Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 43, 2018, 991–1021

THE EXISTENCE AND CONCENTRATION BEHAVIOR OF GROUND STATE SOLUTIONS FOR FRACTIONAL SCHRÖDINGER–KIRCHHOFF TYPE EQUATIONS

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Abstract. In this paper, we study the following fractional Schrödinger Kirchhoff type problem

$$(Q_\epsilon) \qquad \begin{cases} \mathfrak{L}_\epsilon^s u = K(x)f(u) & \text{in } \mathbf{R}^3, \\ u \in H^s(\mathbf{R}^3), \end{cases}$$

where $\mathfrak{L}_\epsilon^s$ is a nonlocal operator defined by

$$\mathfrak{L}_\epsilon^s u = M\left(\frac{1}{\epsilon^{3-2s}} \iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \frac{1}{\epsilon^3}\int_{\mathbf{R}^3} V(x)u^2\,dx\right)[\epsilon^{2s}(-\Delta)^s u + V(x)u],$$

ϵ is a small positive parameter, $\frac{3}{4} < s < 1$ is a fixed constant, the operator $(-\Delta)^s$ is the fractional Laplacian of order s, M, V, K and f are continuous functions. Under proper assumptions on M, V, K and f, we prove the existence and concentration phenomena of solutions of the problem (Q_ϵ). With minimax theorems and the Ljusternik–Schnirelmann theory, we also obtain multiple solutions of problem (Q_ϵ) by employing the topology of the set where the potentials $V(x)$ attains its minimum and $K(x)$ attains its maximum.

1. Introduction and main results

In this paper, we consider the following fractional Schrödinger Kirchhoff type problem

$$(Q_\epsilon) \qquad \begin{cases} \mathfrak{L}_\epsilon^s u = K(x)f(u) & \text{in } \mathbf{R}^3, \\ u \in H^s(\mathbf{R}^3), \end{cases}$$

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ϵ is a small positive parameter, $\frac{3}{4} < s < 1$ is a fixed constant, the operator $(-\Delta)^s$ is the fractional Laplacian of order s, which can be defined by the Fourier transform $(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u)$. $M\colon [0,\infty) \to \mathbf{R}^+$, $f\colon \mathbf{R} \to \mathbf{R}$, $V\colon \mathbf{R}^3 \to \mathbf{R}$ and $K\colon \mathbf{R}^3 \to$

https://doi.org/10.5186/aasfm.2018.4361
2010 Mathematics Subject Classification: Primary 35J20, 35J25, 35J60.
Key words: Schrödinger–Kirchhoff type equations, concentration behavior, ground state solution, variational methods.
This work was supported by Natural Science Foundation of China (Grant No. 11771166), Hubei Key Laboratory of Mathematical Sciences and Program for Changjiang Scholars and Innovative Research Team in University #IRT_17R46.
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$\mathbf{R}$ are given continuous functions satisfying (M_1)–(M_5), (f_1)–(f_4) and (P_0)–(P_2) given below. The feature of (Q_ϵ) is that the term

$$M\left(\frac{1}{\epsilon^{3-2s}}\iint_{\mathbf{R}^3\times\mathbf{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy+\frac{1}{\epsilon^3}\int_{\mathbf{R}^3}V(x)u^2\,dx\right)$$

makes (Q_ϵ) a nonlocal problem.

First, we collect some useful results about the fractional order Sobolev spaces (see [7, 20, 24]). For any $0<s<1$, the fractional Sobolev space $H^s(\mathbf{R}^3)$ is defined by

$$H^s(\mathbf{R}^3)=\left\{u\in L^2(\mathbf{R}^3)\colon \frac{|u(x)-u(y)|}{|x-y|^{\frac{3+2s}{2}}}\in L^2(\mathbf{R}^3\times\mathbf{R}^3)\right\} \tag{1.1}$$

endowed with the norm

$$\|u\|_{H^s(\mathbf{R}^3)}=\left(\int_{\mathbf{R}^3}|u|^2dx+\iint_{\mathbf{R}^3\times\mathbf{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy\right)^{\frac{1}{2}}, \tag{1.2}$$

where the term

$$[u]_s=[u]_{H^s(\mathbf{R}^3)}=\left(\iint_{\mathbf{R}^3\times\mathbf{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy\right)^{\frac{1}{2}} \tag{1.3}$$

is the so-called Gagliardo seminorm of u. Also, in light of [[7], Proposition 3.4 and Proposition 3.6], we have

$$\|(-\Delta)^{\frac{s}{2}}u\|_2^2=\int_{\mathbf{R}^3}|\xi|^{2s}|\hat{u}(\xi)|^2\,d\xi=\frac{1}{2}C(s)\iint_{\mathbf{R}^3\times\mathbf{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy, \tag{1.4}$$

where $\hat{u}$ stands for the Fourier transform of u and

$$C(s)=\left(\int_{\mathbf{R}^3}\frac{1-cos\zeta_1}{|\zeta|^{3+2s}}\,d\zeta\right)^{-1},\quad \zeta=(\zeta_1,\zeta_2,\zeta_3). \tag{1.5}$$

In particular, if $s=1$, then

$$\|\nabla u\|_2^2=\|(-\Delta)^{\frac{1}{2}}u\|_2^2=\frac{1}{2}C(1)\iint_{\mathbf{R}^3\times\mathbf{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^5}\,dx\,dy.$$

Moreover, $(-\Delta)^s u$ can be equivalently represented as (see [[7], Lemma 3.2])

$$(-\Delta)^s u(x)=-\frac{C(s)}{2}\int_{\mathbf{R}^3}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{3+2s}}\,dy,\quad \forall\, x\in\mathbf{R}^3. \tag{1.6}$$

We denote $\|\cdot\|_{H^s}$ by $\|\cdot\|$ in the sequel for convenience.

If $a>0$, $b>0$, $M(t)=a+\frac{bC(1)}{2}t$ where C(1) is given in (1.5), $\epsilon=1$, $s=1$ and $V(x)=0$, then (1.4) shows that (Q_ϵ) with $\mathbf{R}^3$ replaced by Ω reduces to

$$\begin{cases}-(a+b\int_\Omega|\nabla u|^2)\Delta u=f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases} \tag{1.7}$$

where $\Omega\subset\mathbf{R}^3$ is a bounded domain. In recent years, a great interest has been devoted to problem (1.7), which is related to the stationary analogue of the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2\right)\frac{\partial^2 u}{\partial x^2}=0,$$

proposed by Kirchhoff in [13] as a model of the classical D'Alembert's wave equations for free vibration of elastic strings. In [17], Lions introduced an abstract functional analysis framework to the following equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2\,dx\right)\Delta u = f(x,u). \tag{1.8}$$

After that, (1.8) received much attention, see [1, 2, 3, 5, 6] and the references therein.

Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. In (1.7), u denotes the displacement, $f(x,u)$ the external force and b the initial tension while a is related to the intrinsic properties of the string, such as Young's modulus. We have to point out that such nonlocal problems also appear in other fields as biological systems, where u describes a process which depends on the average of itself, for example, population density. For more mathematical and physical background of the problem (1.7), we refer the readers to the papers [1, 3, 5, 10, 12, 13, 15] and the references therein.

When $M \equiv 1$, (Q_ϵ) becomes the fractional Schrödinger equation

$$\begin{cases} \epsilon^{2s}(-\Delta)^s u + V(x)u = f(x,u) & \text{in } \mathbf{R}^3, \\ u \in H^s(\mathbf{R}^3). \end{cases} \tag{1.9}$$

Solutions of the problem (1.9) are related to the existence of standing wave solutions of the form $\psi(x,t) = e^{\frac{-iEt}{\epsilon}}u(x)$ to the following fractional Schrödinger equation of the form

$$i\epsilon\frac{\partial\psi}{\partial t} = \epsilon^{2s}(-\Delta)^s\psi + V(x)\psi - f(x,|\psi|), \quad \forall\, x \in \mathbf{R}^3,$$

where E is a constant and $u(x)$ is a solution of the problem (1.9). The fractional Schrödinger equation is a fundamental equation in fractional quantum mechanics. It was introduced by Laskin [14] as a fundamental equation of fractional quantum mechanics in the study of particles on the stochastic fields modeled by Levy process. We refer to [7] for more physical background.

Now, we give some hypotheses about M, V, K and f.

- (P_0) V, $K \in L^\infty(\mathbf{R}^3)$ are uniformly continuous on $\mathbf{R}^3$ and there exist $\tilde{x} \in \mathbf{R}^3$, $\hat{x} \in \mathbf{R}^3$ such that $V(\tilde{x}) = V_{\min} = \min_{x\in\mathbf{R}^3} V > 0$, $K(\hat{x}) = K_{\max} = \max_{x\in\mathbf{R}^3} K > 0$ and $K_{\inf} = \inf_{x\in\mathbf{R}^3} K > 0$;

Set

$$\mathcal{V} = \{x \in \mathbf{R}^3 \colon V(x) = V_{\min} = \min_{x\in\mathbf{R}^3} V\}, \quad V_\infty = \liminf_{|x|\to\infty} V(x),$$

$$\mathcal{K} = \{x \in \mathbf{R}^3 \colon K(x) = K_{\max} = \max_{x\in\mathbf{R}^3} K\}, \quad K_\infty = \limsup_{|x|\to\infty} K(x).$$

- (P_1) $V_{\min} < V_\infty < +\infty$ and there exists an $x_1 \in \mathcal{V}$ such that $K(x_1) \geq K(x)$ for $|x| \geq R$ with $R > 0$ sufficiently large;
- (P_2) $K_{\max} > K_\infty \geq \inf K > 0$ and there exists an $x_2 \in \mathcal{K}$ such that $V(x_2) \leq V(x)$ for $|x| \geq R$ with $R > 0$ sufficiently large;
- (P_3) $\mathcal{V} \cap \mathcal{K} \neq \emptyset$;

- (P_4) We assume that V and K are $\mathbf{Z}^3$-periodic functions, that is, $V(x+y) = V(x)$ and $K(x+y) = K(x)$ for all $x \in \mathbf{R}^3$ and for all $y \in \mathbf{Z}^3$.

Obviously, if (P_1) holds, we can assume $K(x_1) = \max_{x\in\mathcal{V}} K(x)$, and set

$$\mathcal{H}_1 = \{x \in \mathcal{V} \colon K(x) = K(x_1)\} \cup \{x \notin \mathcal{V} \colon K(x) > K(x_1)\}.$$

If (P_2) holds, we can assume $V(x_2)=\min_{x\in\mathcal{K}}V(x)$, and set

$$\mathcal{H}_2=\{x\in\mathcal{K}\colon V(x)=V(x_2)\}\cup\{x\notin\mathcal{K}\colon V(x)<V(x_2)\}.$$

Clearly, $\mathcal{H}_1$ and $\mathcal{H}_2$ are bounded sets. Moreover, if $\mathcal{V}\cap\mathcal{K}\neq\emptyset$, then $\mathcal{H}_1=\mathcal{H}_2=\mathcal{V}\cap\mathcal{K}$.

(V_1) $V\in C(\mathbf{R}^3,\mathbf{R})$ and $\inf\limits_{x\in\mathbf{R}^3}V(x)=V_0>0$;

(V_2) For each $\delta>0$ there is an open and bounded set $\Lambda=\Lambda(\delta)\subset\mathbf{R}^3$ depending on δ such that

$$V_0:=\inf_{x\in\mathbf{R}^3}V(x)<\min_{z\in\partial\Lambda}V(z),\quad \Pi=\{x\in\Lambda\colon V(x)=V_0\}\neq\emptyset,$$

and

$$\Pi_\delta=\{x\in\mathbf{R}^N\colon \operatorname{dist}(x,\Pi)\le\delta\}\subset\Lambda.$$

(M_1) $M\in C(\mathbf{R}_0^+,\mathbf{R}^+)$ and $\inf\limits_{t\in\mathbf{R}_0{}^+}M(t)\ge a>0$, where $a>0$ is a constant;

(M_2) The function $t\to M(t)$ is increasing on $[0,+\infty)$;

(M_3) The function $t\to\frac{M(t)}{t}$ is nonincreasing in $(0,+\infty)$;

(M_4) There exists a $\theta\in(4,2_s^*)$ such that

$$\lim_{t\to\infty}\frac{1}{t}\left[\frac{1}{2}\widehat{M}(t^2)-\frac{1}{\theta}M(t^2)t^2\right]=+\infty,$$

where $\widehat{M}(t)=\int_0^t M(s)\,ds$, $2_s^*=\frac{6}{3-2s}$;

(M_5) The function $t\to\frac{1}{2}\widehat{M}(t^2)-\frac{1}{4}M(t^2)t^2$ is increasing on $[0,+\infty)$;

$(\widehat{M}_3)$ For all $t_1\ge t_2>0$,

$$\frac{M(t_1)}{t_1}-\frac{M(t_2)}{t_2}\le a\left(\frac{1}{t_1}-\frac{1}{t_2}\right).$$

From (M_1)–(M_3), there exists an $r>0$ such that

$$M(t)\le r(1+t),\quad \forall t\ge 0. \tag{1.10}$$

A typical example of a function satisfying the conditions (M_1)-(M_5) is given by $M(t)=a+bt$, where $a>0$ and $b>0$. In this case, (Q_ϵ) becomes the standard Kirchhoff equation.

$(\widehat{f}_1)$ $\lim\limits_{t\to0^+}\dfrac{f(t)}{t^3}=0$;

$(\widehat{f}_2)$ There is a $p\in(4,6)$ such that

$$\lim_{t\to\infty}\frac{f(t)}{t^{p-1}}=0;$$

$(\widehat{f}_3)$ There is a $\vartheta\in(4,6)$ such that

$$0<\vartheta F(t)\le f(t)t,\quad \forall t>0;$$

(f_1) $f(t)=o(t)$ as $t\to0$, $f(t)t>0$ for all $t\neq0$ and $f(t)=0$ for all $t\le0$;

(f_2) There exist constants σ, $q\in(4,2_s^*)$, $C_0>0$ such that

$$f(t)\ge C_0t^{\sigma-1},\quad \text{for all } t\ge0,\quad \text{and } \lim_{t\to\infty}\frac{f(t)}{t^{q-1}}=0, \tag{1.11}$$

where $2_s^*=\frac{6}{3-2s}$;

(f_3)

$$0<\theta F(t)\le f(t)t,\quad \forall t>0, \tag{1.12}$$

where $F(t)=\int_0^t f(s)\,ds$, θ is given by (M_4);

(f_4) The application

$$t \to \frac{f(t)}{t^3}$$

is increasing in $(0,\infty)$.

A typical example of a function satisfying the conditions (f_1)–(f_4) is

$$f(t) = \sum_{i=1}^{N} C_i (t^+)^{q_i - 1}$$

where $4 < q_i < 2_s^* = \frac{6}{3-2s}$ and $C_i \geq 0$ for all i with $1 \leq i \leq N$ and $C_i > 0$ for at least one i. The hypotheses (P_0)–(P_2) appeared in [11] and [24], (V_1)–(V_2), $(\widehat{M}_3)$, $(\widehat{f}_1)$–$(\widehat{f}_3)$ and (f_4) appeared in [9] and (M_1)–(M_5) were given in [2].

Recently, the following Kirchhoff type equation

$$\begin{cases} -\Big(a + b\int_{\mathbf{R}^3} |\nabla u|^2 dx\Big)\Delta u + u = f(x,u) & \text{in } \mathbf{R}^3, \\ u \in H^1(\mathbf{R}^3) \end{cases} \tag{1.13}$$

has been studied extensively by many researchers where $f \in C(\mathbf{R}^3 \times \mathbf{R}, \mathbf{R})$, $a, b > 0$ are constants.

He and Zou in [10] studied (1.13) under the conditions that $f(x,u) := f(u) \in C^1(\mathbf{R}^+, \mathbf{R}^+)$ satisfies the Ambrosetti–Rabinowitz condition ((AR) condition in short):

$$\exists \mu > 4, \quad 0 < \mu \int_0^u f(s)\,ds \leq f(u)u,$$

$\lim_{|u|\to 0} f(u)/|u|^3 = 0$, $\lim_{|u|\to\infty} f(u)/|u|^q = 0$ for some $3 < q < 5$ and $f(u)/u^3$ is strictly increasing for $u > 0$, that is, $f(u)$ behaves like $|u|^{p-2}u$ $(4 < p < 6)$. They showed that the Mountain Pass Theorem and the Nehari manifold can be used directly to obtain a positive ground state solution to (1.13).

