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LIPSCHITZ EQUIVALENCE OF SELF-SIMILAR SETS WITH EXACT OVERLAPS

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Abstract. In this paper, we study a class $\mathcal{A}(\lambda, n, m)$ of self-similar sets with m exact overlaps generated by n similitudes of the same ratio λ. We obtain a necessary condition for a self-similar set in $\mathcal{A}(\lambda, n, m)$ to be Lipschitz equivalent to a self-similar set satisfying the strong separation condition, i.e., there exists an integer $k \geq 2$ such that $x^{2k} - mx^k + n$ is reducible, in particular, m belongs to $\{a^i : a \in \mathbf{N} \text{ with } i \geq 2\}$.

1. Introduction

Recall that a compact subset K of Euclidean space is said to be a self-similar set [6], if $K = \bigcup_{i=1}^n S_i(K)$ is generated by contractive similitudes $\{S_i\}_i$ with ratio set $\{r_i\}_i \subset (0,1)$ satisfying $|S_i(x) - S_i(y)| = r_i|x-y|$ for all x, y. The classical dimension result under the open set condition (OSC) is

$$\dim_H K = s \quad \text{with} \quad \sum\nolimits_{i=1}^n (r_i)^s = 1. \tag{1.1}$$

In particular, K is said to be *dust-like* when the strong separation condition (SSC) holds, i.e., $S_i(K) \cap S_j(K) = \emptyset$ for all $i \neq j$, then the open set condition holds and thus (1.1) is valid.

The self-similar sets with overlaps have complicated structures, for example, Hochman [5] studied the self-similar sets

$$E_\theta = E_\theta/3 \cup (E_\theta/3 + \theta/3) \cup (E_\theta/3 + 2/3)$$

and obtained $\dim_H E_\theta = 1$ for any θ irrational. If θ is rational, Kenyon [8] obtained that the OSC is fulfilled for E_θ if and only if $\theta = p/q \in \mathbf{Q}$ with $p \equiv q \not\equiv 0 \pmod 3$. Rao and Wen [11] also discussed the structure of E_θ with $\theta \in \mathbf{Q}$ using the key idea "graph-directed structure" introduced by Mauldin and Williams [9].

Recently, Jiang, Wang and Xi [7] investigated a class $\mathcal{A}(\lambda, n, m)$ of self-similar sets with exact overlaps where $\lambda \in (0,1)$ and $m, n \in \mathbf{N}$ with $1 \leq m \leq n-2$. Let $f_i(x) = \lambda x + b_i$ with $0 = b_1 < b_2 < \cdots < b_n = 1 - \lambda$. Write $I = [0,1]$ and $I_i = f_i(I)$. Assume that

$$\frac{|I_i \cap I_{i+1}|}{|I_i|} \in \{0, \lambda\} \ \text{ if } \ I_i \cap I_{i+1} \neq \emptyset, \quad \text{and} \quad \sharp\left\{i : \frac{|I_i \cap I_{i+1}|}{|I_i|} = \lambda\right\} = m.$$

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We call $E = \cup_{i=1}^n f_i(E)$ a self-similar set with exact overlap, denoted by $E \in \mathcal{A}(\lambda, n, m)$. It is proved in [7] that $\dim_H E = \frac{\log\beta}{-\log\lambda}$ where the P.V. number $\beta > 1$ is a root of the irreducible polynomial $x^2 - nx + m = (x-\beta)(x-\beta')$ with $|\beta'| < 1 < \beta$.

In this paper, we will compare self-similar sets in $\mathcal{A}(\lambda, n, m)$ with dust-like self-similar sets in terms of Lipschitz equivalence.

