FREDHOLM THEORY OF TOEPLITZ OPERATORS
ON STANDARD WEIGHTED FOCK SPACES

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Abstract. We study the Fredholm properties of Toeplitz operators with bounded symbols of vanishing mean oscillation in the complex plane. In particular, we prove that the Toeplitz operator with such a symbol is Fredholm on a standard weighted Fock space if and only if the Berezin transform of the symbol is bounded away from zero outside a sufficiently large disk in the complex plane. We also show that the Fredholm index of the Toeplitz operator can be computed via the winding of the symbol along a sufficiently large circle. We finish by considering Toeplitz operators with matrix-valued symbols.

1. Introduction

Toeplitz operators T_f form one of the most important classes of nonselfadjoint operators and it is remarkable that so much can be said about their spectral properties. In Hardy spaces H^p of the unit circle, each Toeplitz operator has a simple representation as an infinite matrix constant along the parallels to its main diagonal; in particular, the (i,j)-entry of T_a is given by the Fourier coeffient a_{i-j} of the measurable function a. In general, such representations do not exist for Toeplitz operators acting on other function spaces, such as Bergman spaces A^p of the unit disk $\mathbf{D}$ and Fock spaces F_α^p of the entire complex plane $\mathbf{C}$.

Our focus here is on the Fredholm properties of Toeplitz operators T_f on F_α^p, including the computation of the index formula and descriptions of the essential spectrum in terms (of the Berezin transform $\tilde{f}$) of the bounded symbol f that generates T_f. In particular, we generalize the results of Berger and Coburn [3] and Stroethoff [17] on the essential spectra of Toeplitz operators on the unweighted Fock space F^2 with bounded symbols f for which the corresponding Hankel operator H_f is compact; that is, we show that for $f \in L^\infty \cap VMO^1$

$$(1) \qquad \sigma_{\mathrm{ess}}(T_f) = \bigcap_{R>0} \mathrm{cl}\, \tilde{f}(\mathbf{C} \setminus R\mathbf{D}),$$

where $\mathrm{cl}\, X$ denotes the closure of the set X and VMO^1 stands for the functions of vanishing mean oscillation defined below. We also observe that

$$(2) \qquad \{f \in L^\infty \colon H_f \text{ is compact}\} = L^\infty \cap VMO^1,$$

which was proved in [3, 17, 19] for Hankel operators on the Fock–Hilbert spaces F_α^2.

The importance of Fredholm theory, as a fundamental part of spectral theory, stems from the role it plays in (global) analysis, many other parts of mathematics and various applications. The study of the Fredholm properties of Toeplitz operators dates back to the 1950s when Hartman and Wintner showed that the essential range of a bounded symbol $a\colon \partial\mathbf{D} \to \mathbf{C}$ is contained in the essential spectrum of T_a acting on the Hardy–Hilbert space H^2. Since then there has been a steady increase in the study of spectral properties of Toeplitz operators on Hardy spaces and Bergman spaces, and in particular there is a good understanding of their spectral properties; for example, geometric descriptions of the spectra and essential spectra of Toeplitz operators have been obtained for several classes of symbols, such as continuous, piecewise continuous, (locally) sectorial, (semi) almost periodic, and Fisher–Hartwig symbols.

The situation regarding Toeplitz operators on Fock spaces is very different. While in the last ten years a considerable effort has been devoted to the study of boundedness, compactness and Schatten class properties of Toeplitz and Hankel operators on Fock spaces, see [10, 11, 12, 13, 14, 15, 16, 19], much less attention has been paid to the other important properties of these operators in the setting of Fock spaces. In fact, there are only *two* articles, both published more than two decades ago, which deal with the Fredholm properties of Toeplitz operators acting on the standard Fock space F^2, see [3, 17], and one recent work in the other (weighted) Fock spaces F_α^p; see [7]. The understanding of the Fredholm properties of Toeplitz operators in Hardy and Bergman spaces is significantly superior and we only refer to [5] for Hardy spaces and [18] for the Bergman space, which are the two leading *monographs* on the subject of Toeplitz operators and their Fredholm properties in these two function spaces.

Our goal is to improve the level of understanding of Fredholm theory of Toeplitz operators, review some known results that may be useful for further advances, and list open problems. We hope this will stimulate further research into Fredholm theory in Fock spaces and lead to discoveries of new symbol classes for which meaningful criteria for Fredholmness can be established as in Hardy and Bergman spaces. Our main results and their proofs are given in Section 4.

2. Preliminaries

For $\alpha > 0$ and $1 \le p < \infty$, we write $L_\alpha^p = L^p(\mathbf{C}, d\mu_{p\alpha/2})$ for the space of Lebesgue measurable functions with respect to the Gaussian measure

$$(3) \qquad\qquad d\mu_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2} dA(z),$$

where dA is the standard area measure on the complex plane. Obviously, we have $f \in L_\alpha^p$ if and only if the function $f(z)e^{-\alpha|z|^2/2}$ is in $L^p(\mathbf{C}, dA)$. Also observe that for $f \in L_\alpha^p$,

$$(4) \qquad \|f\|_{p,\alpha}^p = \frac{p\alpha}{2\pi} \int_{\mathbf{C}} |f(z)|^p \, d\mu_{p\alpha/2}(z) = \frac{p\alpha}{2\pi} \int_{\mathbf{C}} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} \, dA(z).$$

The standard weighted *Fock space* F_α^p is defined by

$$(5) \qquad\qquad F_\alpha^p = \{f \in L_\alpha^p \colon f \text{ is entire}\}.$$

We note that each Fock space F_α^p is a closed subspace of L_α^p and hence itself a Banach space. If $0 < p < q < \infty$, then $F_\alpha^p \subset F_\alpha^q$ and

$$(6) \qquad\qquad \|f\|_{q,\alpha} \le (q/p)^{1/q} \|f\|_{p,\alpha}$$

for $f \in F_\alpha^p$ (see Theorem 2.10 of [19]).