Similarly, Wang et al. in [22], He, Li and Peng in [12] and Li and Ye in [16] used the same arguments as in [10] to prove the existence of a positive ground state solution for (1.13) when $f(x,u) := \lambda f(u) + |u|^4 u$, which exhibits a critical growth, where $\lim_{|u|\to 0} f(u)/|u|^3 = 0$, $f(u)u \geq 0$, $f(u)/u^3$ is strictly increasing for $u > 0$ and $|f(u)| \leq C(1+|u|^q)$ for some $3 < q < 5$, that is, $f(x,u) \sim \lambda|u|^{p-2}u + |u|^4 u (4 < p < 6)$. For the case $f(x,u) = |u|^{p-2}u$ $(3 < p \leq 4)$, Li and Ye in [15] used the constrained minimization on a new manifold which is obtained by combining the Nehari manifold and the corresponding Pohozaev's identity to get a positive ground state solution to (1.13).

Recently, He and Li in [11] studied the problem

$$\begin{cases} -\epsilon^2 \Delta u + V(x)u - \epsilon^2 \Delta(u^2)u = K(x)u^{q-1} + u^{2\cdot 2^* - 1}, & x \in \mathbf{R}^N, \\ u > 0, & x \in \mathbf{R}^N, \end{cases} \tag{1.14}$$

where $\epsilon > 0$ is a small positive parameter, $N \geq 3$, $2^* = \frac{2N}{N-2}$, $4 < q < 2 \cdot 2^*$, V and K are bounded locally Hölder continuous functions. Under the assumptions (P_0), (P_1) and (P_2), they proved the existence and concentration phenomena of soliton solutions of the problem (1.14). With minimax theorems and Ljusternik–Schnirelmann theory, they also obtained multiple soliton solutions by employing the topology of the set where the potentials $V(x)$ attains its minimum and $K(x)$ attains its maximum.

Some authors studied problems of the type (Q_ϵ). For example, in [9], Figueiredo and Santos proved a result of multiplicity and concentration behavior of positive solutions of the following problem

$$(1.15)\qquad \begin{cases} \mathfrak{A}_\epsilon u = f(u), & x \in \mathbf{R}^3, \\ u > 0, & x \in \mathbf{R}^3, \\ u \in H^1(\mathbf{R}^3), \end{cases}$$

where ϵ is a small positive parameter, $\mathfrak{A}_\epsilon$ is a nonlocal operator defined by

$$\mathfrak{A}_\epsilon u = M\left(\frac{1}{\epsilon}\int_{\mathbf{R}^3} |\nabla u|^2\,dx + \frac{1}{\epsilon^3}\int_{\mathbf{R}^3} V(x)u^2\,dx\right)[-\epsilon^2\Delta u + V(x)u],$$

the potential V satisfies (V_1)–(V_2), the function $M\colon \mathbf{R}^+ \to \mathbf{R}^+$ satisfies (M_1), (M_2), $(\widehat{M_3})$, and the nonlinearity $f \in C(\mathbf{R}^3, \mathbf{R})$ satisfies $(\widehat{f_1})$–$(\widehat{f_3})$ and (f_4).

In [24], Yu, Zhao and Zhao studied the following fractional Schrödinger–Poisson system

$$(1.16)\qquad \begin{cases} \epsilon^{2s}(-\Delta)^s u + V(x)u + \phi u = K(x)u^{p-2}u, & x \in \mathbf{R}^3, \\ \epsilon^{2s}(-\Delta)^s \phi = u^2, & x \in \mathbf{R}^3, \end{cases}$$

where $\epsilon > 0$ is a small parameter, $\frac{3}{4} < s < 1$, $4 < p < 2_s^* = \frac{6}{3-2s}$, $V(x) \in C(\mathbf{R}^3) \cap L^\infty(\mathbf{R}^3)$ has a positive global minimum, and $K(x) \in C(\mathbf{R}^3) \cap L^\infty(\mathbf{R}^3)$ is positive and has a global maximum. Under the assumptions (P_0), (P_1) and (P_2), they proved the existence of a positive ground state solution by using variational methods for each $\epsilon > 0$ sufficiently small, and they determined a concrete set related to the potentials V and K as the concentration position of these ground state solutions as $\epsilon \to 0$. Moreover, they considered some properties of these ground state solutions, such as convergence and decay estimate.

Motivated by the works described above, particularly, by the results in [9, 11, 24], we will study the existence and concentration phenomena of positive ground state solutions of the problem (Q_ϵ) via variational methods in this paper. Moreover, we will prove the existence of multiple solutions to problem (Q_ϵ) by using the Ljusternik–Schnirelmann theory.

We define

$$E = \left\{ u \in H^s(\mathbf{R}^3) \mid \int_{\mathbf{R}^3} V(x)u^2\,dx < \infty \right\}$$

with the norm

$$\|u\|_E = \left(\iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbf{R}^3} V(x)|u|^2\,dx \right)^{\frac{1}{2}}.$$

It is easy to see that $(E, \|\cdot\|_E)$ is a real Banach space.

We call $u \in E$ a weak solution to (Q_ϵ) if for any $\varphi \in E$, it holds that

$$M\left(\frac{1}{\epsilon^{3-2s}}[u]_s^2 + \frac{1}{\epsilon^3}\int_{\mathbf{R}^3} V(x)u^2\,dx\right)\left[\epsilon^{2s}\iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{3+2s}}\,dx\,dy \right.$$
$$\left. + \int_{\mathbf{R}^3} V(x)u\varphi\,dx\right] = \int_{\mathbf{R}^3} K(x)f(u)\varphi\,dx.$$

We recall that, if Y is a closed set of a topological space X, $cat_X(Y)$ is the Ljusternik–Schnirelmann category of Y in X, namely the least number of closed and contractible sets in X which cover Y.

For $I \in C^1(E, \mathbf{R})$, we say that $\{u_n\} \subset E$ is a Palais–Smale (PS) sequence at level c (henceforth denoted $(PS)_c$) for I if $\{u_n\}$ satisfies

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as } n \to \infty.$$

Moreover, I satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence possesses a convergent subsequence.

Our main results are as follows.

Theorem 1.1. *Assume that* (P_0), (M_1)–(M_5) *and* (f_1)–(f_4) *hold.*

(A) *Suppose* (P_1) *holds, then for all small* $\epsilon > 0$.

  (i) *The problem* (Q_ϵ) *has a positive ground state solution* u_ϵ*;*

  (ii) *The solution* u_ϵ *obtained in (i) possesses a global maximum point* x_ϵ *such that, up to a subsequence,* $x_\epsilon \to x_0$ *as* $\epsilon \to 0$, $\lim_{\epsilon\to 0} dist(x_\epsilon, \mathcal{H}_1) = 0$, *and* $v_\epsilon(x) = u_\epsilon(\epsilon x + x_\epsilon)$ *converges in* $H^s(\mathbf{R}^3)$ *to a positive ground state solution of*

$$M\Big(\iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbf{R}^3} V(x_0)u^2\,dx\Big)[(-\Delta)^s u + V(x_0)u] = K(x_0)f(u). \tag{1.17}$$

In particular, if $\mathcal{V}\cap\mathcal{K} \neq \emptyset$, *then* $\lim_{\epsilon\to 0} dist(x_\epsilon, \mathcal{V}\cap\mathcal{K}) = 0$, *and up to a subsequence,* v_ϵ *converges in* $H^s(\mathbf{R}^3)$ *to a positive ground state solution of*

$$M\left(\iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbf{R}^3} V_{\min}u^2\,dx\right)[(-\Delta)^s u + V_{\min}u] = K_{\max}f(u); \tag{1.18}$$

  (iii) *There exists a constant* $C > 0$ *such that*

$$u_\epsilon(x) \le \frac{C\epsilon^{3+2s}}{\epsilon^{3+2s} + |x-x_\epsilon|^{3+2s}}, \quad \forall\, x \in \mathbf{R}^3. \tag{1.19}$$

(B) *Suppose* (P_2) *holds and replace* $(\mathcal{H}_1)$ *by* $(\mathcal{H}_2)$, *then all the conclusions of* (A) *remain true.*

We denote by

$$(\mathcal{V}\cap\mathcal{K})_\delta = \{x \in \mathbf{R}^3\colon\ \mathrm{dist}(x, \mathcal{V}\cap\mathcal{K}) \le \delta\}$$

the closed δ-neighborhood of $\mathcal{V}\cap\mathcal{K}$, and we have the following multiplicity result:

Theorem 1.2. *Suppose that* (P_0), (P_1) *(or* (P_2)*),* (P_3)–(P_4), (M_1)–(M_5) *and* (f_1)–(f_4) *hold. Then, for any given* $\delta > 0$, *there exists a* $\epsilon_\delta > 0$ *such that, for any* $\epsilon \in (0, \epsilon_\delta)$*:*

 (i) *The problem* (Q_ϵ) *has at least* $cat_{(\mathcal{V}\cap\mathcal{K})_\delta}(\mathcal{V}\cap\mathcal{K})$ *solutions;*

 (ii) *If* u_ϵ *denotes one of these solutions, then* u_ϵ *possesses a global maximum point* x_ϵ *such that, up to a subsequence,* $x_\epsilon \to x_0$ *as* $\epsilon \to 0$, $\lim_{\epsilon\to 0} dist(x_\epsilon, \mathcal{V}\cap\mathcal{K}) = 0$, *and* $v_\epsilon(x) = u_\epsilon(\epsilon x + x_\epsilon)$ *converges in* $H^s(\mathbf{R}^3)$ *to a positive ground state solution of*

$$M\left(\iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbf{R}^3} V_{\min}u^2\,dx\right)[(-\Delta)^s u + V_{\min}u] = K_{\max}f(u); \tag{1.20}$$

(iii) *There exists a constant $C > 0$ such that*

$$u_\epsilon(x) \le \frac{C\epsilon^{3+2s}}{\epsilon^{3+2s} + |x - x_\epsilon|^{3+2s}}, \quad \forall\, x \in \mathbf{R}^3. \tag{1.21}$$

Remark 1.3. Theorem 1.1 generalizes the main result in [24], Theorem 1.2 generalizes the main result in [11], respectively. Here, we explain why the condition $s \in (\frac{3}{4}, 1)$ was imposed. We use the Mountain Pass Theorem to get a critical point of the energy functional $I_\epsilon(u)$ of the problem $(\widetilde{Q}_\epsilon)$. To guarantee that I_ϵ possesses the mountain-pass geometry, the condition (f_2) is required. If $I_\epsilon(u)$ possesses the mountain-pass geometry, we can get a $(PS)_c$ sequence of $I_\epsilon(u)$. To prove that the $(PS)_c$ sequence is bounded, the condition (f_3) ((AR) condition) and (M_4) are required. The conditions (M_4), (f_2) and (f_3) hold only when $s \in (\frac{3}{4}, 1)$. In fact, we can easily see from (M_4), (f_2) and (f_3) that $4 < \sigma, \theta < 2_s^* = \frac{6}{3-2s}$, that is $4 < 2_s^* = \frac{6}{3-2s}$, so we get $s > \frac{3}{4}$, note that $0 < s < 1$, thus $s \in (\frac{3}{4}, 1)$.

The proof of our main results is based on variational methods. We would like to emphasize that the main difficulties are the appearance of the non-local term involving the function M and the lack of compactness due to the unboundedness of the domain $\mathbf{R}^3$. As M is a more general function than those in [10], [12] and [15], we have an additional difficulty: to verify that the weak limit of a Palais–Smale sequence is a weak solution of the related autonomous problem. Our assumptions, which are similar to those given in [2], and careful analysis make us possible to overcome this difficulty. Moreover, as the function f is only continuous, we cannot use standard arguments on the Nehari manifold as in the papers [11, 24]. To overcome this difficulty, we use some variants of critical point theorems of Szulkin and Weth [21]. As we will see later, the competing effect of the nonlocal term with the nonlinearity $f(u)$ and the lack of compactness of the Sobolev's embedding prevent us from using the variational methods in a standard way. Finally, as in [24], there is a competition between the potentials V and K: each would try to attract ground states to their minimum and maximum points, respectively. This causes difficulties in determining the concentration position of the solutions.

Now we sketch the proof of our main results. The problem (Q_ϵ) is equivalent to the following problem

$$(\widetilde{Q}_\epsilon) \qquad \begin{cases} M\left(\iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\, dx\, dy + \int_{\mathbf{R}^3} V(\epsilon x)u^2\, dx\right) [(-\Delta)^s u + V(\epsilon x)u] \\ = K(\epsilon x) f(u), \\ u \in H^s(\mathbf{R}^3) \end{cases}$$

by using the change of variable $v(x) = u(\epsilon x)$. The corresponding energy functional associated with problem $(\widetilde{Q}_\epsilon)$ is defined by

$$I_\epsilon(u) = \frac{1}{2}\widehat{M}(\|u\|_\epsilon^2) - \int_{\mathbf{R}^3} K(\epsilon x) F(u)\, dx \tag{1.22}$$

where $\widehat{M}(t) = \int_0^t M(s)\, ds$ and $F(t) = \int_0^t f(s)\, ds$, which are well defined on the space H_ϵ given by

$$H_\epsilon = \left\{ u \in H^s(\mathbf{R}^3) \colon \int_{\mathbf{R}^3} V(\epsilon x) u^2\, dx < \infty \right\}.$$

The norm of $u \in H_\epsilon$ is defined as $\|u\|_\epsilon = (\iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\, dx\, dy + \int_{\mathbf{R}^3} V(\epsilon x)|u|^2\, dx)^{\frac{1}{2}}$ and H_ϵ is a Banach space under the norm $\|\cdot\|_\epsilon$ given above. The energy functional I_ϵ

is well defined in the fractional Sobolev space $H^s(\mathbf{R}^3)$ and it possesses the Mountain Pass geometry, hence a bounded $(P.S.)_c$ sequence is obtained. First, we obtain a positive ground state solution of $(\widetilde{Q}_\epsilon)$ via variational method for each $\epsilon > 0$ small enough. Next, we establish the L^∞ and decay estimate of these solutions to study the concentration behavior of these solutions as $\epsilon \to 0$. Finally, we determine a concrete set related to the potentials V and K as the concentration position of these solutions.

The proof of Theorem 1.2 is mainly based on the Ljusternik–Schnirelmann theory. Using the technique introduced by Benci and Cerami in [4] (see also [11]), we establish a relation between the category of the set $\mathcal{V} \cap \mathcal{K}$ and the number of critical points of I_ϵ.

Throughout this paper, we use standard notations. $L^p = L^p(\mathbf{R}^3)$ $(1 \le p \le \infty)$ is the usual Lebesgue space with the standard norm $|\cdot|_p$. We use " $\to$ " and " $\rightharpoonup$ " to denote the strong and weak convergence in the related function spaces, respectively. C and C_i will denote positive constants unless specified. $B_R(x) := \{y \in \mathbf{R}^3 \,|\, |y-x| < R, x \in \mathbf{R}^3\}$. $\langle \cdot, \cdot \rangle$ denote the dual pair for any Banach space and its dual space. The Fourier transform and its inverse in $\mathbf{R}^3$ are denoted by $\mathcal{F}$ and $\mathcal{F}^{-1}$, respectively.

The paper is organized as follows. In Section 2, we present some preliminary results and the main embedding results for fractional Sobolev spaces. In Section 3, we study the limit problem of (Q_ϵ) and prove the existence of positive ground state solutions. In Section 4, we study the concentration phenomenon and convergence of ground state solutions. In Section 5, we obtain the decay estimate of solutions while the multiplicity of solutions are given in Section 6.

