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THE n-LINEAR EMBEDDING THEOREM FOR DYADIC RECTANGLES

Hitoshi Tanaka and Kôzô Yabuta

National University Corporation Tsukuba University of Technology
Research and Support Center on Higher Education for the Hearing and Visually Impaired
Kasuga 4-12-7, Tsukuba 305-8521, Japan; htanaka@k.tsukuba-tech.ac.jp

Kwansei Gakuin University, Research center for Mathematical Sciences
Gakuen 2-1, Sanda 669-1337, Japan; kyabuta3@kwansei.ac.jp

Abstract. Let σ_i, $i=1,\dots,n$, be reverse doubling weights on $\mathbf{R}^d$, $\mathcal{DR}(\mathbf{R}^d)$ be the set of all dyadic rectangles on $\mathbf{R}^d$ (Cartesian products of usual dyadic intervals) and $K\colon \mathcal{DR}(\mathbf{R}^d)\to[0,\infty)$ be a map. In this paper we give the n-linear embedding theorem for dyadic rectangles. That is, we prove that the n-linear embedding inequality for dyadic rectangles

$$\sum_{R\in\mathcal{DR}(\mathbf{R}^d)} K(R)\prod_{i=1}^n\left|\int_R f_i\,\mathrm{d}\sigma_i\right| \le C\prod_{i=1}^n \|f_i\|_{L^{p_i}(\sigma_i)}$$

can be characterized by simple testing condition

$$K(R)\prod_{i=1}^n\sigma_i(R)\le C\prod_{i=1}^n\sigma_i(R)^{\frac{1}{p_i}}\quad R\in\mathcal{DR}(\mathbf{R}^d),$$

in the range $1<p_i<\infty$ with $\sum_{i=1}^n\frac{1}{p_i}>1$. As a corollary to this theorem, for reverse doubling weights, we verify a necessary and sufficient condition for which weighted norm inequality for multilinear strong positive dyadic operator and for multilinear strong fractional integral operator to hold.

1. Introduction

The purpose of this paper is to prove the n-linear embedding theorem for dyadic rectangles. We will denote by $\mathcal{DQ}(\mathbf{R}^d)$ the family of all dyadic cubes $Q=2^{-k}(m+[0,1)^d)$, $k\in\mathbf{Z}$, $m\in\mathbf{Z}^d$, and by $\mathcal{DR}(\mathbf{R}^d)$ the family of all dyadic rectangles on $\mathbf{R}^d$. By dyadic rectangle we mean the Cartesian product of the dyadic intervals $\mathcal{DQ}(\mathbf{R})$. Throughout this paper the letter n stands for an integer which is greater than one.

In a series of works [3, 4, 8, 11, 13, 14, 15, 16], the n-linear embedding inequality has been characterized for dyadic cubes. Let σ_i, $i=1,\dots,n$, denote positive Borel measures on $\mathbf{R}^d$ and let $K\colon\mathcal{DQ}(\mathbf{R}^d)\to[0,\infty)$ be a map. The n-linear embedding inequality for dyadic cubes

$$\sum_{Q\in\mathcal{DQ}(\mathbf{R}^d)} K(Q)\prod_{i=1}^n\left|\int_Q f_i\,\mathrm{d}\sigma_i\right|\le C\prod_{i=1}^n\|f_i\|_{L^{p_i}(\mathrm{d}\sigma_i)} \tag{1.1}$$

can be characterized in the full range $1<p_i<\infty$. The n-linear embedding theorem (1.1), either can be reduced to the (localized) $(n-1)$-linear embedding theorems, or

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characterized by certain n-weight discrete Wolff potential conditions. The division line is whether the exponents $p_1, \ldots, p_n$ are in the super-dual range $\sum_{i=1}^n \frac{1}{p_i} \geq 1$ or in the strictly sub-dual range $\sum_{i=1}^n \frac{1}{p_i} < 1$. The methods of each range seem to be rather different (see [15]). The main technique used is that of "parallel corona" decomposition from the work of Lacey et al. [7] on the two-weight boundedness of the Hilbert transform. However, this powerful technique deeply depends on the tree structure of dyadic cubes and one can not apply it to the case of dyadic rectangles. Even though, it is natural to consider what happens in the case $\mathcal{DR}(\mathbf{R}^d)$ and the partial answer is given in this paper.

By "weights" we will always mean nonnegative, locally integrable functions which are positive on a set of positive measure. Given a measurable set E and a weight ω, we use $\omega(E)$ to denote the quantity $\int_E \omega \, \mathrm{d}x$ and use 1_E to denote the characteristic function of E.

