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A GENERALIZATION OF THE PARAMETRIZED MODULAR GROUP

Christian Pommerenke and Margarita Toro*

Technische Universität Berlin, Institut für Mathematik
D-10623 Berlin, Germany; pommeren@math.tu-berlin.de

Universidad Nacional de Colombia, Sede Medellín, Escuela de Matemáticas
Calle 59a N. 63-20, Medellín, Colombia; mmtoro@unal.edu.co

Abstract. Given a set of polynomials $p_1, \dots, p_m \in \mathbf{C}[\xi]$ we introduce the group $\Pi = \Pi[p_1, \dots, p_m] = \langle A(p_1), \dots, A(p_m), B \rangle$ where $A(z)$ is the parabolic matrix $\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)$ and B is the elliptic matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. This group unifies the definitions of several groups that often appear in the literature. For instance, $\Pi[1]$ is the modular group and $\Pi[\xi]$ is the parametrized modular group introduced in [MPT15]. For $m = 2$, $p_1 = 1$, $p_2 = i$ we have the Picard group $\Pi[1, i] = \mathrm{SL}(2, \mathbf{Z}[i])$. An important feature is the existence of a simple algorithm to obtain the elements of Π. We discuss several concrete examples, namely the euclidean Bianchi groups and a group from discrete relativity theory, furthermore the subgroup Π_1 of index 4 and its applications to knot theory.

1. Introduction

We will introduce groups built from the parabolic matrix $A(z) = \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)$ and the elliptic matrix $B = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ which appear very often in the literature. This is similar to the approach of Cohn [Coh68]. We define

$$\Pi = \Pi[p_1, \dots, p_m] := \langle A(p_1), \dots, A(p_m), B \rangle \tag{1.1}$$

where the p_μ are polynomials in $\mathbf{C}[\xi]$. For $m = 1$ and $p_1 = \xi$ we obtain the parametrized modular group introduced in [MPT15]. In most of our examples the p_μ will be complex numbers, and therefore, we will obtain subgroups of $\mathrm{SL}(2, \mathbf{C})$; and $\mathrm{PSL}(2, \mathbf{C})$ is isomorphic to the group of orientation preserving isometries of the hiporbolic space H^3. Our applications to knot theory use the fact that many knots K have representations in $\mathrm{PSL}(2, \mathbf{C})$ and therefore $S^3 - K$ admits the structura of a hiperbolic 3-manifold, [Ril82]. The use of the indeterminate ξ however allows us to arrange matrix elements according to the degree of polynomials.

All matrices W of Π can be written as

$$W = B^\kappa U_n B^\lambda, \quad U_n = A(q_n) B \cdots A(q_1) B \tag{1.2}$$

with $\kappa = 0, 1, 2, 3$ and $\lambda = 0, 1$ and with

$$U_n = \begin{pmatrix} \alpha_n & \beta_n \\ \alpha_{n-1} & \beta_{n-1} \end{pmatrix}, \quad \alpha_n = q_n \alpha_{n-1} - \alpha_{n-2}, \quad \beta_n = q_n \beta_{n-1} - \beta_{n-2} \tag{1.3}$$

where the q_n are integral linear combinations of the p_μ.

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In Section 3 we introduce the subgroup Π_1 of index 4 which is generated by the parabolic matrices $A(z)$ and $C(z) = BA(z)B^{-1}$. For $m = 1$ and $p_1 = \xi$ this generalizes the group studied in [PT11a]. As an example, we consider two-bridge and three-bridge knots [BZ85, HMTT12]. Using an idea of Riley [Ril72] we show that at least some of these knots lead to subgroups of Π_1 generated by four or less parabolic matrices.

In Section 4 we adjoin two elliptic matrices as further generators, namely $\left(\begin{smallmatrix} z & 0 \\ 0 & 1/z \end{smallmatrix}\right)$ with $z = i$ and $z = e^{\pi i/4}$. This extension can be applied to the euclidean Bianchi groups [Swa71] and allows us to study the Schild group of discrete relativity theory, see [Sch49, JP17].