Two compact subsets X_1, X_2 of Euclidean spaces are said to be Lipschitz equivalent, denoted by $X_1 \simeq X_2$, if there is a bijection $f\colon X_1 \to X_2$ and a constant $C > 0$ such that for all $x, y \in X_1$,

$$C^{-1}|x-y| \le |f(x) - f(y)| \le C|x-y|.$$

Cooper and Pignataro [1], Falconer and Marsh [3], David and Semmes [2] and Wen and Xi [12] showed that two self-similar sets need not be Lipschitz equivalent although they have the same Hausdorff dimension.

We concern the Lipschitz equivalence between two self-similar sets with the **SSC** and with overlaps respectively.

(1) David and Semmes [2] posed the $\{1,3,5\}$-$\{1,4,5\}$ problem. Let $H_1 = (H_1/5) \cup (H_1 + 2/5) \cup (H_1 + 4/5)$ and $H_2 = (H_2/5) \cup (H_2 + 3/5) \cup (H_2 + 4/5)$ be $\{1,3,5\}$, $\{1,4,5\}$ self-similar sets respectively. The problem asks about the Lipschitz equivalence between H_1 (with the SSC) and H_2 (with the touched structure). Rao, Ruan and Xi [10] proved that H_1 and H_2 are Lipschitz equivalent.

(2) Guo et al. [4] studied the Lipschitz equivalence for $K_n = (\lambda K_n) \cup (\lambda K_n + \lambda^n(1-\lambda)) \cup (\lambda K_n + 1 - \lambda)$ with overlaps and proved that $K_n \simeq K_m$ for all $n, m \ge 1$. In particular, for $n = 1$, $K_1 \in \mathcal{A}(\lambda, 3, 1)$ is Lipschitz equivalent to a dust-like set $F = (\lambda F) \cup (\lambda^{1/2} F + 1 - \lambda^{1/2})$.

We will state our main result.

Theorem 1. *Suppose $E \in \mathcal{A}(\lambda, n, m)$ and $P(x) = x^2 - nx + m$. If there is a dust-like self-similar set F such that $E \simeq F$, then there exists an integer $k \ge 2$ such that*

$$P(x^k) = x^{2k} - nx^k + m \text{ is reducible in } \mathbf{Z}[x].$$

In particular, we have

$$m \in \{a^i \mid a \in \mathbf{N} \text{ and } i \in \mathbf{N} \text{ with } i \ge 2\}.$$

By this theorem, if $m \in \{2,3,5,6,7,10,11,12,13,14,15,17,\cdots\}$, then we cannot find a dust-like self-similar set to be Lipschitz equivalent to $E \in \mathcal{A}(\lambda, n, m)$.

Example 1. For $n = 3$ and $m = 1$, we have $P(x) = x^2 - 3x + 1$ and an example $K_1 \simeq F = (\lambda F) \cup (\lambda^{1/2} F + 1 - \lambda^{1/2})$ in [4] as above. Now, $P(x^2) = (x^2 - x - 1)(x^2 + x - 1)$ is reducible and $1 \in \{a^i \mid a \in \mathbf{N}$ and $i \in \mathbf{N}$ with $i \ge 2\}$.

The paper is organized as follows. In Section 2 we show any self-similar set in $\mathcal{A}(\lambda, n, m)$ has graph-directed structure and obtain the logarithmic commensurability of ratios for the dust-like self-similar set by the approach of Falconer and Marsh [3]. Using the dimension polynomials and their irreducibility, we give the proof of Theorem 1 in Section 3.

2. Logarithmic commensurability of ratios

At first, we show that any self-similar set with exact overlaps will generate a graph-directed construction.