We write F^p for the *standard (unweighted) Fock spaces* $F^p_{1/2}$. We say that $f \in L^\infty_\alpha$ if f is Lebesgue measurable on $\mathbf{C}$ and

$$(7) \qquad \|f\|_{\infty,\alpha} = \operatorname*{ess\,sup}_{z \in \mathbf{C}} |f(z)| e^{-\frac{\alpha}{2}|z|^2} < \infty.$$

The Fock space F^2_α is a Hilbert space with inner product given by

$$(8) \qquad (f,g)_\alpha = \int_{\mathbf{C}} f(z) \overline{g(z)} \, d\mu_\alpha(z).$$

By the Riesz representation theorem, for $w \in \mathbf{C}$, there is a unique function K_w in F^2_α such that

$$(9) \qquad f(w) = (f, K_w) \quad \text{for all } f \in F^2_\alpha.$$

The function $K_\alpha(z,w) = K_w(z)$ is called the *reproducing kernel* of F^2_α. For $z \in \mathbf{C}$, we define

$$(10) \qquad k_z(w) = k_{z,\alpha}(w) = k_\alpha(w,z) = \frac{K(w,z)}{\sqrt{K(z,z)}},$$

which is called the *normalized* reproducing kernel function at z. It is well known that $\|k_z\|_{p,\alpha} = 1$ for all $z \in \mathbf{C}$ and $1 \le p < \infty$. Note that the reproducing kernel $K_\alpha(z,w)$ of F^2_α has the explicit form

$$(11) \qquad K_\alpha(z,w) = e^{\alpha z \bar{w}} \quad \text{and so} \quad k_z(w) = e^{\alpha \bar{z} w - \frac{\alpha}{2}|z|^2}.$$

We denote by P_α the orthogonal projection of L^2_α onto F^2_α. The projection P_α can be expressed as the following integral operator

$$(12) \qquad Pf(z) = \int_{\mathbf{C}} K_\alpha(z,w) f(w) \, d\mu_\alpha(w)$$

for $z \in \mathbf{C}$. For any $\alpha > 0$ and $1 \le p \le \infty$, the integral operator P_α is a bounded projection of L^p_α onto F^p_α; the boundedness in the endpoint cases is in stark contrast with the situation in Hardy and Bergman spaces.

For $f \in L^\infty$, $\alpha > 0$, and $1 < p < \infty$, we define the *Toeplitz operator* $T_a \colon F^p_\alpha \to F^p_\alpha$ and the *Hankel operator* $H_a \colon F^p_\alpha \to L^p_\alpha$ by

$$(13) \qquad T_a f = P(af) \quad \text{and} \quad H_a f = (I - P)(af)$$

for $f \in F^p_\alpha$. Clearly, T_a and H_a are bounded linear operators, and

$$(14) \qquad T_a f(z) = \int_{\mathbf{C}} K_w(z) a(w) f(w) \, d\mu_\alpha(w)$$

for $z \in \mathbf{C}$. We only deal with Toeplitz and Hankel operators with bounded symbols and only consider them on the Fock spaces F^p_α with $1 < p < \infty$. Some references are given for unbounded symbols and generalized Fock spaces in the next section.

In order to derive useful formulas for Toeplitz and Hankel operators at the kernel function, we define three analytic self-maps of the complex plane. For $\lambda \in \mathbf{C}$, set

$$(15) \qquad t_\lambda(z) = z + \lambda, \quad \tau_\lambda(z) = z - \lambda, \quad \varphi_\lambda(z) = \lambda - z.$$

The mapping t_λ is called the translation by λ, and clearly $\tau_\lambda = t_{-\lambda} = t_\lambda^{-1}$ and $\varphi_\lambda^{-1} = \varphi_\lambda$. The following useful formulas follow from a change of variables:

$$
\begin{aligned}
\int_{\mathbf{C}} |f \circ t_\lambda(w)|^p \, d\mu_{p\alpha/2}(w) &= \frac{p\alpha}{2\pi} \int_{\mathbf{C}} |f(w+\lambda)|^p \, e^{-\frac{p\alpha}{2}|w|^2} \, dA(w) \\
&= \frac{p\alpha}{2\pi} \int_{\mathbf{C}} |f(z)|^p \, e^{-\frac{p\alpha}{2}|z-\lambda|^2} \, dA(z) \\
&= \frac{1}{\|K_\lambda\|_{p,\alpha}^p} \int_{\mathbf{C}} |f(z)|^p \, |K_\lambda(z)|^p \, d\mu_{p\alpha/2}(z) \\
&= \int_{\mathbf{C}} |f(z)|^p \, |k_\alpha(z,\lambda)|^p \, d\mu_{p\alpha/2}(z).
\end{aligned}
$$

(16)

For $\lambda \in \mathbf{C}$, using (11), we get

$$
K_z(w) = K_z(\lambda)K_z(w-\lambda) = \overline{K_\lambda(z)}K_{z-\lambda}(w-\lambda)\frac{K_\lambda(w)}{K_\lambda(\lambda)},
$$

and further, using (16),

$$
\begin{aligned}
T_f K_\lambda(z) &= \int_{\mathbf{C}} \overline{K_z(w)} f(w) K_\lambda(w) \, d\mu_\alpha(w) \\
&= \int_{\mathbf{C}} K_\lambda(z)\overline{K_{z-\lambda}(w-\lambda)}\frac{\overline{K_\lambda(w)}}{K_\lambda(\lambda)} f(w) K_\lambda(w) \, d\mu_\alpha(w) \\
&= K_\lambda(z) \int_{\mathbf{C}} |k_\lambda(w)|^2 \overline{K_{z-\lambda}(w-\lambda)} f(w) \, d\mu_\alpha(w) \\
&= K_\lambda(z) \int_{\mathbf{C}} f(t_\lambda(w))\overline{K_{z-\lambda}(w)} \, d\mu_\alpha(w) = K_\lambda(z)P(f \circ t_\lambda)(z-\lambda)
\end{aligned}
$$

and $T_f k_\lambda(z) = k_\lambda(z)P(f \circ t_\lambda)(z-\lambda)$. Therefore, for $f \in L^\infty$ and $\lambda \in \mathbf{C}$,

(17) $$ T_f(K_\lambda) = K_\lambda P(f \circ t_\lambda) \circ \tau_\lambda $$

and

(18) $$ H_f(K_\lambda) = K_\lambda(f - P(f \circ t_\lambda) \circ \tau_\lambda). $$

Similarly,

(19) $$ T_f k_\lambda = k_\lambda P(f \circ \varphi_\lambda) \circ \varphi_\lambda, $$

which will be used later together with the pointwise estimate

(20) $$ |P(f \circ \varphi_\lambda)(z)| \leq \text{const} \, e^{\frac{\alpha}{4}|z|^2} $$

for $z \in \mathbf{C}$ (see Lemma 6.25 of [19]).

