THICK SELF SIMILAR SETS ARE ASYMPTOTICALLY GENERIC

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Abstract. This paper studies the property of thickness for a class of self-similar sets that satisfy a non overlapping condition introduced by Strichartz. We show that in a suitable sense thickness is a generic property.

1. Introduction

Over the last thirty or more years, a subject of interest in the discrete geometry of unbounded sets has been the solution of "point configuation problems". By this we mean the very general problem of determining when a given combinatorial geometric invariant, defined in terms of simplices whose endpoints belong to $\mathcal{F}$, must assume infinitely many *distinct* values. The most studied example is, of course, the distance of 1-simplices between pairs of distinct points. Less frequently studied invariants are the angle between two vectors (see [AS, IMP]) or volume of an n-simplex (see [BMP, GIM]).

A natural approach to address the problem is to introduce a large parameter x that plays the role of the radius of a ball $B(x)$ with center the origin in $\mathbf{R}^n$. One then restricts attention to the finite set of k-simplices whose endpoints belong to $\mathcal{F} \cap B(x)$, and tries to estimate from below the number of distinct invariant values formed by such simplices as $x \to \infty$. Typically the idea is to use the dimension of $\mathcal{F}$ to help detect when the number of distinct invariant values grows without bound as $x \to \infty$.

Implicit here is a choice for what one means by dimension. To a large extent this depends upon the method used to address the problem in the first place (also see [BT] where several possible definitions are introduced and compared). Approaches that are rooted in harmonic analysis and geometric measure theory work with the Hausdorff dimension. Since this is most commonly defined for bounded and nondiscrete sets, it is first necessary to find a suitable way of connecting the intersections $\mathcal{F} \cap B(x)$ to such sets. This is done in [IRU] by normalizing and thickening $\mathcal{F}$ to obtain a family of bounded sets that depend upon x.

Under suitable conditions, the Hausdorff dimension of such sets is shown not to depend upon x. The challenge then becomes one of applying Fourier analytical techniques to show that some lower bound on this common Hausdorff dimension forces the number of distinct invariants of simplices, determined by the points in $\mathcal{F} \cap B(x)$, to grow as $x \to \infty$. So far, this has required that an additional condition called "s-adaptability" be satisfied.

A more recent, and rather different, approach to such problems is presented in [EL-1, EL-2, EL-3]. This showed how zeta function methods (in one or more complex variables) can be used to study point configuration problems. The foundation of this work was given in [EL-1], whose principal goal was the proof of basic analytical properties of "fractal" zeta functions, determined by a discrete self-similar subset $\mathcal{F} \subset \mathbf{R}^n$ that satisfied a discrete analogue of Moran's "finite overlap property" [M]. We were inspired to do this by a careful study of the pioneering work in [LF] that studied "fractal strings" where $n = 1$ (see also [LRZ] for an interesting generalization that overlaps with [IRU]). In this setting, the appropriate notion of dimension would seem to be the upper Minkowski dimension. This is due to the simple observation that the boundary of analyticity of the associated zeta function equals the upper Minkowski dimension of $\mathcal{F}$ (see Definition 2 in §2).

One of the metric invariants we studied in [EL-2, EL-3] was the volume of n-simplices. Our approach required assuming a condition we called "thickness" (see Definition 4 §2 below). This property is analytic in nature, and is the subject of the article.

In [EL-2] a condition we needed to impose in order to use zeta function methods was the algebraic condition of "compatibility" of the self-similar set. This requires that the underlying similarities pairwise commute. In the more recent [EL-3], we were able to eliminate compatibility as a hypothesis, thereby extending the zeta function method to a much larger class of self-similar sets.

We will, however, impose the condition of compatibility in this article since the proof of our main result, Theorem 1, seems to be significantly more difficult when $\mathcal{F}$ is not compatible.

A feature that motivates studying thickness is that a point configuration problem for volumes can be solved in a reasonably general (and translation invariant) way when $\mathcal{F} \subset \mathbf{Z}^n$ ($n \geq 2$) as follows.

Define for any vector $\boldsymbol{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_{n+1}) \in \mathcal{F}^{n+1}$, the volume

$$|\Sigma_n(\boldsymbol{m})| = |\det(\mathbf{m}_1 - \mathbf{m}_{n+1}, \ldots, \mathbf{m}_n - \mathbf{m}_{n+1})|,$$

of the n-simplex

$$\Sigma_n(\mathbf{m}) = \langle \mathbf{m}_1 - \mathbf{m}_{n+1}, \ldots, \mathbf{m}_k - \mathbf{m}_{n+1} \rangle := \left\{ \sum_\ell \eta_\ell \mathbf{m}_\ell : \sum_\ell \eta_\ell = 1 \text{ and } \eta_\ell \geq 0 \ \forall \ell \right\},$$

and set

(1) $$Vol_\mathcal{F}(x) := \#\{|\Sigma_n(\mathbf{m})|; \ \mathbf{m} \in (\mathcal{F} \cap B(x))^{n+1}\}.$$

Then we have:

Theorem. [EL-2, Theorem 4] *Assume $\mathcal{F} \subset \mathbf{Z}^n$ is thick, satisfies Moran's finite overlap property, and has an upper Minkowski dimension $e_\mathcal{F}$ such that $e_\mathcal{F} > n - 1$. Then for all sufficiently small $\varepsilon > 0$:*

(2) $$\mathrm{Vol}_\mathcal{F}(x) \gg_\varepsilon \left[\#\big(\mathcal{F} \cap B(x)\big) \right]^{1 - \frac{n-1}{e_\mathcal{F}} - \varepsilon} \quad \text{as } x \to +\infty.$$

In other words, if $e_\mathcal{F}$ is larger than the threshold value $n - 1$, then the number of distinct volumes of n-simplices $\Sigma_n(\mathbf{m})$ ($\mathbf{m} \in (\mathcal{F} \cap B(x))^{n+1}$) must grow without bound as $x \to \infty$.

One class of examples of thick sets to which this Theorem applies are the Pascal triangles modulo any prime p

$$\text{Pas}(p) := \left\{ (m_1, m_2) \in \mathbf{N}_0^2; \ m_1 \geq m_2 \text{ and } \binom{m_1}{m_2} \not\equiv 0 \ (\text{mod } p) \right\}.$$

A basic result of [E] used zeta function methods to prove that the upper Minkowski dimension of these self-similar sets is as follows:

$$e_{\text{Pas}(p)} = \ln(p(p+1)/2)/\ln p > 1.$$

Two other examples of thick self-similar sets in $\mathbf{R}^n (n \geq 2)$ were also presented in [op.cit.]. These included the "Pascal pyramid mod p", defined analogously to $Pas(p)$, where a multinomial coefficient replaces a binomial coefficient prior to reduction mod p, and self-similar sets symmetric under the group of permutations of $\{1, \ldots, n\}$.

These results suggest that thickness justifies further study. To this end we hope to convince the reader of this by showing that thick sets are, in a suitably asymptotic sense, *generic*. This is the content of our main result Theorem 1 (see §3.2). In addition, we believe it useful to point out an intriguing analogy between thick self-similar sets and Salem sets. This is explained in a Concluding remark.

Before beginning to read this article, the reader should appreciate that Theorem 1 is a purely theoretic result. Its purpose is to show only that thickness occurs reasonably frequently in a sense made precise by the statement of the theorem. The reader should not expect to find anything herein that helps show that a particular self-similar set is, or is not, thick. That is a question of a very different nature, for which quite different, and often ad hoc, techniques would typically be needed. The three examples mentioned above illustrate this point. What Theorem 1 does say is that it is not unreasonable to try and show that a particular self-similar set is thick since this property is "asymptotically generic".

To state the defining property of thickness (see §2), we first introduce a "determinant zeta function" $\zeta_{\text{det}}(\mathcal{F}, \mathbf{s})$ (see (7)). This is represented in some product of half planes as a convergent Dirichlet series on $\mathbf{C}^n$, summed over the points $(\mathbf{m}_1, \ldots, \mathbf{m}_n) \in \mathcal{F}^n$, and with coefficients equal to $\det^2(\mathbf{m}_1, \ldots, \mathbf{m}_n)$. It is a non trivial result that this series admits a meromorphic extension to all of $\mathbf{C}^n$. Among the set of possible poles, there is one evident candidate $\mathbf{D}_{\mathcal{F}}$ (see (5)) that sits both on the diagonal in $\mathbf{R}^n$ and on the boundary of an a priori domain of absolute convergence of the series. Of special interest about this point is that its coordinate depends explicitly upon the upper Minkowski dimension of $\mathcal{F}$.

We then say (see Definition 4) that $\mathcal{F}$ is "thick" if two properties hold. The first is that this particular candidate pole is an *actual* pole of the meromorphic extension of $\zeta_{\text{det}}(\mathcal{F}, \mathbf{s})$. The second, which does *not* follow from the first, is that the iterated residue at this pole *does not vanish*. Proving thickness therefore requires showing this non vanishing property.

In general this is not easy since the iterated residue equals an infinite series (see (9), (10), (13)). To prove that the series is not zero, we have to control the absolute values of its terms with good precision. This is what the discussion in §3, 4 accomplishes.

To this end, our first idea is to introduce a global "configuration space" (see (16)) $\mathcal{M}_\mu$ of pairs $(\mathcal{D}, \mathcal{F})$, where $\mathcal{D}$ is a finite "data set" (see (14)) which specifies the similarities that determine the points of a self-similar set $\mathcal{F}$ of upper Minkowski dimension μ.

Our second idea is to introduce both an appropriate space of *bounded* parameter vectors (see Definition 5) and a pair of *unbounded* parameters (denoted (c, A) in Definition 6) in terms of which we control the norms of the elements of any data set $\mathcal{D}$. This is done in §3.1.

Our third idea is essential. We can detect an explicit candidate for a single "main term" of the iterated residue (see (26)) by allowing the pair (c, A) to grow without bound. This is what we mean by "asymptotic". The intuition here is that we can always isolate a *single candidate main term* for the iterated residue of $\zeta_{\det}(\mathcal{F}, s)$ at $\mathbf{D}_{\mathcal{F}}$ whenever $(\mathcal{D}, \mathcal{F})$ belongs to a suitable neighborhood of $\mathcal{M}_\mu$ at infinity.

For our purposes here, this candidate main term equals an *algebraic function* in the elements of $\mathcal{D}$. As a result, its non vanishing specifies a *generic* condition in the space of data sets $\mathcal{D}$ (see Remark 1 in §3.2 for a complete description of the generic condition we impose upon the elements of a data set $\mathcal{D}$).

Combining these two ideas is what we mean when we say that thickness is an "asymptotically generic" property.

As noted above, Theorem 1 should not be understood as giving an effective recipe for finding thick self-similar sets. It does indicate, however, that if one does look for thick sets, it should not be too difficult to find them. A useful problem for further work, it seems to us, would be to find more intrinsic characterizations of thickness for *particular classes* of self-similar sets, connecting it to underlying symmetries, as in [EL-2]. This should also help identify additional examples of thick self-similar sets that occur naturally in combinatorial geometry or number theory. We suspect such examples will have interesting features.

2. Definition of thickness and other preliminaries

We first recall in Definition 3 the basic notion of what we mean by a non overlapping compatible self-similar subset of $\mathbf{R}^n$ (see [EL-2]).

We fix throughout a Euclidean space (E, q), where $\dim_{\mathbf{R}} E = n$, and q the *standard* Euclidean norm $\|\cdot\| = q^{1/2}$ whose bilinear form $B(x, y) = \langle x, y \rangle$ is the usual scalar product.

Definition 1. Let T_i $(i \in I)$ be a set of pairwise commuting orthogonal linear transformations of (E, q), and $f_i = c_i T_i + \mathbf{b}_i$ $(i \in I)$ a related set of similarities of E. We then say that the f_i are compatible. The constants c_i are the "scale factors" of the similarities.

Definition 2. Let $\mathcal{F}$ be an unbounded discrete subset of E. Define the upper Minkowski dimension of $\mathcal{F}$ by

$$e_{\mathcal{F}} := \varlimsup_{R \to \infty} \frac{\ln \left(\#(\mathcal{F} \cap B(\mathbf{0}, R)) \right)}{\ln R} \in [0, \infty],$$

where $B(\mathbf{0}, R) := \{ \mathbf{m} \in E : \|\mathbf{m}\| < R \}$.

In this case the zeta function of $\mathcal{F}$ is a series summed over $\mathcal{F}' := \mathcal{F} - \{\mathbf{0}\}$

$$(3) \qquad\qquad \zeta(\mathcal{F}, s) := \sum_{\mathbf{m} \in \mathcal{F}'} \|\mathbf{m}\|^{-s}$$

that converges absolutely in the halfplane $\sigma(:= \Re s) > e_{\mathcal{F}}$, and $e_{\mathcal{F}}$ is its abscissa of convergence.

Definition 3. An unbounded discrete subset $\mathcal{F} \subset E$ is said to be a non overlapping compatible self-similar set if these two properties are satisfied:

(1) $0 < e_{\mathcal{F}} < \infty$.