Lemma 1. *There are graph-directed sets $\{E_i\}_{i=1}^u$ with ratio λ satisfying the SSC and $E_1 = E$.*

Proof. Consider the set G in the following form

$$G = \bigcup_{i=1}^{k} (E + a_i) \quad \text{with} \quad 0 = a_1 < a_2 < \cdots < a_k \quad \text{and} \quad k \le n - 1$$

such that $(I + a_i) \cap (I + a_{i+1}) \neq \emptyset$ with $I = [0, 1]$ for all $i \le k - 1$ satisfying

$$|(I + a_i) \cap (I + a_{i+1})| = 0 \text{ or } \lambda.$$

Let $\mathcal{G}$ be the collection of all sets in the form as above. For every $G \in \mathcal{G}$, considering the natural decomposition at the touched point ($|(I+a_i)\cap(I+a_{i+1})| = 0$) or on the exact overlapping ($|(I + a_i) \cap (I + a_{i+1})| = \lambda$), we have the decomposition

$$G = \bigcup_{G' \in \mathcal{G}} \bigcup_{i} (\lambda G' + b_{i,G,G'})$$

which is a disjoint union. That means we obtain a graph directed construction satisfying the SSC. In fact, we only need to choose a subgraph generated by E with $k = 1$. □

The main result of this section is the following Proposition 1. We will use the approach by Falconer and Marsh [3]. In [3], the authors discussed the dust-like self-similar sets, now we will deal with the graph-directed sets.

Proposition 1. *Suppose $E \in \mathcal{A}(\lambda, n, m)$ and $F = \bigcup_{j=1}^{t} g_j(F)$ is a dust-like self-similar set such that $E \simeq F$. Assume r_j is the contractive ratio of g_j for any j. Then there is a ratio $r \in (0, 1)$ and positive integers k and $k_1 \le k_2 \le \cdots \le k_t$ such that*

$$\lambda = r^k, \quad r_1 = r^{k_1}, \quad r_2 = r^{k_2}, \cdots, \quad r_t = r^{k_t}.$$

Without loss of generality, we only need to show that

$$\frac{\log r_j}{\log \lambda} \in \mathbf{Q},$$

or $\frac{\log (r_j)^s}{\log \lambda^s} \in \mathbf{Q}$ with $s = \dim_H E = \dim_H F$. Suppose $f\colon F \to E$ is a bi-Lipschitz bijection and $c \ge 1$ is a constant satisfying

$$c^{-1}|x - y| \le |f(x) - f(y)| \le c|x - y| \quad \text{for all} \quad x, y \in F.$$

Denote $\Sigma^* = \bigcup_{k \ge 0} \{1, \cdots, t\}^k$. For any $\mathbf{j} = j_1 \cdots j_k \in \Sigma^*$, we write $F_{\mathbf{j}} = g_{j_1 \cdots j_k}(F)$.

Suppose $\mathbf{e}$ is an admissible path of length $|\mathbf{e}|$ in the directed graph beginning at vertex $v = b(\mathbf{e})$, then

$$|E_{\mathbf{e}}| = \lambda^{|\mathbf{e}|}|E_v| \quad \text{and} \quad \mathcal{H}^s(E_{\mathbf{e}}) = \lambda^{s|\mathbf{e}|}\mathcal{H}^s(E_v) = \lambda^{s|\mathbf{e}|}\mathcal{H}^s(E_{b(\mathbf{e})}). \tag{2.1}$$

Because of the SSC on F, we assume that there is a constant $\xi > 0$ such that

$$d(F_{\mathbf{j}}, F \backslash F_{\mathbf{j}}) \ge \xi |F_{\mathbf{j}}| \quad \text{for all} \quad \mathbf{j} \in \mathbf{\Sigma}^*, \tag{2.2}$$

and

$$\xi |E_{\mathbf{e_j}}| \le |F_{\mathbf{j}}| \le \xi^{-1} |E_{\mathbf{e_j}}| \quad \text{for all} \quad \mathbf{j} \in \mathbf{\Sigma}^*, \tag{2.3}$$

where we denote by $E_{\mathbf{e_j}} (\subset E)$ the smallest copy containing $f(F_{\mathbf{j}})$.