3. Boundedness and compactness

In this section we study the fundamental properties of Toeplitz and Hankel operators that are needed in the study of their Fredholm theory. In particular, we recall some results on their boundedness and compactness in F_α^2, and then generalize some of the results to the other Fock spaces F_α^p.

First, it is clear that if the symbol f is in L^∞, then both the Toeplitz operator T_a and the Hankel operator H_a are bounded on F_α^p, and it is easy to see that

$$ \|T_f\| \leq \|P\|\|f\|_\infty \quad \text{and} \quad \|H_f\| \leq \|P\|\|f\|_\infty. $$

For unbounded symbols and measures as symbols, see [15].

The following result of [2] provides a complete characterization of compact Toeplitz operators on F_α^p with bounded symbols. The characterization is given in terms of the *Berezin transform* $\tilde{f}$ of the symbol f defined by

$$
\tilde{f}(z) = \int_{\mathbf{C}} |k_z(w)|^2 f(w) \, d\mu_\alpha(w) = \frac{\alpha}{\pi} \int_{\mathbf{C}} f(w) e^{-\alpha|z-w|^2} \, dA(w)
$$

(21)

$$
= \int_{\mathbf{C}} f(z \pm w) \, d\mu_\alpha(w).
$$

For Toeplitz operators on the Fock space F^2 with symbols in BMO^1, see [6], and for operators on unweighted Fock spaces F^p with $0 < p < \infty$, see [9].

Theorem 1. *Let $f \in L^\infty$, $1 < p < \infty$ and $\alpha > 0$. Then T_f is compact on F_α^p if and only if $\tilde{f}$ vanishes at infinity.*

Proof. See [2]. $\qquad\square$

The following alternate characterization for Toeplitz operators on F^2 was proved in [17]. We generalize it to the other values of p.

Theorem 2. *Let $f \in L^\infty$, $1 < p < \infty$ and $\alpha > 0$. Then the following conditions are equivalent:*

(a) *T_f is compact on F_α^p;*
(b) *$\|P(f \circ t_\lambda)\|_{p,\alpha} \to 0$ as $|\lambda| \to \infty$.*

Proof. Suppose that T_f is compact on F_α^p. If g is a polynomial, then

$$
(g, k_\lambda) = \int_{\mathbf{C}} g(z) \overline{k_\lambda(z)} \, d\lambda_\alpha(z) = K_\lambda(\lambda)^{-1/2} \int_{\mathbf{C}} g(z) \overline{K_\lambda(z)} \, d\lambda_\alpha(z) = \frac{g(\lambda)}{e^{\frac{\alpha}{2}|\lambda|^2}} \to 0
$$

as $|\lambda| \to \infty$. Therefore, since the polynomials are dense in F_α^p (see [19, Proposition 2.9]), it follows that $k_\lambda \to 0$ weakly as $|\lambda| \to \infty$, and hence $\|T_f k_\lambda\|_p \to 0$ as $|\lambda| \to \infty$. Using (17), we get

$$
\|T_f k_\lambda\|_{p,\alpha}^p = \int_{\mathbf{C}} |k_\alpha(z,\lambda)|^p |P(f \circ t_\lambda)(\tau_\lambda(z))|^p \, d\mu_{\alpha p/2}(z)
$$

$$
= \int_{\mathbf{C}} |P(f \circ t_\lambda)(w)|^p \, d\lambda_{\alpha p/2}(w) = \|P(f \circ t_\lambda)\|_{p,\alpha}^p,
$$

where we also used a change of variables according to (16). Thus, $\|P(f \circ t_\lambda)\|_{p,\alpha} \to 0$ as $|\lambda| \to \infty$.

Suppose that (b) holds. Then, using (17), we have

$$
|\tilde{f}(\lambda)| = |\langle T_f k_\lambda, k_\lambda \rangle_\alpha| \leq \|T_f k_\lambda\|_{p,\alpha} \|k_\lambda\|_{p',\alpha} = \|P(f \circ t_\lambda)\|_{p,\alpha} \to 0,
$$

where $1/p + 1/p' = 1$. By the previous theorem, T_f is compact on F_α^p. $\qquad\square$

A similar result holds for Hankel operators as seen in the following theorem, which was proved in [17] for Hankel operators from F^2 to L^2 and in [19] for Hankel operators on F_α^2 to L_α^2.

Theorem 3. *Let $f \in L^\infty$ and $\alpha > 0$. Then the following conditions are equivalent:*

(a) *$H_f \colon F_\alpha^2 \to L_\alpha^2$ is compact;*
(b) *$\|f \circ t_\lambda - P(f \circ t_\lambda)\|_{2,\alpha} \to 0$ as $|\lambda| \to \infty$.*

As mentioned in [17], when Hankel operators are acting on F^2, the equivalence of (a) and (b) in the following theorem was obtained in [3], while condition (d) was proved in [4]. An elementary proof of the theorem was given in [17].

Theorem 4. *Let $f \in L^\infty$ and $\alpha > 0$. Then the following statements are equivalent:*

(a) $H_f \colon F_\alpha^2 \to L_\alpha^2$ *is compact;*
(b) $H_{\bar{f}} \colon F_\alpha^2 \to L_\alpha^2$ *is compact;*
(c) $\|f \circ t_\lambda - \tilde{f}(\lambda)\|_{2,\alpha} \to 0$ *as $\lambda \to \infty$;*
(d) $\widetilde{|f|^2}(\lambda) - |\tilde{f}(\lambda)|^2 \to 0$ *as $\lambda \to \infty$.*

One remarkable difference between Fock, Bergman and Hardy spaces is that the equivalence of (a) and (b) in the preceding theorem is only true in Fock spaces. This is often explained by the lack of bounded analytic functions in F_α^2.