Let $1 \leq p < \infty$ and ω be a weight. We define the weighted Lebesgue space $L^p(\omega)$ to be a Banach space equipped with the norm

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbf{R}^d} |f|^p \, \mathrm{d}\omega \right)^{\frac{1}{p}},$$

where we have used the notation $\mathrm{d}\omega := \omega \, \mathrm{d}x$. Given $1 < p < \infty$, $p' = \frac{p}{p-1}$ will denote the conjugate exponent of p.

Let $\mathcal{R}(\mathbf{R}^d)$ denote the set of all rectangles in $\mathbf{R}^d$ with sides parallel to the coordinate axes. We say that a weight ω is "reverse doubling weight" if it satisfies that there is a constant $\beta > 1$ such that $\beta\omega(R') \leq \omega(R)$ for any $R', \ R \in \mathcal{R}(\mathbf{R}^d)$ where R' is any one of the two equal divisions of R. We shall prove the following theorem.

Theorem 1.1. *Let $1 < p_i < \infty$ with $\sum_{i=1}^n \frac{1}{p_i} > 1$. Let $K\colon \mathcal{DR}(\mathbf{R}^d) \to [0, \infty)$ be a map and let σ_i, $i = 1, \ldots, n$, be reverse doubling weights on $\mathbf{R}^d$. The following statements are equivalent:*

(a) *The n-linear embedding inequality for dyadic rectangles*

$$\sum_{R \in \mathcal{DR}(\mathbf{R}^d)} K(R) \prod_{i=1}^n \left| \int_R f_i \, \mathrm{d}\sigma_i \right| \leq c_1 \prod_{i=1}^n \|f_i\|_{L^{p_i}(\sigma_i)} \tag{1.2}$$

holds for all $f_i \in L^{p_i}(\sigma_i)$, $i = 1, \ldots, n$;

(b) *The testing condition*

$$K(R) \prod_{i=1}^n \sigma_i(R) \leq c_2 \prod_{i=1}^n \sigma_i(R)^{\frac{1}{p_i}} \tag{1.3}$$

holds for all dyadic rectangles $R \in \mathcal{DR}(\mathbf{R}^d)$.

Moreover, the least possible constants c_1 and c_2 are equivalent.

Corollary 1.2. *Let $1 < p_i < \infty$ and $1 < q < \infty$ with $\sum_{i=1}^n \frac{1}{p_i} > \frac{1}{q}$. Let $K\colon \mathcal{DR}(\mathbf{R}^d) \to [0, \infty)$ be a map and let σ_i, $i = 1, \ldots, n$, and ω be reverse doubling weights on $\mathbf{R}^d$. The following statements are equivalent:*

(a) *The weighted norm inequality for multilinear strong positive operator*

$$\|T_K(f_1, \ldots, f_n)\|_{L^q(\omega)} \leq c_1 \prod_{i=1}^n \|f_i\|_{L^{p_i}(\sigma_i^{1-p_i})} \tag{1.4}$$

holds for all $f_i \in L^{p_i}(\sigma_i^{1-p_i})$, $i = 1, \dots, n$; *Here,*

$$T_K(f_1, \dots, f_n) := \sum_{R \in \mathcal{DR}(\mathbf{R}^d)} K(R) 1_R \prod_{i=1}^n \int_R f_i \,\mathrm{d}x.$$

(b) *The testing condition*

$$K(R)\omega(R)^{\frac{1}{q}} \prod_{i=1}^n \sigma_i(R) \le c_2 \prod_{i=1}^n \sigma_i(R)^{\frac{1}{p_i}} \tag{1.5}$$

holds for all dyadic rectangles $R \in \mathcal{DR}(\mathbf{R}^d)$.

Moreover, the least possible constants c_1 *and* c_2 *are equivalent.*

Remark. We notice that (1.5) is equivalent to the condition that

$$K(R)\omega(R)^{\frac{1}{q}} \prod_{i=1}^n \sigma_i(R)^{\frac{1}{p_i'}} \le c_2$$

holds for all dyadic rectangles $R \in \mathcal{DR}(\mathbf{R}^d)$. This is known as the Fefferman–Phong-type condition which was first observed in [2]. In [1], the corresponding results were established for the multilinear fractional strong maximal operator. In [5], two-weight inequalities of various type for the strong fractional maximal functions and potentials with multiple kernels defined on $\mathbf{R}^2$ were also established. In the recent paper [12], using our iteration method (see Lemma 2.2 below), Sawyer and Wang proved that the inequality (1.4) holds if the weights satisfy the theta bump condition.

In the last section (Section 5) we shall apply Corollary 1.2 to multilinear strong fractional integral operator.

The letter C will be used for constants that may change from one occurrence to another. Constants with subscripts, such as C_1, C_2, do not change in different occurrences. By $A \approx B$ we mean that $c^{-1}B \le A \le cB$ with some positive finite constant c independent of appropriate quantities.

2. Lemmas

We need two lemmas and we will give their proofs for the sake of completeness.