Lemma 2. *There is a positive integer N such that for any copy $F_{\mathbf{j}}$ of F and smallest copy $E_{\mathbf{e_j}} (\subset E)$ containing $f(F_{\mathbf{j}})$, there is a set $\Delta_{\mathbf{j}}$ composed of pathes $\mathbf{e}'$ with length N satisfying*

$$f(F_{\mathbf{j}}) = \bigcup_{\mathbf{e}' \in \Delta_{\mathbf{j}}} E_{\mathbf{e_j} * \mathbf{e}'}.$$

Proof. Now let $N = [\frac{\log c^{-1}\xi^2(n-1)^{-1}}{\log\lambda}]+1$. It suffices to show that if $z \in E_{\mathbf{e_j}*\mathbf{e}'}$ with $E_{\mathbf{e_j}*\mathbf{e}'} \cap f(F_{\mathbf{j}}) \neq \emptyset$ then $z \in f(F_{\mathbf{j}})$. In fact, if $z \in f(F\backslash F_{\mathbf{j}})$ and $z' \in E_{\mathbf{e_j}*\mathbf{e}'} \cap f(F_{\mathbf{j}})$, by (2.2)–(2.3) we have

$$|z-z'| \geq d(f(F_{\mathbf{j}}), f(F\backslash F_{\mathbf{j}})) \geq c^{-1}\xi|F_{\mathbf{j}}| \geq c^{-1}\xi^2|E_{\mathbf{e_j}}|.$$

On the other hand, using (2.1) and the fact that $1 = |E| \leq |E_v| \leq n-1$, we have

$$|z-z'| \leq |E_{\mathbf{e_j}*\mathbf{e}'}| \leq \lambda^N(n-1)|E_{\mathbf{e_j}}| < c^{-1}\xi^2|E_{\mathbf{e_j}}|,$$

this is a contradiction. □

For any Borel set $B \subset F$, we let

$$h(B) = \frac{\mathcal{H}^s(f(B))}{\mathcal{H}^s(B)}.$$

Since $f\colon F \to E$ is bi-Lipschitz, we have

$$d = \sup_{\mathbf{j}\in\Sigma^*} h(F_{\mathbf{j}}) < \infty.$$

Lemma 3. *There is a finite set* Λ *such that*

$$\frac{h(F_{\mathbf{j}*j})}{h(F_{\mathbf{j}})} \in \Lambda$$

for all $\mathbf{j} \in \Sigma^*$ *and all* $j \in \{1, \cdots, t\}$.

Proof. We note that

$$\frac{h(F_{\mathbf{j}*j})}{h(F_{\mathbf{j}})} = \frac{\mathcal{H}^s(f(F_{\mathbf{j}*j}))/\mathcal{H}^s(F_{\mathbf{j}*j})}{\mathcal{H}^s(f(F_{\mathbf{j}}))/\mathcal{H}^s(F_{\mathbf{j}})} = \frac{\mathcal{H}^s(F_{\mathbf{j}})}{\mathcal{H}^s(F_{\mathbf{j}*j})} \cdot \frac{\lambda^{s|\mathbf{e}_{\mathbf{j}*j}|}}{\lambda^{s|\mathbf{e_j}|}} \cdot \frac{\mathcal{H}^s(f(F_{\mathbf{j}*j}))/\lambda^{s|\mathbf{e}_{\mathbf{j}*j}|}}{\mathcal{H}^s(f(F_{\mathbf{j}}))/\lambda^{s|\mathbf{e_j}|}}.$$

Now, $\frac{\mathcal{H}^s(F_{\mathbf{j}})}{\mathcal{H}^s(F_{\mathbf{j}*j})} \in \{(r_j)^{-s}\}_{j=1}^t$. Suppose M is a upper bound for difference of lengths of $\mathbf{e}_{\mathbf{j}*j}$ and $\mathbf{e_j}$, we have

$$\frac{\lambda^{s|\mathbf{e}_{\mathbf{j}*j}|}}{\lambda^{s|\mathbf{e_j}|}} \in \{\lambda^{sk}\colon k \leq M\}$$