We will study the Fredholm properties of Toeplitz operators with bounded symbols that satisfy an additional condition on their (mean) oscillation at infinity. These types of conditions are intimately related to the boundedness and compactness properties of Hankel operators. For $r > 0$, define the *local oscillation* of a continuous function $f \colon \mathbf{C} \to \mathbf{C}$ at $z \in \mathbf{C}$ by

$$\omega_r(f)(z) = \sup_{w \in B(z,r)} |f(z) - f(w)|.$$

We say that f is of *bounded oscillation* and write $f \in BO_r$ if f is continuous and $\omega_r(f)$ is bounded on $\mathbf{C}$.

We define the *mean average* $\hat{f}_r(z)$ of a locally integrable function f over $B(z,r)$ by

$$\hat{f}_r(z) = \frac{1}{\pi r^2} \int_{D(z,r)} f(w)\, dA(w),$$

and say that f is of *bounded average* and write $f \in BA_r^p$ if $\widehat{|f|_r^p}(z)$ is bounded on $\mathbf{C}$; that is, if

$$(22) \qquad \sup_{z \in \mathbf{C}} \frac{1}{\pi r^2} \int_{B(z,r)} |f(w)|^p\, dA(w) < \infty.$$

We say that f is of *vanishing oscillation* and write $f \in VO_r$ if $f \in BO_r$ and

$$(23) \qquad \lim_{z \to \infty} \omega_r(f) = 0.$$

Similarly, we say that f is of *vanishing average* and write $f \in VA_r^p$ if

$$(24) \qquad \lim_{z \to \infty} \frac{1}{\pi r^2} \int_{B(z,r)} |f(w)|^p\, dA(w) = 0.$$

Finally, we say that a locally integrable function f is of *bounded mean oscillation* and write $f \in BMO_r^p$ if

$$\sup_{z \in \mathbf{C}} \int_{B(z,r)} |f(w) - \hat{f}_r(z)|^p\, dA(w) < \infty.$$

Similarly, we say that $f \in BMO_r^p$ is of *vanishing mean oscillation* and write $f \in VMO_r^p$ if

$$\lim_{z \to \infty} \int_{B(z,r)} |f(w) - \hat{f}_r(z)|^p\, dA(w) = 0.$$

The spaces $BO_r, BA_r^p, VO_r, VA_r^p, BMO_r^p, VMO_r^p$ are all independent of r and we often omit the subscript. However, they do depend on the value p, unlike in the setting of Hardy spaces; in particular, for $1 \leq p < q$, we have $BMO^q \subset BMO^p$ and $VMO^q \subset VMO^p$ with proper inclusions.

It is also well know that for $p \geq 1$,

(25) $$BMO^p = BO + BA^p \quad \text{and} \quad VMO^p = VO + VA^p$$

and each f in BMO^p can be decomposed as the sums

(26) $$f = \tilde{f} + (f - \tilde{f}) \quad \text{or} \quad f = \hat{f} + (f - \hat{f}),$$

where $\tilde{f}, \hat{f} \in BO$ and $f - \tilde{f}, f - \hat{f} \in BA^p$. Analogous decompositions hold for functions in VMO^p (see, e.g., [13] or [19]).

Proposition 5. *For $1 < p < \infty$,*

(27) $$L^\infty \cap BMO^p = L^\infty \cap BMO^1 \text{ and } L^\infty \cap VMO^p = L^\infty \cap VMO^1.$$

Proof. Let $f \in L^\infty \cap VMO^1$. Then

$$|f(w) - \hat{f}_r(z)| \leq 2\|f\|_\infty$$

for all $w, z \in \mathbf{C}$. Thus,

$$\int_{B(z,r)} |f(w) - \hat{f}_r(z)|^p \, dA(w) \leq (2\|f\|_\infty)^{p-1} \int_{B(z,r)} |f(w) - \hat{f}_r(z)| \, dA(w) \to 0$$

as $|z| \to \infty$, which implies that $f \in VMO^p$. The other case is analogous. $\square$

The following theorem characterizes simultaneous boundedness (and compactness) of Hankel operators H_a and $H_{\bar{a}}$; see [13]. It was first proved for Hankel operators on F^2 in [1]; see also [4]. Further generalizations to other Fock spaces F_ϕ^p can be found in [10]. The requirement that $fK_w \in L_\alpha^p$ for all $w \in \mathbf{C}$ in the theorem is natural because linear combinations of the kernel functions K_w form a dense subset of F_α^p. Observe that this condition is satisfied for all $p \in (1, \infty)$ and $\alpha > 0$ if we have $f \in L^\infty$.

Theorem 6. *Let $1 < p < \infty$, $\alpha > 0$ and suppose that $fK_w \in L_\alpha^p$. Then*

(a) H_f *and* $H_{\bar{f}}$ *are both bounded* $F_\alpha^p \to L_\alpha^p$ *if and only if* $f \in BMO^p$;
(b) H_f *and* $H_{\bar{f}}$ *are both compact* $F_\alpha^p \to L_\alpha^p$ *if and only if* $f \in VMO^p$.

Proposition 7. *For $\alpha > 0$,*

$$\{f \in L^\infty : H_f \in K(F_\alpha^2, L_\alpha^2)\} = L^\infty \cap VMO^2.$$

Proof. Apply Theorems 4 and 6. $\square$

Proposition 8. *The space $L^\infty \cap VMO^1$ equipped with $\|\cdot\|_\infty$ is a Banach algebra.*

Proof. Let $f, g \in L^\infty \cap VMO^1 = L^\infty \cap VMO^2$ (see Proposition 5). Then

$$H_{fg} = QM_{fg} = QM_f(P + Q)M_g = QM_f PM_g + QM_f QM_g$$

is compact from F_α^2 to L_α^2, and $fg \in L^\infty$. Therefore, by the previous proposition, it follows that the function fg is in $L^\infty \cap VMO^1$. $\square$

4. Fredholm theory

In this section we state and prove our main results. We start with the definition of Fredholm operators. Let T be a bounded linear operator on a Banach space X. We say that T is a *Fredholm operator* on X if

$$d_1 = \dim \ker T < \infty \quad \text{and} \quad d_2 = \dim (X/T(X)),$$

in which case we define the *index* of T to be $\operatorname{ind} T = d_1 - d_2$. The set of all Fredholm operators on X is denoted by $\Phi(X)$. The definition of Fredholm operators immediately gives rise to the concept of the *essential spectrum* $\sigma_{\mathrm{ess}}(T)$ of a bounded operator T on X defined by

$$\sigma_{\mathrm{ess}}(T) = \{\lambda \in \mathbf{C} \colon T - \lambda I \notin \Phi(X)\},$$

where I is the identity operator on X. Clearly $\sigma_{\mathrm{ess}}(T) \subset \sigma(T)$. A useful characterization of Fredholm operators can be given in terms of invertibility in the Calkin algebra; that is, T is Fredholm if and only if $T + K(X)$ is invertible in $B(X)/K(X)$. We also recall that the index function is stable under small perturbations.

