathbf{D}$ and $0 < r < 1$, $\Delta(a,r)$ will denote the pseudo-hyperbolic disc of center a and radius r:

$$\Delta(a,r) = \{z \in \mathbf{D} \colon \varrho(z,a) < r\}.$$

Using Lemma 3.5 of [18] we see that there exist $\varepsilon > 0$ and $\beta \in (0,1)$ such that the discs $\{\Delta(a_n,\beta) \colon n = 1,2,3,\dots\}$ are pairwise disjoint and so that

$$|B'(z)| \ge \frac{\varepsilon}{1-|a_n|^2}, \quad z \in \Delta(a_n,\beta), \quad n = 1,2,3,\dots.$$

This implies that

$$\bigcup_{n=1}^{\infty} \Delta(a_n,\beta) \subset \Omega_\varepsilon(B). \tag{3.9}$$

Using the fact that the discs $\{\Delta(a_n,\beta)\}$ are pairwise disjoint and (3.9), we obtain

$$\sum_{n=1}^{\infty} \int_{\Delta(a_n,\beta)} \frac{dA(z)}{(1-|z|^2)^{p-\alpha}} = \int_{\cup_{n=1}^{\infty}\Delta(a_n,\beta)} \frac{dA(z)}{(1-|z|^2)^{p-\alpha}} \le \int_{\Omega_\varepsilon(B)} \frac{dA(z)}{(1-|z|^2)^{p-\alpha}}. \tag{3.10}$$

Now, (see [31, p. 69]) it is well known that

$$(1-|z|^2) \asymp (1-|a_n|^2), \quad \text{as long as } z \in \Delta(a_n,\beta),$$

and that the area $A(\Delta(a_n,\beta))$ of $\Delta(a_n,\beta)$ satisfies $A(\Delta(a_n,\beta)) \asymp (1-|a_n|^2)^2$. These two facts imply that

$$\sum_{n=1}^{\infty}(1-|a_n|^2)^{2-(p-\alpha)} \asymp \sum_{n=1}^{\infty} \int_{\Delta(a_n,\beta)} \frac{dA(z)}{(1-|z|^2)^{p-\alpha}}.$$

This, together with (3.10) and (3.8), implies that $\sum_{n=1}^{\infty}(1-|a_n|^2)^{2-(p-\alpha)} < \infty$. □

Acknowledgements. We wish to express our gratitude to the referees for their helpful comments and suggestions which have lead us to improve the paper.

References

[1] Anderson, J. M., J. Clunie, and Ch. Pommerenke: On Bloch functions and normal functions. - J. Reine Angew. Math. 270, 1974, 12–37.

[2] Anderson, J. M.: Bloch functions: the basic theory. - In: Operators and function theory (Lancaster, 1984), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 153, Reidel, Dordrecht, 1985, 1–17.

[3] Arazy, J., S. D. Fisher, and J. Peetre: Möbius invariant function spaces. - J. Reine Angew. Math. 363, 1985, 110–145.

[4] Baernstein, A., D. Girela, and J. A. Peláez: Univalent functions, Hardy spaces and spaces of Dirichlet type. - Illinois J. Math. 48:3, 2004, 837–859.

[5] Bao, G., and N. G. Göğüş: On the closures of Dirichlet type spaces in the Bloch space. - Complex Anal. Oper. Theory (to appear).

[6] Buckley, S. M., P. Koskela, and D. Vukotić: Fractional integration, differentiation, and weighted Bergman spaces. - Math. Proc. Cambridge Philos. Soc. 126:2, 1999, 369–385.

[7] Chatzifountas, Ch., D. Girela, and J. A. Peláez: Multipliers of Dirichlet subspaces of the Bloch space. - J. Operator Theory 72:1, 2014, 159–191.

[8] Donaire, J. J., D. Girela, and D. Vukotić: On univalent functions in some Möbius invariant spaces. - J. Reine Angew. Math. 553, 2002, 43–72.

[9] Duren, P.Ł.: Theory of H^p spaces. - Academic Press, New York-London, 1970.