which is a finite set. By Lemma 2, we also obtain that

$$\frac{\mathcal{H}^s(f(F_{\mathbf{j}}))}{\lambda^{s|\mathbf{e_j}|}} = \frac{\sum_{\mathbf{e}'\in\Delta_{\mathbf{j}}} \mathcal{H}^s(E_{\mathbf{e_j}*\mathbf{e}'})}{\lambda^{s|\mathbf{e_j}|}} = \lambda^{s(|\mathbf{e_j}|+N)}\frac{\sum_{\mathbf{e}'\in\Delta_{\mathbf{j}}} \mathcal{H}^s(E_{b(\mathbf{e}')})}{\lambda^{s|\mathbf{e_j}|}}$$
$$\in \lambda^{sN}\left\{\sum_{\mathbf{e}'\in\Delta} \mathcal{H}^s(E_{b(\mathbf{e}')})\colon \Delta \subset \{\mathbf{e}'\colon |\mathbf{e}'| = N\}\right\}$$

which is also a finite set. □

Lemma 4. *There is a copy* $F_{j_1\cdots j_{k^*}}$ *of* F *and a constant* $\bar{d} > 0$ *such that*

$$\frac{\mathcal{H}^s(f(B))}{\mathcal{H}^s(B)} = \bar{d} \tag{2.4}$$

for Borel set $B \subset F_{j_1\cdots j_{k^*}}$.

Proof. Suppose $\alpha = \max_{x\in(-\infty,1)\cap\Lambda} x < 1$ or $\alpha = 1/2$ if $(-\infty,1)\cap\Lambda = \emptyset$. Take $\epsilon > 0$ such that

$$\max_i(\alpha r_i^s + (1+\epsilon)(1-r_i^s)) < 1. \tag{2.5}$$

Let $d = \sup_{\mathbf{j}\in\Sigma^*} h(F_{\mathbf{j}}) < \infty$ and take a sequence $\mathbf{j} = j_1 \cdots j_{k^*}$ such that $\frac{d}{h(F_{\mathbf{j}})} < 1+\epsilon$. We notice that

$$\bar{d} \hat{=} h(F_{\mathbf{j}}) = \sum_j \frac{\mathcal{H}^s(F_{\mathbf{j}*j})}{\mathcal{H}^s(F_{\mathbf{j}})} h(F_{\mathbf{j}*j}) \quad \text{with} \quad \sum_j \frac{\mathcal{H}^s(F_{\mathbf{j}*j})}{\mathcal{H}^s(F_{\mathbf{j}})} = \sum_j (r_j)^s = 1,$$

i.e., we have

$$1 = \sum_j (r_j)^s \frac{h(F_{\mathbf{j}*j})}{h(F_{\mathbf{j}})} \quad \text{with} \quad \sum_j (r_j)^s = 1. \tag{2.6}$$

We will first show that $h(F_{\mathbf{j}*j}) \geq h(F_{\mathbf{j}})$ for all j. Otherwise, without loss of generality, we assume that $\frac{h(F_{\mathbf{j}*1})}{h(F_{\mathbf{j}})} < 1$. Then

$$\frac{h(F_{\mathbf{j}*1})}{h(F_{\mathbf{j}})} \leq \alpha \quad \text{and} \quad \frac{h(F_{\mathbf{j}*j})}{h(F_{\mathbf{j}})} \leq \frac{d}{h(F_{\mathbf{j}})} < 1+\epsilon \quad \text{for} \quad j \geq 2.$$

It follows from (2.5) that

$$1 = \sum_j (r_j)^s \frac{h(F_{\mathbf{j}*j})}{h(F_{\mathbf{j}})} \leq \alpha r_1^s + (1+\epsilon)(1-r_1^s) < 1,$$

this is a contradiction. Now $h(F_{\mathbf{j}*j}) \geq h(F_{\mathbf{j}})$ for all j, by (2.6) we obtain that

$$h(F_{\mathbf{j}*j}) = h(F_{\mathbf{j}}) = \bar{d} \quad \text{for all} \quad j.$$