The following "approximation" result plays an important role in our analysis; cf. [3]. We say that a function $f\colon \mathbf{C} \to \mathbf{C}$ is *bounded away from zero* if there is a $R > 0$ and an $m > 0$ such that

$$(28) \qquad\qquad\qquad\qquad |f(z)| \geq m \text{ for } |z| \geq R.$$

Note also that in what follows, we only need the case of functions of vanishing oscillation. However, the other case may turn out to be of interest in further considerations of the essential spectra of Toeplitz operators.

Proposition 9. *Let $f\colon \mathbf{C} \to \mathbf{C}$ be a continuous function in A, where $A = L^\infty \cap VO$ or $A = L^\infty \cap VMO^1$. Then f is bounded away from zero on $\mathbf{C} \setminus B(0, R)$ for some $R > 0$ if and only if there is a continuous function $g \in A$ such that $g(z)f(z) \to 1$ as $z \to \infty$.*

Proof. Suppose first that such a function g exists. Then $gf - 1$ vanishes at infinity. If f is not bounded away from zero for any $R > 0$, then there are $z_k \in \mathbf{C}$ such that $|z_k| \to \infty$ and $|f(z_k)| \to 0$, and so $|1 + g(z_k)f(z_k) - 1| \leq \|g\|_\infty |f(z_k)| \to 0$, which is a contradiction.

Conversely, suppose that there is an $m > 0$ such that $|f(z)| \geq m$ for $|z| \geq R$. Denote by κ the winding number of f along the circle $C_R = \{|z| = R\}$. Consider the function h defined by

$$h(z) = f(z)(\bar{z}/|z|)^\kappa$$

for $|z| \geq R$. Since $\operatorname{wind} h = 0$ along $\partial B(0, R)$, the function extends to a bounded continuous function on the whole complex plane which is bounded away from zero on $\mathbf{C}$.

We show that $h \in A$. Suppose first that $A = L^\infty \cap VO$. For $|z - w| \leq 1$,

$$(29) \qquad\qquad\qquad |(\bar{z}/|z|)^\kappa - (\bar{w}/|w|)^\kappa| = O(1/|z|)$$

as $z \to \infty$. Therefore, since $f \in L^\infty \cap VO$ and

$$|h(z) - h(w)| = |f(z)| \, |(\bar{z}/|z|)^\kappa - (\bar{w}/|w|)^\kappa| + |f(z) - f(w)|,$$

it follows that $h \in L^\infty \cap VO$. Suppose then that $A = L^\infty \cap VMO^1$. For $w \in D(z,1)$,

$$
h(w) - \hat{h}(z) = |h(w) - \frac{1}{\pi} \int_{D(z,1)} h(u) \, dA(u)
$$

$$
= \frac{1}{\pi} \int_{D(z,1)} \Big(h(w) - f(w)(\bar{z}/|z|)^\kappa
$$

$$
+ f(w)(\bar{z}/|z|)^\kappa - f(u)(\bar{z}/|z|)^\kappa + f(u)(\bar{z}/|z|)^\kappa - h(u) \Big) dA(u)
$$

$$
= \frac{1}{\pi} \int_{D(z,1)} \Big(f(w) \big((\bar{w}/|w|)^\kappa - (\bar{z}/|z|)^\kappa \big)
$$

$$
+ \big(f(w) - f(u) \big)(\bar{z}/|z|)^\kappa + f(u) \big((\bar{z}/|z|)^\kappa - (\bar{u}/|u|)^\kappa \big) \Big) dA(u).
$$

Therefore,

$$
\frac{1}{\pi} \int_{D(z,1)} |h(w) - \hat{h}(z)| \, dA(z) = O(1/z) + \frac{1}{\pi} \int_{D(z,1)} |f(w) - \hat{f}(z)| \, dA(z) + O(1/z),
$$

which shows that $h \in VMO^1$. Finally, we define a function F on $\mathbf{C}$ by setting

$$
F(z) = \begin{cases} (\bar{z}/|z|)^\kappa, & |z| \geq R, \\ (\bar{z})/R)^\kappa, & |z| < R \end{cases}
$$

and let $g = F/h$. Then $g \in A$ and $g(z)f(z) = 1$ for $|z| \geq R$. $\qquad\square$

In 1987 Berger and Coburn [3] characterized Fredholm Toeplitz operators on the standard Fock space F^2 for bounded symbols of vanishing oscillation at infinity. Their approach was based on heavy machinery employing Hilbert space and C^*-algebra techniques, making use of the properties of the Weyl operator, of the Berezin symbol, and of the Heisenberg group. These are by no means ideal techniques to be used in the Banach space context. In 1992 Stroethoff [17] gave a more elementary proof of Berger and Coburn's result using ideas which are more applicable in the setting of the standard weighted Fock spaces F_α^p. Before we state our main result we need the following results, which allows us to use the Berezin transform in our analysis.