Lemma 2.1. *Given a weight* σ *in* $\mathbf{R}^d$ *and* $1 < p < q < \infty$, *the following statements are equivalent:*

(a) *The Carleson type embedding inequality for dyadic cubes*

$$\sum_{Q \in \mathcal{DQ}(\mathbf{R}^d)} \sigma(Q)^{\frac{q}{p}} \left(\frac{1}{\sigma(Q)} \int_Q f \,\mathrm{d}\sigma \right)^q \le c_1 \left(\int_{\mathbf{R}^d} f^p \,\mathrm{d}\sigma \right)^{\frac{q}{p}} \tag{2.1}$$

holds for all nonnegative function $f \in L^p(\sigma)$;

(b) *The testing condition*

$$\sum_{\substack{Q' \in \mathcal{DQ}(\mathbf{R}^d) \\ Q' \subset Q}} \sigma(Q')^{\frac{q}{p}} \le c_2 \sigma(Q)^{\frac{q}{p}} \tag{2.2}$$

holds for all cubes $Q \in \mathcal{DQ}(\mathbf{R}^d)$.

Moreover, the least possible constants c_1 *and* c_2 *are equivalent.*

Proof. The necessity of (2.2) follows at once if we substitute the test function $f = 1_Q$ into inequality (2.1). To show that inequality (2.2) is sufficient, we fix a (big enough) dyadic cube $Q_0 \in \mathcal{DQ}(\mathbf{R}^d)$ and we prove the inequality

$$\sum_{\substack{Q\in\mathcal{DQ}(\mathbf{R}^d)\\ Q\subset Q_0}} \sigma(Q)^{\frac{q}{p}} \left(\frac{1}{\sigma(Q)}\int_Q f\,\mathrm{d}\sigma\right)^q \le Cc_2 \left(\int_{Q_0} f^p\,\mathrm{d}\sigma\right)^{\frac{q}{p}}. \tag{2.3}$$

We define the collection of principal cubes $\mathcal{F}$ for the pair (f,σ). Namely,

$$\mathcal{F} := \bigcup_{k=0}^{\infty} \mathcal{F}_k,$$

where $\mathcal{F}_0 := \{Q_0\}$,

$$\mathcal{F}_{k+1} := \bigcup_{F\in\mathcal{F}_k} \mathrm{ch}_{\mathcal{F}}(F)$$

and $\mathrm{ch}_{\mathcal{F}}(F)$ is defined by the set of all maximal dyadic cubes $Q \subset F$ such that

$$\frac{1}{\sigma(Q)}\int_Q f\,\mathrm{d}\sigma > \frac{2}{\sigma(F)}\int_F f\,\mathrm{d}\sigma.$$

Observe that

$$\begin{aligned}\sum_{F'\in\mathrm{ch}_{\mathcal{F}}(F)} \sigma(F') &\le \left(\frac{2}{\sigma(F)}\int_F f\,\mathrm{d}\sigma\right)^{-1} \sum_{F'\in\mathrm{ch}_{\mathcal{F}}(F)} \int_{F'} f\,\mathrm{d}\sigma\\ &\le \left(\frac{2}{\sigma(F)}\int_F f\,\mathrm{d}\sigma\right)^{-1} \int_F f\,\mathrm{d}\sigma = \frac{\sigma(F)}{2},\end{aligned}$$

and, hence,

$$\sigma(E_{\mathcal{F}}(F)) := \sigma\left(F \setminus \bigcup_{F'\in\mathrm{ch}_{\mathcal{F}}(F)} F'\right) \ge \frac{\sigma(F)}{2}, \tag{2.4}$$

where the sets in the collection $\{E_{\mathcal{F}}(F)\colon F\in\mathcal{F}\}$ are pairwise disjoint.

We further define the stopping parent, for $Q \in \mathcal{DQ}(\mathbf{R}^d)$,

$$\pi_{\mathcal{F}}(Q) := \min\{F \supset Q\colon F\in\mathcal{F}\}.$$

Then we can rewrite the series in (2.3) as follows:

$$\begin{aligned}\sum_{Q\subset Q_0} \sigma(Q)^{\frac{q}{p}} \left(\frac{1}{\sigma(Q)}\int_Q f\,\mathrm{d}\sigma\right)^q &= \sum_{F\in\mathcal{F}} \sum_{Q\colon \pi_{\mathcal{F}}(Q)=F} \sigma(Q)^{\frac{q}{p}} \left(\frac{1}{\sigma(Q)}\int_Q f\,\mathrm{d}\sigma\right)^q\\ &\le \sum_{F\in\mathcal{F}} \left(\frac{2}{\sigma(F)}\int_F f\,\mathrm{d}\sigma\right)^q \sum_{Q\colon \pi_{\mathcal{F}}(Q)=F} \sigma(Q)^{\frac{q}{p}}\\ &\le 2^q c_2 \sum_{F\in\mathcal{F}} \left(\frac{1}{\sigma(F)}\int_F f\,\mathrm{d}\sigma\right)^q \sigma(F)^{\frac{q}{p}},\end{aligned}$$

where we have used the condition (2.2).