2. Preliminaries

In this section, we give some preliminary results which will be used in the paper. Making the change of variable $x \mapsto \epsilon x$, we can rewrite the problem (Q_ϵ) as the following equivalent problem

$$(\widetilde{Q}_\epsilon) \quad \begin{cases} M\left(\displaystyle\int_{\mathbf{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx + \int_{\mathbf{R}^3} V(\epsilon x) u^2 \, dx\right)[(-\Delta)^s u + V(\epsilon x) u] = K(\epsilon x) f(u), \\ u \in H^s(\mathbf{R}^3). \end{cases}$$

If u is a solution of the problem $(\widetilde{Q}_\epsilon)$, then $v(x) = u(\frac{x}{\epsilon})$ is a solution of the problem (Q_ϵ). Thus, to study the problem (Q_ϵ), it suffices to study the problem $(\widetilde{Q}_\epsilon)$. In view of the presence of the potential $V(x)$, we introduce the space

$$H_\epsilon = \left\{ u \in H^s(\mathbf{R}^3) \colon \int_{\mathbf{R}^3} V(\epsilon x) u^2 \, dx < \infty \right\}, \tag{2.1}$$

which is a Hilbert space equipped with the inner product

$$(u,v)_\epsilon = \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2s}} \, dx \, dy + \int_{\mathbf{R}^3} V(\epsilon x) uv \, dx, \tag{2.2}$$

and the equivalent norm

$$\|u\|_\epsilon^2 = (u,u)_\epsilon = \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} \, dx \, dy + \int_{\mathbf{R}^3} V(\epsilon x) u^2 \, dx. \tag{2.3}$$

Moreover, it can be proved that $u \in H_\epsilon$ is a solution of problem $(\widetilde{Q}_\epsilon)$ if and only if $u \in H_\epsilon$ is a critical point of the functional $I_\epsilon \colon H_\epsilon \to \mathbf{R}$ defined as

$$I_\epsilon(u) = \frac{1}{2} \widehat{M}(\|u\|_\epsilon^2) - \int_{\mathbf{R}^3} K(\epsilon x) F(u) \, dx. \tag{2.4}$$

It is clear that the functional I_ϵ is well-defined for every $u \in H_\epsilon$ and belongs to $C^1(H_\epsilon, \mathbf{R})$. Moreover, for any $u, v \in H_\epsilon$, we have

$$\langle I'_\epsilon(u), v\rangle = M(\|u\|_\epsilon^2)(u,v)_\epsilon - \int_{\mathbf{R}^3} K(\epsilon x) f(u) v \, dx. \tag{2.5}$$

Let us define the Nehari manifold of $(\widetilde{Q}_\epsilon)$ by

$$\mathcal{N}_\epsilon = \{u \in H_\epsilon \backslash \{0\} \colon \langle I'_\epsilon(u), u\rangle = 0\}. \tag{2.6}$$

Next we review the main embedding result for this class of fractional Sobolev spaces.

Lemma 2.1. [7, Theorem 6.5] *Let $0 < s < 1$. Then there exists a constant $C = C(3,s) > 0$, such that*

$$|u|_{2_s^*} \le C\|u\|_{H^s(\mathbf{R}^3)} \tag{2.7}$$

for every $u \in H^s(\mathbf{R}^3)$, where $2_s^* = \frac{6}{3-2s}$ is the fractional Sobolev critical exponent. Moreover, the embedding $H^s(\mathbf{R}^3) \subset L^p(\mathbf{R}^3)$ is continuous for any $p \in [2, 2_s^*]$, and is locally compact whenever $p \in [2, 2_s^*)$.

Lemma 2.2. [19, Lemma II.4] *Assume that $\{u_n\}$ is bounded in $H^s(\mathbf{R}^3)$ and*

$$\lim_{n\to\infty} \sup_{y \in \mathbf{R}^3} \int_{B_R(y)} |u_n(x)|^2 \, dx = 0, \tag{2.8}$$

where $R > 0$. Then $u_n \to 0$ in $L^p(\mathbf{R}^3)$ for every $2 < p < 2_s^*$.

Lemma 2.3. *The functional I_ϵ possesses the mountain pass geometry:*

(i) *There are α, $\rho > 0$, such that*

$$I_\epsilon(u) \ge \alpha, \quad \textit{with } \|u\|_\epsilon = \rho;$$

(ii) *There is an $e \in H_\epsilon \backslash B_\rho(0)$ with $I_\epsilon(e) < 0$.*

Proof. (i) For any $u \in H_\epsilon \backslash \{0\}$, it follows from (M_1), (f_1) and (f_2) that given $\epsilon \in (0, a)$, there exists a $C_\epsilon > 0$ such that

$$I_\epsilon(u) \ge \frac{a-\epsilon}{2}\|u\|_\epsilon^2 - C_\epsilon K_{\max} \int_{\mathbf{R}^3} |u|^q \, dx. \tag{2.9}$$

From the Sobolev inequality, we derive

$$I_\epsilon(u) \ge \frac{a-\epsilon}{2}\|u\|_\epsilon^2 - C\|u\|_\epsilon^q,$$

for some positive constant C. Since $4 < q < 2_s^*$, the result follows. (ii) Fix $v_0 \in C_0^\infty(\mathbf{R}^3)\backslash\{0\}$ with $v_0 \ge 0$ in $\mathbf{R}^3$ and $\|v_0\|_\epsilon = 1$. By (1.10) and (f_2), there exists a $C_0 > 0$ verifying

$$I_\epsilon(tv_0) \le \frac{r}{2}(t^2 + t^4) - C_0 t^\sigma K_{\inf} \int_{\mathbf{R}^3} v_0^\sigma \, dx. \tag{2.10}$$

Since $4 < \sigma < 2_s^*$, the result follows by letting $e = t_* v_0$ for some $t_* > 0$ large enough. □

Lemma 2.4. *I_ϵ is coercive on $\mathcal{N}_\epsilon$, i.e., $I_\epsilon(u) \to \infty$ as $\|u\|_\epsilon \to \infty$, $u \in \mathcal{N}_\epsilon$.*

Proof. Arguing by contradiction, suppose there exists a sequence $\{u_n\} \subset \mathcal{N}_\epsilon$, such that $\|u_n\|_\epsilon \to \infty$ and $I_\epsilon(u_n) \le d$ for some $d > 0$. It follows from (M_4) and (f_3) that

$$\frac{d}{\|u_n\|_\epsilon} + 1 \ge \frac{1}{\|u_n\|_\epsilon}\left[\frac{1}{2}\widehat{M}(\|u_n\|_\epsilon^2) - \frac{1}{\theta}M(\|u_n\|_\epsilon^2)\|u_n\|_\epsilon^2\right] \to +\infty$$

a contradiction. This completes the proof. □

Lemma 2.5. *For each $u \in H_\epsilon\backslash\{0\}$, there exists a unique $t_\epsilon = t_\epsilon(u) > 0$ such that $t_\epsilon u \in \mathcal{N}_\epsilon$. Moreover, $I_\epsilon(t_\epsilon u) = \max_{t\ge0} I_\epsilon(tu)$ and there exist $T_1 > T_2 > 0$ independent of $\epsilon > 0$ such that $T_2 \le t_\epsilon \le T_1$.*

Proof. For $u \in H_\epsilon\backslash\{0\}$ and $t > 0$, let

$$g(t) = I_\epsilon(tu) = \frac{1}{2}\widehat{M}(\|tu\|_\epsilon^2) - \int_{\mathbf{R}^3} K(\epsilon x)F(tu)\,dx. \tag{2.11}$$

Clearly, $g(0) = 0$, $g(t) > 0$ when $t > 0$ is small and $g(t) < 0$ when $t > 0$ is large. Hence g has a positive maximum at $t_\epsilon = t_\epsilon(u) > 0$. So that $g'(t_\epsilon) = 0$ and $t_\epsilon u \in \mathcal{N}_\epsilon$. The condition $g'(t) = 0$ is equivalent to

$$\frac{M(\|tu\|_\epsilon^2)}{\|tu\|_\epsilon^2} = \frac{1}{\|u\|_\epsilon^4}\int_{\mathbf{R}^3} K(\epsilon x)\frac{f(tu)}{(tu)^3}u^4\,dx. \tag{2.12}$$

Suppose that there exist $t_1 > t_2 > 0$ such that t_1u, $t_2u \in \mathcal{N}_\epsilon$. By (2.12), we have

$$\frac{M(\|t_1u\|_\epsilon^2)}{\|t_1u\|_\epsilon^2} - \frac{M(\|t_2u\|_\epsilon^2)}{\|t_2u\|_\epsilon^2} = \frac{1}{\|u\|_\epsilon^4}\int_{\mathbf{R}^3} K(\epsilon x)\left(\frac{f(t_1u)}{(t_1u)^3} - \frac{f(t_2u)}{(t_2u)^3}\right)u^4\,dx,$$

which contradicts to (M_3) and (f_4).

By $t_\epsilon u \in \mathcal{N}_\epsilon$, we deduce from (1.10) and (f_2) that

$$\begin{aligned} r(t_\epsilon^2\|u\|_\epsilon^2 + t_\epsilon^4\|u\|_\epsilon^4) \ge t_\epsilon^2 M(\|t_\epsilon u\|_\epsilon^2)\|u\|_\epsilon^2 &= \int_{\mathbf{R}^3} K(\epsilon x)f(t_\epsilon u)t_\epsilon u\,dx \\ &\ge C_1 t_\epsilon^\sigma \int_{\mathbf{R}^3} |u|^\sigma\,dx. \end{aligned}$$

Since $\sigma \in (4, 2_s^*)$, there exists a $T_1 > 0$ independent of ϵ such that $t_\epsilon \le T_1$.

On the other hand, using $t_\epsilon u \in \mathcal{N}_\epsilon$ again. We conclude from (M_1), (f_1), (f_2) and Sobolev inequalities that

$$at_\epsilon^2\|u\|_\epsilon^2 \le t_\epsilon^2 M(\|t_\epsilon u\|_\epsilon^2)\|u\|_\epsilon^2 = \int_{\mathbf{R}^3} K(\epsilon x)f(t_\epsilon u)t_\epsilon u\,dx \le \xi t_\epsilon^2\|u\|_\epsilon^2 + C_\xi t_\epsilon^q\|u\|_\epsilon^q, \tag{2.13}$$

so, there exists a $T_2 > 0$ independent of ϵ such that $t_\epsilon \ge T_2$. □

Lemma 2.6. *For any $\epsilon > 0$,*

(i) *there exists a $\rho > 0$ such that $c_\epsilon = \inf_{\mathcal{N}_\epsilon} I_\epsilon \ge \inf_{S_\rho} I_\epsilon > 0$, where $S_\rho = \{u \in H_\epsilon\colon \|u\|_\epsilon = \rho\}$;*

(ii) *$M(\|u\|_\epsilon^2)\|u\|_\epsilon^2 \ge 2c_\epsilon > 0$ for all $u \in \mathcal{N}_\epsilon$.*

Proof. (i) For $\epsilon > 0$ and $u \in H_\epsilon\backslash\{0\}$, it follows from (M_1), (f_1), (f_2) and Sobolev inequality that

$$I_\epsilon(u) \ge \frac{a-\epsilon}{2}\|u\|_\epsilon^2 - C\|u\|_\epsilon^q. \tag{2.14}$$

Thus, for sufficiently small ρ, we have $\inf_{S_\rho} I_\epsilon > 0$. For every $u \in \mathcal{N}_\epsilon$, there is a $t > 0$ such that $tu \in S_\rho$ and $c_\epsilon = \inf_{\mathcal{N}_\epsilon} I_\epsilon \ge \inf_{S_\rho} I_\epsilon > 0$ by Lemma 2.5.

(ii) For $u \in \mathcal{N}_\epsilon$, we deduce from (M_2) that

$$c_\epsilon \le I_\epsilon(u) = \frac{1}{2}\widehat{M}(\|u\|_\epsilon^2) - \int_{\mathbf{R}^3} K(\epsilon x)F(u)\,dx \le \frac{1}{2}M(\|u\|_\epsilon^2)\|u\|_\epsilon^2. \tag{2.15}$$

The conclusion follows. □

Lemma 2.7. *If $\mathcal{W}$ is a compact subset of $H_\epsilon\backslash\{0\}$, then there exists an $R > 0$ such that $I_\epsilon(tu) \le 0$ for each $u \in \mathcal{W}$ and $t \ge 0$ with $t\|u\|_\epsilon > R$.*

Proof. Since $\mathcal{W}$ is a compact subset of $H_\epsilon\backslash\{0\}$, there are positive constants C_1 and C_2 such that $C_1 \le \|u\|_\epsilon \le C_2$ for each $u \in \mathcal{W}$. Also, there is a constant $C > 0$ such that $\int_{\mathbf{R}^3} |u|^\sigma dx \ge C$ for each $u \in \mathcal{W}$.

Just suppose that the conclusion were not true, then for each n, there exist $u_n \in \mathcal{W}$ and $t_n \ge 0$ with $t_n\|u_n\|_\epsilon > n$ such that $I_\epsilon(t_n u_n) \ge 0$. By (f_2) and $t_n > \frac{n}{\|u_n\|_\epsilon} \ge \frac{n}{C_2} \to +\infty$, we have

$$\begin{aligned}
I_\epsilon(t_n u_n) &= \frac{1}{2}\widehat{M}(\|t_n u_n\|_\epsilon^2) - \int_{\mathbf{R}^3} K(\epsilon x)F(t_n u_n)\,dx \\
&\le \frac{r}{2}(t_n^2\|u_n\|_\epsilon^2 + t_n^4\|u_n\|_\epsilon^4) - t_n^\sigma K_{\inf}\int_{\mathbf{R}^3} |u_n|^\sigma\,dx \\
&= t_n^\sigma\left(\frac{r}{2}t_n^{2-\sigma}\|u_n\|_\epsilon^2 + \frac{r}{2}t_n^{4-\sigma}\|u_n\|_\epsilon^4 - K_{\inf}\int_{\mathbf{R}^3}|u_n|^\sigma\,dx\right) \\
&\le t_n^\sigma\left(\frac{r}{2}t_n^{2-\sigma}C_2^2 + \frac{r}{2}t_n^{4-\sigma}C_2^4 - K_{\inf}C\right) \to -\infty,
\end{aligned} \tag{2.16}$$

which is a contradiction. This completes the proof. □

The following assumptions and propositions come from [21, page 9, Chacter 3, A_1, A_2 and A_3)]. Let E^* be a Banach space such that the unit sphere S in E^* is a submanifold of class (at least) C^1 and let $\Phi \in C^1(E^*, \mathbf{R})$ and $\Phi(0) = 0$. The corresponding Nehari manifold is $\mathcal{N} := \{u \in E^*\backslash\{0\} : \langle \Phi'(u), u\rangle = 0\}$. A function $\varphi \in C(\mathbf{R}_+, \mathbf{R}_+)$ is said to be a normalization function if $\varphi(0) = 0$, φ is strictly increasing and $\varphi(t) \to \infty$ as $t \to \infty$. The authors in [21] give the following further assumptions:

(A_1) There exists a normalization function φ such that

$$u \mapsto \psi(u) := \int_0^{\|u\|} \varphi(t)\,dt \in C^1(E^*\backslash\{0\}, \mathbf{R}),$$

$J := \psi'$ is bounded on bounded sets and $\langle J(w), w\rangle = 1$ for all $w \in S$;

(A_2) For each $w \in E^*\backslash\{0\}$ there exists a s_w such that if $\alpha_w(s) := \Phi(sw)$, then $\alpha_w'(s) > 0$ for $0 < s < s_w$ and $\alpha_w'(s) < 0$ for $s > s_w$;

(A_3) There exists $\delta > 0$ such that $s_w \ge \delta$ for all $w \in S$ and for each compact subset $\mathcal{W} \subset S$ there exists a constant $C_{\mathcal{W}}$ such that $s_w \le C_{\mathcal{W}}$ for all $w \in \mathcal{W}$.

Proposition 2.8. [21, Proposition 8, Proposition 9, Corollary 10]

(I) *Define the mappings $\hat{m} := E^*\backslash\{0\} \to \mathcal{N}$ and $m : S \to \mathcal{N}$ by setting*

$$\hat{m}(w) := s_w w \quad \textit{and} \quad m := \hat{m}|_S.$$

Suppose Φ satisfies (A_2) and (A_3). Then:

(a) *The mapping $\hat{m}$ is continuous;*

(b) *The mapping* m *is a homeomorphism between* S *and* $\mathcal{N}$, *and the inverse of* m *is given by* $m^{-1}(u) = u/\|u\|$.

(II) *Set the functionals* $\hat{\Psi}\colon E^*\backslash\{0\} \to \mathbf{R}$ *and* $\Psi\colon S \to \mathbf{R}$ *defined by*

$$\hat{\Psi}(w) := \Phi(\hat{m}(w)) \quad \textit{and: } \Phi := \hat{\Psi}|_S.$$

Suppose E^* *is a Banach space satisfying* (A_1). *If* Φ *satisfies* (A_2) *and* (A_3), *then* $\hat{\Psi} \in C^1(E^*\backslash\{0\}, \mathbf{R})$ *and*

$$\langle \hat{\Psi}'(w), z\rangle = \frac{\|\hat{m}(w)\|}{\|w\|}\langle \Phi'(\hat{m}(w)), z\rangle$$

for all w, $z \in E^*$, $w \neq 0$.

(III) *Suppose* E^* *is a Banach space satisfying* (A_1). *If* Φ *satisfies* (A_2) *and* (A_3), *then*

(c) $\Psi \in C^1(S, \mathbf{R})$ *and*

$$\langle \Psi'(w), z\rangle = \|m(w)\|\langle \Phi'(m(w)), z\rangle$$

for all $z \in T_w(S) = \{z \in E^*\colon \langle J(w), z\rangle = 0\}$;

(d) *If* $\{w_n\}$ *is a Palais–Smale sequence for* Ψ, *then* $\{m(w_n)\}$ *is a Palais–Smale sequence for* Φ. *If* $\{u_n\} \subset \mathcal{N}$ *is a bounded Palais–Smale sequence for* Φ, *then* $\{m^{-1}(u_n)\}$ *is a Palais–Smale sequence for* Ψ;

(e) w *is a critical point of* Ψ *if and only if* $m(w)$ *is a nontrivial critical point of* Φ. *Moreover, the corresponding values of* Ψ *and* Φ *coincide and* $\inf_S \Psi = \inf_{\mathcal{N}} \Phi$.