Lemma 10. *If $f \in VO \cap L^\infty$, then $f(z) - \tilde{f}(z) \to 0$ as $|z| \to \infty$.*

Proof. Let $\epsilon > 0$. By direct computation, there is an $M > 0$ such that

$$
|f(z) - \tilde{f}(z)| \leq \frac{\alpha}{\pi} \int_{\mathbf{C}} |f(z) - f(w)| e^{-\alpha|z-w|^2} \, dA(w)
$$

$$
= \frac{\alpha}{\pi} \int_{\mathbf{C}} |f(z) - f(z+w)| e^{-\alpha|w|^2} \, dA(w)
$$

$$
\leq \frac{\alpha}{\pi} 2\|f\|_\infty \int_{|w| \geq M} e^{-\alpha|w|^2} \, dA(w)
$$

$$
+ \frac{\alpha}{\pi} \int_{|w| < M} |f(z) - f(z+w)| e^{-\alpha|w|^2} \, dA(w)
$$

$$
\leq \epsilon + \frac{\alpha}{\pi} \int_{|w| < M} |f(z) - f(z+w)| e^{-\alpha|w|^2} \, dA(w).
$$

Since $f \in VO$, the last integral goes to zero as $|z| \to \infty$, and the proof is complete.
$\qquad\square$

Corollary 11. *Let $f \in VO$. Then $\tilde{f}(z) \to 0$ as $|z| \to \infty$ if and only if $f(z) \to 0$ as $|z| \to \infty$.*

Proposition 12. *Let $f \in L^\infty \cap VMO^1$, $1 < p < \infty$, and $\alpha > 0$. Then $T_{f-\tilde{f}}$ is compact on F_α^p.*

Proof. This follows from Theorem 1. □

We can now characterize Fredholmness of Toeplitz operators in terms of their symbols.

Theorem 13. *Let $\alpha > 0$, $1 < p < \infty$, and suppose that f is in $L^\infty \cap VO$. Then the Toeplitz operator T_f is Fredholm on F_α^p if and only the symbol f is bounded away from zero; see (28).*

Proof. Suppose that f is bounded away from zero. We use Proposition 9 to construct a regularizer. Indeed, there is a function g in $L^\infty \cap VO$ such that $f(z)g(z) - 1 \to 0$ as $z \to \infty$. By Theorem 6, H_f and H_g are compact, and by Theorem 1, T_{fg-1} is compact. Therefore,

$$T_f T_g = P M_f P M_g = P M_f (I - Q) M_g = T_{fg} - P M_f Q M_g$$
$$= T_{fg} - P M_f H_g = I + T_{fg-1} - P M_f H_g = I + K_1$$

for some compact operator K_1. Similarly, $T_g T_f = I + T_{fg-1} - P M_g H_f = I + K_2$ for K_2 compact. Thus, T_g is a regularizer for T_f and hence T_f is Fredholm.

Conversely, suppose that there are $\lambda_k \in \mathbf{C}$ such that $f(\lambda_k) \to 0$ when $|\lambda_k| \to \infty$. By Lemma 11, $\tilde{f}(\lambda_k) \to 0$ as $k \to \infty$. By (19), we have

$$(30) \qquad \|T_f k_{\lambda_k}\|_{p,\alpha}^p = \|P_\alpha(f \circ t_{\lambda_k})\|_{p,\alpha}^p = \text{const} \int_{\mathbf{C}} |P(f \circ t_{\lambda_k})(z)|^p \, d\mu_{\alpha p/2}(z).$$

By Lemma 6.26 of [19], for $z \in \mathbf{C}$, $P(f \circ \varphi_{\lambda_k})(z) \to 0$ as $k \to \infty$. Therefore, by (20),

$$\left| k_z(\lambda_k - z) P(f \circ \varphi_{\lambda_k})(z) e^{-\frac{1}{2}|\lambda_k - z|^2} \right| = e^{-\frac{\alpha}{2}|z|^2} |P(f \circ \varphi_{\lambda_k})(z)| \leq \text{const} \, e^{-\frac{\alpha}{4}|z|^2},$$

and so by the Lebesgue dominated convergence theorem,

$$\|T_f k_{\lambda_k}\|_{p,\alpha}^p = \text{const} \int_{\mathbf{C}} \left| k_{\lambda_k}(z) P(f \circ \varphi_{\lambda_k})(\lambda_k - z) e^{\frac{\alpha}{2}|z|^2} \right|^p \, dA(z)$$
$$= \int_{\mathbf{C}} \left| k_z(\lambda_k - z) P(f \circ \varphi_{\lambda_k})(z) e^{-\frac{1}{2}|\lambda_k - z|^2} \right|^p \, dA(z) \to 0$$

as $k \to \infty$. Thus, $T_f + K$ is not invertible in the Calkin algebra $L(F_\alpha^p)/K(F_\alpha^p)$, and hence T_f is not Fredholm on F_α^p. □

Corollary 14. *Let $f \in L^\infty \cap VO$. Then the essential spectrum of T_f on F_α^p with $1 < p < \infty$ and $\alpha > 0$ is given by*

$$(31) \qquad \sigma_{\text{ess}}(T_f) = \bigcap_{R>0} \text{cl} \, f(\mathbf{C} \setminus R\mathbf{D}).$$

By using the result on the compactness of Toeplitz operators, we can consider a slightly larger class of symbols.

Corollary 15. *Let $\alpha > 0$, $1 < p < \infty$, and suppose that f is in $L^\infty \cap VMO^1$. Then the Toeplitz operator T_f is Fredholm on F_α^p if and only if $\tilde{f}$ is bounded away from zero for some $R > 0$; see (28).*

Proof. Write $f = \tilde{f} + (f - \tilde{f})$ as in (26). Then $\tilde{f} \in VO$ and $f - \tilde{f} \in VA^1$. Since $T_{f-\tilde{f}}$ is compact, the standard Fredholm theory implies that T_f is Fredholm if and only if $T_{\tilde{f}}$ is Fredholm, which is equivalent to $\tilde{f}$ being bounded away from zero by the preceding theorem. $\qquad\square$

Corollary 16. *Let* $f \in L^\infty \cap VMO^1$. *Then the essential spectrum of* T_f *on* F_α^p *with* $1 < p < \infty$ *and* $\alpha > 0$ *is given by*

(32) $$\sigma_{\text{ess}}(T_f) = \bigcap_{R>0} \text{cl}\, \tilde{f}(\mathbf{C} \setminus R\mathbf{D}).$$

Proof. This follows from Theorem 13 and the fact that $\sigma_{\text{ess}}(T_f) = \sigma_{\text{ess}}(T_{\tilde{f}})$, which is easy to see using Proposition 12 and standard Fredholm theory. $\qquad\square$

The description of the essential spectrum of T_f in terms of Berezin transform $\tilde{f}$ above comes useful because $\tilde{f}$ is continuous and hence the sets $\text{cl}\, \tilde{f}(\mathbf{C} \setminus R\mathbf{D})$ are compact and connected, which immediately proves the following result.