Using $\|\cdot\|_{l^p} \geq \|\cdot\|_{l^q}$, for $0 < p \leq q < \infty$, and (2.4) we can proceed further that

$$
\begin{aligned}
\sum_{Q\subset Q_0} \sigma(Q)^{\frac{q}{p}} \left(\frac{1}{\sigma(Q)}\int_Q f\,\mathrm{d}\sigma\right)^q &\leq Cc_2 \left\{\sum_{F\in\mathcal{F}} \left(\frac{1}{\sigma(F)}\int_F f\,\mathrm{d}\sigma\right)^p \sigma(F)\right\}^{\frac{q}{p}} \\
&\leq Cc_2 \left\{\sum_{F\in\mathcal{F}} \left(\frac{1}{\sigma(F)}\int_F f\,\mathrm{d}\sigma\right)^p \sigma(E_{\mathcal{F}}(F))\right\}^{\frac{q}{p}} \\
&\leq Cc_2 \left(\int_{Q_0} M^{\sigma}_{\mathcal{DQ}}[f1_{Q_0}]^p\,\mathrm{d}\sigma\right)^{\frac{q}{p}} \\
&\leq Cc_2 \left(\int_{Q_0} f^p\,\mathrm{d}\sigma\right)^{\frac{q}{p}},
\end{aligned}
$$

where $M^{\sigma}_{\mathcal{DQ}}$ stands for the dyadic Hardy–Littlewood maximal operator with respect to the measure $\mathrm{d}\sigma$ and we have used its $L^p(\sigma)$-boundedness. This completes the proof. $\square$

We denote by P_i, $i = 1,\dots,d$, the projection on the x_i-axis. For the dyadic rectangle $R \in \mathcal{DR}(\mathbf{R}^d)$, the dyadic interval $I \in \mathcal{DQ}(\mathbf{R})$ and $j = 1,\dots,d$, we define the dyadic rectangle

$$
[R;\, I, j] := \left(\prod_{i=1}^{j-1} P_i(R)\right) \times I \times \left(\prod_{i=j+1}^{d} P_i(R)\right).
$$

Lemma 2.2. *Given a weight σ in $\mathbf{R}^d$ and $1 < p < q < \infty$, the following statements are equivalent:*

(a) *The Carleson type embedding inequality for rectangles*

$$
\sum_{R\in\mathcal{DR}(\mathbf{R}^d)} \sigma(R)^{\frac{q}{p}} \left(\frac{1}{\sigma(R)}\int_R f\,\mathrm{d}\sigma\right)^q \leq c_1 \left(\int_{\mathbf{R}^d} f^p\,\mathrm{d}\sigma\right)^{\frac{q}{p}} \tag{2.5}
$$

holds for all nonnegative function $f \in L^p(\sigma)$;

(b) *The testing condition*

$$
\sum_{\substack{I\in\mathcal{DQ}(\mathbf{R})\\ I\subset P_j(R)}} \sigma([R;\, I, j])^{\frac{q}{p}} \leq c_2\sigma(R)^{\frac{q}{p}} \tag{2.6}
$$

holds for all dyadic rectangles $R \in \mathcal{DR}(\mathbf{R}^d)$ and $j = 1,\dots,d$.

Moreover, the least possible constants c_1 and c_2 enjoy $c_1 \leq Cc_2^d$ and $c_2 \leq c_1$.

Proof. The necessity is clear, so we shall prove the sufficiency. We use induction on the dimension d. To do this, we assume that the lemma is true for the case $d-1$. We assume that the weight σ in $\mathbf{R}^d$ satisfies the testing condition (2.6) (d-dimensional case). We will write $x = (x_1,\dots,x_{d-1},x_d) = (\overline{x}, x_d)$.

We need two observations. First, we verify that, for any dyadic interval $I_d \in \mathcal{DQ}(\mathbf{R})$, if we let

$$
v_{I_d}(\overline{x}) := \int_{I_d} \sigma(\overline{x}, x_d)\,\mathrm{d}x_d,
$$

then $v_{I_d}(\overline{x})$ satisfies the testing condition (2.6) ($(d-1)$-dimensional case). Indeed, for any $\overline{R} \in \mathcal{DR}(\mathbf{R}^{d-1})$, setting $R = \overline{R} \times I_d$, we have that, for $j = 1, \dots, d-1$,

$$\sum_{\substack{I \in \mathcal{DQ}(\mathbf{R}) \\ I \subset P_j(\overline{R})}} v_{I_d}([\overline{R};\, I, j])^{\frac{q}{p}} = \sum_{\substack{I \in \mathcal{DQ}(\mathbf{R}) \\ I \subset P_j(R)}} \sigma([R;\, I, j])^{\frac{q}{p}} \le c_2 \sigma(R)^{\frac{q}{p}} = c_2 v_{I_d}(\overline{R})^{\frac{q}{p}}.$$