In the same way, we have

$$h(F_{\mathbf{j}*j_1*j_2}) = h(F_{\mathbf{j}}) = \bar{d} \quad \text{for all} \quad j_1, j_2.$$

Again and again, we obtain

$$h(F_{\mathbf{j}'}) = \bar{d} \quad \text{for any} \quad \mathbf{j}' \quad \text{with prefix} \quad \mathbf{j}.$$

Then (2.4) follows. □

Proof of Proposition 1. Take $\mathbf{j} = j_1 \cdots j_{k^*}$ in Lemma 4. For any j, we consider the sequence $\mathbf{j}[j]^k = \mathbf{j} * [j]^k$ where the sequence $[j]^k$ is composed of k successive digits j. Then

$$\frac{h(F_{\mathbf{j}[j]^{k'}})}{h(F_{\mathbf{j}[j]^k})} = 1 \quad \text{with} \quad k > k'.$$

Hence we obtain that

$$\begin{aligned}(r_j^s)^{k-k'} &= \frac{\mathcal{H}^s(F_{\mathbf{j}[j]^k})}{\mathcal{H}^s(F_{\mathbf{j}[j]^{k'}})} = \frac{h(F_{\mathbf{j}[j]^{k'}})}{h(F_{\mathbf{j}[j]^k})} \cdot \frac{\sum_{\mathbf{e}'\in\Delta_{\mathbf{j}[j]^k}} \mathcal{H}^s(E_{b(\mathbf{e}')})}{\sum_{\mathbf{e}'\in\Delta_{\mathbf{j}[j]^{k'}}} \mathcal{H}^s(E_{b(\mathbf{e}')})} \cdot \lambda^{s(|\mathbf{e}_{\mathbf{j}[j]^k}|-|\mathbf{e}_{\mathbf{j}[j]^{k'}}|)} \\ &= \frac{\sum_{\mathbf{e}'\in\Delta_{\mathbf{j}[j]^k}} \mathcal{H}^s(E_{b(\mathbf{e}')})}{\sum_{\mathbf{e}'\in\Delta_{\mathbf{j}[j]^{k'}}} \mathcal{H}^s(E_{b(\mathbf{e}')})} \cdot \lambda^{s(|\mathbf{e}_{\mathbf{j}[j]^k}|-|\mathbf{e}_{\mathbf{j}[j]^{k'}}|)}.\end{aligned}$$

From the finiteness, we can find $k \neq k'$ such that $\Delta_{\mathbf{j}[j]^k} = \Delta_{\mathbf{j}[j]^{k'}}$ then

$$(r_j^s)^{k-k'} = \lambda^{s(|\mathbf{e}_{\mathbf{j}[j]^k}|-|\mathbf{e}_{\mathbf{j}[j]^{k'}}|)},$$

that means $(r_j)^{k-k'} = \lambda^{|\mathbf{e}_{\mathbf{j}[j]^k}|-|\mathbf{e}_{\mathbf{j}[j]^{k'}}|}$, i.e.,

$$\log r_j / \log \lambda \in \mathbf{Q}$$

for all j. Then Proposition 1 is proved. □

3. Proof of Theorem

3.1. Dimension polynomials. From [7] we have

$$P(x) = x^2 - nx + m = (x - \beta)(x - \beta') \quad \text{with} \quad |\beta'| < 1 < \beta.$$

Using notations in Proposition 1, we consider the following two polynomials

$$\bar{P}(x) = P(x^k) \quad \text{and} \quad \bar{Q}(x) = x^{k_t} - \sum_{i=1}^{t} x^{k_t - k_i}. \tag{3.1}$$