(2) There exists a finite set $\mathbf{f} = \{f_i\}_{i=1}^r$ of compatible affine similarities with each scale factor $c_i > 1$ such that[1]

(4)
$$\mathcal{F} \equiv \bigcup_{i=1}^r f_i(\mathcal{F}) \quad \text{and} \quad f_i(\mathcal{F}) \cap f_{i'}(\mathcal{F}) = \emptyset \ \text{ if } i \neq i'.$$

Remark. We define

(5)
$$D_{\mathcal{F}} := e_{\mathcal{F}} + 2; \quad \mathbf{D}_{\mathcal{F}} := (D_{\mathcal{F}}, \dots, D_{\mathcal{F}}) \in \mathbf{R}^n.$$

Notations. We fix an orthonormal basis $\mathcal{B} = \{\mathbf{g}_1, \dots \mathbf{g}_n\}$ of $E_{\mathbf{C}}$, the complexification of E, with respect to which each T_j is diagonalizable, and each element $\mathbf{m} \in \mathcal{F}$ is written as a linear combination $\mathbf{m} = \sum_j m_j \mathbf{g}_j$. It follows that

$$\exists \boldsymbol{\lambda}_j = (\lambda_{1,j}, \dots, \lambda_{n,j}) \in (S^1)^n \ \text{ such that } \ T_j^*(\mathbf{g}_k) = \lambda_{k,j} \mathbf{g}_k \ \forall k = 1, \dots, n$$
$$\text{and} \ \forall j = 1, \dots, r.$$

We denote by $\mathbf{e}_1, \dots, \mathbf{e}_n$ the standard unit basis for E.

We set $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, and for any $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}_0^n$, $\mathbf{m} = (m_1, \dots, m_n) \in \mathbf{C}^n$, we define the weight $|\boldsymbol{\alpha}| = \sum_i \alpha_i$, and set $\mathbf{m}^{\boldsymbol{\alpha}} := \prod_j m_j^{\alpha_j}$. We also define the (a priori formal) series (see Lemma 1)

(6)
$$\zeta_{\mathcal{F}}(s, \boldsymbol{\alpha}) := \sum_{\mathbf{m} \in \mathcal{F}'} \frac{\mathbf{m}^{\boldsymbol{\alpha}}}{\|\mathbf{m}\|^s} \quad (s = \sigma + i\tau).$$

We denote the sign of a permutation $\omega \in \mathcal{S}_n$ by $\operatorname{sgn} \omega$. We define (formally) the determinant zeta function associated to $\mathcal{F}$ as a function of $\mathbf{s} = (s_1, \dots, s_n)$ as follows:

(7)
$$\boldsymbol{\zeta}_{\det}(\mathcal{F}, \mathbf{s}) := \sum_{\mathbf{m}_1, \dots, \mathbf{m}_n \in \mathcal{F}'} \frac{\det^2(\mathbf{m}_1, \dots, \mathbf{m}_n)}{\|\mathbf{m}_1\|^{s_1} \dots \|\mathbf{m}_n\|^{s_n}}.$$

The basic analytical properties of $\boldsymbol{\zeta}_{\det}$ we need are as follows.

Proposition 1. *The determinant zeta function $\boldsymbol{\zeta}_{\det}(\mathcal{F}, \mathbf{s})$ of $\mathcal{F}$ converges absolutely in the domain $\bigcap_{i=1}^n \{\sigma_i = \Re(s_i) > D_{\mathcal{F}}\}$ and has a meromorphic extension with moderate growth[2] to $\mathbf{C}^n$.*

Idea of Proof. It follows from Hadamard's inequality (i.e. $|det(\mathbf{m}_1, \dots, \mathbf{m}_n)| \leq \|\mathbf{m}_1\| \dots \|\mathbf{m}_n\|$) that $\boldsymbol{\zeta}_{det}(\mathcal{F}, \mathbf{s})$ converges absolutely on the set $\bigcup_{i=1}^n \{\sigma_i > D_{\mathcal{F}}\}$. Applying the formula

(8)
$$\boldsymbol{\zeta}_{det}(\mathcal{F}, \mathbf{s}) = \sum_{\omega_1, \omega_2 \in \mathcal{S}_n} \operatorname{sgn}(\omega_1 \omega_2) \prod_{i=1}^n \zeta_{\mathcal{F}}(s_i, \mathbf{e}_{\omega_1(i)} + \mathbf{e}_{\omega_2(i)}).$$

and using the method of analytic continuation from [EL-1] to construct the meromorphic extension of any $\zeta_{\mathcal{F}}(s, \boldsymbol{\alpha})$ finishes the proof. $\square$

The only properties about the meromorphic extension of $\zeta_{\det}(\mathcal{F}, \mathbf{s})$ that we will need for this article are as follows. Their proofs can be found in [EL-2, Section 2].

Lemma 1. *Assume $\mathcal{F}$ is non overlapping and compatible, and let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}_0^n$ have weight $|\boldsymbol{\alpha}| = 2$.*

(1) *If $e_{\mathcal{F}} + |\boldsymbol{\alpha}|$ is a pole of $\zeta_{\mathcal{F}}(s, \boldsymbol{\alpha})$, then it is simple and $\boldsymbol{\lambda}_j^{\boldsymbol{\alpha}} = 1$ for all $j = 1, \dots, r$.*

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[1]The notation $F \equiv G$ means that $(F \setminus G) \cup (G \setminus F)$ is a finite set.

[2]A meromorphic function $F(\mathbf{s})$ with polar locus $\mathcal{P}$ has moderate growth on a domain $\mathcal{D} \subset \mathbf{C}^n$ if there exists $a, b > 0$ such that $\forall \delta > 0$ and $\forall \mathbf{s} \in \{d(\mathbf{s}, \mathcal{P} \cap \mathcal{D}) \geq \delta\}$, $F(\boldsymbol{\sigma} + i\boldsymbol{\tau}) \ll_{\boldsymbol{\sigma}, \delta} 1 + |\boldsymbol{\tau}|^{a|\boldsymbol{\sigma}| + b}$ (see [EL-1]).

(2) *The residue* $\mathcal{B}(\mathcal{F}, \boldsymbol{\alpha}) := \lim\limits_{s \to e_{\mathcal{F}}+|\boldsymbol{\alpha}|} \left(s - (e_{\mathcal{F}} + |\boldsymbol{\alpha}|) \right) \zeta_{\mathcal{F}}(s, \boldsymbol{\alpha})$ *is given by*

$$(9) \qquad \mathcal{B}(\mathcal{F}, \boldsymbol{\alpha}) = \left(\sum_{j=1}^{r} c_j^{-e_{\mathcal{F}}} \ln c_j \right)^{-1} \sum_{j=1}^{r} c_j^{-e_{\mathcal{F}}} \Delta(\boldsymbol{\alpha}, \mathbf{u}_j)$$

where $\forall j = 1, \ldots, r$:

$$(10) \quad \mathbf{u}_j := c_j^{-1} T_j^{-1}(\mathbf{b}_j) \quad and \quad \Delta(\boldsymbol{\alpha}, \mathbf{u}_j) := \sum_{\mathbf{m} \in \mathcal{F}} \left(\frac{(\mathbf{m} + \mathbf{u}_j)^{\boldsymbol{\alpha}}}{\|\mathbf{m} + \mathbf{u}_j\|^{e_{\mathcal{F}}+|\boldsymbol{\alpha}|}} - \frac{\mathbf{m}^{\boldsymbol{\alpha}}}{\|\mathbf{m}\|^{e_{\mathcal{F}}+|\boldsymbol{\alpha}|}} \right).$$

(3) *The following bound holds:*

$$(11) \qquad \frac{(\mathbf{m} + \mathbf{u}_j)^{\boldsymbol{\alpha}}}{\|\mathbf{m} + \mathbf{u}_j\|^{e_{\mathcal{F}}+|\boldsymbol{\alpha}|}} - \frac{\mathbf{m}^{\boldsymbol{\alpha}}}{\|\mathbf{m}\|^{e_{\mathcal{F}}+|\boldsymbol{\alpha}|}} \ll_{\sigma} (1 + |\tau|) \frac{\|\mathbf{u}_j\|}{\|\mathbf{m}\|^{\sigma-1}}.$$

Definition 4. The set $\mathcal{F}$ is *thick* if the point $\mathbf{D}_{\mathcal{F}}$ (see (5)) is a pole of $\boldsymbol{\zeta}_{\det}(\mathcal{F}, \mathbf{s})$, and the iterated residue

$$\mathrm{Res}_{s_1=D_{\mathcal{F}}} \cdots \mathrm{Res}_{s_n=D_{\mathcal{F}}} \left(\boldsymbol{\zeta}_{\det} \right) \neq 0.$$

Defining

$$V(\mathcal{F}; \mathbf{D}_{\mathcal{F}}) := \lim_{s_1 \to D_{\mathcal{F}}} \cdots \lim_{s_n \to D_{\mathcal{F}}} \left(\prod_{i=1}^{n} (s_i - D_{\mathcal{F}}) \boldsymbol{\zeta}_{\det}(\mathcal{F}, s_1, \ldots, s_n) \right)$$

it follows that

$$(12) \qquad \mathcal{F} \text{ is thick iff } V(\mathcal{F}; \mathbf{D}_{\mathcal{F}}) \neq 0.$$

From Lemma 1 we also see that

$$(13) \qquad V(\mathcal{F}; \mathbf{D}_{\mathcal{F}}) = \sum_{(\omega_1, \omega_2) \in \mathcal{S}_n(\mathcal{F})} \mathrm{sgn}(\omega_1 \omega_2) \prod_{i=1}^{n} \mathcal{B}(\mathcal{F}; \mathbf{e}_{\omega_1(i)} + \mathbf{e}_{\omega_2(i)})$$

where

$$\mathcal{S}_n(\mathcal{F}) = \{(\omega_1, \omega_2) \in \mathcal{S}_n^2 : \boldsymbol{\lambda}_j^{\mathbf{e}_{\omega_1(i)} + \mathbf{e}_{\omega_2(i)}} = 1 \ \ \forall j \ \forall i\}.$$

3. A procedure to construct thick $\mathcal{F}$ and statements of Lemma 2 and Theorem 1

Let $\mathcal{F}$ be a compatible non overlapping self-similar subset of $\mathbf{R}^n$ and satisfying the following (entirely for convenience) condition

$$\mathbf{0} \notin \mathcal{F}.$$

Using notations from §1, we should think of the set

$$(14) \qquad \begin{aligned} \mathcal{D} = \Big\{ &r, c_1, \ldots, c_r, \mathbf{b}_1, \ldots, \mathbf{b}_r, T_1, \ldots, T_r : \\ &\{f_j := c_j T_j + \mathbf{b}_j\}_1^r \text{ is a compatible family of similarities} \Big\} \end{aligned}$$

as basic data for the set of all such $\mathcal{F}$.

A fixed point of a similarity $f = cT + \mathbf{b}$ (when $c > 1$) is evidently equal to $P = (\mathrm{Id} - cT)^{-1}(\mathbf{b})$. Strichartz [St] has shown that the points of a non-overlapping

$\mathcal{F}$ with similarities $\{f_j\}_1^r$ are determined by a subset of the fixed points as follows:

$$\exists\, K \subset \{1, \ldots, r\} \text{ such that}$$

$$(15) \qquad \mathcal{F} = \bigsqcup_{k \in K} \bigsqcup_{t \geq 1} \bigsqcup_{\substack{(j_1, \ldots, j_t) \in \{1, \ldots, r\}^t \\ f_{j_t}(P_k) \neq P_k}} \{f_{j_1} \circ \cdots \circ f_{j_t}(P_k)\} \bigsqcup \{P_k \mid k \in K\}$$

(where $P_k = (\mathrm{Id} - c_k T_k)^{-1}(\mathbf{b}_k)$ and $\bigsqcup$ denotes that the union is disjoint).

Combining $\mathcal{D}$ with the parametrization (15), we now form a configuration space for the set of *non-overlapping compatible self similar sets* with fixed upper Minkowski dimension μ by setting:

$$\mathfrak{M}_\mu = \Big\{ \mathcal{D} \times \mathcal{F} :$$

$$(16) \qquad \mathcal{F} = \bigsqcup_{k \in K} \bigsqcup_{t \geq 1} \bigsqcup_{\substack{(j_1, \ldots, j_t) \in \{1, \ldots, r\}^t \\ f_{j_t}(P_k) \neq P_k}} \{f_{j_1} \circ \cdots \circ f_{j_t}(P_k)\} \bigsqcup \{P_k \mid k \in K\} :$$

$$f_j = c_j T_j + \mathbf{b}_j \quad \text{when } c_j, \mathbf{b}_j, T_j \in \mathcal{D},$$

$$\text{and } \mathcal{F} \text{ has upper Minkowski dimension } \mu \Big\}.$$

We then say that any such $\mathcal{D}$ resp. $\mathcal{F}$ is a *data set* resp. *self similar set* component of $\mathfrak{M}_\mu$, and subsequently will always use the notation μ in place of $e_\mathcal{F}$, as in §1,2, in the rest of the article.

Given a self-similar component $\mathcal{F}$ of $\mathfrak{M}_\mu$, a basic observation is that Lemma 1 implies an expression for the iterated residue of the determinant zeta function at the particular point $\mathbf{D}_\mu = (\mu + 2, \ldots, \mu + 2)$ (see Definition 4) as an infinite sum over the points appearing in the Strichartz parametrization of $\mathcal{F}$. To show that such a sum is non zero does not seem to be an easy task in general. However, by thinking of the problem in an asymptotic manner, we are able to identify an *expected main term* for the iterated residue at $\mathbf{D}_\mu$.