2. The new group Π

2.1. Definition of the group. Let $m \in \mathbf{N}$ and let

$$p_\mu \in \mathbf{C}[\xi], \quad p_\mu \neq 0 \quad (\mu = 1, \dots, m), \tag{2.1}$$

that is, the p_μ are nonzero polynomials with complex coefficients and a indeterminate ξ which is the same for all μ. Our basic matrices are

$$A(z) := \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.2}$$

where A is parabolic and B is elliptic of order 4. We will study the group

$$\Pi = \Pi[p_1, \dots, p_m] := \langle A(p_1), \dots, A(p_m), B \rangle \subset \mathrm{SL}(2, \mathbf{C}[\xi]); \tag{2.3}$$

we only consider generators and not relations. Since $A(a)A(b) = A(a+b)$ it follows from (2.2) that

$$A(p_1)^{k_1} \cdots A(p_m)^{k_m} = \begin{pmatrix} 1 & k_1 p_1 + \ldots + k_m p_m \\ 0 & 1 \end{pmatrix} \quad \text{for } k_\mu \in \mathbf{Z}. \tag{2.4}$$

If $m = 1$ and $p_1 = \xi$ then $\Pi = \Pi[\xi]$ is the parametrized modular group [MPT15]. Another simple example is the group $\Pi[1, \xi]$ generated by $A(1), A(\xi), B$.

For $\zeta \in \mathbf{C}$, the notation

$$\Pi(\zeta) := \Pi[p_1, \dots, p_m](\zeta) \in \mathrm{SL}(2, \mathbf{C}) \quad (\zeta \in \mathbf{C}), \tag{2.5}$$

means that the polynomials p_μ are evaluated at ζ. If $W = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then, for instance, $a = a(\xi)$ is a polynomial whereas $a(\zeta)$ is a complex number.

2.2. The recursive evaluation.

Theorem 2.1. *Let $W \in \Pi = \Pi[p_1, \dots, p_m]$. Then there is $n \in \mathbf{N}_0$ such that*

$$W = B^\kappa U_n B^\lambda, \quad U_n := A(q_n)B \cdots A(q_2)B \cdot A(q_1)B, \tag{2.6}$$

where $\kappa = 0, 1, 2, 3$ and $\lambda = 0, 1$ and where

$$q_\nu := k_{1,\nu} p_1 + \ldots + k_{m,\nu} p_m \quad (k_{\mu,\nu} \in \mathbf{Z}). \tag{2.7}$$

Conversely, every W of the form (2.6) belongs to Π.

The matrices U_n in (2.6) depend on the choice of the polynomials $q_1, \dots, q_n$ defined in (2.7). In general these polynomials are not uniquely determined by U_n. However, we have uniqueness under some special conditions on the p_μ, see Proposition 2.5.

Proof. By the definition of generators, W is the product of all powers of the $A(p_1), \dots, A(p_m), B$ in any order. We have $B^k = \pm B, \pm I$ for $k \in \mathbf{Z}$. Since $B^2 = \pm I$

commutes with every matrix we can rewrite W such that only the B inside, B^κ at the beginning and B^λ at the end of the word remain. The remaining inside symbols B separate blocks of powers of $A(p_\mu)$. Using (2.4) we obtain (2.6) with (2.7). As an example, $W = BA(p_1)^2B^2A(p_1)BA(p_2)B^2$ becomes $B^3 \cdot A(3p_1)BA(p_2)B \cdot B$. Finally, by the definition (2.3) and by (2.7) it is clear that (2.6) represents an element of Π. $\square$

Let the q_ν be as in (2.7). We define α_n and β_n recursively by

$$\begin{aligned} &\alpha_0 = 1, \quad \alpha_1 = q_1, \quad \alpha_{n+1} = q_{n+1}\alpha_n - \alpha_{n-1}, \\ &\beta_0 = 0, \quad \beta_1 = -1, \quad \beta_{n+1} = q_{n+1}\beta_n - \beta_{n-1}. \end{aligned} \tag{2.8}$$

Proposition 2.2. *The U_n defined in* (2.6) *satisfy*

$$U_n = \begin{pmatrix} \alpha_n & \beta_n \\ \alpha_{n-1} & \beta_{n-1} \end{pmatrix}. \tag{2.9}$$

Proof. By (2.2) we have

$$A(z)B = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.10}$$

We prove (2.9) by induction. The case $n = 1$ is trivial by (2.8). Let (2.9) be true for n. Then, by (2.6),

$$U_{n+1} = \begin{pmatrix} q_{n+1} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_n & \beta_n \\ \alpha_{n-1} & \beta_{n-1} \end{pmatrix} = \begin{pmatrix} q_{n+1}\alpha_n - \alpha_{n-1} & q_{n+1}\beta_n - \beta_{n-1} \\ \alpha_n & \beta_n \end{pmatrix},$$

and (2.9) follows from (2.8) for $n + 1$. $\square$

By (2.8) the first expressions α_n and β_n are

$$\begin{aligned} &\alpha_2 = q_1q_2 - 1, \quad \alpha_3 = q_1q_2q_3 - (q_1 + q_3), \\ &\alpha_4 = q_1q_2q_3q_4 - (q_1q_2 + q_1q_4 + q_3q_4) + 1, \\ &\beta_2 = -q_2, \quad \beta_3 = -q_2q_3 + 1, \quad \beta_4 = -q_2q_3q_4 + (q_2 + q_4). \end{aligned} \tag{2.11}$$