[10] Duren, P. L., and A. P. Schuster: Bergman spaces. - Math. Surveys Monogr. 100, Amer. Math. Soc., Providence, Rhode Island, 2004.

[11] Flett, T. M.: The dual of an inequality of Hardy and Littlewood and some related inequalities. - J. Math. Anal. Appl. 38, 1972, 746–765.

[12] Galanopoulos, P., D. Girela, and J. A. Peláez: Multipliers and integration operators on Dirichlet spaces. - Trans. Amer. Math. Soc. 363:4, 2011, 1855–1886.

[13] Galanopoulos, P., N. Monreal Galán, and J. Pau: Closure of Hardy spaces in the Bloch space. - J. Math. Anal. Appl. 429:2, 2015, 1214–1221.

[14] Ghatage, P. G., and D. Zheng: Analytic functions of bounded mean oscillation and the Bloch space. - Integral Equations Operator Theory 17:4, 1993, 501–515.

[15] Girela, D.: Analytic functions of bounded mean oscillation. - In: Complex Function Spaces, Mekrijärvi 1999, editor R. Aulaskari, Univ. Joensuu Dept. Math. Rep. Ser. 4, Univ. Joensuu, Joensuu, 2001, 61–170.

[16] Girela, D., and J. A. Peláez: Carleson measures for spaces of Dirichlet type. - Integral Equations Operator Theory 55:3, 2006, 415–427.

[17] Girela, D., and J. A. Peláez: Growth properties and sequences of zeros of analytic functions in spaces of Dirichlet type. - J. Australian Math. Soc. 80, 2006, 397–418.

[18] Girela, D., J. A. Peláez, and D. Vukotić: Integrability of the derivative of a Blaschke product. - Proc. Edinb. Math. Soc. (2) 50:3, 2007, 673–687.

[19] Hedenmalm, H., B. Korenblum, and K. Zhu: Theory of Bergman spaces. - Grad. Texts in Math. 199, Springer, New York, Berlin, 2000.

[20] Horowitz, C.: Zeros of functions in the Bergman spaces. - Duke Math. J. 41, 1974, 693–710.

[21] Littlewood, J. E., and R. E. A. C. Paley: Theorems on Fourier series and power series (II). - Proc. London Math. Soc. 42, 1936, 52–89.

[22] Monreal Galán, N., and A. Nicolau: The closure of the Hardy space in the Bloch norm. - Algebra i Analiz 22:1, 2010, 75–81; transl. in St. Petersburg Math. J. 22:1, 2011, 55–59.

[23] Peláez, J. A., F. Pérez-González, and J. Rättyä: Operator theoretic differences between Hardy and Dirichlet-type spaces. - J. Math. Anal. Appl. 418:1, 2014, 387–402.

[24] Protas, D.: Blaschke products with derivative in H^p and B^p. - Michigan Math. J. 20, 1973, 393–396.

[25] Rudin, W.: The radial variation of analytic functions. - Duke Math. J. 22, 1955, 235–242.

[26] Tjani, M.: Distance of a Bloch function to the little Bloch space. - Bull. Austral. Math. Soc. 74:1, 2006, 101–119.

[27] Vinogradov, S. A.: Multiplication and division in the space of analytic functions with area integrable derivative, and in some related spaces. - Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 222, 1995, Issled. po Linein. Oper. i Teor. Funktsii 23, 45–77, 308 (in Russian); transl. in J. Math. Sci. (New York) 87:5, 1997, 3806–3827.

[28] Wu, Z.: Carleson measures and multipliers for Dirichlet spaces. - J. Funct. Anal. 169, 1999, 148–163.

[29] Zhao, R.: Distances from Bloch functions to some Möbius invariant spaces. - Ann. Acad. Sci. Fenn. Math. 33:1, 2008, 303–313.

[30] Zhu, K.: Analytic Besov spaces. - J. Math. Anal. Appl. 157, 1991, 318–336.

[31] Zhu, K.: Operator theory in function spaces. Second edition. - Math. Surveys Monogr. 138, Amer. Math. Soc., Providence, Rhode Island, 2007.

Received 18 March 2018 • Accepted 24 July 2018