Now, we define the mappings $\widetilde{m}_\epsilon\colon H_\epsilon\backslash\{0\} \to \mathcal{N}_\epsilon$ and $m_\epsilon\colon S_\epsilon \to \mathcal{N}_\epsilon$ by setting

$$\widetilde{m}_\epsilon(w) = t_w w \quad \text{and} \quad m_\epsilon = \widetilde{m}_\epsilon|_{S_\epsilon}. \tag{2.17}$$

Thus, the hypotheses of (A_1), (A_2) and (A_3) are satisfied by Lemma 2.5, 2.6 and 2.7. We also consider the functionals $\widetilde{\Upsilon}_\epsilon\colon H_\epsilon\backslash\{0\} \to \mathbf{R}$ and $\Upsilon_\epsilon\colon S_\epsilon \to \mathbf{R}$ defined by

$$\widetilde{\Upsilon}_\epsilon(u) = I_\epsilon(\widetilde{m}_\epsilon(u)) \quad \text{and} \quad \Upsilon_\epsilon = \widetilde{\Upsilon}_\epsilon|_{S_\epsilon}. \tag{2.18}$$

By Proposition 2.8, we have the following result.

Lemma 2.9. *Suppose* (M_1)–(M_5) *and* (f_1)–(f_4) *hold, for each* $\epsilon > 0$, *then*

(i) *the mapping* $\widetilde{m}_\epsilon$ *is continuous; mapping* m_ϵ *is a homeomorphism between* S_ϵ *and* $\mathcal{N}_\epsilon$, *and the inverse of* m_ϵ *is given by* $m_\epsilon^{-1}(u) = u/\|u\|_\epsilon$;

(ii) $\Upsilon_\epsilon \in C^1(S_\epsilon, \mathbf{R})$ *and* $\langle \Upsilon_\epsilon'(w), z\rangle = \|m_\epsilon(w)\|\langle I_\epsilon'(m_\epsilon(w)), z\rangle$ *for all* $z \in T_w(S_\epsilon) = \{v \in H_\epsilon\colon \langle w, v\rangle = 0\}$;

(iii) *If* $\{w_n\}$ *is a PS sequence for* Υ_ϵ, *then* $\{m_\epsilon(w_n)\}$ *is a PS sequence for* I_ϵ. *If* $\{u_n\} \subset \mathcal{N}_\epsilon$ *is a bounded PS sequence for* I_ϵ, *then* $\{m_\epsilon^{-1}(u_n)\}$ *is a PS sequence for* Υ_ϵ;

(iv) w *is a nontrivial critical point of* Υ_ϵ *if and only if* $m_\epsilon(w)$ *is a nontrivial critical point of* I_ϵ. *Moreover, the corresponding values of* Υ_ϵ *and* I_ϵ *coincide and* $\inf_{S_\epsilon} \Upsilon_\epsilon = \inf_{\mathcal{N}_\epsilon} I_\epsilon$.

Remark 2.10. As in [21], we have the following minimax characterization of the infimum of I_ϵ over $\mathcal{N}_\epsilon$:

$$c_\epsilon = \inf_{u\in\mathcal{N}_\epsilon} I_\epsilon(u) = \inf_{w\in H_\epsilon\backslash\{0\}} \max_{t>0} I_\epsilon(tw) = \inf_{w\in S_\epsilon} \max_{t>0} I_\epsilon(tw). \tag{2.19}$$

3. The limit problem

In this section, we study the autonomous problem

$$(Q_{\mu\nu}) \qquad \begin{cases} \mathfrak{L}_\mu u = \nu f(u), & \text{in } \mathbf{R}^3, \\ u \in H^s(\mathbf{R}^3), \end{cases}$$

where

$$\mathfrak{L}_\mu u = M\left(\iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbf{R}^3} \mu u^2\,dx\right)[(-\Delta)^s u + \mu u],$$

$\mu > 0$ and $\nu > 0$. The corresponding energy functional is defined by

$$I_{\mu\nu}(u) = \frac{1}{2}\widehat{M}\left(\iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbf{R}^3} \mu u^2\,dx\right) - \nu\int_{\mathbf{R}^3} F(u). \tag{3.1}$$

$I_{\mu\nu} \in C^1(H_\mu, \mathbf{R})$ is well defined on the Hilbert space $H_\mu = H^s(\mathbf{R}^3)$ with the inner product

$$(u,v)_\mu = \iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbf{R}^3} \mu uv\,dx$$

and the norm

$$\|u\|_\mu^2 = \iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbf{R}^3} \mu u^2\,dx.$$

The associated Nehari manifold $\mathcal{N}_{\mu\nu}$ of $I_{\mu\nu}$ is given as

$$\mathcal{N}_{\mu\nu} = \{u \in H_\mu\backslash\{0\} \colon \langle I'_{\mu\nu}(u), u\rangle = 0\}. \tag{3.2}$$

We define the ground state energy associated with $(Q_{\mu\nu})$ by

$$c_{\mu\nu} = \inf_{u\in\mathcal{N}_{\mu\nu}} I_{\mu\nu}(u). \tag{3.3}$$

The number $c_{\mu\nu}$ and the Nehari manifold $\mathcal{N}_{\mu\nu}$ have properties similar to those of c_ϵ and $\mathcal{N}_\epsilon$ such as Lemmas 2.4–2.7. Hence, for each $u \in H_\mu\backslash\{0\}$, there exists an unique $t_u > 0$ such that $t_u u \in \mathcal{N}_{\mu\nu}$. Recall that $S_\mu = \{u \in H_\mu \colon \|u\|_\mu = 1\}$ and define the mapping $\widetilde{m}_{\mu\nu} \colon H_\mu\backslash\{0\} \to \mathcal{N}_{\mu\nu}$ by $\widetilde{m}_{\mu\nu}(u) = t_u u$, and $m_{\mu\nu} = \widetilde{m}_{\mu\nu}|_{S_\mu}$. Moreover, the inverse of $m_{\mu\nu}$ is given by $m_{\mu\nu}^{-1}(u) = u/\|u\|_\mu$. Define the functional $\widetilde{\Upsilon}_{\mu\nu} \colon H_\mu\backslash\{0\} \to \mathbf{R}$ by

$$\widetilde{\Upsilon}_{\mu\nu}(u) = I_{\mu\nu}(\widetilde{m}_{\mu\nu}(u)) \quad \text{and} \quad \Upsilon_{\mu\nu} = \widetilde{\Upsilon}_{\mu\nu}|_{S_\mu}. \tag{3.4}$$

We have similar results as Lemma 2.9 for functional $\Upsilon_{\mu\nu}$. Moreover,

$$c_{\mu\nu} = \inf_{u\in\mathcal{N}_{\mu\nu}} I_{\mu\nu}(u) = \inf_{u\in H_\mu\backslash\{0\}} \max_{t>0} I_{\mu\nu}(tu) = \inf_{u\in S_\mu} \max_{t>0} I_{\mu\nu}(tu). \tag{3.5}$$

It is easy to see that $I_{\mu\nu}$ possesses the Mountain–Pass geometry.

Lemma 3.1. *Let $\mu_j > 0$ and $\nu_j > 0$, $j = 1, 2$, with $\mu_1 \le \mu_2$ and $\nu_1 \ge \nu_2$. Then $c_{\mu_1\nu_1} \le c_{\mu_2\nu_2}$. In particular, if $\nu_1 > \nu_2$ or $\mu_1 < \mu_2$, then $c_{\mu_1\nu_1} < c_{\mu_2\nu_2}$.*

Proof. Let $u \in \mathcal{N}_{\mu_2\nu_2}$ be such that

$$c_{\mu_2\nu_2} = I_{\mu_2\nu_2}(u) = \max_{t>0} I_{\mu_2\nu_2}(tu). \tag{3.6}$$

Let $u_0 = t_1 u$ be such that $I_{\mu_1\nu_1}(u_0) = \max_{t>0} I_{\mu_1\nu_1}(tu)$. We have

$$\begin{aligned} c_{\mu_2\nu_2} &= I_{\mu_2\nu_2}(u) \geq I_{\mu_2\nu_2}(u_0) \\ &= I_{\mu_1\nu_1}(u_0) + \frac{1}{2}\int_{\|u_0\|^2_{\mu_1}}^{\|u_0\|^2_{\mu_2}} M(t)\,dt + (\nu_1 - \nu_2)\int_{\mathbf{R}^3} F(u_0)\,dx \geq c_{\mu_1\nu_1}. \end{aligned} \tag{3.7}$$

This completes the proof. □

Proposition 3.2. *Suppose* (M_1)–(M_5) *and* (f_1)–(f_4) *hold. Then for any* $\mu > 0$ *and* $\nu > 0$*, problem* $(Q_{\mu\nu})$ *has a nonnegative ground state solution.*

Proof. Let $\{w_n\} \subset S_\mu$ be a minimizing sequence of $\Upsilon_{\mu\nu}$. By Ekeland's variational principle [23, Theorem 2.4], we may assume $\Upsilon_{\mu\nu}(w_n) \to c_{\mu\nu}$, $\Upsilon'_{\mu\nu}(w_n) \to 0$. Set $u_n = m_{\mu\nu}(w_n) \in \mathcal{N}_{\mu\nu}$ for all $n \in \mathbf{N}$. Then

$$I_{\mu\nu}(u_n) \to c_{\mu\nu} \quad \text{and} \quad I'_{\mu\nu}(u_n) \to 0 \tag{3.8}$$

as $n \to \infty$. Arguing as in the proof of Lemma 2.4, we see that $I_{\mu\nu}$ is coercive on $\mathcal{N}_{\mu\nu}$. Thus, $\{u_n\}$ is bounded in $H^s(\mathbf{R}^3)$, then, up to a subsequence, there exist $u \in H_\mu$ and $\rho_0 \geq 0$ such that

$$\begin{aligned} &\|u_n\|_\mu \to \rho_0 \text{ in } \mathbf{R}, \quad u_n \rightharpoonup u \text{ in } H^s(\mathbf{R}^3), \quad u_n(x) \to u(x) \text{ a.e. } \mathbf{R}^3, \\ &u_n \to u \text{ in } L^t_{\text{loc}}(\mathbf{R}^3) \text{ for all } t \in (2, 2^*_s). \end{aligned}$$

We claim that there is a sequence $\{y_n\} \subset \mathbf{R}^3$ and constants R, $\eta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^2\,dx \geq \eta > 0. \tag{3.9}$$

If not, for any $R > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbf{R}^3} \int_{B_R(y)} |u_n|^2\,dx = 0.$$

By Lemma 2.2, we get $u_n \to 0$ in $L^p(\mathbf{R}^3)$, $2 < p < 2^*_s$. We deduce from (f_1) and (f_2) that

$$\lim_{n\to\infty} \nu \int_{\mathbf{R}^3} f(u_n)u_n\,dx = 0. \tag{3.10}$$

Since $M(\|u_n\|^2_\mu) \geq a$ and $\langle I'_{\mu\nu}(u_n), u_n\rangle = 0$, we have

$$u_n \to 0 \quad \text{in } H^s(\mathbf{R}^3). \tag{3.11}$$

This leads to $c_{\mu\nu} = 0$, a contradiction.

Thus, there exist $\{y_n\}$, R and η such that (3.9) holds. So we can choose $R' > R > 0$ and a sequence $\{y_n\} \subset \mathbf{R}^3$ such that

$$\liminf_{n\to\infty} \int_{B_{R'}(y_n)} |u_n|^2\,dx \geq \frac{\eta}{2} > 0.$$

Since $I_{\mu\nu}$ and $\mathcal{N}_{\mu\nu}$ are invariant under translations of the form $u \to u(\cdot + k)$ with $k \in \mathbf{Z}^3$, we assume that $\{y_n\}$ is bounded in $\mathbf{R}^3$. Then $u_n \rightharpoonup u \neq 0$.

From the continuity of M, we get

$$M(\|u_n\|^2_\mu) \to M(\rho_0^2),$$

since $I'_{\mu\nu}(u_n) = o(1)$, we obtain that u is a positive solution of the problem

$$M(\rho_0^2)[(-\Delta)^s u + \mu u] = \nu f(u) \quad \text{in } \mathbf{R}^3,\ u \in H^s(\mathbf{R}^3).$$

To conclude our proof, we need to prove that

$$M(\rho_0^2) = M(\|u\|_\mu^2). \tag{3.12}$$

In order to prove (3.12), as $u_n \rightharpoonup u$ in $H^s(\mathbf{R}^3)$, we have

$$\liminf_{n\to\infty} \|u_n\|_\mu \geq \|u\|_\mu.$$

So, given $\varsigma > 0$, there exists $n_0 \in \mathbf{N}$ such that, for $n \geq n_0$,

$$\|u_n\|_\mu \geq \|u\|_\mu - \varsigma.$$

We conclude from (M_2) that $M(\|u_n\|_\mu^2) \geq M((\|u\|_\mu - \varsigma)^2)$ for $n \geq n_0$. Letting $n \to \infty$, and after $\varsigma \to 0$, we obtain

$$M(\rho_0^2) \geq M(\|u\|_\mu^2).$$

Arguing by contradiction, suppose that $M(\rho_0^2) > M(\|u\|_\mu^2)$, we deduce that

$$M(\|u\|_\mu^2)\|u\|_\mu^2 < M(\rho_0^2)\|u\|_\mu^2 = \nu \int_{\mathbf{R}^3} f(u)u\,dx. \tag{3.13}$$

(3.13) implies that $\langle I'_{\mu\nu}(u), u\rangle < 0$. Hence, there exists a $\bar{t} \in (0,1)$ such that $\bar{t}u \in \mathcal{N}_{\mu\nu}$. Combining this information with the characterization of mountain pass level, we have

$$\begin{aligned} c_{\mu\nu} &\leq I_{\mu\nu}(\bar{t}u) = I_{\mu\nu}(\bar{t}u) - \frac{1}{4}\langle I'_{\mu\nu}(\bar{t}u), \bar{t}u\rangle \\ &= \left[\frac{1}{2}\widehat{M}(\|\bar{t}u\|_\mu^2) - \frac{1}{4}M(\|\bar{t}u\|_\mu^2)\|\bar{t}u\|_\mu^2\right] + \nu \int_{\mathbf{R}^3} \left[\frac{1}{4}f(\bar{t}u)\bar{t}u - F(\bar{t}u)\right] dx. \end{aligned} \tag{3.14}$$

It follows from (M_5) and (f_4) that

$$c_{\mu\nu} < \left[\frac{1}{2}\widehat{M}(\|u\|_\mu^2) - \frac{1}{4}M(\|u\|_\mu^2)\|u\|_\mu^2\right] + \nu \int_{\mathbf{R}^3} \left[\frac{1}{4}f(u)u - F(u)\right] dx. \tag{3.15}$$

On the other hand, by Fatou's Lemma, we obtain

$$\begin{aligned} &\liminf_{n\to\infty} \left[I_{\mu\nu}(u_n) - \frac{1}{4}\langle I'_{\mu\nu}(u_n), u_n\rangle\right] \\ &\geq \left[\frac{1}{2}\widehat{M}(\|u\|_\mu^2) - \frac{1}{4}M(\|u\|_\mu^2)\|u\|_\mu^2\right] + \nu \int_{\mathbf{R}^3} \left[\frac{1}{4}f(u)u - F(u)\right] dx. \end{aligned} \tag{3.16}$$

So, $c_{\mu\nu} < \liminf_{n\to\infty}[I_{\mu\nu}(u_n) - \frac{1}{4}\langle I'_{\mu\nu}(u_n), u_n\rangle] = c_{\mu\nu}$, a contradiction. This way, $M(\rho_0^2) = M(\|u\|_\mu^2)$. Hence, $u_n \to u$ in H_μ.