Proposition 2. *Let $s = \dim_H E = \dim_H F$ and r the ratio in Proposition 1. Then*

$$\bar{P}(r^{-s}) = \bar{Q}(r^{-s}) = 0.$$

Proof. It follows from [7] that for $s = \dim_H E$,

$$(\lambda^{-s})^2 - n(\lambda^{-s}) + m = 0.$$

On the other hand, for $s = \dim_H F$, by the SSC we have

$$\sum\nolimits_{i=1}^{t} (r_i)^s = 1.$$

Then the proposition follows the relations in Proposition 1. □

3.2. Irreducibility of polynomial.

Proposition 3. *For any $Q \in \{x^p - \sum_{i=0}^{p-1} b_i x^i : p \geq 1, b_i \in \mathbf{Z}$ and $b_i \geq 0\}$, we have*

$$P(x^q) \nmid Q(x).$$

Proof. Let $Q(x) = (\sum a_i x^i)\,(x^{2q} - nx^q + m)$. Suppose

$$\sum a_i x^i = P_0 + P_1 + \cdots + P_{q-1}$$

where $P_v = \sum_{i \equiv v (\mathrm{mod}\, q)} a_i x^i$ for $v = 0, 1, \cdots, (q-1)$. Then we have

$$Q(x) = P_0 P(x^q) \oplus P_1 P(x^q) \oplus \cdots \oplus P_{q-1} P(x^q),$$

where $\oplus$ means the orthogonality of above polynomials in the basis $\{1, x, x^2, \cdots\}$.

Without loss of generality, we assume that

$$\deg\left(\sum a_i x^i\right) \equiv u \pmod q \quad \text{with} \quad 0 \leq u \leq q-1.$$

Let $c_i = a_{qi+u}$, then

$$P_u = x^u (c_0 + c_1 x^q + c_2 x^{2q} + \cdots + c_l x^{lq}) = x^u U(x^q).$$

Since $p \equiv 2q + \deg(\sum a_i x^i) \equiv u \pmod q$, we have

$$x^u U(x^q) P(x^q) = x^p - \sum_{j \equiv u (\mathrm{mod}\, q)} b_j x^j,$$

which implies

$$U(x) P(x) = x^{p'} - \sum_{i=0}^{p'} b'_i x^i \quad \text{with} \quad b'_i \in \mathbf{Z} \quad \text{and} \quad b'_i \geq 0.$$

Therefore we obtain that

$$(x^2 - nx + m)(c_0 + c_1 x + c_2 x^2 + \cdots + c_l x^l) = x^{l+2} - \sum_{i=0}^{l+1} b'_i x^i,$$

where

$$c_l = 1. \tag{3.2}$$

We recall that

$$x^2 - nx + m = (x-\beta)(x-\beta') \text{ with } \beta > 1 > |\beta'|.$$

Now, we have the following

Claim 1. *For any* $0 \le i \le l-1$,

$$c_{i+1} \le c_i\beta^{-1} \le 0. \tag{3.3}$$

We will verify (3.3) by induction.
(1) For $i = 0$, we have $c_0 m = -b'_0 \le 0$ and thus $c_0 \le 0$.
(2) For $i = 1$, we have $-c_0 n + mc_1 = -b'_1 \le 0$ and thus

$$c_1 \le \frac{n}{m}c_0 \le \beta^{-1}c_0 \le 0$$

here $\frac{n}{m} > 1 > \beta^{-1}$.
(3) Assume that (3.3) is true for $i-1$, i.e., we have $c_i \le c_{i-1}\beta^{-1} \le 0$. Hence

$$mc_{i+1} - nc_i + \beta c_i \le mc_{i+1} - nc_i + c_{i-1} = -b'_{i+1} \le 0,$$

which implies

$$mc_{i+1} \le \frac{(n-\beta)}{m}c_i = \beta^{-1}c_i \le 0$$

due to $\frac{(n-\beta)}{m} = \beta^{-1}$. Then (3.3) is verified. In particular, we have