As a result, we will look for thick self-similar sets (with a fixed upper Minkowski dimension μ) near the 'boundary of infinity" of $\mathfrak{M}_\mu$, which we can approach by allowing some of the defining data set elements to grow without bound. We can then detect a thick set by showing that this expected main term is not zero.

3.1. Parameters (bounded and unbounded) for elements of $\mathfrak{M}_\mu$. Intuitively, we can approach the boundary of $\mathfrak{M}_\mu$ at infinity in many different ways. We do so by first identifying a few parameters, in terms of which elements of any component data set $\mathcal{D}$ can be expressed, and then making the parameters grow without bound and independently of one another. To make our work as simple as possible, the number of such parameters should be as small as possible. In this subsection we define the two unbounded parameters that we will need, as well as the remaining set of bounded parameters.

One of the unbounded parameters can be easily specified without any additional discussion. We define a common scale c and positive ratios β_j, in terms of which the different scale factors $c_j \in \mathcal{D}$ are defined, by setting

$$(17) \qquad\qquad c_j = \beta_j c \quad \text{for each } j = 1, \ldots, r.$$

We will think of c as an *unbounded* parameter.

Recalling that $K \subset \{1, \ldots, r\}$ indexes the fixed points of those similarities that determine the Strichartz parametrization (15) of a self similar component of $\mathfrak{M}_\mu$, a

helpful technical condition concerning the β_k $(k \in K)$, is the following

(18)

K-distinct property:

the vectors $|\widehat{\mathbf{b}}_k| := \beta_k^{-1}(|b_{1,k}|, \ldots, |b_{n,k}|)$, $k \in K$, are distinct.

Note. We will always assume throughout the rest of the article that the K-distinct property is satisfied for any self similar component of an element of $\mathfrak{M}_\mu$.

Remark. (18) implies that the two quantities

(19)
$$\Omega_K := \inf_{k \neq k' \in K} \left\{ \sum_{i=1}^n \left| \frac{|b_{i,k}|}{\beta_k} - \frac{|b_{i,k'}|}{\beta_{k'}} \right| \right\} \quad \text{and}$$

$$\rho_K := \inf\{1, \Omega_K\} \quad \text{are both positive.}$$

The set of *bounded* parameters can be conveniently organized into vectors as follows.

Definition 5. An "admissible vector" consists of five parameters:

(20)
$$\mathcal{Q} := \big((G_1 \times G_2), \alpha_1, \alpha_2, N, (\varepsilon, \delta, \theta)\big)$$

where

- $G_1 \subset \mathbf{R}^n - \{\mathbf{0}\}$ and $G_2 \subset (0, \infty)$ are compact sets;
- $0 < \alpha_1 \leq \dfrac{1}{2\sqrt{n}} \cdot \dfrac{\inf\{\|\mathbf{u}\| : \mathbf{u} \in G_1\}}{\sup\{\|\mathbf{u}\| : \mathbf{u} \in G_1\}} \cdot \inf G_2 \quad$ and

  $\alpha_2 \geq 1 + 2\sqrt{n} \cdot \dfrac{\sup\{\|\mathbf{u}\| : \mathbf{u} \in G_1\}}{\inf\{\|\mathbf{u}\| : \mathbf{u} \in G_1\}} \cdot \sup G_2;$
- $N \in \mathbf{N}$;
- $\varepsilon > 0$, $\theta \in (0, \frac{1}{3})$, and $\delta \in (0, 1 - \theta)$.

The second preliminary definition is that of a (c, A) permissible element $(\mathcal{D}, \mathcal{F}) \in \mathfrak{M}_\mu$.

Definition 6. Let $(\mathcal{D}, \mathcal{F}) \in \mathfrak{M}_\mu$, where $\mathcal{D} = \{r, c_1, \ldots, c_r, \mathbf{b}_1, \ldots, \mathbf{b}_r, T_1, \ldots, T_r\}$, and K indexes the fixed points in the Strichartz parametrization of $\mathcal{F}$. We say that $(\mathcal{D}, \mathcal{F})$ is "(c, A) permissible" for an admissible vector $\mathcal{Q}$, as in (20), if the following five conditions are satisfied:

1. Each $\mathbf{b}_j \in G_1$;
2. $A \geq \alpha_2$ and $c \geq C(A, \Omega_K, \mathcal{Q})$ where

(21)
$$C(A, \Omega_K, \mathcal{Q}) := \max \left\{ A^{\frac{1}{\mu - v}}, \; \rho_K^{-1/\delta}, \; r^\varepsilon, \; 2^{-\frac{1}{1-\theta}}, \; (2\rho_K)^{-1} \right\} \text{ and}$$

$$v := \max\{\mu\delta, \mu + 2\theta - 1\};$$

3. There is a partition $\{1, \ldots, r\} \setminus K = S(\mathcal{Q}) \sqcup L(\mathcal{Q})$ such that

(22)
$$k \in K \quad \text{iff} \quad \beta_k \in G_2; \quad j \in L(\mathcal{Q}) \quad \text{iff} \quad \beta_j \geq A; \quad \text{and}$$

$$j \in S(\mathcal{Q}) \quad \text{iff} \quad \beta_j \in [c^{-\theta}, \alpha_1];$$

4. $\#K \leq N$;
5. There exist $\eta > 0$ and $\theta' > \theta$ such that

(23)
$$c \geq C(A, \Omega_K, \mathcal{Q}) \quad \text{implies} \quad \#S(\mathcal{Q}) \leq \eta \, c^{(1-\theta')\mu}.$$

Remark. Part 4 implies that $\#K$ is uniformly bounded in c. This is *not obvious* since for any fixed μ, if c is allowed to grow without bound, then r must also. To see

this, it suffices to recall from [EL-1] that

$$(24) \qquad \sum_{j=1}^{r} c_j^{-\mu} = 1.$$

So, in principle, this could also force $\#K$ to increase without bound. Thus, requiring $\#K$ to be bounded uniformly in $c \geq C(A, \Omega_K, \mathcal{Q})$ is an implicit constraint in Theorem 1. This can also be thought of as a restriction upon how we choose to approach the boundary of $\mathfrak{M}_\mu$ at infinity.

The reason for imposing this property is explained in Remark 1 (see §3.2). $\qquad \square$

When given both an admissible $\mathcal{Q}$ and (c, A) permissible $(\mathcal{D}, \mathcal{F})$, it will also be useful to specify the following shorthand notation $O_{\mathcal{Q}}$.

Notation. An expression $f(c) = O_{\mathcal{Q}}(g(c))$ means that there exist a constant $B_{\mathcal{Q}}$, depending solely upon (some of) the components of $\mathcal{Q}$, such that

$$(25) \qquad |f(c)| \leq B_{\mathcal{Q}} \cdot |g(c)| \quad \text{for all } c > C(A, \Omega_K, \mathcal{Q}).$$

3.2. Statements of Lemma 2 and Theorem 1. As above, we start with a pair $(\mathcal{D}, \mathcal{F}) \in \mathfrak{M}_\mu$, and define (see (13), (17)):

$$(26) \qquad \Psi_K := \sum_{(\omega_1, \omega_2) \in \mathcal{S}_n(\mathcal{F})} \text{sgn}(\omega_1 \omega_2) \cdot \prod_{i=1}^{n} \left(\sum_{k \in K} \beta_k^\mu \frac{\mathbf{b}_k^{\mathbf{e}_{\omega_1(i)} + \mathbf{e}_{\omega_2(i)}}}{\|\mathbf{b}_k\|^{\mu+2}} \right).$$

Our main result gives the following criteria for $\mathcal{F}$ to be thick.

Theorem 1. *Let $\mathcal{Q} = \big((G_1 \times G_2), \alpha_1, \alpha_2, N, (\varepsilon, \delta, \theta)\big)$ be an admissible vector (20). Then $\mathcal{F}$ is thick if:*

1. $\Psi_K \neq 0$;
2. *For some $A \geq \alpha_2$ there exists (see (21))*

$$C^* = C^*(A, \Psi_K, \Omega_K, \mathcal{Q}) \geq C(A, \Omega_K, \mathcal{Q})$$

such that $(\mathcal{D}, \mathcal{F})$ is (c, A)-permissible whenever $c > C^*$.

It is also interesting to note the following variant of Theorem 1 for planar self-similar sets.

Corollary 1. *If $n = 2$ and the similarities T_j $(j = 1, \ldots, r)$ are simultaneously diagonalizable over $\mathbf{R}$, then $\mathcal{F}$ is thick if, in addition to property 2 above, the following replaces the hypothesis $\Psi_K \neq 0$:*

The two vectors $(b_{1,k})_{k \in K}$ and $(b_{2,k})_{k \in K}$ are linearly independent.

The key lemma we need, and from which Theorem 1 is an easy consequence, is the following.

Lemma 2. *Let $(\mathcal{D}, \mathcal{F}) \in \mathfrak{M}_\mu$ be (c, A) permissible for an admissible vector $\mathcal{Q}$. Then, for any $\boldsymbol{\alpha} \in \mathbf{N}_0^n$ satisfying $|\boldsymbol{\alpha}| = 2$ and $\boldsymbol{\lambda}_j^{\boldsymbol{\alpha}} = 1$ $(\forall j = 1, \ldots, r)$, it follows that (see (25))*

$$(27) \qquad \left(\sum_{j=1}^{r} c_j^{-\mu} \ln c_j \right) \mathcal{B}(\mathcal{F}, \boldsymbol{\alpha}) = \sum_{k \in K} \left(\beta_k^\mu \frac{\mathbf{b}_k^{\boldsymbol{\alpha}}}{\|\mathbf{b}_k\|^{\mu+2}} \right) c^\mu + O_{\mathcal{Q}} \left(\left(\frac{2}{A} + \frac{\#S(\mathcal{Q})}{c^{(1-\theta)\mu}} \right) c^\mu \right).$$

Remark 1. Since Ψ_K equals a generalized algebraic expression in the $\mathbf{b}_k, \beta_k$ $(k \in K)$, the two conditions that depend solely upon K (see (19))

$$\Psi_K \neq 0 \quad \text{and} \quad \Omega_K > 0$$

just mean that $(\mathbf{b}_k, \beta_k)_{k \in K}$ should be a "generic" vector in the sense that it lies outside a generalized hypersurface in the compact set $G_1^{\#K} \times G_2^{\#K}$ of bounded dimension.

In this sense of generic, Theorem 1 tells us that thickness of $\mathcal{F}$ is an "asymptotically generic" property (in c, A). Precisely, this means:

If $(\mathcal{D}, \mathcal{F}) \in \mathfrak{M}_\mu$ is (c, A) permissible for an admissible $\mathcal{Q}$, and the bounded vector $(\mathbf{b}_k, \beta_k)_k$ is generic, then:

$$\mathcal{F} \text{ is thick whenever } A \geq \alpha_2 \text{ and } c > C^*(A, \Psi_K, \Omega_K, \mathcal{Q}).$$

As noted in the Remark following Definition 6, the assumption that $\#K$ is bounded is explicitly a property of the admissible vector $\mathcal{Q}$. The reason for this condition can now be seen. It simply means that Ψ_K is a generalized rational function in a *bounded* number of variables. Thus, the generic set is the complement of the zero locus of such a function that is defined on a Euclidean space of bounded dimension.

Were this hypothesis on $\#K$ *not* imposed it would require only that we delete the component N from $\mathcal{Q}$, and interpret the generic set as lying inside the inverse limit

$$\varprojlim_K \{\Psi_K \neq 0\}.$$

Although this set lies in an infinite dimensional space, we can think of it as being a generic set in the sense that it lies outside a generalized proalgebraic subset whose projection to any finite dimensional subspace is generic in the usual sense of the term.

The reader should therefore not interpret the boundedness hypothesis of $\#K$ as a significant constraint upon the generality of our principal result. Its purpose is to simplify our discussion by eliminating the need to specify what is meant by a generic subset of a space of infinite dimension.