It remains to prove that the ground state solution is nonnegative. Put $u^\pm = \max\{\pm u, 0\}$ the positive (negative) part of u. We note that all the calculations above can be repeated word by word, replacing $I^+_{\mu\nu}(u)$ with the functional

$$I^+_{\mu\nu}(u) = \frac{1}{2}\widehat{M}(\|u\|_\mu^2) - \nu \int_{\mathbf{R}^3} F(u^+)\,dx. \tag{3.17}$$

In this way we get a ground state solution u of the equation

$$M(\|u\|_\mu^2)[(-\Delta)^s u + \mu u] = \nu f(u^+), \quad x \in \mathbf{R}^3. \tag{3.18}$$

Multiplying (3.18) by u^- and integrating by parts, we have

$$M(\|u\|_\mu^2) \int_{\mathbf{R}^3} \left((-\Delta)^s u \cdot u^- + \mu |u^-|^2\right) dx = 0,$$

so

$$\int_{\mathbf{R}^3} (-\Delta)^s u \cdot u^-\,dx = -\mu \int_{\mathbf{R}^3} |u^-|^2\,dx \leq 0.$$

But we know

$$
\begin{aligned}
\int_{\mathbf{R}^3} (-\Delta)^s u \cdot u^- \, dx &= \frac{C(s)}{2} \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{3+2s}} \, dx \, dy \\
&\geq \frac{C(s)}{2} \Bigg(\iint_{\{u>0\}\times\{u<0\}} \frac{(u(x)-u(y))(-u^-(y))}{|x-y|^{3+2s}} \, dx \, dy \\
&\quad + \iint_{\{u<0\}\times\{u<0\}} \frac{(u^-(x)-u^-(y))^2}{|x-y|^{3+2s}} \, dx \, dy \\
&\quad + \iint_{\{u<0\}\times\{u>0\}} \frac{(u(x)-u(y))u^-(x)}{|x-y|^{3+2s}} \, dx \, dy \Bigg) \geq 0.
\end{aligned} \tag{3.19}
$$

Thus, $u^- = 0$ and $u \geq 0$. Moreover, if $u(x_0) = 0$ for some $x_0 \in \mathbf{R}^3$, then $(-\Delta)^s u(x_0) = 0$ and by (1.6), we have

$$
(-\Delta)^s u(x_0) = -\frac{C(s)}{2} \int_{\mathbf{R}^3} \frac{u(x_0+y)+u(x_0-y)-2u(x_0)}{|y|^{3+2s}} \, dy,
$$

so,

$$
\int_{\mathbf{R}^3} \frac{u(x_0+y)+u(x_0-y)}{|y|^{3+2s}} \, dy = 0,
$$

yielding $u \equiv 0$, a contradiction. Therefore, u is a positive solution of the problem $(\widetilde{Q}_{\mu\nu})$ and the proof is completed. □

Lemma 3.3. *Let $\{u_n\} \subset \mathcal{N}_\epsilon$ be such that $I_\epsilon(u_n) \to c$ and $u_n \rightharpoonup 0$ in H_ϵ. Then, one of the following conslusions holds:*

(i) *$u_n \to 0$ in H_ϵ, or*

(ii) *there exists a sequence $\{y_n\} \subset \mathbf{R}^3$, and constants R, $\eta > 0$ such that*

$$
\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^2 \, dx \geq \eta > 0.
$$

Proof. By Lemma 2.4, we know that $\{u_n\}$ is bounded in H_ϵ. Suppose (ii) does not occur. We deduce from Lemma 2.2 that $u_n \to 0$ in $L^q(\mathbf{R}^3)$ for $q \in (2, 2_s^*)$. By (M_1), (f_1) and (f_2), we have

$$
0 \leq a\|u_n\|_\epsilon \leq \int_{\mathbf{R}^3} K(\epsilon x) f(u_n) u_n \, dx = o(1). \tag{3.20}
$$

Therefore (i) is true. □

Without loss of generality, up to a translation, we may assume that

$$
x_1 = 0 \in \mathcal{V},
$$

so

$$
V(0) = V_{\min} \quad \text{and} \quad \kappa := K(0) \geq K(x) \quad \text{for all } |x| \geq R.
$$

Lemma 3.4.

$$
\limsup_{\epsilon\to 0} c_\epsilon \leq c_{V_{\min}\kappa}. \tag{3.21}
$$

Proof. Set $V^z(x) = \max\{z, V(x)\}$, $K^d(x) = \min\{d, K(x)\}$, $V^z_\epsilon(x) = V^z(\epsilon x)$ and $K^d_\epsilon(x) = K^d(\epsilon x)$, where z, d are positive constants. For any $u \in H^s(\mathbf{R}^3)$, define

$$\begin{aligned} I^{zd}_\epsilon(u) = \frac{1}{2}\widehat{M}\left(\iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbf{R}^3} V^z_\epsilon(x)u^2\,dx\right) \\ - \int_{\mathbf{R}^3} K^d_\epsilon(x)F(u)\,dx, \end{aligned} \tag{3.22}$$

which implies that $I_{zd}(u) \le I^{zd}_\epsilon(u)$, and thus $c_{zd} \le c^{zd}_\epsilon$, where c^{zd}_ϵ is the least energy of I^{zd}_ϵ. By the definition of $V_{\min}$ and $K_{\max}$, we get $V^{V_{\min}}_\epsilon(x) = V(\epsilon x)$, $K^{K_{\max}}_\epsilon(x) = K(\epsilon x)$. Therefore, we have

$$I^{V_{\min}K_{\max}}_\epsilon(u) = I_\epsilon(u), \tag{3.23}$$

and $V^{V_{\min}}_\epsilon(x) \to V(0) = V_{\min}$, $K^{K_{\max}}_\epsilon(x) \to K(0) = \kappa$ uniformly on bounded sets of x as $\epsilon \to 0$.

Now, we claim $\limsup_{\epsilon\to 0} c^{V_{\min}K_{\max}}_\epsilon \le c_{V_{\min}\kappa}$. Indeed, take $w \in H^s(\mathbf{R}^3)$ such that $I_{V_{\min}\kappa}(w) = c_{V_{\min}\kappa}$, then there exists a $t_\epsilon > 0$ such that $t_\epsilon w \in \mathcal{N}^{V_{\min}K_{\max}}_\epsilon$, where $\mathcal{N}^{V_{\min}K_{\max}}_\epsilon$ is the Nehari manifold of the functional $I^{V_{\min}K_{\max}}_\epsilon$. Thus

$$c^{V_{\min}K_{\max}}_\epsilon \le I^{V_{\min}K_{\max}}_\epsilon(t_\epsilon w) = \max_{t\ge 0} I^{V_{\min}K_{\max}}_\epsilon(tw). \tag{3.24}$$

We have

$$\begin{aligned} I^{V_{\min}K_{\max}}_\epsilon(t_\epsilon w) = I_{V_{\min}\kappa}(t_\epsilon w) + \frac{1}{2}\int_{[t_\epsilon w]_s^2+\int_{\mathbf{R}^3} V_{\min}|t_\epsilon w|^2}^{[t_\epsilon w]_s^2+\int_{\mathbf{R}^3} V(\epsilon x)|t_\epsilon w|^2} M(t)\,dt \\ + \int_{\mathbf{R}^3} (\kappa - K^{K_{\max}}_\epsilon(x))F(t_\epsilon w)\,dx. \end{aligned} \tag{3.25}$$

By Lemma 2.5, we can assume that $t_\epsilon \to t_0$ as $\epsilon \to 0$. Since $w \in L^2(\mathbf{R}^3)$, for any $\eta > 0$, there exists a $R > 0$ such that

$$\int_{\mathbf{R}^3\setminus B_R(0)} |w|^2\,dx < \eta.$$

So,

$$\begin{aligned} \left|\int_{[t_\epsilon w]_s^2+\int_{\mathbf{R}^3} V_{\min}|t_\epsilon w|^2}^{[t_\epsilon w]_s^2+\int_{\mathbf{R}^3} V(\epsilon x)|t_\epsilon w|^2} M(t)\,dt\right| &\le C\int_{\mathbf{R}^3} (V(\epsilon x) - V_{\min})|t_\epsilon w|^2\,dx \\ &\le Ct_0^2\eta + o(1), \end{aligned} \tag{3.26}$$

here use the fact that $V^{V_{\min}}_\epsilon(x) \to V_{\min}$ uniformly in $x \in B_R(0)$, we obtain

$$\int_{\mathbf{R}^3} (V(\epsilon x) - V_{\min})|t_\epsilon w|^2\,dx = o(1).$$

Similarly, we have

$$\int_{\mathbf{R}^3} (\kappa - K(\epsilon x))F(t_\epsilon w)\,dx = o(1).$$

Thus, by (3.25), we have

$$I^{V_{\min}K_{\max}}_\epsilon(t_\epsilon w) = I_{V_{\min}\kappa}(t_\epsilon w) + o(1) \to I_{V_{\min}\kappa}(t_0 w) \tag{3.27}$$

as $\epsilon \to 0$. Consequently,

$$\begin{aligned} c^{V_{\min}K_{\max}}_\epsilon \le I^{V_{\min}K_{\max}}_\epsilon(t_\epsilon w) \to I_{V_{\min}\kappa}(t_0 w) \le \max_{t\ge 0} I_{V_{\min}\kappa}(tw) \\ = I_{V_{\min}\kappa}(w) = c_{V_{\min}\kappa}. \end{aligned} \tag{3.28}$$

From (3.23), we obtain $c_\epsilon^{V_{\min}K_{\max}} = c_\epsilon$. This completes the proof. □

Proposition 3.5. *c_ϵ is attained at some positive $u_\epsilon \in H_\epsilon$ for small $\epsilon > 0$.*

Proof. By Lemma 2.6, we have $c_\epsilon > 0$ for each $\epsilon > 0$. Suppose $u_\epsilon \in \mathcal{N}_\epsilon$ satisfies $I_\epsilon(u_\epsilon) = c_\epsilon$, then $\Upsilon_\epsilon(m_\epsilon^{-1}(u_\epsilon)) = c_\epsilon$, and $m_\epsilon^{-1}(u_\epsilon)$ is a critical point of Υ_ϵ. We deduce from Lemma 2.9 that u_ϵ is a critical point of I_ϵ. Next we show that there exists a minimizer $u_\epsilon \in \mathcal{N}_\epsilon$ of I_ϵ.

By Ekeland's variational principle [[23], Theorem 2.4], there exists a sequence $\{w_n\} \subset S_\epsilon$ with $\Upsilon_\epsilon(w_n) \to c_\epsilon$ and $\Upsilon'_\epsilon(w_n) \to 0$ as $n \to \infty$. Let $u_n = m_\epsilon(w_n)$, we know that $u_n \in \mathcal{N}_\epsilon$ for all $n \in \mathbf{N}$. It follows from Lemma 2.9 that $I_\epsilon(u_n) \to c_\epsilon$ and $I'_\epsilon(u_n) \to 0$ as $n \to \infty$. By Lemma 2.4, we know that $\{u_n\}$ is bounded in H_ϵ. Suppose that $u_n \rightharpoonup u_\epsilon$ in H_ϵ, if $u_\epsilon \neq 0$, then by an argument similar to the proof of Proposition 3.2, we obtain that $u_n \to u_\epsilon$ in H_ϵ.

Next we show that $u_\epsilon \neq 0$ for small $\epsilon > 0$. Arguing by contradiction, suppose that there exists a sequence $\epsilon_j \to 0$ such that $u_{\epsilon_j} = 0$, then

$$u_n \rightharpoonup 0 \text{ in } H_\epsilon, \quad u_n \to 0 \text{ in } L^t_{\text{loc}}(\mathbf{R}^3),\ t \in [1, 2_s^*), \quad u_n(x) \to 0 \text{ a.e. in } \mathbf{R}^3.$$

By (P_1), take $\tau \in (V_{\min}, V_\infty)$ and consider the functional $I_{\epsilon_j}^{\tau\kappa}$. Let $t_n > 0$ be such that $t_n u_n \in \mathcal{N}_{\epsilon_j}^{\tau\kappa}$, it follows from Lemma 2.5 that $\{t_n\}$ is bounded. Assume $t_n \to t_0$ as $n \to \infty$. By (P_1), we know that the set $O_\epsilon = \{x \in \mathbf{R}^3\colon V_\epsilon(x) < \tau \text{ or } K_\epsilon(x) \geq \kappa\}$ is bounded. Since $I_{\epsilon_j}(t_n u_n) \leq I_{\epsilon_j}(u_n)$, we obtain

$$\begin{aligned}
c_{\epsilon_j}^{\tau\kappa} &\leq I_{\epsilon_j}^{\tau\kappa}(t_n u_n) \\
&= I_{\epsilon_j}(t_n u_n) + \frac{1}{2}\int_{[t_n u_n]_s^2 + \int_{\mathbf{R}^3} V(\epsilon_j x)|t_n u_n|^2}^{[t_n u_n]_s^2 + \int_{\mathbf{R}^3} V_{\epsilon_j}^{\tau}(x)|t_n u_n|^2} M(t)\,dt \\
&\quad + \int_{\mathbf{R}^3} (K(\epsilon_j x) - K_{\epsilon_j}^{\kappa}(x)) F(t_n u_n)\,dx \\
&= I_{\epsilon_j}(t_n u_n) + \frac{1}{2}\int_{[t_n u_n]_s^2 + \int_{O_{\epsilon_j}} V(\epsilon_j x)|t_n u_n|^2}^{[t_n u_n]_s^2 + \int_{O_{\epsilon_j}} \tau|t_n u_n|^2} M(t)\,dt \\
&\quad + \int_{\mathbf{R}^3} (K(\epsilon_j x) - \kappa) F(t_n u_n)\,dx \\
&\leq I_{\epsilon_j}(t_n u_n) + o(1) \leq I_{\epsilon_j}(u_n) + o(1) = c_{\epsilon_j}.
\end{aligned} \tag{3.29}$$

Notice that $c_{\tau\kappa} \leq c_{\epsilon_j}^{\tau\kappa}$, hence $c_{\tau\kappa} \leq c_{\epsilon_j}$. By Lemma 3.4, letting $\epsilon_j \to 0$, we have

$$c_{\tau\kappa} \leq c_{V_{\min}\kappa},$$

which is impossible since $c_{V_{\min}\kappa} < c_{\tau\kappa}$. Hence, c_ϵ is attained at some $u_\epsilon \neq 0$ for small $\epsilon > 0$. Arguing as in the proof of Proposition 3.2, we get $u_\epsilon > 0$. □

4. Concentration and convergence of ground state solutions

In this section, we study the concentration behavior of the ground state solutions u_ϵ as $\epsilon \to 0$. Our main result is the following:

Theorem 4.1. *Let u_ϵ be a solution of the problem $(\widetilde{Q}_\epsilon)$ given by Proposition 3.5, then u_ϵ possesses a global maximum point y_ϵ such that, up to a subsequence, $\epsilon y_\epsilon \to x_0$*

as $\epsilon \to 0$, $\lim_{\epsilon\to 0} \operatorname{dist}(\epsilon y_\epsilon, \mathcal{H}_1) = 0$ *and* $v_\epsilon(x) = u_\epsilon(x + y_\epsilon)$ *converges in* $H^s(\mathbf{R}^3)$ *to a positive ground state solution of*

$$(Q_0) \qquad \begin{cases} M\left(\iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbf{R}^3} V(x_0)u^2\,dx\right)[(-\Delta)^s u + V(x_0)u] \\ = K(x_0)f(u), \\ u \in H^s(\mathbf{R}^3). \end{cases}$$

In particular, if $\mathcal{V}\cap\mathcal{K} \neq \emptyset$, *then* $\lim_{\epsilon\to 0} \operatorname{dist}(\epsilon y_\epsilon, \mathcal{V}\cap\mathcal{K}) = 0$, *and up to a subsequence,* $v_\epsilon(x)$ *converges in* $H^s(\mathbf{R}^3)$ *to a positive ground state solution of*

$$(Q_{\min}) \qquad \begin{cases} M\left(\iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbf{R}^3} V_{\min}u^2\,dx\right)[(-\Delta)^s u + V_{\min}u] \\ = K_{\max}f(u), \\ u \in H^s(\mathbf{R}^3). \end{cases}$$

To prove Theorem 4.1, we need several lemmas.