We need the following identities where we write $W = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

$$WB = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}, \quad BWB = \begin{pmatrix} -d & c \\ b & -a \end{pmatrix}, \quad BW = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}. \tag{2.12}$$

Corollary 2.3. *Let the α_n, β_n be as in* (2.8) *and let $W \in \Pi[p_1, \dots, p_m]$. Then there is $n \in \mathbf{N}$ such that W has one of the eight forms*

$$\begin{aligned} &\pm\begin{pmatrix} \alpha_n & \beta_n \\ \alpha_{n-1} & \beta_{n-1} \end{pmatrix}, \quad \pm\begin{pmatrix} \beta_n & -\alpha_n \\ \beta_{n-1} & -\alpha_{n-1} \end{pmatrix}, \\ &\pm\begin{pmatrix} -\beta_{n-1} & \alpha_{n-1} \\ \beta_n & -\alpha_n \end{pmatrix}, \quad \pm\begin{pmatrix} -\alpha_{n-1} & -\beta_{n-1} \\ \alpha_n & \beta_n \end{pmatrix}, \end{aligned} \tag{2.13}$$

and conversely these matrices belong to Π.

Proof. Every $W \in \Pi$ has the form (2.6). Therefore (2.13) follows from Proposition 2.2 and from (2.12) in the same order. Conversely, let W be one of the matrices (2.13). Since B is a generator of Π and $B^2 = -I$ we can apply (2.12) to show that $W \in \Pi$. $\square$

In the following two results we need the additional assumption

$$(2.14)\qquad \begin{aligned} &p_\mu = t_\mu \xi + s_\mu \quad \text{with} \quad t_\mu, s_\mu \in \mathbf{C} \quad (\mu = 1, \dots, m),\\ &k_1 t_1 + \ldots + k_m t_m = 0 \implies k_1 = \ldots = k_m = 0. \end{aligned}$$

Proposition 2.4. *Let (2.14) be satisfied. If the q_ν are given by (2.7) then $\deg(\alpha_n) = n$, $\deg(\beta_n) = n - 1$ and*

$$(2.15)\qquad \alpha_n = r_1 \cdots r_n \xi^n + \ldots, \qquad \beta_n = r_2 \cdots r_n \xi^{n-1} + \ldots$$

where $r_1, \dots, r_n$ are nonzero complex numbers.

Proof. We obtain from (2.7) and (2.14) that, for $\nu \in \mathbf{N}$,

$$q_\nu = k_{\nu,1}(t_1\xi + s_1) + \ldots + k_{\nu,m}(t_m\xi + s_m) = r_\nu \xi + h_\nu$$

with $r_\nu \neq 0$. Hence (2.15) follows from (2.8) by induction. □

The following uniqueness result extends [MPT15, Lem.2.2].

Proposition 2.5. *Let (2.14) be satisfied with $s_\mu = 0$ and let $W \in \Pi[r_1\xi, \dots, r_m\xi]$ with $W \neq \pm I, \pm B$. Then κ, λ, n in (2.6) and $r_\nu := k_{1,\nu}t_1 + \ldots + k_{m,\nu}t_m$ are uniquely determined.*

Proof. Since $W \neq \pm I, \pm B$ we have $n \geq 1$ in (2.6). It follows from Proposition 2.4 that the polynomials α_n has the highest degree among the elements of U_n. By (2.12) and (2.13), W is obtained from U_n by the rotations effected by B^κ and B^λ. It follows that κ and λ are uniquely determined and therefore also n and U_n.