We next verify that, for a.e. $\overline{x} \in \mathbf{R}^{d-1}$, if we let

$$v_{\overline{x}}(x_d) = \sigma(\overline{x}, x_d),$$

then $v_{\overline{x}}(x_d)$ satisfies the testing condition (2.2) (one-dimensional case). We must prove that the inequality

$$\sum_{\substack{I \in \mathcal{DQ}(\mathbf{R}) \\ I \subset I_d}} v_{\overline{x}}(I)^{\frac{q}{p}} \le c_2 v_{\overline{x}}(I_d)^{\frac{q}{p}} \tag{2.7}$$

holds for any $I_d \in \mathcal{DQ}(\mathbf{R})$. For a cube $\overline{Q} \in \mathcal{DQ}(\mathbf{R}^{d-1})$, it follows by setting $R = \overline{Q} \times I_d$ that

$$\sum_{\substack{I \in \mathcal{DQ}(\mathbf{R}) \\ I \subset P_d(R)}} \sigma([R;\, I, d])^{\frac{q}{p}} \le c_2 \sigma(R)^{\frac{q}{p}}.$$

Dividing the both sides by the volume $|\overline{Q}|^{\frac{q}{p}}$, we have that

$$\sum_{\substack{I \in \mathcal{DQ}(\mathbf{R}) \\ I \subset P_d(R)}} \left(\frac{1}{|\overline{Q}|} \int_{\overline{Q} \times I} \sigma(\overline{x}, x_d)\, \mathrm{d}x_d\, \mathrm{d}\overline{x} \right)^{\frac{q}{p}} \le c_2 \left(\frac{1}{|\overline{Q}|} \int_{\overline{Q} \times I_d} \sigma(\overline{x}, x_d)\, \mathrm{d}x_d\, \mathrm{d}\overline{x} \right)^{\frac{q}{p}}.$$

In the both sides of this inequality, considering the Lebesgue point $\overline{y}$ with respect to the integral averages over $\overline{Q}$, which exists a.e. in $\mathbf{R}^{d-1}$ because our argument is countable, and shrinking $\overline{Q}$ to $\overline{y}$, we obtain

$$\sum_{\substack{I \in \mathcal{DQ}(\mathbf{R}) \\ I \subset I_d}} \left(\int_I \sigma(\overline{y}, x_d)\, \mathrm{d}x_d \right)^{\frac{q}{p}} \le c_2 \left(\int_{I_d} \sigma(\overline{y}, x_d)\, \mathrm{d}x_d \right)^{\frac{q}{p}},$$

which means (2.7).

By the use of these two observations we can prove the lemma. Fix a nonnegative function $f \in L^p(\sigma)$. We shall evaluate

$$\text{(i)} := \sum_{I_d \in \mathcal{DQ}(\mathbf{R})} \sum_{\overline{R} \in \mathcal{DR}(\mathbf{R}^{d-1})} \sigma(R)^{\frac{q}{p}} \left(\frac{1}{\sigma(R)} \int_R f\, \mathrm{d}\sigma \right)^q,$$

where we have used $R = \overline{R} \times I_d$.

There holds

$$\begin{aligned} \text{(i)} = {} & \sum_{I_d \in \mathcal{DQ}(\mathbf{R})} \sum_{\overline{R} \in \mathcal{DR}(\mathbf{R}^{d-1})} v_{I_d}(\overline{R})^{\frac{q}{p}} \\ & \times \left(\frac{1}{v_{I_d}(\overline{R})} \int_{\overline{R}} \left(\int_{I_d} f(\overline{x}, x_d) \sigma(\overline{x}, x_d)\, \mathrm{d}x_d v_{I_d}(\overline{x})^{-1} \right) v_{I_d}(\overline{x})\, \mathrm{d}\overline{x} \right)^q. \end{aligned}$$

Since $v_{I_d}(\overline{x})$ satisfies the testing condition (2.6) ($(d-1)$-dimensional case), by our induction assumption, we have that

$$\begin{aligned}
\text{(i)} &\le Cc_2^{d-1} \sum_{I_d \in \mathcal{DQ}(\mathbf{R})} \left(\int_{\mathbf{R}^{d-1}} \left(\int_{I_d} f(\overline{x}, x_d)\sigma(\overline{x}, x_d)\, \mathrm{d}x_d v_{I_d}(\overline{x})^{-1} \right)^p v_{I_d}(\overline{x})\, \mathrm{d}\overline{x} \right)^{\frac{q}{p}} \\
&= Cc_2^{d-1} \left[\{\cdots\cdots\}^{\frac{p}{q}} \right]^{\frac{q}{p}}.
\end{aligned}$$