Lemma 4.2. *There exists a* $\epsilon^* > 0$ *such that, for all* $\epsilon \in (0, \epsilon^*)$, *there exist* $\{y_\epsilon\} \subset \mathbf{R}^3$ *and* $\widetilde{R}$, $\tau > 0$ *such that*

$$\int_{B_{\widetilde{R}}(y_\epsilon)} u_\epsilon^2\,dx \geq \tau.$$

Proof. Arguing by contradiction, suppose that there exists a sequence $\epsilon_j \to 0$ as $j \to \infty$, such that for any $R > 0$,

$$\lim_{j\to\infty} \sup_{y\in\mathbf{R}^3} \int_{B_R(y)} u_{\epsilon_j}^2\,dx = 0.$$

We conclude from Lemma 2.2 that

$$u_{\epsilon_j} \to 0 \ \text{ in } \ L^q(\mathbf{R}^3) \ \text{ for } \ 2 < q < 2_s^*.$$

So,

$$a\|u_{\epsilon_j}\|_{\epsilon_j}^2 \leq M(\|u_{\epsilon_j}\|_{\epsilon_j}^2)\|u_{\epsilon_j}\|_{\epsilon_j}^2 = \int_{\mathbf{R}^3} K(\epsilon_j x)f(u_{\epsilon_j})u_{\epsilon_j}\,dx \to 0 \quad \text{as } j \to \infty.$$

Hence, $I_{\epsilon_j}(u_{\epsilon_j}) \to 0$ as $j \to \infty$, which contradicts $I_{\epsilon_j}(u_{\epsilon_j}) \to c_{\epsilon_j} > 0$. □

Set $v_\epsilon(x) = u_\epsilon(x + y_\epsilon)$, then v_ϵ satisfies

$$(4.1) \qquad \begin{aligned} &M\left([v_\epsilon]_s^2 + \int_{\mathbf{R}^3} V(\epsilon(x+y_\epsilon))v_\epsilon^2\,dx\right)[(-\Delta)^s v_\epsilon + V(\epsilon(x+y_\epsilon))v_\epsilon] \\ &= K(\epsilon(x+y_\epsilon))f(v_\epsilon) \end{aligned}$$

with energy

$$(4.2) \qquad \begin{aligned} J_\epsilon(v_\epsilon) &= \frac{1}{2}\widehat{M}\left([v_\epsilon]_s^2 + \int_{\mathbf{R}^3} V(\epsilon(x+y_\epsilon))v_\epsilon^2\,dx\right) - \int_{\mathbf{R}^3} K(\epsilon(x+y_\epsilon))F(v_\epsilon)\,dx \\ &= I_\epsilon(u_\epsilon) = c_\epsilon. \end{aligned}$$

We may assume $v_\epsilon \rightharpoonup u \neq 0$ in H_ϵ, and $v_\epsilon \to u$ in $L^t_{loc}(\mathbf{R}^3)$ for $t \in [1, 2_s^*)$.

By V, $K \in L^\infty(\mathbf{R}^3)$, we may assume that $V(\epsilon y_\epsilon) \to V_0$ and $K(\epsilon y_\epsilon) \to K_0$ as $\epsilon \to 0$.

Lemma 4.3. *u satisfies the following results:*

(i) *u is a positive ground state solution of*

$$M\left(\iint_{\mathbf{R}^3\times\mathbf{R}^3}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dx\,dy+\int_{\mathbf{R}^3}V_0u^2\,dx\right)[(-\Delta)^s u+V_0u]=K_0f(u). \tag{4.3}$$

(ii) *$v_\epsilon\to u$ in $H^s(\mathbf{R}^3)$.*

Proof. Since V, K are uniformly continuous, we have

$$|V(\epsilon(x+y_\epsilon))-V(\epsilon y_\epsilon)|\to 0 \quad\text{and}\quad |K(\epsilon(x+y_\epsilon))-K(\epsilon y_\epsilon)|\to 0 \ \text{ as } \epsilon\to 0$$

uniformly on bounded sets of $x\in\mathbf{R}^3$. Therefore, $V(\epsilon(x+y_\epsilon))\to V_0$ and $V(\epsilon(x+y_\epsilon))\to K_0$ as $\epsilon\to 0$ uniformly on bounded sets of $x\in\mathbf{R}^3$. By (4.1), using the similar proof in Proposition 3.5, we obtain that $v_\epsilon\to u$ in $H^s(\mathbf{R}^3)$ and therefore u solves (4.3) with energy

$$I_{V_0K_0}(u)=\frac{1}{2}\widehat{M}\left([u]_s^2+\int_{\mathbf{R}^3}V_0u^2\,dx\right)-\int_{\mathbf{R}^3}K_0F(u)\,dx\ge c_{V_0K_0}. \tag{4.4}$$

Arguing as in the proof of Lemma 3.4, by Fatou's lemma, we have

$$\begin{aligned} c_{V_0K_0}&\le I_{V_0K_0}(u)=I_{V_0K_0}(u)-\frac{1}{\theta}\langle I'_{V_0K_0}(u),u\rangle\\ &=\left(\frac{1}{2}\widehat{M}(\|u\|_{V_0}^2)-\frac{1}{\theta}M(\|u\|_{V_0}^2)\|u\|_{V_0}^2\right)+K_0\left(\frac{1}{\theta}\int_{\mathbf{R}^3}f(u)u\,dx-\int_{\mathbf{R}^3}F(u)\,dx\right)\\ &\le\liminf_{\epsilon\to 0}\left[\left[\frac{1}{2}\widehat{M}\left([v_\epsilon]_s^2+\int_{\mathbf{R}^3}V(\epsilon(x+y_\epsilon))v_\epsilon^2\,dx\right)\right.\right.\\ &\quad\left.-\frac{1}{\theta}M\left([v_\epsilon]_s^2+\int_{\mathbf{R}^3}V(\epsilon(x+y_\epsilon))v_\epsilon^2\,dx\right)\left([v_\epsilon]_s^2+\int_{\mathbf{R}^3}V(\epsilon(x+y_\epsilon))v_\epsilon^2\,dx\right)\right]\\ &\quad\left.+K(\epsilon(x+y_\epsilon))\left(\frac{1}{\theta}\int_{\mathbf{R}^3}f(v_\epsilon)v_\epsilon\,dx-\int_{\mathbf{R}^3}F(v_\epsilon)\,dx\right)\right]\\ &=\liminf_{\epsilon\to 0}J_\epsilon(v_\epsilon)\le\limsup_{\epsilon\to 0}I_\epsilon(u_\epsilon)\le c_{V_0K_0}. \end{aligned} \tag{4.5}$$

Hence,

$$\lim_{\epsilon\to 0}J_\epsilon(v_\epsilon)=\lim_{\epsilon\to 0}c_\epsilon=I_{V_0K_0}(u)=c_{V_0K_0}. \tag{4.6}$$

So, u is a ground state solution of problem (4.3). As in the proof of Proposition 3.2, u is positive. □

Lemma 4.4. *$\{\epsilon y_\epsilon\}$ is bounded.*

Proof. Arguing by contradiction, suppose that, after passing to a subsequence, $|\epsilon y_\epsilon|\to\infty$. By V, $K\in L^\infty(\mathbf{R}^3)$, we may assume that $V(\epsilon y_\epsilon)\to V_0$ and $K(\epsilon y_\epsilon)\to K_0$ as $\epsilon\to 0$. Since $V(0)=V_{\min}$ and $\kappa=K(0)\ge K(x)$ for all $|x|\ge R$, we deduce that $V_0>V_{\min}$ and $K_0\le\kappa$. It follows from Lemma 3.1 that $c_{V_0K_0}>c_{V_{\min}\kappa}$.

However, we conclude from (4.6) and Lemma 3.4 that $c_\epsilon\to c_{V_0K_0}\le c_{V_{\min}\kappa}$, which is a contradiction. Therefore, $\{\epsilon y_\epsilon\}$ is bounded. □

After passing to a subsequence, we may assume $\epsilon y_\epsilon\to x_0$ as $\epsilon\to 0$, then $V_0=V(x_0)$ and $K_0=K(x_0)$.

Lemma 4.5.

$$\lim_{\epsilon\to 0}\operatorname{dist}(\epsilon y_\epsilon, \mathcal{H}_1) = 0.$$

Proof. It suffices to show that $x_0 \in \mathcal{H}_1$. Arguing by contradiction, if $x_0 \notin \mathcal{H}_1$, then by (P_1) and Lemma 3.1, we deduce that $c_{V(x_0)K(x_0)} > c_{V_{\min}\kappa}$. So, by Lemma 3.4, we have

$$\lim_{\epsilon\to 0} c_\epsilon = c_{V(x_0)K(x_0)} > c_{V_{\min}\kappa} \geq \lim_{\epsilon\to 0} c_\epsilon,$$

which is a contration. □

To establish the L^∞-estimate of ground state solutions, we recall the following result which can be found in [24, Lemma 4.6].

Lemma 4.6. *Suppose that $f_* \in C^1(\mathbf{R},\mathbf{R})$ is convex, f_* and f_*' are Lipschitz continuous with the Lipschitz constant L, $f_*(0)$, $f_*'(0) = 0$. Then for each $u \in H^s(\mathbf{R}^3)$, $f_*(u)$, $f_*'(u) \in H^s(\mathbf{R}^3)$ and*

$$(-\Delta)^s f_*(u) \leq f_*'(u)(-\Delta)^s u \tag{4.7}$$

in the weak sense.

Lemma 4.7. *Let $\epsilon_n \to 0$ and v_{ϵ_n} be a solution of the following problem*

$$\begin{aligned} &M\left(\iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|v_{\epsilon_n}(x)-v_{\epsilon_n}(y)|^2}{|x-y|^{3+2s}}\,dx\,dy + \int_{\mathbf{R}^3} V(\epsilon_n(x+y_{\epsilon_n}))v_{\epsilon_n}^2\,dx\right)[(-\Delta)^s v_{\epsilon_n} \\ &\quad + V(\epsilon_n(x+y_{\epsilon_n}))v_{\epsilon_n}] = K(\epsilon_n(x+y_{\epsilon_n}))f(v_{\epsilon_n}) \quad \text{in } \mathbf{R}^3, \end{aligned} \tag{4.8}$$

where y_{ϵ_n} is given in Lemma 4.2. Then $v_{\epsilon_n} \in L^\infty(\mathbf{R}^3)$ and there exists $C > 0$ such that

$$|v_{\epsilon_n}|_\infty \leq C \quad \textit{uniformly in } n \in \mathbf{N}.$$

Moreover, $v_{\epsilon_n} \to u$ in $L^p(\mathbf{R}^3)$, $\forall\, p \in [2,+\infty)$.

Proof. We denote v_{ϵ_n} and y_{ϵ_n} by v_n and y_n, respectively. Define

$$h(x,v_n) = \frac{1}{a}K(\epsilon_n(x+y_{\epsilon_n}))f(v_n) - V(\epsilon_n(x+y_{\epsilon_n}))v_n.$$

For $\phi \in C_0^\infty(\mathbf{R}^3,\mathbf{R})$ with $\phi \geq 0$, by the assumption of v_{ϵ_n}, we have

$$\begin{aligned} &a\left[\int_{\mathbf{R}^3}(-\Delta)^s v_{\epsilon_n}\,dx + \int_{\mathbf{R}^3} V(\epsilon_n(x+y_{\epsilon_n}))v_{\epsilon_n}\phi\,dx\right] \\ &\leq M\left([v_{\epsilon_n}]_s^2 + \int_{\mathbf{R}^3} V(\epsilon_n(x+y_{\epsilon_n}))v_{\epsilon_n}^2\,dx\right)\left[\int_{\mathbf{R}^3}(-\Delta)^s v_{\epsilon_n}\phi\,dx\right. \\ &\quad \left. + \int_{\mathbf{R}^3} V(\epsilon_n(x+y_{\epsilon_n}))v_{\epsilon_n}\phi\,dx\right] \\ &= \int_{\mathbf{R}^3} K(\epsilon_n(x+y_{\epsilon_n}))f(v_{\epsilon_n})\phi\,dx \quad \text{in } \mathbf{R}^3. \end{aligned} \tag{4.9}$$

So,

$$(-\Delta)^s v_{\epsilon_n} \leq h(x,v_n) \quad \text{in the weak sense.} \tag{4.10}$$

From Lemma 4.3, we know that $\{v_n\}$ is bounded in $H^s(\mathbf{R}^3)$, hence, for any $p \in [2, 2_s^*]$, there exists some $C > 0$ such that

$$|v_n|_p \leq C$$

uniformly in n. We also have

$$|h(x,v_n)| \leq C(|v_n| + |v_n|^{q-1}) \leq C(1+|v_n|^{2_s^*-1}). \tag{4.11}$$

Arguing as in the proof of [24, Lemma 4.7], we obtain $v_n \in L^\infty(\mathbf{R}^3)$ and $|v_n|_\infty \le C$ uniformly in $n \in \mathbf{N}$. Moreover, we have $v_n \to u$ in $L^q(\mathbf{R}^3)$, $\forall\, q \in [2, +\infty)$. □

Lemma 4.8. [8, pages 1242–1243] *Assume that $u \in H^s(\mathbf{R}^N)$ satisfies the equation*

$$(-\Delta)^s u + u = g, \quad x \in \mathbf{R}^N,$$

and $g \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, then,

$$u = \mathcal{K} * g,$$

where $\mathcal{K}$ is the Bessel kernel

$$\mathcal{K}(x) = \mathcal{F}^{-1}\left(\frac{1}{1+|\xi|^{2s}}\right).$$

Lemma 4.9.

$$v_n(x) \to 0 \quad \text{as } |x| \to \infty \;\; \text{uniformly in } n.$$

Proof. Since v_n satisfies the equation

$$(-\Delta)^s v_n + v_n = \Theta_n, \quad x \in \mathbf{R}^3,$$

where

$$\Theta_n(x) = v_n(x) - V(\epsilon_n(x+y_n))v_n(x) + \frac{K(\epsilon_n(x+y_n))f(v_n)}{M\left([v_n]_s^2 + \int_{\mathbf{R}^3} K(\epsilon_n(x+y_n))v_n^2\,dx\right)}, \quad x \in \mathbf{R}^3.$$

Putting $\Theta(x) = u(x) - V(x_0)u(x) + \frac{K(x_0)f(u)}{M\left([u]_s^2 + \int_{\mathbf{R}^3} V(x_0)u^2\,dx\right)}$, we deduce from Lemma 4.7 that

$$\Theta_n \to \Theta \quad \text{in } L^q(\mathbf{R}^3), \; \forall\, q \in [2, +\infty),$$

and there exists a $C_2 > 0$ such that

$$|\Theta_n|_\infty \le C_2, \; \forall\, n \in \mathbf{N}.$$

By Lemma 4.8, we have that

$$v_n(x) = \mathcal{G} * \Theta_n = \int_{\mathbf{R}^3} \mathcal{G}(x-y)\Theta_n(y)\,dy,$$

where $\mathcal{G}$ is the Bessel Kernel

$$\mathcal{G}(x) = \mathcal{F}^{-1}\left(\frac{1}{1+|\xi|^{2s}}\right).$$

Now argue as in the proof of [[24], lemma 4.8], we deduce that

$$v_n(x) \to 0 \quad \text{as } |x| \to \infty \tag{4.12}$$

uniformly in $n \in \mathbf{N}$. □

Proof of Theorem 4.1. First we claim that there exists a $\rho_0 > 0$ such that $|v_n|_\infty \ge \rho_0$, $\forall\, n \in \mathbf{N}$. Otherwise, suppose that $|v_n|_\infty \to 0$ as $n \to \infty$. Let $\epsilon_0 = \frac{V_{\min}}{2}$, then there exists a $n_0 \in \mathbf{N}$ such that

$$K_{\max}|v_n|_\infty^{p-2} < \frac{V_{\min}}{2} \quad \text{for } n > n_0.$$

So, it follows from (f_1) and (f_2) that given $\varsigma \in (0, \frac{a}{K_{\max}})$, there is a $C_\varsigma > 0$ such that

$$\begin{aligned} & a\left([v_n]_s^2 + \int_{\mathbf{R}^3} V(\epsilon_n(x+y_{\epsilon_n}))v_n^2\,dx\right) \\ & \le M\left([v_n]_s^2 + \int_{\mathbf{R}^3} V(\epsilon_n(x+y_{\epsilon_n}))v_n^2\,dx\right)\left([v_n]_s^2 + \int_{\mathbf{R}^3} V(\epsilon_n(x+y_{\epsilon_n}))v_n^2\,dx\right) \\ & = \int_{\mathbf{R}^3} K(\epsilon_n(x+y_{\epsilon_n}))f(v_n)v_n\,dx \le K_{\max}\left(\varsigma\int_{\mathbf{R}^3} v_n^2\,dx + C_\varsigma\int_{\mathbf{R}^3} v_n^q\,dx\right) \\ & \le \varsigma K_{\max}\int_{\mathbf{R}^3} v_n^2\,dx + C_\varsigma K_{\max}|v_n|_\infty^{q-2}\int_{\mathbf{R}^3} v_n^2\,dx. \end{aligned} \tag{4.13}$$

This implies that $\|v_n\| = 0$ for $n > n_0$, which is impossible because $v_n \to u$ in $H^s(\mathbf{R}^3)$ and $u \neq 0$.