Corollary 17. *Let* $f \in L^\infty \cap VMO^1$, $1 < p < \infty$, *and* $\alpha > 0$. *Then the essential spectrum* $\sigma_{\text{ess}}(T_f)$ *of* T_f *on* F_α^p *is connected.*

We reduce the computation of the index of T_f to that of $T_{\bar{z}/|z|}$. To compute the index of the latter simple operator, we use the following two lemmas on weighted shift operators.

Lemma 18. *Let* $\alpha > 0$ *and* $1 < p < \infty$. *The Toeplitz operator* $T_{z/|z|}$ *is a weighted shift operator on* F_α^p; *that is,* $T_{z/|z|}e_k = \alpha_k e_{k+1}$, *where* $e_k = \sqrt{\frac{\alpha^k}{k!}}z^k$ *and* $\alpha_k \neq 0$.

Proof. Set $g(z) = z/|z|$ for $z \in \mathbf{C}$. If $w = re^{i\theta}$, then $dA(w) = r\,dr\,d\theta$ and

$$T_g e_k(z) = P(ge_k)(z) = \frac{\alpha}{\pi} \int_{\mathbf{C}} e^{\alpha z \bar{w} - \alpha|w|^2}(e^{i\theta}e_k)\,dA(w)$$

$$= \frac{\alpha}{\pi} \int_{\mathbf{C}} e^{\alpha z \bar{w} - \alpha|w|^2}\, e^{i\theta} \sqrt{\frac{\alpha^k}{k!}}z^k\,dA(w)$$

$$= \frac{\alpha}{\pi}\sqrt{\frac{\alpha^k}{k!}} \int_0^\infty r^{k+1}e^{-\alpha r^2}\left[\int_0^{2\pi} e^{\alpha z r e^{-i\theta}}\, e^{i\theta(k+1)}\,d\theta\right]dr$$

$$= \frac{\alpha}{\pi}\sqrt{\frac{\alpha^k}{k!}} \int_0^\infty r^{k+1}e^{-\alpha r^2}\left[\int_{|\lambda|=1} (-i)e^{\alpha z r \lambda}\, \frac{d\lambda}{\lambda^{k+2}}\right]dr,$$

where $e^{-i\theta} = \lambda$. By the residue theorem,

$$\int_{|\lambda|=1} \frac{e^{\mu\lambda}}{\lambda^{k+2}}\,d\lambda = \frac{2\pi i \mu^{k+1}}{(k+1)!}.$$

Therefore,

$$T_g e_k(z) = 2\alpha\sqrt{\frac{\alpha^k}{k!}} \frac{1}{(k+1)!} \alpha^{k+1} z^{k+1} \int_0^\infty r^{2k+2}\, e^{-\alpha r^2}\,dr = \alpha_k e_{k+1}(z),$$

where

$$\alpha_k = \frac{\alpha^k \sqrt{\alpha}}{k!\sqrt{k+1}} \int_0^\infty r^{2k+2}\, e^{-\alpha r^2}\,dr \neq 0$$

for $k \in \mathbf{N}$. $\qquad\square$

Next we compute the index of $T_{\bar{z}/|z|}$ using the properties of the diagonal operator.

Lemma 19. *Let $\alpha > 0$ and $1 < p < \infty$. Then $\operatorname{ind} T_{\bar{z}/|z|} = 1$.*

Proof. Let $g \in F_\alpha^p$, then $g(z) = \sum_{k=0}^\infty \lambda_k e_k$. Using Lemma 18, we get

$$T_{\frac{z}{|z|}}(g)(z) = \sum_{k=0}^\infty \lambda_k T_{\frac{z}{|z|}} e_k = \sum_{k=0}^\infty \lambda_k \alpha_k e_{k+1}, \alpha_k \neq 0 \quad \text{for all } k \geq 0.$$

Thus, $g \in \ker T_{\frac{z}{|z|}}$ if and only if $\sum_{k=0}^\infty \lambda_k \alpha_k e_{k+1} = 0$. Hence, $g \in \ker T_{\frac{z}{|z|}}$ if and only if $\lambda_k = 0$ for all $k \geq 0$. This implies that $\ker T_{\frac{z}{|z|}} = 0$. Moreover, if $h(z) = ae_0$ then $h \notin \operatorname{Range} T_{\frac{z}{|z|}}$, this means that $\operatorname{Codim} T_{\frac{z}{|z|}} = 1$. Therefore, $T_{\frac{z}{|z|}}$ is Fredholm with index 1. This implies that $T_{\frac{z}{|z|}}^* = T_{\frac{\bar{z}}{|z|}}$ is Fredholm with index -1. $\square$

We can now prove the index formula for Toeplitz operators on F_α^p with continuous symbols of vanishing oscillation.

Theorem 20. *Let $f \in L^\infty \cap VO$, $\alpha > 0$, and $1 < p < \infty$. If f is bounded away from zero, then*

$$\operatorname{ind} T_f = -\operatorname{wind}(f|_{|z|=R}),$$

where R is any positive number for which $|f(z)| > m > 0$ for $|z| \geq R$ and $\operatorname{wind} f|_{\{|z|=R\}}$ is the winding of the curve $f(\{|z| = R\})$ around the origin.