By integral version of Minkowski's inequality,

$$\begin{aligned}
\text{(ii)} &= \left\{ \sum_{I_d \in \mathcal{DQ}(\mathbf{R})} \left(\int_{\mathbf{R}^{d-1}} \left(\int_{I_d} f(\overline{x}, x_d)\sigma(\overline{x}, x_d)\, \mathrm{d}x_d v_{I_d}(\overline{x})^{-1} \right)^p v_{I_d}(\overline{x})\, \mathrm{d}\overline{x} \right)^{\frac{q}{p}} \right\}^{\frac{p}{q}} \\
&\le \int_{\mathbf{R}^{d-1}} \left\{ \sum_{I_d \in \mathcal{DQ}(\mathbf{R})} \left(\int_{I_d} f(\overline{x}, x_d)\sigma(\overline{x}, x_d)\, \mathrm{d}x_d v_{I_d}(\overline{x})^{-1} \right)^q v_{I_d}(\overline{x})^{\frac{q}{p}} \right\}^{\frac{p}{q}} \mathrm{d}\overline{x} \\
&= \int_{\mathbf{R}^{d-1}} \left\{ \sum_{I_d \in \mathcal{DQ}(\mathbf{R})} v_{\overline{x}}(I_d)^{\frac{q}{p}} \left(\frac{1}{v_{\overline{x}}(I_d)} \int_{I_d} f(\overline{x}, x_d) v_{\overline{x}}(x_d)\, \mathrm{d}x_d \right)^q \right\}^{\frac{p}{q}} \mathrm{d}\overline{x}.
\end{aligned}$$

Since $v_{\overline{x}}(x_d)$ satisfies (2.2) (one-dimensional case), by Lemma 2.1

$$\text{(ii)} \le c_2^{\frac{p}{q}} \int_{\mathbf{R}^{d-1}} \int_{\mathbf{R}} f(\overline{x}, x_d)^p \sigma(\overline{x}, x_d)\, \mathrm{d}x_d\, \mathrm{d}\overline{x} = c_2^{\frac{p}{q}} \int_{\mathbf{R}^d} f^p\, \mathrm{d}\sigma.$$

Altogether, we obtain

$$\text{(i)} \le Cc_2^d \left(\int_{\mathbf{R}^d} f^p\, \mathrm{d}\sigma \right)^{\frac{q}{p}}.$$

This proves the lemma. □

3. Proof of Theorem 1.1

In what follows we shall prove Theorem 1.1. We first notice that, if σ is a reverse doubling weight on $\mathbf{R}^d$ with $\beta > 1$, then it satisfies the testing condition (2.6). Indeed, for the dyadic rectangles $R \in \mathcal{DR}(\mathbf{R}^d)$ and $j = 1, \ldots, d$, we have that

$$\begin{aligned}
\sum_{\substack{I \in \mathcal{DQ}(\mathbf{R}) \\ I \subset P_j(R)}} \sigma([R;\, I, j])^{\frac{q}{p}} &= \sum_{k=0}^{\infty} \sum_{\substack{I \in \mathcal{DQ}(\mathbf{R}) \\ I \subset P_j(R),\, |I| = 2^{-k}|P_j(R)|}} \sigma([R;\, I, j])^{\frac{q}{p}-1} \sigma([R;\, I, j]) \\
&\le \sum_{k=0}^{\infty} \left(\frac{1}{\beta^k} \right)^{\frac{q}{p}-1} \sigma(R)^{\frac{q}{p}-1} \sum_{\substack{I \in \mathcal{DQ}(\mathbf{R}) \\ I \subset P_j(R),\, |I| = 2^{-k}|P_j(R)|}} \sigma([R;\, I, j]) \\
&= \sigma(R)^{\frac{q}{p}} \sum_{k=0}^{\infty} \left(\frac{1}{\beta^k} \right)^{\frac{q}{p}-1} = C\sigma(R)^{\frac{q}{p}}.
\end{aligned}$$

The necessity of (1.3) follows at once if we substitute the test functions $f_i = 1_R$, $i = 1, \ldots, n$, into inequality (1.2). To show that inequality (1.3) is sufficient, we take