4. Proofs of Lemma 2 and Theorem 1

Proof of Lemma 2. For any $J = (j_1, \ldots, j_t) \in \{1, \ldots, r\}^t$, and $k \in K$ we set (see (10))

(28)
$$\mathbf{m}(k; J) := f_{j_1} \circ \cdots \circ f_{j_t}(P_k),$$
$$\Xi_j(\mathbf{m}(k; J), \boldsymbol{\alpha}) = \frac{(\mathbf{m}(k; J) + \mathbf{u}_j)^{\boldsymbol{\alpha}}}{\|\mathbf{m}(k; J) + \mathbf{u}_j\|^{\mu+2}} - \frac{\mathbf{m}(k; J)^{\boldsymbol{\alpha}}}{\|\mathbf{m}(k; J)\|^{\mu+2}},$$

and define, similarly,

$$\Xi_j(P_k, \boldsymbol{\alpha}) = \frac{(P_k + \mathbf{u}_j)^{\boldsymbol{\alpha}}}{\|P_k + \mathbf{u}_j\|^{\mu+2}} - \frac{P_k^{\boldsymbol{\alpha}}}{\|P_k\|^{\mu+2}}.$$

Combining the right side of (9) with (15) and Part 3 of Definition 6, it follows that

(29) $$\Delta(\boldsymbol{\alpha}, \mathbf{u}_j) = \Delta^{(1)}(\boldsymbol{\alpha}, \mathbf{u}_j) + \Delta^{(2)}(\boldsymbol{\alpha}, \mathbf{u}_j) + \Delta^{(3)}(\boldsymbol{\alpha}, \mathbf{u}_j) + \Delta^{(4)}(\boldsymbol{\alpha}, \mathbf{u}_j)$$

where

$$\Delta^{(1)}(\boldsymbol{\alpha}, \mathbf{u}_j) := \sum_{k \in K} \Xi_j(P_k, \boldsymbol{\alpha}),$$

$$\Delta^{(2)}(\boldsymbol{\alpha}, \mathbf{u}_j) := \sum_{k \in K} \sum_{t \in \mathbf{N}} \sum_{\substack{J = (j_1, \ldots, j_t) \in \{1, \ldots, r\}^t \\ f_{j_t}(P_k) \neq P_k \text{ and } j_t \in L(\mathcal{Q})}} \Xi_j(\mathbf{m}(k; J), \boldsymbol{\alpha}),$$

$$\Delta^{(3)}(\boldsymbol{\alpha}, \mathbf{u}_j) := \sum_{k \in K} \sum_{t \in \mathbf{N}} \sum_{\substack{J=(j_1,\ldots,j_t) \in \{1,\ldots,r\}^t \\ f_{j_t}(P_k) \neq P_k \text{ and } j_t \in S(\mathcal{Q})}} \Xi_j(\mathbf{m}(k; J), \boldsymbol{\alpha}),$$

$$\Delta^{(4)}(\boldsymbol{\alpha}, \mathbf{u}_j) := \sum_{k \in K} \sum_{t \in \mathbf{N}} \sum_{\substack{J=(j_1,\ldots,j_t) \in \{1,\ldots,r\}^t \\ f_{j_t}(P_k) \neq P_k \text{ and } j_t \in K}} \Xi_j(\mathbf{m}(k; J), \boldsymbol{\alpha}).$$

Remark 2. The first issue we must address is how to sharpen the estimate (11) when the role of $\mathbf{m}$ is played by $\mathbf{m}(k; J)$. This depends upon whether j_t belongs to $S(\mathcal{Q})$, $L(\mathcal{Q})$ or K.

Step 1. We prove that

(30) $\quad \|\mathbf{m}(k; J)\| \asymp_{\mathcal{Q}} c^{t-1} \beta_{j_1} \cdots \beta_{j_t} \quad$ uniformly in $t \geq 1$, and J such that $j_t \in L(\mathcal{Q})$.

Remark. This statement means that there exist positive constants $\kappa_1 = \kappa_1(\mathcal{Q})$, $\kappa_2 = \kappa_2(\mathcal{Q})$ and $C_1 = C_1(\mathcal{Q})$ such that $c \geq C_1$ implies

$$\kappa_1 < \|\mathbf{m}(k; J)\|/c^{t-1} \beta_{j_1} \cdots \beta_{j_t} < \kappa_2 \quad \text{for all } t \geq 1$$

and

$$J = (j_1, \ldots, j_t) \quad \text{such that } j_t \in L(\mathcal{Q}).$$

It will be clear from Steps 4–7 why we need such uniformity.

To avoid having to repeat certain phrases, we use the convention that t refers to any index at least 1. Whenever the exponent of c equals $t-2$, it is implicitly assumed that the expression containing c^{t-2} is set equal to 0 if $t = 1$.

Proof of (30). First we note that

$$\mathbf{m}(k; J) = c_{j_1} \cdots c_{j_t} T_{j_1} \circ \cdots \circ T_{j_t}(P_k) + c_{j_1} \cdots c_{j_{t-1}} T_{j_1} \circ \cdots \circ T_{j_{t-1}}(\mathbf{b}_{j_t})$$

$$+ \sum_{h=1}^{t-2} c_{j_1} \cdots c_{j_h} T_{j_1} \circ \cdots \circ T_{j_h}(\mathbf{b}_{j_{h+1}})$$

$$= c_{j_1} \cdots c_{j_t} T_{j_1} \circ \cdots \circ T_{j_t}(Id - c_k T_k)^{-1}(\mathbf{b}_k) + c_{j_1} \cdots c_{j_{t-1}} T_{j_1} \circ \cdots \circ T_{j_{t-1}}(\mathbf{b}_{j_t})$$

$$+ \sum_{h=1}^{t-2} c_{j_1} \cdots c_{j_h} T_{j_1} \circ \cdots \circ T_{j_h}(\mathbf{b}_{j_{h+1}}),$$

with the convention that the sum over h reduces to $\mathbf{b}_{j_1}$ if $t = 2$, and is empty if $t = 1$, in which case, the middle term reduces to $\mathbf{b}_{j_1}$.

Denoting the coordinates of each $\mathbf{b}_k$ in the basis $\{\mathbf{g}_1, \ldots, \mathbf{g}_n\}$ for $E_{\mathbf{C}}$ by $(b_{1,k}, \ldots, b_{n,k})$, the fact that each $I - c_k T_k$ is non singular evidently implies that the coordinates for $P_k = (I - c_k T_k)^{-1}(\mathbf{b}_k)$ in the same basis are

$$P_k = \left(\frac{b_{1,k}}{1 - c_k \lambda_{1,k}}, \ldots, \frac{b_{n,k}}{1 - c_k \lambda_{n,k}} \right).$$

As a result, setting $\mathbf{m}(k; J) = (m_1(k; J), \ldots, m_n(k; J))$ to denote the coordinates of each $\mathbf{m}(k; J)$ in this basis, it then follows that for each $i = 1, \ldots, n$

$$m_i(k; J) = c_{j_1} \cdots c_{j_t} \lambda_{i,j_1} \cdots \lambda_{i,j_t} \frac{b_{i,k}}{1 - c_k \lambda_{i,k}} + c_{j_1} \cdots c_{j_{t-1}} \lambda_{i,j_1} \cdots \lambda_{i,j_{t-1}} b_{i,j_t}$$

$$+ \sum_{h=1}^{t-2} c_{j_1} \cdots c_{j_h} \lambda_{i,j_1} \cdots \lambda_{i,j_h} b_{i,j_{h+1}}$$

$$= c_{j_1} \cdots c_{j_t} \lambda_{i,j_1} \cdots \lambda_{i,j_t} \left(\frac{b_{i,j_t}}{\lambda_{i,j_t} c_{j_t}} - \frac{b_{i,k}}{\lambda_{i,k} c_k} \right) + O_{\mathcal{Q}} \left(\frac{c_{j_1} \cdots c_{j_t}}{c_k^2} + \sum_{h=1}^{t-2} c_{j_1} \cdots c_{j_h} \right)$$

$$= c^{t-1} \beta_{j_1} \cdots \beta_{j_t} \lambda_{i,j_1} \cdots \lambda_{i,j_t} \left(\frac{b_{i,j_t}}{\lambda_{i,j_t} \beta_{j_t}} - \frac{b_{i,k}}{\lambda_{i,k} \beta_k} \right)$$

(31)
$$+ O_{\mathcal{Q}} \left(c^{t-2} \frac{\beta_{j_1} \cdots \beta_{j_t}}{\beta_k^2} + \sum_{h=1}^{t-2} c^h \beta_{j_1} \cdots \beta_{j_h} \right)$$

$$= c^{t-1} \beta_{j_1} \cdots \beta_{j_t} \lambda_{i,j_1} \cdots \lambda_{i,j_t} \left(\frac{b_{i,j_t}}{\lambda_{i,j_t} \beta_{j_t}} - \frac{b_{i,k}}{\lambda_{i,k} \beta_k} \right)$$

$$+ O_{\mathcal{Q}} \left(c^{t-2} \frac{\beta_{j_1} \cdots \beta_{j_t}}{\beta_k^2} + \beta_{j_1} \cdots \beta_{j_t} \sum_{h=1}^{t-2} \frac{c^h}{\beta_{j_{h+1}} \cdots \beta_{j_{t-1}}} \right)$$

$$= c^{t-1} \beta_{j_1} \cdots \beta_{j_t} \lambda_{i,j_1} \cdots \lambda_{i,j_t} \left(\frac{b_{i,j_t}}{\lambda_{i,j_t} \beta_{j_t}} - \frac{b_{i,k}}{\lambda_{i,k} \beta_k} \right)$$

$$+ O_{\mathcal{Q}} \left(c^{t-2} \beta_{j_1} \cdots \beta_{j_t} + \beta_{j_1} \cdots \beta_{j_t} \sum_{h=1}^{t-2} c^h c^{\theta(t-h-1)} \right)$$

where we have used the property that each $\beta_j \geq c^{-\theta}$ in the last line. Thus, for each i

$$m_i(k; J)$$
(32)
$$= c^{t-1} \beta_{j_1} \cdots \beta_{j_t} \lambda_{i,j_1} \cdots \lambda_{i,j_t} \left(\frac{b_{i,j_t}}{\lambda_{i,j_t} \beta_{j_t}} - \frac{b_{i,k}}{\lambda_{i,k} \beta_k} \right) + O_{\mathcal{Q}} \left(c^{t-2+\theta} \beta_{j_1} \cdots \beta_{j_t} \right).$$

Assume now that $j_t \in L(\mathcal{Q})$. Since $(\mathcal{D}, \mathcal{F})$ is (c, A)-permissible and $A \geq \alpha_2$ it follows from (22) and the fact that each $|\lambda_{i,j}| = 1$ that

(33)
$$1 \gg_{\mathcal{Q}} \sum_{i=1}^{n} \left| \frac{b_{i,j_t}}{\lambda_{i,j_t} \beta_{j_t}} - \frac{b_{i,k}}{\lambda_{i,k} \beta_k} \right| \geq \sum_{i=1}^{n} \frac{|b_{i,k}|}{\beta_k} - \frac{|b_{i,j_t}|}{\beta_{j_t}} \geq \frac{\|\mathbf{b}_k\|}{\beta_k} - \frac{\sqrt{n}\|\mathbf{b}_{j_t}\|}{\beta_{j_t}},$$

where the notation "$1 \gg_{\mathcal{Q}} \square$" (resp. "$1 \ll_{\mathcal{Q}} \square$"—see below) means that the quantity $\square$ is bounded above (resp. below) by a constant $c_1(\mathcal{Q})$.

Setting

(34)
$$M' = \inf\{\|\mathbf{u}\| : \mathbf{u} \in G_1\}, \quad M = \sup\{\|\mathbf{u}\| : \mathbf{u} \in G_1\}$$

the lower bounds $A \geq \alpha_2 \geq 1 + 2\sqrt{n}\beta_k \cdot M/M'$ imply

$$\frac{\|\mathbf{b}_k\|}{\beta_k} - \frac{\sqrt{n}\|\mathbf{b}_{j_t}\|}{\beta_{j_t}} \geq \frac{M'}{\beta_k} - \frac{\sqrt{n}M}{\beta_{j_t}} = \frac{M}{\beta_k} \left(\frac{M'}{M} - \frac{\sqrt{n}\beta_k}{\beta_{j_t}} \right)$$

$$\geq \frac{M}{\beta_k} \left(\frac{M'}{M} - \frac{\sqrt{n}\beta_k}{A} \right) \geq \frac{M'}{2\beta_k} \gg_{\mathcal{Q}} 1.$$

Combining this with (32) completes the proof of (30). $\square$

Step 2. We prove that $\rho_K c^\delta \geq 1$ implies

$$(35) \qquad \begin{aligned} c^{t-1}(\beta_{j_1} \cdots \beta_{j_{t-1}}) &\gg_{\mathcal{Q}} \|\mathbf{m}(k; J)\| \\ &\geq_{\mathcal{Q}} c^{t-1}(\beta_{j_1} \cdots \beta_{j_{t-1}}) \, \rho_K \left(1 + O_{\mathcal{Q}}\left(c^{-(1-\theta-\delta)}\right)\right), \end{aligned}$$

uniformly in t, J such that $j_t \in K \setminus \{k\}$. If $j_t \in K \setminus \{k\}$, it follows from (32), the hypothesis $\rho_K c^\delta \geq 1$, and the fact that each $|\lambda_{i,j}| = 1$ that

$$\begin{aligned} c^{t-1}(\beta_{j_1} \cdots \beta_{j_t}) \gg_{\mathcal{Q}} \|\mathbf{m}(k; J)\| &\geq \frac{1}{\sqrt{n}} c^{t-1}(\beta_{j_1} \cdots \beta_{j_t}) \, \rho_K + O_{\mathcal{Q}}\left(c^{t-2+\theta} \beta_{j_1} \cdots \beta_{j_t}\right) \\ &\geq \frac{1}{\sqrt{n}} c^{t-1}(\beta_{j_1} \cdots \beta_{j_t}) \, \rho_K \left(1 + O_{\mathcal{Q}}\left(c^{-(1-\theta)} \rho_K^{-1}\right)\right). \end{aligned}$$

(35) now follows, whenever $t \geq 2$, by combining this with the facts that $j_t \in K$ and $\delta + \theta < 1$. If, however, $t = 1$, then the $O_{\mathcal{Q}}$ term is not present and the lower bound is understood to equal ρ_K.