We conclude from Lemma 4.7 and Lemma 4.9 that v_n has a global maximum point p_n and $p_n \in B_{R_0}(0)$ for some $R_0 > 0$. Hence, u_{ϵ_n} has a global maximum point $p_n + y_n$. Define $\psi_n(x) = u_{\epsilon_n}(x + p_n + y_n)$, where $v_n(x) = u_{\epsilon_n}(x + y_n)$. Since $\{p_n\} \subset B_{R_0}(0)$ is bounded, then $\{\epsilon_n(p_n + y_n)\}$ is bounded and $\epsilon_n(p_n + y_n) \to x_0 \in \mathcal{H}_1$. Since $\{u_{\epsilon_n}\}$ is bounded in $H^s(\mathbf{R}^3)$, we know that $\{\psi_n\}$ is bounded in $H^s(\mathbf{R}^3)$, we may assume that $\psi_n \rightharpoonup \psi$ in $H^s(\mathbf{R}^3)$, $\psi_n \to \psi$ in $L^p_{loc}(\mathbf{R}^3)$ for $p \in [1, 2_s^*)$. On the other hand, we deduce from Lemma 4.2 that

$$\int_{B_{\widetilde{R}+R_0}(0)} \psi_n^2(x)\,dx \ge \int_{|x+p_n|<\widetilde{R}} \psi_n^2(x)\,dx = \int_{B_{\widetilde{R}}(y_n)} u_{\epsilon_n}^2(x)\,dx \ge \tau,$$

so we obtain $\psi \neq 0$. Moreover, similar to the argument above, we know that ψ is a ground state solution of (4.3) and $\psi_n \to \psi$ in $H^s(\mathbf{R}^3)$. Thus, ψ_n possesses same properties as v_n, and we can assume that y_n is a global maximum point of u_{ϵ_n}. We can then prove Theorem 4.1 by Lemmas 4.2–4.5. □

5. Decay estimates

In this section, we estimate the decay properties of the solution v_n to (4.1).

Lemma 5.1. *There exists a $C > 0$ such that*

$$v_n(x) \le \frac{C}{1+|x|^{3+2s}}, \ \forall\, x \in \mathbf{R}^3.$$

Proof. According to [[8], Lemma 4.3], there exists a continuous function $\overline{\omega}$ such that

$$0 < \overline{\omega}(x) \le \frac{C}{1+|x|^{3+2s}}, \tag{5.1}$$

and

$$(-\triangle)^s\overline{\omega} + \frac{V_{\min}}{2}\overline{\omega} = 0, \quad \text{in } \mathbf{R}^3\backslash B_{\bar{R}}(0) \tag{5.2}$$

for some suitable $\bar{R} > 0$. From Lemma 4.9, we know that $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in n. Therefore, by (M_1), (f_1) and (f_2), for $\epsilon > 0$ small enough and some large $R_1 > 0$, we obtain

$$
\begin{aligned}
& M\left([v_n]_s^2 + \int_{\mathbf{R}^3} V(\epsilon_n(x+y_{\epsilon_n}))v_n^2\,dx\right)\left((-\Delta)^s v_n + \frac{V_{\min}}{2}v_n\right) \\
&= M\left([v_n]_s^2 + \int_{\mathbf{R}^3} V(\epsilon_n(x+y_{\epsilon_n}))v_n^2\,dx\right)\Bigg((-\Delta)^s v_n + V(\epsilon_n(x+y_{\epsilon_n}))v_n \\
&\quad - \left(V(\epsilon_n(x+y_{\epsilon_n})) - \frac{V_{\min}}{2}\right)v_n\Bigg) \\
&= K(\epsilon_n(x+y_{\epsilon_n}))f(v_n) - M\left([v_n]_s^2 + \int_{\mathbf{R}^3} V(\epsilon_n(x+y_{\epsilon_n}))v_n^2\,dx\right) \\
&\quad \cdot \left(\left(V(\epsilon_n(x+y_{\epsilon_n})) - \frac{V_{\min}}{2}\right)v_n\right) \\
&\le \left(\epsilon K_{\max} + C_\epsilon K_{\max}|v_n|^{q-1} - \frac{V_{\min}}{2}\right)v_n \le 0
\end{aligned} \tag{5.3}
$$

for $x \in \mathbf{R}^3 \backslash B_{R_1}(0)$. Now we take $R_2 = \max\{\bar{R}, R_1\}$ and set

$$z_n = (m+1)\overline{\omega} - bv_n,$$

where $m = \sup_{n\in\mathbf{N}} |v_n|_\infty < \infty$ and $b = \min_{\bar{B}_{R_2}(0)} \overline{\omega} > 0$. Arguing as in the proof of [24, Lemma 5.1], we obtain

$$v_n(x) \le \frac{C}{1+|x|^{3+2s}},\ \forall\, x \in \mathbf{R}^3.$$

□

Proof of Theorem 1.1. Define $\omega_n(x) = u_n(\frac{x}{\epsilon_n})$, then ω_n is a positive ground state solution of problem (Q_ϵ) and $x_{\epsilon_n} = \epsilon_n y_n$ is a maximum point of ω_n. We conclude from Theorem 4.1 that the Theorem 1.1(A)(i), (ii) hold. Furthermore, we have, $\forall$ $x \in \mathbf{R}^3$,

$$
\begin{aligned}
\omega_n(x) &= u_n\left(\frac{x}{\epsilon_n}\right) = v_n\left(\frac{x}{\epsilon_n} - y_n\right) \le \frac{C}{1+|\frac{x}{\epsilon_n} - y_n|^{3+2s}} \\
&= \frac{C\epsilon_n^{3+2s}}{\epsilon_n^{3+2s} + |x - \epsilon_n y_n|^{3+2s}} = \frac{C\epsilon_n^{3+2s}}{\epsilon_n^{3+2s} + |x - x_{\epsilon_n}|^{3+2s}}.
\end{aligned} \tag{5.4}
$$

Hence, the proof of Theorem 1.1(A) is completed.

The proof of Theorem 1.1(B) is similar. □

6. Multiplicity of solutions to (Q_ϵ)

In this section, we apply the Ljusternik–Schnirelmann category theory to prove a multiplicity result for equation (Q_ϵ).

Let $w \in \mathcal{N}_{V_{\min}K_{\max}}$ satisfying $I_{V_{\min}K_{\max}}(w) = c_{V_{\min}K_{\max}}$ by Proposition 3.2. Let us consider a smooth nonincreasing cut-off function η with $0 \le \eta \le 1$, $\eta = 1$ on $B_1(0)$, $\eta = 0$ on $\mathbf{R}^3 \backslash B_2(0)$, $|\nabla \eta| \le C$. For any $y \in \mathcal{V} \cap \mathcal{K}$, we define the function

$$\psi_{\epsilon,y}(x) = \eta(\epsilon x - y)w\left(\frac{\epsilon x - y}{\epsilon}\right)$$

and $t_\epsilon > 0$ satisfying $\max_{t\ge 0} I_\epsilon(t\psi_{\epsilon,y}) = I_\epsilon(t_\epsilon\psi_{\epsilon,y})$ and $\frac{dI_\epsilon(t\psi_{\epsilon,y})}{dt}|_{t=t_\epsilon>0} = 0$.

Define $\Phi_\epsilon\colon \mathcal{V}\cap\mathcal{K}\to\mathcal{N}_\epsilon$ by $\Phi_\epsilon(y)=t_\epsilon\psi_{\epsilon,y}$. By construction, $\Phi_\epsilon(y)$ has a compact support for any $y\in\mathcal{V}\cap\mathcal{K}$.

Proposition 6.1. *Uniformly for $y\in\mathcal{V}\cap\mathcal{K}$, we have*

$$\lim_{\epsilon\to 0} I_\epsilon(\Phi_\epsilon(y)) = c_{V_{\min}K_{\max}}. \tag{6.1}$$

Proof. Arguing by contradiction. Suppose that there exist some $\rho>0$, $\{y_n\}\subset\mathcal{V}\cap\mathcal{K}$ and $\epsilon_n\to 0$ such that

$$|I_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) - c_{V_{\min}K_{\max}}| \geq \rho > 0. \tag{6.2}$$

First we claim that $\lim_{n\to\infty} t_{\epsilon_n}=1$. In fact, by the definition of t_{ϵ_n} and Lemma 2.6, we have

$$\begin{aligned} 2c_\epsilon &\leq t_{\epsilon_n}^2 M\left([t_{\epsilon_n}\psi_{\epsilon_n y_n}]_s^2 + t_{\epsilon_n}^2\int_{\mathbf{R}^3} V(\epsilon_n x)\psi_{\epsilon_n y_n}^2\,dx\right)\\ &\quad\cdot\left([\psi_{\epsilon_n y_n}]_s^2 + \int_{\mathbf{R}^3} V(\epsilon_n x)\psi_{\epsilon_n y_n}^2\,dx\right)\\ &= t_{\epsilon_n}\int_{\mathbf{R}^3} K(\epsilon_n x) f(t_{\epsilon_n}\psi_{\epsilon_n y_n})\psi_{\epsilon_n y_n}\,dx. \end{aligned} \tag{6.3}$$

It follows from (f_1), (f_2) and (6.3) that $t_{\epsilon_n}\not\to 0$, then $t_{\epsilon_n}\geq t_0>0$ for some $t_0>0$. Denote

$$\Lambda_n^2 = [\psi_{\epsilon_n y_n}]_s^2 + \int_{\mathbf{R}^3} V(\epsilon_n x)\psi_{\epsilon_n y_n}^2\,dx. \tag{6.4}$$

If $t_{\epsilon_n}\to\infty$, by the boundedness of $\psi_{\epsilon_n y_n}$, we have

$$\frac{M(t_{\epsilon_n}^2\Lambda_n^2)}{t_{\epsilon_n}^2\Lambda_n^2} = \frac{1}{\Lambda_n^4}\int_{\mathbf{R}^3} K(\epsilon_n x)\frac{f(t_{\epsilon_n}\psi_{\epsilon_n y_n})}{(t_{\epsilon_n}\psi_{\epsilon_n y_n})^3}\psi_{\epsilon_n y_n}^4\,dx \tag{6.5}$$

as $n\to\infty$, which is a contradiction. Hence, $0<t_0<t_{\epsilon_n}\leq C$. We assume that $t_{\epsilon_n}\to T$.

Next we show that $T=1$. By [[18], Lemma 5], we have

$$\lim_{n\to\infty}\iint_{\mathbf{R}^3\times\mathbf{R}^3}\frac{|\psi_{\epsilon_n y_n}(x)-\psi_{\epsilon_n y_n}(y)|^2}{|x-y|^{3+2s}}\,dx\,dy = \iint_{\mathbf{R}^3\times\mathbf{R}^3}\frac{|w(x)-w(y)|^2}{|x-y|^{3+2s}}\,dx\,dy. \tag{6.6}$$

Denote $\Lambda_w^2=[w]_s^2+\int_{\mathbf{R}^3} V_{\min}w^2\,dx$, it follows from Lebesgue's theorem and (6.3) that

$$\frac{M(T^2\Lambda_w^2)}{T^2\Lambda_w^2} = \frac{1}{\Lambda_w^4}\int_{\mathbf{R}^3} K_{\max}\frac{f(Tw)}{(Tw)^3}w^4\,dx. \tag{6.7}$$

Moreover, by the definition of w, we have

$$M(\Lambda_w^2)\Lambda_w^2 = \int_{\mathbf{R}^3} K_{\max} f(w)w\,dx. \tag{6.8}$$

We conclude from (6.7) and (6.8) that

$$\left(\frac{M(T^2\Lambda_w^2)}{T^2\Lambda_w^2} - \frac{M(\Lambda_w^2)}{\Lambda_w^2}\right) = \frac{1}{\Lambda_w^4}\int_{\mathbf{R}^3} K_{\max}\left(\frac{f(Tw)}{(Tw)^3}-\frac{f(w)}{(w)^3}\right)w^4\,dx. \tag{6.9}$$

By (M_3) and (f_3), we obtain that $T=1$. It follows from (6.6) and Lebesgue's theorem that

$$\lim_{n\to\infty} I_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) = I_{V_{\min}K_{\max}}(w) = c_{V_{\min}K_{\max}},$$

which contradicts to (6.2). This completes the proof. □

Consider $\delta > 0$ and choose $\rho = \rho(\delta) > 0$ such that $(\mathcal{V} \cap \mathcal{K})_\delta \subset B_\rho(0)$. Let $\Gamma\colon \mathbf{R}^3 \to \mathbf{R}^3$ be defined as $\Gamma(x) = x$ for $|x| \le \rho$ and $\Gamma(x) = \rho x/|x|$ for $|x| \ge \rho$, and consider the map $\beta_\epsilon\colon \mathcal{N}_\epsilon \to \mathbf{R}^3$ given by

$$\beta_\epsilon(u) = \frac{\displaystyle\int_{\mathbf{R}^3} \Gamma(\epsilon x) u^2\, dx}{\displaystyle\int_{\mathbf{R}^3} u^2\, dx}.$$

Lemma 6.2. *Uniformly in $y \in \mathcal{V} \cap \mathcal{K}$, we have*

$$\lim_{\epsilon \to 0} \beta_\epsilon(\Phi_\epsilon(y)) = y \quad \text{uniformly for } y \in \mathcal{V} \cap \mathcal{K}. \tag{6.10}$$

Proof. Arguing by contradiction, suppose that there exist $\delta_0 > 0$, $\{y_n\} \subset \mathcal{V} \cap \mathcal{K}$ and $\epsilon_n \to 0$ such that

$$|\beta_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) - y_n| \ge \delta_0.$$

Using the change of variables $z = \frac{\epsilon_n x - y_n}{\epsilon_n}$, we have

$$\beta_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) = y_n + \frac{\displaystyle\int_{\mathbf{R}^3} (\Gamma(\epsilon_n z + y_n) - y_n) |\eta(\epsilon_n z) w(z)|^2\, dx}{\displaystyle\int_{\mathbf{R}^3} |\eta(\epsilon_n z) w(z)|^2\, dx}.$$

Since $\{y_n\} \subset \mathcal{V} \cap \mathcal{K} \subset B_\rho(0)$ and $\Gamma|_{B_\rho(0)} \equiv id$, we deduce that

$$|\beta_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) - y_n| = o(1),$$

which is a contradiction and the lemma is proved. □

Lemma 6.3. *Let $\{u_n\} \subset \mathcal{N}_{V_{\min}K_{\max}}$ such that $I_{V_{\min}K_{\max}}(u_n) \to c_{V_{\min}K_{\max}}$. Then, either $\{u_n\}$ has a subsequence strongly convergent in $H^s(\mathbf{R}^3)$ or there exists a sequence $\{y_n\} \subset \mathbf{R}^3$ such that the sequence $v_n(x) = u_n(x + y_n)$ converges strongly in $H^s(\mathbf{R}^3)$. In particular, there exists a minimizer for $c_{V_{\min}K_{\max}}$.*

Proof. Arguing as in the proof of Lemma 2.4, we know that $I_{V_{\min}K_{\max}}$ is coercive on $\mathcal{N}_{V_{\min}K_{\max}}$. So, $\{u_n\}$ is a bounded sequence in $H^s(\mathbf{R}^3)$. It follows from Lemma 2.9 that $\{m^{-1}_{V_{\min}K_{\max}}(u_n)\}$ is a minimizer sequence of $\Upsilon_{V_{\min}K_{\max}}$. Let $w_n = m^{-1}_{V_{\min}K_{\max}}(u_n)$. By Ekeland's variational principle [23, Theorem 2.4], we may assume that $\Upsilon_{V_{\min}K_{\max}}(w_n) \to c_{V_{\min}K_{\max}}$ and $\Upsilon'_{V_{\min}K_{\max}}(w_n) \to 0$. Then, we have

$$I_{V_{\min}K_{\max}}(u_n) \to c_{V_{\min}K_{\max}},\ I'_{V_{\min}K_{\max}}(u_n) \to 0 \ \text{ and } \ \langle I'_{V_{\min}K_{\max}}(u_n), u_n\rangle = 0, \tag{6.11}$$

where $u_n = m_{V_{\min}K_{\max}}(w_n)$. Hence, passing to a subsequence, we may assume that $u_n \rightharpoonup u$ in $H^s(\mathbf{R}^3)$.

We distinguish the following two cases.

Case 1. $u \ne 0$. In this case, by the same arguments used in the proof of Proposition 3.2, it is easy to check that $u_n \to u$ in $H^s(\mathbf{R}^3)$.