$q_i > p_i$, $i = 1, \ldots, n$, with $\sum_{i=1}^n \frac{1}{q_i} = 1$. This is possible because $\sum_{i=1}^n \frac{1}{p_i} > 1$. It follows from testing condition (1.3) and Hölder's inequality that

$$\begin{aligned}\sum_{R\in\mathcal{DR}(\mathbf{R}^d)} K(R)\prod_{i=1}^n \left|\int_R f_i\,\mathrm{d}\sigma_i\right| &\le c_2 \sum_{R\in\mathcal{DR}(\mathbf{R}^d)} \prod_{i=1}^n \sigma_i(R)^{\frac{1}{p_i}}\left(\frac{1}{\sigma_i(R)}\int_R |f_i|\,\mathrm{d}\sigma_i\right)\\ &\le c_2\prod_{i=1}^n\left(\sum_{R\in\mathcal{DR}(\mathbf{R}^d)} \sigma_i(R)^{\frac{q_i}{p_i}}\left(\frac{1}{\sigma_i(R)}\int_R |f_i|\,\mathrm{d}\sigma_i\right)^{q_i}\right)^{\frac{1}{q_i}}\\ &\le Cc_2\prod_{i=1}^n \|f_i\|_{L^{p_i}(\sigma_i)},\end{aligned}$$

where we have used Lemma 2.2 by noticing that every σ_i satisfies the testing condition (2.6). This completes the proof. □

4. Proof of Corollary 1.2

In what follows we shall prove Corollary 1.2. The necessity of (1.5) follows at once if we substitute the test functions $f_i = 1_R\sigma_i$, $i = 1, \ldots, n$, into inequality (1.4). To show that inequality (1.5) is sufficient, we notice that the condition

$$\sum_{i=1}^n \frac{1}{p_i} > \frac{1}{q}$$

leads to the condition

$$\frac{1}{q'} + \sum_{i=1}^n \frac{1}{p_i} > 1.$$

By Theorem 1.1 we have that the inequality

$$\sum_{R\in\mathcal{DR}(\mathbf{R}^d)} K(R)\int_R g\,\mathrm{d}\omega \prod_{i=1}^n \int_R f_i\,\mathrm{d}\sigma_i \le C\|g\|_{L^{q'}(\omega)}\prod_{i=1}^n \|f_i\|_{L^{p_i}(\sigma_i)} \tag{4.1}$$

holds for all nonnegative functions $g \in L^{q'}(\omega)$ and $f_i \in L^{p_i}(\sigma_i)$, provided that the testing condition

$$K(R)\omega(R)\prod_{i=1}^n \sigma_i(R) \le C\omega(R)^{\frac{1}{q'}}\prod_{i=1}^n \sigma_i(R)^{\frac{1}{p_i}} \tag{4.2}$$

holds for all dyadic rectangles $R \in \mathcal{DR}(\mathbf{R}^d)$.

Since (4.2) is equivalent to our assumption (1.5), the inequality (4.1) is proper. Rewrite $f_i\sigma_i = h_i$ in (4.1), then

$$\sum_{R\in\mathcal{DR}(\mathbf{R}^d)} K(R)\int_R g\,\mathrm{d}\omega \prod_{i=1}^n \int_R h_i\,\mathrm{d}x \le Cc_2\|g\|_{L^{q'}(\omega)}\prod_{i=1}^n \|h_i\|_{L^{p_i}(\sigma_i^{1-p_i})}.$$

This means that

$$\int_{\mathbf{R}^d} gT_K(h_1, \ldots, h_n)\,\mathrm{d}\omega \le Cc_2\|g\|_{L^{q'}(\omega)}\prod_{i=1}^n \|h_i\|_{L^{p_i}(\sigma_i^{1-p_i})}$$

and, by duality,

$$\|T_K(h_1, \ldots, h_n)\|_{L^q(\omega)} \le Cc_2\prod_{i=1}^n \|h_i\|_{L^{p_i}(\sigma_i^{1-p_i})},$$

which yields the proof. □

5. Applications

In what follows we give some applications to multilinear strong fractional integral operator.

For a number $c > 0$ and a rectangle $R \in \mathcal{R}$, we will use cR to denote the rectangle with the same center as R but with c times the side-lengths of R. Let f_i, $i = 1, \dots, n$, be locally integrable functions on R^d. Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_d)$ with $0 < \alpha_j < n$. We define the following "multilinear strong fractional integral operator" $\widetilde{I}_{\vec{\alpha}}(f_1, \dots, f_n)(x)$, $x \in \mathbf{R}^d$, by

$$\widetilde{I}_{\vec{\alpha}}(f_1, \dots, f_n)(x) := \int_{y_1,\dots,y_n \in \mathbf{R}^d} \frac{f_1(y_1) \cdots f_n(y_n) \, \mathrm{d}y_1 \cdots \mathrm{d}y_n}{\Big(\prod_{j=1}^d \max_{i=1}^n |P_j(x) - P_j(y_i)|\Big)^{n-\alpha_j}},$$

where $P_j(x)$, $j = 1, \dots, d$, is the projection on the x_j-axis of the point $x \in \mathbf{R}^d$. This is different from the multilinear fractional integral operator defined in [10].