Step 3. We prove that

$$(36) \qquad \|\mathbf{m}(k; J)\| \asymp_{\mathcal{Q}} c^{t-1} \beta_{j_1} \cdots \beta_{j_{t-1}} \quad \text{uniformly in } t, J \text{ such that } j_t \in S(\mathcal{Q}).$$

Proof of (36). The equation (31) implies that

$$(37) \qquad m_i(k; J) = c^{t-1} \beta_{j_1} \cdots \beta_{j_t} \lambda_{i,j_1} \cdots \lambda_{i,j_t} \left(\frac{b_{i,j_t}}{\lambda_{i,j_t} \beta_{j_t}} - \frac{b_{i,k}}{\lambda_{i,k} \beta_k} \right) + R_i(k; J)$$

where

$$R_i(k; J) = O_{\mathcal{Q}}\left(c^{t-2} \frac{\beta_{j_1} \cdots \beta_{j_t}}{\beta_k^2} + \sum_{h=1}^{t-2} c^h \beta_{j_1} \cdots \beta_{j_h} \right).$$

The condition $j_t \in S(\mathcal{Q})$ implies (by (22)) that $\beta_{j_t} \in [c^{-\theta}, \alpha_1]$. Combining this with the inequality $\beta_j \geq c^{-\theta}$ (for each j) then implies

$$(38) \qquad \begin{aligned} R_i(k; J) &\ll_{\mathcal{Q}} c^{t-2} \beta_{j_1} \cdots \beta_{j_t} + \beta_{j_1} \cdots \beta_{j_t} \sum_{h=1}^{t-2} \frac{c^h}{\beta_{j_{h+1}} \cdots \beta_{j_t}} \\ &\ll_{\mathcal{Q}} c^{t-2} \beta_{j_1} \cdots \beta_{j_t} + \beta_{j_1} \cdots \beta_{j_t} \sum_{h=1}^{t-2} c^h (c^\theta)^{t-h} \\ &\ll_{\mathcal{Q}} c^{t-2} \beta_{j_1} \cdots \beta_{j_t} + c^{t-2+2\theta} \beta_{j_1} \cdots \beta_{j_t} \\ &\ll_{\mathcal{Q}} c^{t-2+3\theta} \beta_{j_1} \cdots \beta_{j_{t-1}}. \end{aligned}$$

It now follows from (37) and (38) that there exists a constant $\Gamma_{\mathcal{Q}}$ such that for each i, k, J

$$|m_i(k; J)| \geq c^{t-1} \beta_{j_1} \cdots \beta_{j_{t-1}} \left(|b_{i,j_t}| - |b_{i,k}| \frac{\beta_{j_t}}{\beta_k} \right) - \Gamma_{\mathcal{Q}} c^{t-2+3\theta} \beta_{j_1} \cdots \beta_{j_{t-1}}.$$

Using, in addition, (34) and the (c, A)-permissibility of $(\mathcal{D}, \mathcal{F})$ (i.e., Definition 6, Parts 1, 3), we conclude (after an elementary argument left to the reader)

$$
\|\mathbf{m}(k; J)\|_1 := \sum_{i=1}^{n} |m_i(k; J)|
$$

$$
\geq c^{t-1} \beta_{j_1} \cdots \beta_{j_{t-1}} \left(\|\mathbf{b}_{j_t}\|_1 - \|\mathbf{b}_k\|_1 \cdot \frac{\beta_{j_t}}{\beta_k} \right) - \Gamma_{\mathcal{Q}} c^{t-2+3\theta} \beta_{j_1} \cdots \beta_{j_{t-1}}
$$

$$
\geq c^{t-1} \beta_{j_1} \cdots \beta_{j_{t-1}} \left(\|\mathbf{b}_{j_t}\| - \sqrt{n} \, \|\mathbf{b}_k\| \cdot \frac{\beta_{j_t}}{\beta_k} \right) - \Gamma_{\mathcal{Q}} c^{t-2+3\theta} \beta_{j_1} \cdots \beta_{j_{t-1}}
$$

$$
\geq c^{t-1} \beta_{j_1} \cdots \beta_{j_{t-1}} \left(M' - \alpha_1 \sqrt{n} \, \frac{\|\mathbf{b}_k\|}{\beta_k} \right) - \Gamma_{\mathcal{Q}} c^{t-2+3\theta} \beta_{j_1} \cdots \beta_{j_{t-1}}
$$

$$
\geq \frac{M'}{2} c^{t-1} \beta_{j_1} \cdots \beta_{j_{t-1}} - \Gamma_{\mathcal{Q}} c^{t-2+3\theta} \beta_{j_1} \cdots \beta_{j_{t-1}} .
$$

Thus, there exist constants $\kappa_1' > 0$ and $C' = C'(\mathcal{Q})$ such that $c > C'$ implies

(39) $\qquad \|\mathbf{m}(k; J)\| \gg \|\mathbf{m}(k; J)\|_1 > \kappa_1' c^{t-1} \beta_{j_1} \cdots \beta_{j_{t-1}}$ uniformly in t, J.

To prove the inequality in the other direction, we note

$$
\|\mathbf{m}(k; J)\| \leq \sqrt{n} \, c^{t-1} \beta_{j_1} \cdots \beta_{j_{t-1}} \left(\|\mathbf{b}_{j_t}\| + \|\mathbf{b}_k\| \frac{\beta_{j_t}}{\beta_k} \right) + O_{\mathcal{Q}} \left(c^{t-2+3\theta} \beta_{j_1} \cdots \beta_{j_{t-1}} \right)
$$

$$
\leq c^{t-1} \beta_{j_1} \cdots \beta_{j_{t-1}} \left(M' + R \frac{\|\mathbf{b}_k\|}{\beta_k} \right) + O_{\mathcal{Q}} \left(c^{t-2+3\theta} \beta_{j_1} \cdots \beta_{j_{t-1}} \right) .
$$

Thus, there exist constants $\kappa_2' > 0$ and $C'' = C''(\mathcal{Q})$ such that $c > C''$ implies

(40) $\qquad \|\mathbf{m}(k; J)\| < \kappa_2' c^{t-1} \beta_{j_1} \cdots \beta_{j_{t-1}}$ uniformly in t, J $(j_t \in S(\mathcal{Q}))$.

Combining (39) and (40) finishes the proof of (36).

Remark. Putting Steps 1–3 together, and using the a priori bound (that follows from (10)),

$$
\|\mathbf{u}_j\| = O_{\mathcal{Q}} \left(\frac{1}{c \beta_j} \right) \quad \text{for all } j \in \{1, \ldots, r\},
$$

we can then bound $\|\mathbf{u}_j\| / \|\mathbf{m}(k; J)\|$ as follows:

$$
\|\mathbf{u}_j\| / \|\mathbf{m}(k; J)\| \ll_{\mathcal{Q}} \begin{cases} c^{-t(1-\theta)} & \text{if } j_t \in S(\mathcal{Q}) \cup L(\mathcal{Q}), \\ c^{-(t-1)(1-\theta)-1} \rho_K^{-1} & \text{if } j_t \in K - \{k\}, \end{cases}
$$

where the implied constant only depends upon the element θ of $\mathcal{Q}$ and is uniformly bounded when $\theta \in (0, 1/3)$ (an easily verified property, left to the reader). Since the exponent of c is strictly negative in either of the two cases (in particular, $\delta < 1 - \theta$ implies this if $j_t \in K$) it follows that

(41) $\qquad \|\mathbf{u}_j\| / \|\mathbf{m}(k; J)\| < 1/2$ in all cases whenever $c \geq \max\{2^{-\frac{1}{1-\theta}}, (2\rho_K)^{-1}\}$.

Step 4. Estimate for $\Delta^{(2)}(\boldsymbol{\alpha}, \mathbf{u}_j)$. We show

(42) $\qquad\qquad\qquad \Delta^{(2)}(\boldsymbol{\alpha}, u_j) \ll_{\mathcal{Q}} c^{\mu-1+2\theta}.$

Proof. By (41), we are justified in using the Mean Value Theorem, applied to the function $\varphi(\mathbf{x}) = \mathbf{x}^{\boldsymbol{\alpha}} / \|\mathbf{x}\|^{\mu+2}$ and point $\mathbf{x} = \mathbf{m}(k, J)$ when restricted to any compact

convex subregion *not* containing the origin. Thus, (30) implies that

$$\frac{(\mathbf{m}(k;J)+\mathbf{u}_j)^{\boldsymbol{\alpha}}}{\|\mathbf{m}(k;J)+\mathbf{u}_j\|^{\mu+2}} - \frac{\mathbf{m}(k;J)^{\boldsymbol{\alpha}}}{\|\mathbf{m}(k;J)\|^{\mu+2}} \quad \ll_{\mathcal{Q}} \quad \frac{\|\mathbf{u}_j\|}{\|\mathbf{m}(k;J)\|^{\mu+1}} \ll_{\mathcal{Q}} \frac{\frac{1}{c\beta_j}}{(c^{t-1}\beta_{j_1}\cdots\beta_{j_t})^{\mu+1}}$$

$$(43) \qquad\qquad\qquad\qquad\qquad\qquad \ll_{\mathcal{Q}} \quad \frac{1}{\beta_j(\beta_{j_1}\cdots\beta_{j_t})^{\mu+1}c^{(t-1)(\mu+1)+1}}$$

uniformly in t, J such that $j_t \in L(\mathcal{Q})$. It follows that

$$\Delta^{(2)}(\boldsymbol{\alpha}, u_j) \ll_{\mathcal{Q}} \sum_{t\in\mathbf{N}} \sum_{\{(j_1,\cdots,j_t):j_t\in L(\mathcal{Q})\}} \frac{1}{\beta_j(\beta_{j_1}\cdots\beta_{j_t})^{\mu+1}c^{(t-1)(\mu+1)+1}}$$

$$\ll_{\mathcal{Q}} \sum_{t=1}^{\infty} \frac{1}{\beta_j c^{(t-1)(\mu+1)+1}} \left(\sum_{h=1}^{r} \frac{1}{\beta_h^{\mu+1}}\right)^t.$$

Since each $\beta_h \geq c^{-\theta}$, we see that (24) implies

$$\left(\sum_{h=1}^{r} \frac{1}{\beta_h^{\mu+1}}\right)^t \leq \left(\sum_{h=1}^{r} \frac{c^\theta}{\beta_h^\mu}\right)^t = c^{(\theta+\mu)t}.$$

Thus, a simple check verifies

$$(44) \qquad \Delta^{(2)}(\boldsymbol{\alpha}, \mathbf{u}_j) \ll_{\mathcal{Q}} \frac{c^\mu}{\beta_j} \sum_{t=1}^{\infty} \frac{1}{c^{(1-\theta)t}} \ll_{\mathcal{Q}} \frac{c^\mu}{\beta_j} \cdot \frac{1}{c^{1-\theta}-1} \ll_{\mathcal{Q}} \frac{c^{\mu-1+\theta}}{\beta_j} \ll_{\mathcal{Q}} c^{\mu-1+2\theta}.$$

Step 5. Estimate for $\Delta^{(3)}(\boldsymbol{\alpha}, \mathbf{u}_j)$. We show

$$(45) \qquad\qquad\qquad\qquad \Delta^{(3)}(\boldsymbol{\alpha}, \mathbf{u}_j) \ll_{\mathcal{Q}} c^{\mu-1+\theta}.$$

Proof. As in Step 4, the estimate of Step 3 implies

$$(46) \qquad \frac{(\mathbf{m}(k;J)+\mathbf{u}_j)^{\boldsymbol{\alpha}}}{\|\mathbf{m}(k;J)+\mathbf{u}_j\|^{\mu+2}} - \frac{\mathbf{m}(k;J)^{\boldsymbol{\alpha}}}{\|\mathbf{m}(k;J)\|^{\mu+2}} \ll_{\mathcal{Q}} \frac{1}{\beta_j(\beta_{j_1}\cdots\beta_{j_{t-1}})^{\mu+1}c^{(t-1)(\mu+1)+1}}.$$

uniformly in $t, J = (j_1,\ldots,j_t)$ such that $j_t \in S(\mathcal{Q})$. Proceeding as in Step 4, it follows that

$$\Delta^{(3)}(\boldsymbol{\alpha}, \mathbf{u}_j) \ll_{\mathcal{Q}} \sum_{t\in\mathbf{N}} \sum_{\{(j_1,\cdots,j_t):j_t\in S(\mathcal{Q})\}} \frac{1}{\beta_j(\beta_{j_1}\cdots\beta_{j_{t-1}})^{\mu+1}c^{(t-1)(\mu+1)+1}}$$

$$\ll_{\mathcal{Q}} \sum_{t=1}^{\infty} \frac{\#S(\mathcal{Q})}{\beta_j c^{(t-1)(\mu+1)+1}} \cdot \left(\sum_{h=1}^{r} \frac{1}{\beta_h^{\mu+1}}\right)^{t-1}$$

$$\ll_{\mathcal{Q}} \sum_{t=1}^{\infty} \frac{(\#S(\mathcal{Q}))\ c^{\theta(t-1)}}{\beta_j c^{(t-1)(\mu+1)+1}} \cdot \left(\sum_{h=1}^{r} \beta_h^{-\mu}\right)^{t-1} \ll_{\mathcal{Q}} \sum_{t=1}^{\infty} \frac{(\#S(\mathcal{Q}))\ c^{(\theta+\mu)(t-1)}}{\beta_j c^{(t-1)(\mu+1)+1}}$$

$$\ll_{\mathcal{Q}} \frac{(\#S(\mathcal{Q}))\ c^{-\theta}}{\beta_j} \sum_{t=1}^{\infty} \frac{1}{c^{(1-\theta)t}} \ll_{\mathcal{Q}} \frac{\#S(\mathcal{Q})}{c\beta_j}.$$

Since (22) also implies $\#S(\mathcal{Q}) \cdot \alpha_1^{-\mu} \leq \sum_{\ell\in\mathcal{S}(\mathcal{Q})} \beta_\ell^{-\mu}$, it follows that

$$\#S(\mathcal{Q}) \leq \alpha_1^\mu \cdot \sum_{\ell\in S(\mathcal{Q})} \beta_\ell^{-\mu} \leq \alpha_1^\mu \cdot \sum_{h=1}^{r} \beta_h^{-\mu} = \alpha_1^\mu \cdot c^\mu.$$

Thus, for each $j = 1, \ldots, r$,

$$\Delta^{(3)}(\boldsymbol{\alpha}, \mathbf{u}_j) \ll_{\mathcal{Q}} \frac{\alpha_1^{\mu} c^{\mu-1}}{\beta_j} \ll_{\mathcal{Q}} c^{\mu-1+\theta}.$$

Remark. In particular, the estimate from Part 5 of Definition 6 is *not needed* for this bound, unlike that for (64) below, since the exponent for c in (45) is already strictly less than μ.