$$c_l \le 0$$

which contradicts to (3.2). □

Proposition 4. *Suppose* $m \notin \{a^i \mid a \in \mathbf{N}$ *and* $i \in \mathbf{N}$ *with* $i \ge 2\}$. *Then*

$$P(x^q) \text{ is irreducible in } \mathbf{Z}[x] \text{ for any } q \ge 1.$$

Proof. Note that $P(x) = P(x^1)$ is irreducible (e.g. see [7]). Without loss of generality, we assume that $q \ge 2$. Let $\omega = e^{2\pi\sqrt{-1}/q}$. Then

$$P(x^q) = \left(\prod_{i=0}^{q-1}(x - \omega^i\beta^{1/q})\right) \cdot \left(\prod_{i=0}^{q-1}(x - \omega^i(\beta')^{1/q})\right).$$

Suppose on the contrary that $P(x^q) = Q_1(x)Q_2(x)$ and $Q_1(x), Q_2(x) \in \mathbf{Z}[x]$ with $\deg Q_1, \deg Q_2 \ge 1$. We note that

$$m = |P(0)| = |Q_1(0)| \cdot |Q_2(0)|,$$

where

$$|Q_1(0)| = |\beta^{u_1}(\beta')^{v_1}|^{1/q} \in \mathbf{N} \text{ and } |Q_2(0)| = |\beta^{u_2}(\beta')^{v_2}|^{1/q} \in \mathbf{N}$$

with $u_1, v_1, u_2, v_2 \ge 1$.

We will show that $u_1 = v_1$. Otherwise by symmetry we may assume that $u_1 > v_1$, then

$$(\beta^{u_1 - v_1}) = \frac{|Q_1(0)|^q}{(\beta\beta')^{v_1}} = \frac{|Q_1(0)|^q}{(m)^{v_1}},$$

which implies

$$R(\beta) = 0 \text{ with } R(x) = m^{v_1}x^{u_1-v_1} - |Q_1(0)|^q \in \mathbf{Z}[x].$$

By [7], we obtain that $P(x) = x^2 - nx + m$ is an irreducible polynomial satisfying $P(\beta) = 0$. Therefore, we have

$$P|R \text{ but } R \text{ only has roots with module } \beta.$$

Now $R(\beta') = P(\beta') = 0$ with $|\beta'| < |\beta|$. This is a contradiction.

In the same way, we have $u_2 = v_2$. Now we obtain that

$$u_1 = v_1 \text{ and } u_2 = v_2.$$

Let $u_1/q = j/i$ with $(i,j) = 1$ and $j < i$ $(i \geq 2)$, then $u_2/q = (i-j)/i$ since $u_1 + u_2 = q$. Hence

$$|Q_1(0)| = m^{\frac{j}{i}} \in \mathbf{N} \text{ and } |Q_2(0)| = m^{\frac{i-j}{i}} \in \mathbf{N}$$

and thus $m^{\frac{1}{i}} = a \in \mathbf{N}$ and $m = a^i$ with $i \geq 2$. This is a contradiction. □

3.3. Proof of Theorem. It follows from Propositions 1-2 that there are $r \in (0,1)$ and $k, k_1 \leq k_2 \leq \cdots \leq k_t \in \mathbf{N}$ such that

$$\bar{P}(r^{-s}) = \bar{Q}(r^{-s}) = 0,$$

where $\bar{P}$ and $\bar{Q}$ are defined in (3.1). Suppose on the contrary that $\bar{P}(x) = P(x^k) = x^{2k} - nx^k + m$ is irreducible in $\mathbf{Z}[x]$, then we have

$$P(x^k)|(x^{k_t} - \sum_{i=1}^{t} x^{k_t - k_i}),$$

which contradicts to Proposition 3. Therefore $P(x^k)$ is reducible in $\mathbf{Z}[x]$, and thus $m \in \{a^i \mid a \in \mathbf{N} \text{ and } i \in \mathbf{N} \text{ with } i \geq 2\}$ by Proposition 4.

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