Remark. We examine the case $n = 1$. For $\vec{\alpha} = (\alpha_1, \dots, \alpha_d)$ with $0 < \alpha_j < 1$, $x = (x_1, \dots, x_d) \in \mathbf{R}^d$, $u = (u_1, \dots, u_d) \in \mathbf{R}^d$ and $f\colon \mathbf{R}^d \to \mathbf{R}$, we have

$$\widetilde{I}_{\vec{\alpha}}(f)(x) = \int_{\mathbf{R}^d} \frac{f(u) \, \mathrm{d}u}{\prod_{j=1}^d |x_j - u_j|^{1-\alpha_j}}.$$

This operator with product type kernel was treated on grand Lebesgue spaces by Kokilashvili and Meskhi [6], where $\alpha_1 = \dots = \alpha_d$ and $\mathbf{R}^d$ is replaced by a bounded rectangle. We are grateful to Eiichi Nakai for this information.

When $n = d = 1$, $\widetilde{I}_{\vec{\alpha}}$ is just the usual one-dimensional fractional integral operator.

We observe that, for $s, t \in \mathbf{R}$ with $s \neq t$, the minimal dyadic interval $I \in \mathcal{DQ}(\mathbf{R})$ such that $I \ni s$ and $3I \ni t$ satisfies

$$\frac{|I|}{2} < |s - t| < 2|I|.$$

This observation and a calculus of geometric series enable us that, for any $y_1, \dots, y_n \neq x$,

$$\sum_{R \in \mathcal{DR}(\mathbf{R}^d)} \prod_{j=1}^d |P_j(R)|^{\alpha_j - n} 1_R(x) \prod_{i=1}^n 1_{3R}(y_i) \approx \left(\prod_{j=1}^d \max_{i=1}^n |P_j(x) - P_j(y_i)|\right)^{\alpha_j - n}.$$

This equation and Fubini's theorem yield the precise point-wise relation

$$\widetilde{I}_{\vec{\alpha}}(f_1, \dots, f_n)(x) \approx \sum_{R \in \mathcal{DR}(\mathbf{R}^d)} \prod_{j=1}^d |P_j(R)|^{\alpha_j - n} 1_R(x) \prod_{i=1}^n \int_{3R} f_i(y_i) \, \mathrm{d}y_i, \quad x \in \mathbf{R}^d. \tag{5.1}$$

Since the right-hand of (5.1) can be controlled by the estimate based upon the finite number of the systems of dyadic rectangles (see, for example, [9]), by Corollary 1.2, we have the following.

Proposition 5.1. *Let $1 < p_i < \infty$, $i = 1, \dots, n$, and $1 < q < \infty$ with $\sum_{i=1}^n \frac{1}{p_i} > \frac{1}{q}$. Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_d)$ with $0 < \alpha_j < n$ and let σ_i, $i = 1, \dots, n$, and ω be reverse doubling weights on $\mathbf{R}^d$. The following statements are equivalent:*

(a) *The weighted norm inequality for multilinear strong fractional integral operator*

$$\|\widetilde{I}_{\vec{\alpha}}(f_1,\dots,f_n)\|_{L^q(\omega)} \le c_1 \prod_{i=1}^n \|f_i\|_{L^{p_i}(\sigma_i^{1-p_i})}$$

holds for all $f_i \in L^{p_i}(\sigma_i^{1-p_i})$, $i=1,\dots,n$;

(b) *The testing condition*

$$\prod_{j=1}^d |P_j(R)|^{\alpha_j-n}\omega(R)^{\frac{1}{q}} \prod_{i=1}^n \sigma_i(R) \le c_2 \prod_{i=1}^n \sigma_i(R)^{\frac{1}{p_i}}$$

holds for all rectangles $R \in \mathcal{R}(\mathbf{R}^d)$.

Moreover, the least possible constants c_1 *and* c_2 *are equivalent.*

Letting $\omega \equiv \sigma_1 \equiv \cdots \equiv \sigma_n \equiv 1$ and $\vec{\alpha} = (\alpha/d,\dots,\alpha/d)$, we have the following Hardy–Littlewood–Sobolev inequality for strong fractional integral operator.

Proposition 5.2. *Let* $1<q<\infty$, $1<p_i<\infty$, $0<\alpha<dn$ *and*

$$\frac{1}{q} = \sum_{i=1}^n \frac{1}{p_i} - \frac{\alpha}{d}.$$

Then the multilinear norm inequality

$$\|\widetilde{I}_{\vec{\alpha}}(f_1,\dots,f_n)\|_{L^q(\mathbf{R}^d)} \le C \prod_{i=1}^n \|f_i\|_{L^{p_i}(\mathbf{R}^d)}$$

holds for all $f_i \in L^{p_i}(\mathbf{R}^d)$, $i=1,\dots,n$.

Remark. About the weighted estimate for multilinear fractional integral operators see, for example, [10]. We could not find any literature about the Hardy–Littlewood–Sobolev inequality for our strong fractional integral operators.

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