Step 6. Estimate of $\Delta^{(4)}(\boldsymbol{\alpha}, \mathbf{u}_j)$. We prove

$$(47) \qquad \delta < 1 - \theta \quad \text{implies for all } j \ \ \Delta^{(4)}(\boldsymbol{\alpha}, \mathbf{u}_j) \ll_{\mathcal{Q}} \frac{1}{c^{1-\theta} \, \rho_K^{\mu+1}}$$

uniformly in c satisfying $c \geq \max\left\{\rho_K^{-1/\delta}, 2^{-\frac{1}{1-\theta}}, (2\rho_K)^{-1}\right\}$.

As in steps 4–5, (35), (41), and the Mean Value Theorem imply that for $c \geq \max\{\rho_K^{-1/\delta}, 2^{-\frac{1}{1-\theta}}, (2\rho_K)^{-1}\}$:

$$\frac{(\mathbf{m}(k; J) + \mathbf{u}_j)^{\boldsymbol{\alpha}}}{\|\mathbf{m}(k; J) + \mathbf{u}_j\|^{\mu+2}} - \frac{\mathbf{m}(k; J)^{\boldsymbol{\alpha}}}{\|\mathbf{m}(k; J)\|^{\mu+2}} \ll_{\mathcal{Q}} \frac{\|\mathbf{u}_j\|}{\|\mathbf{m}(k; J)\|^{\mu+1}}$$

$$(48) \qquad\qquad\qquad\qquad \ll_{\mathcal{Q}} \frac{\frac{1}{c\beta_j}}{(c^{t-1}(\beta_{j_1} \cdots \beta_{j_{t-1}})\rho_K)^{\mu+1}}$$

$$\ll_{\mathcal{Q}} \frac{1}{\beta_j(\beta_{j_1} \cdots \beta_{j_t})^{\mu+1} c^{(t-1)(\mu+1)+1} \, \rho_K^{\mu+1}} \, .$$

Proceeding as in Step 4, it follows that

$$\Delta^{(4)}(\boldsymbol{\alpha}, \mathbf{u}_j) \ll_{\mathcal{Q}} \sum_{t \in \mathbf{N}} \sum_{\{(j_1,\ldots,j_t): j_t \in K \setminus \{k\}\}} \frac{1}{\beta_j(\beta_{j_1} \cdots \beta_{j_{t-1}})^{\mu+1} c^{(t-1)(\mu+1)+1} \, \rho_K^{\mu+1}}$$

$$\ll_{\mathcal{Q}} \sum_{t=1}^{\infty} \frac{1}{\beta_j c^{(t-1)(\mu+1)+1} \, \rho_K^{\mu+1}} \left(\sum_{h=1}^{r} \frac{1}{\beta_h^{\mu+1}} \right)^{t-1}$$

$$\ll_{\mathcal{Q}} \sum_{t=1}^{\infty} \frac{c^{\theta(t-1)}}{\beta_j c^{(t-1)(\mu+1)+1} \, \rho_K^{\mu+1}} \left(\sum_{h=1}^{r} \beta_h^{-\mu} \right)^{t-1} \ll_{\mathcal{Q}} \sum_{t=1}^{\infty} \frac{c^{(\theta+\mu)(t-1)}}{\beta_j c^{(t-1)(\mu+1)+1} \, \rho_K^{\mu+1}}$$

$$\ll_{\mathcal{Q}} \frac{c^{-\theta}}{\beta_j \, \rho_K^{\mu+1}} \sum_{t=1}^{\infty} \frac{1}{c^{(1-\theta)t}} \ll_{\mathcal{Q}} \frac{1}{c \, \beta_j \, \rho_K^{\mu+1}} \ll_{\mathcal{Q}} \frac{1}{c^{1-\theta} \, \rho_K^{\mu+1}}.$$

This completes the proof of (47).

Step 7. Estimate for $\Delta^{(1)}(\boldsymbol{\alpha}, \mathbf{u}_j)$. For given $k \in K$, the estimates we need for each $\Xi_j(P_k, \boldsymbol{\alpha})$ depend upon whether $j = k$, $j \in L(\mathcal{Q})$, $j \in K \setminus \{k\}$, or $j \in S(\mathcal{Q})$. So it is convenient to split the discussion into four cases.

For each such case it is useful to have an explicit expression for the coordinates of $P_k + \mathbf{u}_j$ (in the basis $\{\mathbf{g}_j\}_j$) in powers of c^{-1} (at least to first order). Setting $\mathbf{u}_j = (u_{1,j}, \ldots, u_{n,j})$ it follows that if $j \neq k$, then

$$(49) \qquad P_{i,k} + \mathbf{u}_{i,j} = \frac{b_{i,k}}{1 - c\beta_k\lambda_{i,k}} + \frac{b_{i,j}}{c\beta_j\lambda_{i,j}} = \delta_{j,k}(i) \cdot c^{-1} + O_{\mathcal{Q}}(c^{-2}),$$

where

$$\delta_{j,k}(i) = \frac{b_{i,j}}{\lambda_{i,j}\beta_j} - \frac{b_{i,k}}{\lambda_{i,k}\beta_k} \, .$$

Case 1. $(j = k)$ Here the expression is simple. For each $i = 1, \ldots, n$, it is clear that

$$P_{i,k} + u_{i,k} = \frac{b_{i,k}}{1 - c\beta_k \lambda_{i,k}} + \frac{b_{i,k}}{c\beta_k \lambda_{i,k}} = \left(\frac{b_{i,k}}{\beta_k^2 \lambda_{i,k}^2}\right) \cdot c^{-2} + O_{\mathcal{Q}}(c^{-3}).$$

Using the relation $\lambda_k^{\boldsymbol{\alpha}} = 1$, we then verify (details left to the reader):

$$\frac{(P_k + \mathbf{u}_k)^{\boldsymbol{\alpha}}}{\|P_k + \mathbf{u}_k\|^{\mu+2}} = \left(\beta_k^{2\mu} \frac{\mathbf{b}_k^{\boldsymbol{\alpha}}}{\|\mathbf{b}_k\|^{\mu+2}}\right) c^{2\mu} + O_{\mathcal{Q}}(c^{2\mu-1}).$$

Furthermore, since

$$P_{i,k} = \frac{b_{i,k}}{1 - c\beta_k \lambda_{i,k}} = \frac{-b_{i,k}}{\beta_k \lambda_{i,k}} \cdot c^{-1} + O_{\mathcal{Q}}(c^{-2}),$$

it is clear that

$$\frac{P_k^{\boldsymbol{\alpha}}}{\|P_k\|^{\mu+2}} \asymp_{\mathcal{Q}} c^{\mu}.$$

Thus,

$$(50) \qquad \Xi_k(P_k, \boldsymbol{\alpha}) = \left(\beta_k^{2\mu} \frac{\mathbf{b}_k^{\boldsymbol{\alpha}}}{\|\mathbf{b}_k\|^{\mu+2}}\right) c^{2\mu} + O_{\mathcal{Q}}(c^{2\mu-1} + c^{\mu}).$$

Case 2. $(j \in L(\mathcal{Q}))$ Using (49) we have

$$(51) \qquad \Xi_j(P_k, \boldsymbol{\alpha}) = \frac{\prod_{i=1}^{n}\left(\delta_{j,k}(i) \cdot c^{-1} + O_{\mathcal{Q}}(c^{-2})\right)^{\alpha_i}}{\left\|\left(\delta_{j,k}(1) \cdot c^{-1} + O_{\mathcal{Q}}(c^{-2}), \ldots, \delta_{j,k}(n) \cdot c^{-1} + O_{\mathcal{Q}}(c^{-2})\right)\right\|^{\mu+2}}$$
$$- \frac{\prod_{i=1}^{n}\left([\frac{-b_{i,k}}{\beta_k \lambda_{i,k}}]c^{-1} + O_{\mathcal{Q}}(c^{-2})\right)^{\alpha_i}}{\left\|\left(\frac{-b_{1,k}}{\beta_k \lambda_{1,k}} \cdot c^{-1} + O_{\mathcal{Q}}(c^{-2}), \ldots, \frac{-b_{n,k}}{\beta_k \lambda_{k,n}} \cdot c^{-1} + O_{\mathcal{Q}}(c^{-2})\right)\right\|^{\mu+2}}.$$

Setting $\boldsymbol{\delta}(j, k) = \left(\delta_{j,k}(1), \ldots, \delta_{j,k}(n)\right)$, we note that part 1 of Definition 6, (22), (34), and the fact that $j \in L(\mathcal{Q})$ then tell us

$$1 \gg_{\mathcal{Q}} \|\boldsymbol{\delta}(j, k)\| \gg \sum_{i=1}^{n} \frac{|b_{i,k}|}{\beta_k} - \frac{|b_{i,j}|}{\beta_j} \geq \frac{\|\mathbf{b}_k\|}{\beta_k} - \frac{\sqrt{n}\|\mathbf{b}_j\|}{\beta_j}$$
$$\geq \frac{M'}{\beta_k} - \frac{\sqrt{n}M}{\beta_{j_t}} = \frac{M}{\beta_k}\left(\frac{M'}{M} - \frac{\sqrt{n}\beta_k}{\beta_{j_t}}\right) \geq \frac{M}{\beta_k}\left(\frac{M'}{M} - \frac{\sqrt{n}\beta_k}{A}\right) \geq \frac{M'}{2\beta_k} \gg_{\mathcal{Q}} 1.$$

A routine calculation, left to the reader, now implies

$$(52) \qquad \Xi_j(P_k, \boldsymbol{\alpha}) = \left[\frac{\boldsymbol{\delta}(j, k)^{\boldsymbol{\alpha}}}{\|\boldsymbol{\delta}(j, k)\|^{\mu+2}} - \frac{\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)^{\boldsymbol{\alpha}}}{\|\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)\|^{\mu+2}}\right] c^{\mu} + O_{\mathcal{Q}}(c^{\mu-1}),$$

where

$$\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k) := \beta_k^{-1} \cdot \left(b_{1,k}/\lambda_{1,k}, \ldots, b_{n,k}/\lambda_{n,k}\right).$$

Case 3. $(j \in S(\mathcal{Q}))$ An elementary check shows that for each i

$$|P_{i,k} + u_{i,j}| \geq \left(\frac{|b_{i,j}|}{\beta_j} - \frac{|b_{i,k}|}{\beta_k}\right) \cdot c^{-1} + O_{\mathcal{Q}}(c^{-2}).$$

Thus, using again (22), (34), and the norm $\|\cdot\|_1$ from Step 3, we find:

$$
\begin{aligned}
\|P_k + \mathbf{u}_j\| &\geq \frac{1}{\sqrt{n}}\|P_k + \mathbf{u}_j\|_1 \geq \frac{1}{\sqrt{n}}\left(\frac{\|\mathbf{b}_j\|_1}{\beta_j} - \frac{\|\mathbf{b}_k\|_1}{\beta_k}\right)c^{-1} + O_{\mathcal{Q}}(c^{-2}) \\
&\geq \frac{1}{\sqrt{n}}\left(\frac{\|\mathbf{b}_j\|}{\beta_j} - \sqrt{n}\frac{\|\mathbf{b}_k\|}{\beta_k}\right)c^{-1} + O_{\mathcal{Q}}(c^{-2}) \\
&\geq \frac{1}{\sqrt{n}}\left(\frac{M'}{\alpha_1} - \sqrt{n}\frac{\|b_k\|}{\beta_k}\right)c^{-1} + O_{\mathcal{Q}}(c^{-2}) \geq \frac{M}{\sup G_2}\cdot c^{-1} + O_{\mathcal{Q}}(c^{-2}).
\end{aligned}
$$