Case 2. $u \equiv 0$. Since $\{u_n\} \subset \mathcal{N}_{V_{\min}K_{\max}}$, arguing as in the proof of Lemma 2.6(ii), we have $M(\|u_n\|^2_{V_{\min}})\|u_n\|^2_{V_{\min}} \ge 2c_{V_{\min}K_{\max}} > 0$. So $\|u_n\|^2_{V_{\min}} \not\to 0$ as $n \to \infty$. Since $c_{V_{\min}K_{\max}} \le c_{V_\infty K_\infty}$, using the same arguments used in the proof of Lemma 4.2, there exist $\{y_n\} \subset \mathbf{R}^3$ and constants R, $\sigma > 0$ such that

$$\liminf_{n \to \infty} \int_{B_R(y_n)} |u_n|^2\, dx \ge \sigma > 0. \tag{6.12}$$

Let $v_n(x) = u_n(x + y_n)$, then $I_{V_{\min}K_{\max}}(v_n) \to c_{V_{\min}K_{\max}}$ and $\langle I'_{V_{\min}K_{\max}}(v_n), v_n\rangle = 0$. It is clear that $\{v_n\}$ is bounded in $H^s(\mathbf{R}^3)$ and there exists $v \in H^s(\mathbf{R}^3)$ satisfying

$v_n \rightharpoonup v$ in $H^s(\mathbf{R}^3)$. We deduce from (6.12) and Lemma 2.2 that $v \neq 0$. Then the proof follows from the arguments used in case 1. □

Lemma 6.4. *Let $\epsilon_n \to 0$ and $\{u_n\} \subset \mathcal{N}_{\epsilon_n}$ such that $I_{\epsilon_n}(u_n) \to c_{V_{\min}K_{\max}}$. Then there exists a sequence $\{y_n\} \subset \mathbf{R}^3$ such that $\epsilon_n y_n \to y \in \mathcal{V} \cap \mathcal{K}$.*

Proof. By Lemma 2.4 and 2.6, we know that $\{u_n\}$ is bounded and $\|u_n\|_{\epsilon_n} \not\to 0$. Arguing as in the proof of Lemma 4.2, there exist a sequence $\{y_n\} \subset \mathbf{R}^3$ and positive constants R, $\beta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^2\,dx \geq \beta > 0.$$

Denote $v_n(x) = u_n(x + y_n)$, passing to a subsequence, we may assume that

$$v_n \rightharpoonup v \neq 0 \quad \text{in } H^s(\mathbf{R}^3) \quad \text{and} \quad v_n(x) \to v(x) \text{ a.e. in } \mathbf{R}^3. \tag{6.13}$$

Let $t_n \subset (0,\infty)$ be such that $w_n = t_n v_n \in \mathcal{N}_{V_{\min}K_{\max}}$, where

$$\mathcal{N}_{V_{\min}K_{\max}} = \{u \in H^s(\mathbf{R}^3)\backslash\{0\} \colon \langle I'_{V_{\min}K_{\max}}(u), u\rangle = 0\}.$$

It follows from the definition of $I_{V_{\min}K_{\max}}$ and $c_{V_{\min}K_{\max}}$ that

$$c_{V_{\min}K_{\max}} \leq I_{V_{\min}K_{\max}}(w_n) = I_{V_{\min}K_{\max}}(t_n v_n) \leq I_{\epsilon_n}(t_n u_n) \leq I_{\epsilon_n}(u_n) = c_{V_{\min}K_{\max}} + o(1),$$

so, $\lim_{n\to\infty} I_{V_{\min}K_{\max}}(t_n \tilde{u}_n) \to c_{V_{\min}K_{\max}}$. By Lemma 2.4, we know that $\{w_n\}$ is bounded, it follows from the boundedness of $\{v_n\}$ that $\{t_n\}$ is bounded, we may assume that $t_n \to t_0 \geq 0$. If $t_0 = 0$, by the boundedness of $\{v_n\}$ in $H^s(\mathbf{R}^3)$, we have $w_n = t_n v_n \to 0$ in $H^s(\mathbf{R}^3)$, then $I_{V_{\min}K_{\max}}(w_n) \to 0$, which contradicts $c_{V_{\min}K_{\max}} > 0$. Hence, $t_0 > 0$ and the weak limit of $\{w_n\}$ is different from zero. Thus, up to a subsequence, we have $w_n \rightharpoonup w = t_0 v \not\equiv 0$ in $H^s(\mathbf{R}^3)$ by the uniqueness of the weak limit. We deduce from Lemma 6.3 that $w_n \to w$ in $H^s(\mathbf{R}^3)$. Moreover, $w \in \mathcal{N}_{V_{\min}K_{\max}}$.

Now, we are going to prove that $\epsilon_n y_n \to y \in \mathcal{V} \cap \mathcal{K}$. First, we show that $\{\epsilon_n y_n\}$ is bounded in $\mathbf{R}^3$. Suppose that after passing to a subsequence, $|\epsilon_n y_n| \to \infty$. Then, we deduce from the Fatou's lemma that

$$\begin{aligned}
c_{V_{\min}K_{\max}} &= I_{V_{\min}K_{\max}}(w) < I_{V_\infty K_\infty}(w) = I_{V_\infty K_\infty}(w) - \frac{1}{4}\langle I'_{V_{\min}K_{\max}}(w), w\rangle \\
&= \frac{1}{2}\widehat{M}\left([w]_s^2 + \int_{\mathbf{R}^3} V_\infty w^2\,dx\right) - \frac{1}{\theta}M\left([w]_s^2 + \int_{\mathbf{R}^3} V_{\min} w^2\,dx\right) \\
&\quad \cdot \left([w]_s^2 + \int_{\mathbf{R}^3} V_{\min} w^2\,dx\right) + \int_{\mathbf{R}^3}\left(\frac{K_{\max}}{\theta} f(w)w - K_\infty F(w)\right)dx \\
&\leq \liminf_{n\to\infty}\left[\frac{1}{2}\widehat{M}\left([w_n]_s^2 + \int_{\mathbf{R}^3} V(\epsilon_n x + \epsilon_n y_n) w_n^2\,dx\right)\right. \\
&\quad - \frac{1}{\theta}M\left([w_n]_s^2 + \int_{\mathbf{R}^3} V_{\min} w_n^2\,dx\right) \cdot \left([w_n]_s^2 + \int_{\mathbf{R}^3} V_{\min} w_n^2\,dx\right) \\
&\quad \left. + \int_{\mathbf{R}^3}\left(\frac{K_{\max}}{\theta} f(w_n)w_n - K(\epsilon_n x + \epsilon_n y_n)F(w_n)\right)dx\right] \\
&= \liminf_{n\to\infty} I_{\epsilon_n}(t_n u_n) \leq \liminf_{n\to\infty} I_{\epsilon_n}(u_n) = c_{V_{\min}K_{\max}},
\end{aligned} \tag{6.14}$$

which is a contradiction. Thus, $\{\epsilon_n y_n\}$ is bounded and up to a subsequence, $\epsilon_n y_n \to y$ in $\mathbf{R}^3$. Now it suffices to show that $V(y) = V_{\min}$ and $K(y) = K_{\max}$. Arguing by contradiction again, suppose that $V(y) > V_{\min}$ or $K(y) < K_{\max}$,

$$\begin{aligned} c_{V_{\min}K_{\max}} &\le I_{V_{\min}K_{\max}}(w) < I_{V(y)K(y)}(w) \\ &\le \lim_{n\to\infty} I_{\epsilon_n}(t_n u_n) \le \lim_{n\to\infty} I_{\epsilon_n}(u_n) = c_{V_{\min}K_{\max}}, \end{aligned} \tag{6.15}$$

which does not make sense, thus $V(y) = V_{\min}$, $K(y) = K_{\max}$ and the proof is completed. □

Define

$$\widetilde{\mathcal{N}}_\epsilon = \{u \in \mathcal{N}_\epsilon \colon I_\epsilon(u) \le c_{V_{\min}K_{\max}} + h(\epsilon)\},$$

where $h(\epsilon) = |I_\epsilon(\Phi_\epsilon(y)) - c_{V_{\min}K_{\max}}|$. We can deduce from Proposition 6.1 that $h(\epsilon) \to 0$ as $\epsilon \to 0^+$. By the definition of $h(\epsilon)$, we know that, for any $y \in \mathcal{V} \cap \mathcal{K}$ and $\epsilon > 0$, $\Phi_\epsilon(y) \in \widetilde{\mathcal{N}}_\epsilon$ and $\widetilde{\mathcal{N}}_\epsilon \ne \emptyset$.

Lemma 6.5. *I_ϵ satisfies the $(P.S.)_c$ condition in $\widetilde{\mathcal{N}}_\epsilon$ for $c > 0$.*

Proof. Let $\{u_n\} \subset \widetilde{\mathcal{N}}_\epsilon$ satisfying

$$I_\epsilon(u_n) \to c \quad \text{and} \quad I'_\epsilon(u_n) \to 0. \tag{6.16}$$

By Lemma 2.4, we know that $\{u_n\}$ is bounded, then there exists a function $u \in H_\epsilon$ such that $u_n \rightharpoonup u$ in H_ϵ and $u_n(x) \to u(x)$ a.e. in $\mathbf{R}^3$. Arguing as in the proof of Proposition 3.2, there is a sequence $\{y_n\} \subset \mathbf{R}^3$ and constants R, $\eta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^2 \, dx \ge \eta > 0. \tag{6.17}$$

A direct computation shows that we can assume $\{y_n\} \subset \mathbf{Z}^3$. Considering $v_n(x) = u_n(x + y_n)$, since V and K are $\mathbf{Z}^3$-periodic function, we have that v_n is also bounded in $H^s(\mathbf{R}^3)$ and its weak limit denoted by v is nontrivial, because the last inequality together Sobolev embedding implies that

$$\liminf_{n\to\infty} \int_{B_R(0)} |v|^2 \, dx \ge \eta > 0.$$

Moreover, it follows from (6.16) that

$$I_\epsilon(v_n) \to c \quad \text{and} \quad I'_\epsilon(v_n) \to 0.$$

Hence, we can assume that $u \ne 0$, then by an argument similar to the proof of Proposition 3.2, we obtain that $u_n \to u$ in H_ϵ. □

Lemma 6.6. *For any $\delta > 0$, there holds that*

$$\lim_{\epsilon\to 0^+} \sup_{u\in\widetilde{\mathcal{N}}_\epsilon} \operatorname{dist}(\beta_\epsilon(u), (\mathcal{V} \cap \mathcal{K})_\delta) = 0.$$

Proof. Let $\epsilon_n \to 0$ as $n \to \infty$, for any $n \in \mathbf{N}$, there exists $\{u_n\} \subset \widetilde{\mathcal{N}}_\epsilon$ such that

$$\operatorname{dist}(\beta_{\epsilon_n}(u_n), (\mathcal{V} \cap \mathcal{K})_\delta) = \sup_{u\in\widetilde{\mathcal{N}}_\epsilon} \operatorname{dist}(\beta_{\epsilon_n}(u), (\mathcal{V} \cap \mathcal{K})_\delta) + o(1).$$

So, it suffices to find a sequence $\{\tilde{y}_n\} \subset (\mathcal{V} \cap \mathcal{K})_\delta$ satisfying

$$\lim_{n\to\infty} |\beta_{\epsilon_n}(u_n) - \epsilon_n y_n| = 0. \tag{6.18}$$

Since $\{u_n\} \subset \widetilde{\mathcal{N}}_{\epsilon_n} \subset \mathcal{N}_{\epsilon_n}$, we have

$$c_{V_{\min}K_{\max}} \le c_{V_{\epsilon_n}} \le I_{\epsilon_n}(u_n) \le c_{V_{\min}K_{\max}} + h(\epsilon_n).$$

So, $I_{\epsilon_n}(u_n) \to c_{V_{\min}K_{\max}}$. By Lemma 6.4, we can obtain a sequence $\{y_n\} \subset \mathbf{R}^3$ such that

$$\epsilon_n y_n \to y \in \mathcal{V}\cap\mathcal{K} \subset (\mathcal{V}\cap\mathcal{K})_\delta$$

for n large enough. Thus,

$$\beta_{\epsilon_n}(u_n) = \epsilon_n y_n + \frac{\int_{\mathbf{R}^3} \Gamma(\epsilon_n z + \epsilon_n y_n) u_n^2(z+y_n)\,dx}{\int_{\mathbf{R}^3} u_n^2(z+y_n)\,dx}.$$

Since $\epsilon_n z + \epsilon_n y_n \to y \in \mathcal{V}\cap\mathcal{K}$, we have that the sequence $\{\epsilon_n y_n\}$ satisfies (6.18). This completes the proof. □

Now, we are already to present the proof of the multiplicity results in the following.

Proof of Theorem 1.2. Define $\gamma_\epsilon(y) = m_\epsilon^{-1}(\Phi_\epsilon(y))$ for $y \in \mathcal{V}\cap\mathcal{K}$. It follows from Proposition 6.1 that

$$\lim_{\epsilon\to 0} \Upsilon_\epsilon(\gamma_\epsilon(y)) = \lim_{\epsilon\to 0} I_\epsilon(\Phi_\epsilon(y)) = c_{V_{\min}K_{\max}} \tag{6.19}$$

uniformly in $y \in \mathcal{V}\cap\mathcal{K}$. Let

$$\mathcal{N}_\epsilon^* = \{w \in S_\epsilon \colon \Upsilon_\epsilon(w) \le c_{V_{\min}K_{\max}} + h(\epsilon)\}.$$

We deduce from (6.19) that $\mathcal{N}_\epsilon^* \neq \emptyset$ for $\epsilon > 0$ small.

Given $\delta > 0$, we can use Proposition 6.1, Lemma 6.2 and Lemma 6.6 to obtain some $\epsilon_\delta > 0$ such that for any $\epsilon \in (0, \epsilon_\delta)$, the diagram

$$(\mathcal{V}\cap\mathcal{W}) \xrightarrow{\Phi_\epsilon} \widetilde{\mathcal{N}}_\epsilon \xrightarrow{m_\epsilon^{-1}} \mathcal{N}_\epsilon^* \xrightarrow{m_\epsilon} \widetilde{\mathcal{N}}_\epsilon \xrightarrow{\beta_\epsilon} (\mathcal{V}\cap\mathcal{W})_\delta$$

is well defined. By Lemma 6.2, for ϵ small enough, we can denote by $\beta_\epsilon(\Phi_\epsilon(y)) = y + \theta(y)$ for $y \in \mathcal{V}\cap\mathcal{W}$, where $|\theta(y)| < \frac{\delta}{2}$ uniformly for $y \in \mathcal{V}\cap\mathcal{W}$. Define $S(t,y) = y + (1-t)\theta(y)$. Thus $S\colon [0,1]\times(\mathcal{V}\cap\mathcal{W}) \to (\mathcal{V}\cap\mathcal{W})_\delta$ is continuous. Obviously, $S(0,y) = \beta_\epsilon(\Phi_\epsilon(y))$ and $S(1,y) = y$ for all $y \in \mathcal{V}\cap\mathcal{W}$. That is, $\beta_\epsilon\circ\Phi_\epsilon$ is homotopically equivalent to the map $\mathrm{Id}\colon (\mathcal{V}\cap\mathcal{W}) \to (\mathcal{V}\cap\mathcal{W})_\delta$. By [4, Lemma 4.3], we obtain that

$$\mathrm{cat}_{\mathcal{N}_\epsilon^*}(\mathcal{N}_\epsilon^*) \ge \mathrm{cat}_{(\mathcal{V}\cap\mathcal{W})_\delta}(\mathcal{V}\cap\mathcal{W}).$$

We can conclude from Lemma 6.5 that I_ϵ satisfies the (P.S.) condition in $\widetilde{\mathcal{N}}_\epsilon$ for all small $\epsilon > 0$. It follows from Lemma 2.9 and from the category abstract theorem (see [21], Corollary 28), with $c = c_\epsilon \le c_{V_{\min}K_{\max}} + h(\epsilon) = d$ and $K = \mathcal{N}_\epsilon^*$, that Υ_ϵ has at least $\mathrm{cat}_{\mathcal{N}_\epsilon^*}(\mathcal{N}_\epsilon^*)$ critical points on $\mathcal{N}_\epsilon^*$. By Lemma 2.9 again, we deduce that I_ϵ has at least $\mathrm{cat}_{(\mathcal{V}\cap\mathcal{K})_\delta}(\mathcal{V}\cap\mathcal{K})$ critical points. Arguing as in the proof of Proposition 3.2, we know that $(\widetilde{Q}_\epsilon)$ has at least $\mathrm{cat}_{(\mathcal{V}\cap\mathcal{K})_\delta}(\mathcal{V}\cap\mathcal{K})$ nonnegative solutions and therefore, up to a change of variables, (Q_ϵ) has at least $\mathrm{cat}_{(\mathcal{V}\cap\mathcal{K})_\delta}(\mathcal{V}\cap\mathcal{K})$ nonnegative solutions. Proceeding as we prove Theorem 1.1, we can complete the proof. □

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Received 1 March 2018 • Revised received 8 April 2018 • Accepted 20 April 2018