Proof. By Theorem 13, T_f is Fredholm. Suppose $\operatorname{wind}(f|_{|z|=R}) = n$. By Proposition 9 and its proof there is an H in $L^\infty(\mathbf{C}) \cap VO$ such that H is bounded away from zero on the whole complex plane and

$$\lim_{z \to \infty} [(\bar{z}/|z|)^n f(z) - H(z)] = 0.$$

It follows that $T_{(\bar{z}/|z|)^n f - H}$ is compact by Theorem 1. Therefore, since T_H is Fredholm (see Theorem 13), we have

$$\operatorname{ind}\left(T_{(\bar{z}/|z|)^n} T_f\right) = \operatorname{ind} T_{(\bar{z}/|z|)^n f} = \operatorname{ind} T_H.$$

By the index theorem and Lemma 19,

$$\operatorname{ind} T_f = -n + \operatorname{ind} T_H.$$

It remains to show that $\operatorname{ind} T_H = 0$. Define

$$h_t = \frac{t}{|H|} + 1 - t$$

for $t \in [0, 1]$. Then each T_{h_t} is Fredholm and

$$\|T_{h_t} - T_{h_s}\| \leq \operatorname{const} |t - s| \| |H|^{-1} - 1 \|,$$

so

$$\operatorname{ind} T_{1/|H|} = \operatorname{ind} T_{h_1} = \operatorname{ind} T_{h_0} = 0.$$

Therefore,

$$\operatorname{ind} T_{H/|H|} = \operatorname{ind} T_H + \operatorname{ind} T_{1/|H|} = \operatorname{ind} T_H,$$

and hence we can assume that $|H| = 1$ and write $H = e^{i \arg H}$. For any integer $m > 0$, since H is a continuous function of no winding,

$$\exp\left(i(\arg H)/m\right) \in L^\infty \cap VO.$$

Thus,

$$\operatorname{ind} T_H = \operatorname{ind} T_{\exp(i(\arg H)/m)}^m = m \operatorname{ind} T_{\exp(i(\arg H)/m)},$$

which implies that $\operatorname{ind} T_H = 0$. $\square$

5. Matrix-valued symbols

Let $N \in \mathbf{N}$, $1 \leq p < \infty$, and $\alpha > 0$. We define

$$L^p_{\alpha,N} = \{(f_1, \ldots, f_N)^T : f_k \in L^p_\alpha\}$$

and

$$\|f\|_{\alpha,p,N} = \sum_{k=1}^N \|f_k\|^p_\alpha.$$

The *vectorial Fock* space

$$F^p_{\alpha,N} = \{(f_1, \ldots, f_N)^T : f_k \in F^p_\alpha\}$$

is a closed subspace of $L^p_{\alpha,N}$.

Given $a = (a_{jk})_{1 \leq j,k \leq N}$, where $a_{jk} \in L^\infty$, we define the *block Toeplitz operator* T_a on $F^p_{\alpha,N}$ by

$$
T_a f = P(af) = P\begin{pmatrix} a_{11}f_1 & + & a_{12}f_2 & + & \ldots & + & a_{1N}f_N \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{N1}f_1 & + & a_{N2}f_2 & + & \ldots & + & a_{NN}f_N \end{pmatrix}
$$

$$
= \begin{pmatrix} P(a_{11}f_1) & + & P(a_{12}f_2) & + & \ldots & + & P(a_{1N}f_N) \\ \vdots & & \vdots & & \vdots & & \vdots \\ P(a_{N1}f_1) & + & P(a_{N2}f_2) & + & \ldots & + & P(a_{NN}f_N) \end{pmatrix}
$$

$$
= \left(\sum_{k=1}^N T_{a_{jk}} f_k\right)^T_{1 \leq j \leq N} = \begin{pmatrix} T_{a_{11}} & T_{a_{12}} & \cdots & T_{a_{1N}} \\ \vdots & \vdots & & \vdots \\ T_{a_{N1}} & T_{a_{N2}} & \cdots & T_{a_{NN}} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}
$$

for $f \in F^p_{\alpha,N}$. It is not difficult to see that T_a is bounded on $F^p_{\alpha,N}$.

Lemma 21. *There is a compact operator K such that* $\det T_a = T_{\det a} + K$.

Proof. For an $N \times N$ matrix A, we have

$$\det A = \sum_{\sigma \in S_N} \text{sgn}(\sigma) \prod_{j=1}^N A_{j\sigma(j)},$$

where S_N is the group of N-permutations and $\text{sgn}(\sigma)$ is the sign of each permutation. If $f, g \in L^\infty \cap VMO^1$, then

$$T_f T_g = T_{fg} - PM_f H_g = T_{fg} + K$$

for some compact operator K since H_g is compact by Theorem 6. Therefore, there is a compact operator K such that

$$\det T_a f = \sum_{\sigma \in S_N} \text{sgn}(\sigma) T a_{1\sigma(1)} \cdots T_{N\sigma(N)} = \sum_{\sigma \in S_N} \text{sgn}(\sigma) T_{a_{1\sigma(1)} \cdots a_{N\sigma(N)}} f + Kf$$

$$= P\left(\sum_{\sigma \in S_N} \text{sgn}(\sigma)(a_{1\sigma(1)} \cdots a_{N\sigma(N)}) f\right) + Kf = P\big((\det a)f\big) + Kf$$

$$= T_{\det a} f + Kf. \qquad \square$$

Theorem 22. *Let $1 < p < \infty$ and $\alpha > 0$. If $a \in (L^\infty \cap VMO^1)_{N \times N}$ for some $N \in \mathbf{N}$. Then T_a is Fredholm on $F^p_{\alpha,N}$ if and only if $\widetilde{\det a}$ is bounded away from zero. If $a \in (L^\infty \cap VO)_{N \times N}$ for some $N \in \mathbf{N}$. Then T_a is Fredholm on $F^p_{\alpha,N}$ if and only if $\det a$ is bounded away from zero*

Proof. In the first case, since the entries of T_a commute pairwise modulo compact operators, Theorem 1.14 of [5] implies that T_a is Fredholm if and only if $\det T_a$ is Fredholm. By the previous lemma and Corollary 15, T_a is Fredholm if and only if $\widetilde{\det a}$ is bounded away from zero. The other case is analogous. □

We conjecture that if $a \in (L^\infty \cap VO)_{N \times N}$ and if $\det a$ is bounded away from zero for some $R > 0$, then

$$(33) \qquad \operatorname{ind} T_a = \operatorname{ind} T_{\det a} = -\operatorname{wind}\left((\det a)|_{\{|z|=R\}}\right).$$

Indeed, this true in some particular cases; e.g. when the entries of T_a commute modulo trace class operators, which happens if and only if

$$\int_{\mathbf{C}} \left(\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2\right)^{1/2} dA(z) < \infty$$

(see Theorem 8.26 of [19]). Another particular case in which the index formula (33) holds is provided by Theorem 7.6 of [8]. Analogous index formulas hold for Toeplitz operators on Hardy and Bergman spaces.

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