It follows that

$$
\|P_k + \mathbf{u}_j\| \gg_{\mathcal{Q}} c^{-1}.
$$

We deduce that for $j \in S(\mathcal{Q})$:

$$
\begin{aligned}
(53)\qquad \Xi_j(P_k, \boldsymbol{\alpha}) &= \frac{(P_k + \mathbf{u}_j)^{\boldsymbol{\alpha}}}{\|P_k + \mathbf{u}_j\|^{\mu+2}} - \frac{P_k^{\boldsymbol{\alpha}}}{\|P_k\|^{\mu+2}} \ll_{\mathcal{Q}} \frac{1}{\|P_k + \mathbf{u}_j\|^{\mu}} + \frac{1}{\|P_k\|^{\mu}} \\
&\ll_{\mathcal{Q}} c^{\mu} + \frac{1}{\|P_k\|^{\mu}} \ll_{\mathcal{Q}} c^{\mu}.
\end{aligned}
$$

Case 4. ($j \in K \setminus \{k\}$) Using notation from Case 2, it follows from (19) that

$$
(54)\qquad 1 \gg_{\mathcal{Q}} \|\boldsymbol{\delta}(j,k)\| \gg \sum_{i=1}^{n}\Big|\frac{b_{i,j}}{\lambda_{i,j}\beta_j} - \frac{b_{i,k}}{\lambda_{i,k}\beta_k}\Big| \geq \sum_{i=1}^{n}\Big|\frac{|b_{i,k}|}{\beta_k} - \frac{|b_{i,j}|}{\beta_j}\Big| \geq \rho_K,
$$

uniformly in $j \in K \setminus \{k\}$. As with (52), the expression (51) for $\Xi_j(P_k, \boldsymbol{\alpha})$ gives

$$
(55)\qquad \Xi_j(P_k, \boldsymbol{\alpha}) = \left[\frac{\boldsymbol{\delta}(j,k)^{\boldsymbol{\alpha}}}{\|\boldsymbol{\delta}(j,k)\|^{\mu+2}} - \frac{\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)^{\boldsymbol{\alpha}}}{\|\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)\|^{\mu+2}}\right]c^{\mu} + O_{\mathcal{Q}}\left(c^{\mu-1}\rho_K^{-\mu-2}\right),
$$

Thus, the hypothesis $\rho_K c^{\delta} \geq 1$ then implies

$$
(56)\qquad \Xi_j(P_k, \boldsymbol{\alpha}) = \left[\frac{\boldsymbol{\delta}(j,k)^{\boldsymbol{\alpha}}}{\|\boldsymbol{\delta}(j,k)\|^{\mu+2}} - \frac{\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)^{\boldsymbol{\alpha}}}{\|\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)\|^{\mu+2}}\right]c^{\mu} + O_{\mathcal{Q}}\left(c^{\mu-1+\delta(\mu+2)}\right).
$$

Step 8. Finishing the proof of Lemma 2. We assume that $c \geq C(A, \Psi_K, \mathcal{Q})$, as defined in (21). Combining (29) with Steps 4–6 (i.e., (42), (45), (47)), and the global constraint (24), we have

$$
\begin{aligned}
(57)\qquad \left(\sum_{j=1}^{r} c_j^{-\mu}\ln c_j\right)\mathcal{B}(\mathcal{F}, \boldsymbol{\alpha}) &= \sum_{j=1}^{r} c_j^{-\mu}\Delta(\boldsymbol{\alpha}, \mathbf{u}_j) \\
&= \sum_{j=1}^{r} c_j^{-\mu}\left(\Delta^{(1)}(\boldsymbol{\alpha}, \mathbf{u}_j) + \Delta^{(2)}(\boldsymbol{\alpha}, \mathbf{u}_j) + \Delta^{(3)}(\boldsymbol{\alpha}, \mathbf{u}_j) + \Delta^{(4)}(\boldsymbol{\alpha}, \mathbf{u}_j)\right) \\
&= \sum_{j=1}^{r} c_j^{-\mu}\Delta^{(1)}(\boldsymbol{\alpha}, \mathbf{u}_j) + O_{\mathcal{Q}}\left(\sum_{j=1}^{r} c_j^{-\mu}\left[c^{\mu-1+2\theta} + c^{\mu-1+\theta} + c^{-1+\theta}\rho_K^{-\mu-1}\right]\right) \\
&= \sum_{j=1}^{r} c_j^{-\mu}\Delta^{(1)}(\boldsymbol{\alpha}, \mathbf{u}_j) + O_{\mathcal{Q}}\left(c^{\mu-1+2\theta} + c^{-1+\theta}\rho_K^{-\mu-1}\right).
\end{aligned}
$$

By definition,

$$(58) \quad \sum_{j=1}^{r} c_j^{-\mu} \Delta^{(1)}(\boldsymbol{\alpha}, \mathbf{u}_j) = \sum_{k \in K} c_k^{-\mu} \Xi_k(P_k, \boldsymbol{\alpha}) + \sum_{k \in K} \sum_{j \in L(\mathcal{Q})} c_j^{-\mu} \Xi_j(P_k, \boldsymbol{\alpha})$$
$$+ \sum_{k \in K} \sum_{j \in S(\mathcal{Q})} c_j^{-\mu} \Xi_j(P_k, \boldsymbol{\alpha}) + \sum_{k \in K} \sum_{j \in K \setminus \{k\}} c_j^{-\mu} \Xi_j(P_k, \boldsymbol{\alpha}).$$

Now, Case 1 of Step 7 implies (see (50))

$$(59) \quad \sum_{k \in K} c_k^{-\mu} \Xi_k(P_k, \boldsymbol{\alpha}) = \sum_{k \in K} c_k^{-\mu} \left[\left(\beta_k^{2\mu} \frac{\mathbf{b}_k^{\boldsymbol{\alpha}}}{\|\mathbf{b}_k\|^{\mu+2}} \right) c^{2\mu} + O_{\mathcal{Q}} \left(c^{2\mu-1} + c^\mu \right) \right]$$
$$= \sum_{k \in K} \left(\beta_k^{\mu} \frac{\mathbf{b}_k^{\boldsymbol{\alpha}}}{\|\mathbf{b}_k\|^{\mu+2}} \right) c^\mu + O_{\mathcal{Q}} \left(c^{\mu-1} \right).$$

Case 2 of Step 7 (see (52)) implies

$$(60) \quad \sum_{k \in K} \sum_{j \in L(\mathcal{Q})} c_j^{-\mu} \Xi_j(P_k, \boldsymbol{\alpha})$$
$$= \sum_{k \in K} \sum_{j \in L(\mathcal{Q})} c_j^{-\mu} \left\{ \left[\frac{\boldsymbol{\delta}(j,k)^{\boldsymbol{\alpha}}}{\|\boldsymbol{\delta}(j,k)\|^{\mu+2}} - \frac{\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)^{\boldsymbol{\alpha}}}{\|\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)\|^{\mu+2}} \right] c^\mu + O_{\mathcal{Q}} \left(c^{\mu-1} \right) \right\}.$$

To bound the sum over $j \in L(\mathcal{Q})$, we use the (c, A)-admissibility of $(\mathcal{D}, \mathcal{F})$ and the Mean-Value Theorem, applied to the function $\varphi(\mathbf{x}) = \mathbf{x}^{\boldsymbol{\alpha}} / \|\mathbf{x}\|^{\mu+2}$, when restricted to any compact convex subregion *not* containing the origin. For our purposes, since $\boldsymbol{\delta}(j,k) = \mathbf{y}_j - \mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)$ where

$$(61) \quad \mathbf{y}_j := \beta_j^{-1} \cdot \left(b_{1,j}/\lambda_{1,j}, \ldots, b_{n,j}/\lambda_{n,j} \right),$$

we will need to choose $\mathbf{x}$ to equal $-\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)$ for some $k \in K$.

Since $\|\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)\| \gg 1$ (i.e. uniformly in k), it is clear that there exists $0 < \kappa_1 < \kappa$ so that $\|\mathbf{y}_j\| \le \kappa_1$ implies the line segment connecting $\mathbf{y}_j - \mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)$ to $-\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)$ will not pass through the origin for all k. Indeed, it suffices to choose $\kappa_1 := \frac{1}{2} \cdot \min_k \{\|\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)\|\}$. By construction, it follows that κ_1 *depends only upon* $\mathcal{Q}$.

We now note that any $\mathbf{y}_j$ defined in (61) satisfies this needed property since the definition of (c, A)-permissibility implies $A \ge \alpha_2$. This insures that $AM'/M\beta_k > 2$. Thus, for any $\mathbf{y}_j$, it follows that

$$\|\mathbf{y}_j\| \le \frac{M}{A} = \frac{M\beta_k}{AM'} \cdot \frac{M'}{\beta_k} \le \frac{M\beta_k}{AM'} \cdot \|\mathbf{b}_k^*(\beta_k, \lambda_k)\| < \frac{1}{2} \cdot \|\mathbf{b}_k^*(\beta_k, \lambda_k)\| = \kappa_1 \text{ (for each } k \in K).$$

Since $|\boldsymbol{\alpha}| = 2$ implies that φ is an even function, we now conclude

$$(62) \quad \varphi\left(-\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k) + \mathbf{y}_j \right) - \varphi\left(\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k) \right) \ll_{\kappa_1} \frac{\|\mathbf{y}_j\|}{\|\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)\|^{\mu+1}} \ll_{\mathcal{Q}} \|\mathbf{y}_j\|$$

uniformly in k and $j \in L(\mathcal{Q})$. Thus,

$$\sum_{k \in K} \sum_{j \in L(\mathcal{Q})} c_j^{-\mu} \left\{ \left[\frac{\boldsymbol{\delta}(j,k)^{\boldsymbol{\alpha}}}{\|\boldsymbol{\delta}(j,k)\|^{\mu+2}} - \frac{\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)^{\boldsymbol{\alpha}}}{\|\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)\|^{\mu+2}} \right] \right\} c^\mu \ll_{\mathcal{Q}} \frac{c^\mu}{A} \sum_{k \in K} \sum_{j \in L(\mathcal{Q})} c_j^{-\mu}$$
$$(63) \quad \ll_{\mathcal{Q}} \frac{c^\mu}{A} \left(\sum_{j=1}^{r} c_j^{-\mu} \right) = \frac{c^\mu}{A}.$$

Case 3 of Step 7 (see (53)), combined with Part 5 of Definition 6, implies

$$(64) \quad \sum_{j \in S(\mathcal{Q})} c_j^{-\mu} \Xi_j(P_k, \boldsymbol{\alpha}) = O_{\mathcal{Q}}\Big(\sum_{j \in S(\mathcal{Q})} \beta_j^{-\mu} \Big) = O_{\mathcal{Q}}\big(\#S(\mathcal{Q}) c^{\theta\mu} \big) = O_{\mathcal{Q}}\bigg(\frac{\#S(\mathcal{Q})}{c^{(1-\theta)\mu}} \cdot c^\mu \bigg).$$

Case 4 of Step 7 (see (55)) implies that

$$\sum_{k \in K} \sum_{j \in K \setminus \{k\}} c_j^{-\mu} \, \Xi_j(P_k, \boldsymbol{\alpha}) = \sum_{k \in K} \sum_{j \in K \setminus \{k\}} c_j^{-\mu} \left\{ \left[\frac{\boldsymbol{\delta}(j,k)^{\boldsymbol{\alpha}}}{\|\boldsymbol{\delta}(j,k)\|^{\mu+2}} - \frac{\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)^{\boldsymbol{\alpha}}}{\|\mathbf{b}_k^*(\beta_k, \boldsymbol{\lambda}_k)\|^{\mu+2}} \right] c^\mu \right.$$

$$(65) \qquad\qquad\qquad\qquad \left. + O_{\mathcal{Q}}\big(c^{\mu-1} \rho_K^{-\mu-2} \big) \right\}.$$

Applying (54) we conclude

$$(66) \quad \sum_{k \in K} \sum_{j \in K \setminus \{k\}} c_j^{-\mu} \, \Xi_j(P_k, \boldsymbol{\alpha}) = O_{\mathcal{Q}} \left(\frac{1}{\|\boldsymbol{\delta}(j,k)\|^\mu} + 1 + c^{-1} \rho_K^{-\mu-2} \right)$$

$$= O_{\mathcal{Q}} \left(1 + \rho_K^{-\mu} + c^{-1} \rho_K^{-\mu-2} \right).$$

Combining together (57)–(64) and (66), we see that

$$\left(\sum_{j=1}^r c_j^{-\mu} \ln c_j \right) \mathcal{B}(\mathcal{F}, \boldsymbol{\alpha}) = \sum_{k \in K} \left(\beta_k^\mu \frac{\mathbf{b}_k^{\boldsymbol{\alpha}}}{\|\mathbf{b}_k\|^{\mu+2}} \right) c^\mu + O_{\mathcal{Q}} \left(1 + c^{\mu-1+2\theta} + c^{-1+\theta} \rho_K^{-\mu-1} \right.$$

$$(67) \qquad\qquad\qquad \left. + c^{-1} \rho_K^{-\mu-2} + \rho_K^{-\mu} + \left(\frac{\#S(\mathcal{Q})}{c^{(1-\theta)\mu}} + \frac{1}{A} \right) c^\mu \right).$$

Moreover, the fact that $\rho_K \, c^\delta \geq 1$ implies

$$(68) \quad \left(\sum_{j=1}^r c_j^{-\mu} \ln c_j \right) \mathcal{B}(\mathcal{F}, \boldsymbol{\alpha}) = \sum_{k \in K} \left(\beta_k^\mu \frac{\mathbf{b}_k^{\boldsymbol{\alpha}}}{\|\mathbf{b}_k\|^{\mu+2}} \right) c^\mu + O_{\mathcal{Q}} \left(c^v + \left(\frac{\#S(\mathcal{Q})}{c^{(1-\theta)\mu}} + \frac{1}{A} \right) c^\mu \right),$$

where $v < \mu$ is defined in (21). This implies the estimate asserted in (27) and finishes the proof. $\qquad\square$

Proof of Theorem 1. Given an admissible vector $\mathcal{Q}$ and (c, A)-permissible $(\mathcal{D}, \mathcal{F})$, Lemma 2 gives the following explicit identity (see (13))

$$(69) \qquad \left(\sum_{j=1}^r c_j^{-\mu} \ln c_j \right)^n V(\mathcal{F}, \mathbf{D}_\mu)$$

$$(70) \qquad = \sum_{(\omega_1, \omega_2) \in \mathcal{S}_n(\mathcal{F})} \operatorname{sgn}(\omega_1 \omega_2) \prod_{i=1}^n \left[\left(\sum_{k \in K} \frac{\beta_k^\mu \, \mathbf{b}_k^{\mathbf{e}_{\omega_1(i)} + \mathbf{e}_{\omega_2(i)}}}{\|\mathbf{b}_k\|^{\mu+2}} \right) c^\mu \right.$$

$$\left. + O_{\mathcal{Q}} \left(\left[\frac{2}{A} + \frac{\#S(\mathcal{Q})}{c^{(1-\theta)\mu}} \right] \cdot c^\mu \right) \right]$$

$$(71) \qquad = \Psi_K \cdot c^{n\mu} + O_{\mathcal{Q}} \left(\left[\frac{2}{A} + \frac{\#S(\mathcal{Q})}{c^{(1-\theta)\mu}} \right] \cdot c^{n\mu} \right),$$

where the proof of Lemma 2 has shown that the implicit constant in the $O_{\mathcal{Q}}$ bound is bounded above by an expression of the form

$$\phi_0(\mathcal{Q}) + \phi_1(\mathcal{Q}) c^{-\gamma},$$

with $\gamma = \gamma(\delta, \theta) > 0$, and ϕ_0, ϕ_1 bounded uniformly over all possible (δ, θ) of any admissible $\mathcal{Q}$.

As explained at the end of §3.2 in Remark 1, it follows that $\mathcal{F}$ will be thick once we know that $\Psi_K \neq 0$ and c and A are sufficiently large, which follows from the second condition in the statement of Theorem 1. This completes the proof of the theorem. $\qquad\square$

Proof of Corollary 1. To conclude, by using Theorem 1, it suffices to prove that

$$(72) \quad \mathcal{R}(\mathcal{F}) := \sum_{(\omega_1, \omega_2) \in \mathcal{S}_2(\mathcal{F})} \mathrm{sgn}(\omega_1 \omega_2) \left(\sum_{k \in K} \beta_k^\mu \frac{\mathbf{b}_k^{\mathbf{e}_{\omega_1(1)} + \mathbf{e}_{\omega_2(1)}}}{\|\mathbf{b}_k\|^{\mu+2}} \right) \left(\sum_{k \in K} \beta_k^\mu \frac{\mathbf{b}_k^{\mathbf{e}_{\omega_1(2)} + \mathbf{e}_{\omega_2(2)}}}{\|\mathbf{b}_k\|^{\mu+2}} \right)$$
$$\neq 0.$$

where $\mathcal{S}_2(\mathcal{F}) = \{(\omega_1, \omega_2) \in \mathcal{S}_2^2 \mid \boldsymbol{\lambda}_j^{\mathbf{e}_{\omega_1(i)} + \mathbf{e}_{\omega_2(i)}}$ for all $j = 1, \ldots, r$ and each $i = 1, 2\}$.

Claim. For all $j = 1, \ldots, r$ $(\lambda_{1,j})^2 = (\lambda_{2,j})^2 = 1$ and $\lambda_{1,j} \lambda_{2,j} = \pm 1$.

Proof. We know that $\zeta_\mathcal{F}(2\mathbf{e}_1; s) + \zeta_\mathcal{F}(2\mathbf{e}_2; s) = \zeta_\mathcal{F}(s - 2)$ has a pole at $s = \mu + 2$. It follows that at least one of the two zeta function $\zeta_\mathcal{F}(2\mathbf{e}_1; s)$ or $\zeta_\mathcal{F}(2\mathbf{e}_2; s)$ also has a pole at $s = \mu + 2$. As a result, Lemma 1 Part 1 implies

$$\lambda_{1,j}^2 = 1 \quad \text{or} \quad \lambda_{2,j}^2 = 1 \quad \text{(for each } j\text{)}.$$

Moreover, we also know that $\lambda_{j,1} \lambda_{j,2} = \det(T_j) = \pm 1$ for each j. Thus, $\lambda_{1,j}^2 = \lambda_{2,j}^2 = 1$ and $\lambda_{j,1} \lambda_{j,2} = \pm 1$ (for each j). This ends the proof of the Claim. $\qquad\square$

Now we distinguish two cases:

First case. We assume that $\lambda_{1,j} \lambda_{2,j} = 1$ for each j. It is then clear that in this case we have $\mathcal{S}_2(\mathcal{F}) = \{(\mathrm{id}, \mathrm{id}); (\mathrm{id}, \tau); (\tau, \mathrm{id}); (\tau, \tau)\}$ where $\tau = (12)$ exchanges 1 and 2. It follows from (72) that

$$\mathcal{R}(\mathcal{F}) = 2 \left(\sum_{k \in K} \beta_k^\mu \frac{(b_{1,k})^2}{\|\mathbf{b}_k\|^{\mu+2}} \right) \left(\sum_{k \in K} \beta_k^\mu \frac{(b_{2,k})^2}{\|\mathbf{b}_k\|^{\mu+2}} \right) - 2 \left(\sum_{k \in K} \beta_k^\mu \frac{b_{1,k} b_{2,k}}{\|\mathbf{b}_k\|^{\mu+2}} \right)^2.$$

Cauchy–Schwarz Inequality and the linear independence of the two vectors $(b_{1,k})_{k \in K}$ and $(b_{2,k})_{k \in K}$ of $\mathbf{R}^{\#K}$ imply then that $\mathcal{R}(\mathcal{F}) > 0$. This completes the proof of the corollary in this case.

Second case. We assume that there exists $j = 1, \ldots, r$ such that $\lambda_{1,j} \lambda_{2,j} \neq 1$. It is clear in this case that we have $\mathcal{S}_2(\mathcal{F}) = \{(\mathrm{id}, \mathrm{id}); (\tau, \tau)\}$ where $\tau = (12)$. It follows from (72) that

$$\mathcal{R}(\mathcal{F}) = 2 \left(\sum_{k \in K} \beta_k^\mu \frac{(b_{1,k})^2}{\|\mathbf{b}_k\|^{\mu+2}} \right) \left(\sum_{k \in K} \beta_k^\mu \frac{(b_{2,k})^2}{\|\mathbf{b}_k\|^{\mu+2}} \right) > 0.$$

This completes the proof of Corollary 1. $\qquad\square$

Concluding remark. It seems to us quite intriguing that our solution to a discrete Falconer type problem for volumes turns out to be a natural analogue to the solution recently found for compact Salem sets in [GIM].

We think that this analogy between Salem sets and thick sets merits additional study. Whereas the Salem property implies a maximal rate of decay for the Fourier transform of a Frostman measure supported on the set, thickness implies that the polar locus of a certain "determinant zeta function" is, along the diagonal, as far from the origin as possible. Such extremal analytic behavior often leads to interesting

geometric phenomena exhibited by the sets. More precisely, the Fourier dimension of a compact set $E \subset \mathbf{R}^n$, denoted by $\dim_F E$, is defined [M] as the supremum of $\beta \geq 0$ such that for some probability measure $d\mu$ supported on E, $|\hat{d\mu}(x)| \ll_\beta |x|^{-\beta/2}$ as $|x| \to \infty$. We always have $\dim_F E \leq \dim_H E$. However, several examples (see [M]) show that Hausdorff dimension and Fourier dimension do not agree in general. A set E is said to be a Salem set if its Fourier dimension (which measures an arithmetic property of E) agrees with its Hausdorff dimension (which measures a metric property of E). In the discrete setting, we associate to any discrete self-similar set $\mathcal{F}$ a natural measure ν defined for any subset A of $\mathcal{F}^n$ by

$$\nu(A) := \sum_{(\mathbf{m}_1,\ldots,\mathbf{m}_n)\in A} \frac{\det(\mathbf{m}_1,\ldots,\mathbf{m}_n)^2}{\|\mathbf{m}_1\|^2 \ldots \|\mathbf{m}_n\|^2} \in [0,\infty].$$

This measure is characterized by its characteristic function defined by

$$V_\nu(u_1,\ldots,u_n) := \nu\left(\left(\prod_{j=1}^n B(e_j^u)\right) \cap \mathcal{F}^n\right),$$

where $B(r)$ denotes the ball of radius r. The multivariate Laplace[3] transform of this characteristic function is defined formally by

$$\mathcal{L}v(s_1,\ldots,s_n) = \int_0^\infty \ldots \int_0^\infty e^{-s_1 u_1} \ldots e^{-s_n u_n} \, V_\nu(u_1,\ldots,u_n) \, du_1 \ldots du_n.$$

This transform is related to our multivariate determinantal zeta function (see (7)) by the formula

$$\mathcal{L}v(s_1,\ldots,s_n) = \frac{1}{s_1 \ldots s_n}\boldsymbol{\zeta}_{\det}(\mathcal{F}; s_1+2,\ldots,s_n+2)$$
$$= \frac{1}{s_1 \ldots s_n} \sum_{\mathbf{m}_1,\ldots,\mathbf{m}_n\in\mathcal{F}'} \frac{det^2(\mathbf{m}_1,\ldots,\mathbf{m}_n)}{\|\mathbf{m}_1\|^{s_1+2} \ldots \|\mathbf{m}_n\|^{s_n+2}}.$$

In the one variable setting, the abscissa of convergence, which is also the largest real pole of the zeta function, is called the zeta-dimension or Laplace dimension. In the multivariate setting, the vertices of maximal weight on the boundary of the real part of the domain of convergence, may be considered as the Laplace dimensions if they are indeed poles. The defining property of thickness (see Definition 4) means that there is a Laplace dimension at the point $(\mu,\ldots,\mu)$, where μ is the the upper Minkowski dimension of $\mathcal{F}$.

References

[AS]   APFELBAUM, R., and M. SHARIR: Repeated angles in three and four dimensions. - SIAM J. Discrete Math. 19, 2005, 294–300.

[BT]   BARLOW, M., and S. J. TAYLOR: Defining fractal subsets of $\mathbf{Z}^d$. - Proc. London Math. Soc. 64, 1992, 125–152.

[BMP]   BRASS, P., W. MOSER, and J. PACH: Research problems in discrete geometry. - Springer-Verlag, 2000.

[E]   ESSOUABRI, D.: Zeta functions associated to Pascal's triangle mod p. - Jpn. J. Math. 31, 2005, 157–174.

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[3]The Laplace transform is more convenient in the discrete setting. However, Laplace transform and Fourier transform are related to each other by a simple change of variables.

[EL-1] Essouabri, D., and B. Lichtin: Zeta functions of discrete self-similar sets. - Adv. Math. 232, 2013, 142–187.

[EL-2] Essouabri, D., and B. Lichtin: Zeta functions and solutions of Falconer-type problems for self similar subsets of $\mathbf{Z}^n$. - Proc. London Math. Soc. 111, 2015, 1163–1205.

[EL-3] Essouabri, D., and B. Lichtin: Zeta functions of discrete self similar sets and applications to point configuration and sum-product type problems. - Int. Math. Res. Not. IMRN (to appear).

[GIM] Greenleaf, A., A. Iosevich, and M. Mourgoglou: On volumes determined by subsets of Euclidean space. - Forum Math. 27, 2015, 635–646.

[IMP] Iosevich, A., M. Mourgoglou, and E. Palsson: On angles determined by fractal subsets of the Euclidean space via Sobolev bounds for bilinear operators. - Math. Res. Lett. 23, 2016, 1737–1759.

[IRU] Iosevich, A., M. Rudnev, and I. Uriarte-Tuero: Theory of dimension for large discrete sets and applications. - Math. Model. Nat. Phenom., Harmonic analysis issue, 2012.

[LF] Lapidus, M. L., and M. van Frankenhuysen: Fractal geometry, complex dimensions, and zeta functions: Geometry and spectra of fractal strings. - Springer-Verlag, New York, 2006.

[LRZ] Lapidus, M. L., G. Radunović, and D. Zubrinić: Fractal zeta functions and complex dimensions of relative fractal drums. - J. Fixed Point Theory Appl. 15, 2014, 321–378.

[M] Mockenhaupt, G.: Salem sets and restriction properties of Fourier transforms. - Geom. Funct. Anal. 10, 2000, 1579–1587.

[M] Moran, P. A. P.: Additive functions of intervals and Hausdorff dimension. - Proc. Cambridge Phil. Soc. 42, 1946, 15–23.

[St] Strichartz, R.: Fractals in the large. - Canad. J. Math. 50, 1996, 638–657.