By (2.9) this implies that α_n and α_{n-1} are uniquely determined. By (2.15) we have $\alpha_n = r_n r_{n-1} \cdots r_1 \xi^n + \ldots$ and $\alpha_{n-1} = r_{n-1} \cdots r_1 \xi^{n-1} + \ldots$ with nonzero highest coefficients. We conclude that $r_n = (r_n \cdots r_1)/(r_{n-1} \cdots r_1)$ is uniquely determined. Since $p_\mu = r_\mu \xi$ it follows from (2.6) that $U_n = A(r_n\xi)U_{n-1}$ so that now U_{n-1} is uniquely determined. Continuing the descent we obtain that $r_n, \dots, r_2, r_1$ are all uniquely determined. □

2.3. An example. We choose $m = 2$, $p_1 = 1 + i$, $p_2 = 1 - i$ and therefore consider the group

$$\Pi[(1+i)\xi, (1-i)\xi] = \langle A((1+i)\xi), A((1-i)\xi), B \rangle.$$

Let U_n be as in (2.6). First we are going to show that

$$(2.16)\quad U_{2\nu} = \begin{pmatrix} 1 + 2f_{2\nu} & (1+i)g_{2\nu-1} \\ (1+i)f_{2\nu-1} & 1 + 2g_{2\nu-2} \end{pmatrix}, \quad U_{2\nu+1} = \begin{pmatrix} (1+i)f_{2\nu+1} & 1 + 2g_{2\nu} \\ 1 + 2f_{2\nu} & (1+i)g_{2\nu-1} \end{pmatrix}$$

where the f_n and g_n are polynomials in ξ of degree n with coefficients in the ring $\mathbf{Z}[i] = \{x + iy \colon x, y \in \mathbf{Z}\}$. In our induction proof we use the α_n and β_n given by (2.8), and from these we obtain the U_n using (2.9). We only derive the α_n. The derivation of the β_n is similar. By (2.7) the factors in (2.8) are

$$q = k_1(1+i)\xi + k_2(1-i)\xi = (1+i)r\xi \quad \text{with} \quad r := k_1 - ik_2 \in \mathbf{Z}[i].$$

From (2.8) we obtain $\alpha_0 = 1$, $\alpha_1 = (1+i)r\xi$ and therefore

$$\alpha_2 = (1+i)r_2\xi\alpha_1 - \alpha_0 = (1+i)r_2\xi \cdot (1+i)r_1\xi - 1 = 1 + 2f_2$$

where $f_2 := -ir_1r_2\xi^2 + 1$. Now let (2.16) be true for $n \leq 2\nu$, $\nu \geq 1$. Then we obtain from (2.8) that

$$\alpha_{2\nu+1} = (1+i)r_{2\nu+1}\xi(1 + 2f_{2\nu}) - (1+i)f_{2\nu-1} = (1+i)f_{2\nu+1}$$

where $f_{2\nu+1} := r_{2\nu+1}\xi(1+2f_{2\nu}) - f_{2\nu-1}$ has degree $2\nu+1$ and coefficients in $\mathbf{Z}[i]$ because $r_{2\nu+1} \in \mathbf{Z}[i]$. It follows that

$$\alpha_{2\nu+2} = (1+i)r_{2\upsilon+2}\xi f_{2\nu+1} - 1 - f_{2\nu} = 1 + 2f_{2\nu+2}$$

where $f_{2\nu+2} := -r_{2\nu+2}\xi f_{2\nu+1} - f_{2\nu} - 1$ has degree $2\nu+2$ and coefficients in $\mathbf{Z}[i]$. This proves that U_n is given by (2.16).

By (2.6) all group matrices have the form $B^\kappa U_n B^\lambda$. It follows from (2.12) that the factors B only rotate U_n possibly changing signs. A change from 1 to -1 is handled by writing $-1 = 1-2$. Therefore we have proved:

Proposition 2.6. *Every matrix in* $\Pi[(1+i)\xi,(1-i)\xi]$ *has one of the two forms* (2.16) *where* f_n *and* g_n *are polynomials in* ξ *of degree* n *with coefficientes in* $\mathbf{Z}[i]$.

2.4. The bilateral multiplication and palindromes. If the word W has some kind of symmetry then a symmetric form of multiplication is often more suitable. Let $n \in \mathbf{N}$ and U_n be as in (2.6). We write $n = 2j$ if n is even and $n = 2j+1$ if n is odd. We are now going to define

$$V_\mu := \begin{pmatrix} a_\mu & b_\mu \\ c_\mu & d_\mu \end{pmatrix} \quad (\mu = 0, 1, \ldots, j) \tag{2.17}$$

by induction. We put

$$V_0 := I, \quad \sigma = 1 \text{ if } n = 2j; \quad V_0 := A(q_{j+1})B, \quad \sigma = 2 \text{ if } n = 2j+1. \tag{2.18}$$

For $\mu = 0, \ldots, j-1$ let

$$V_{\mu+1} := A(q_{j+\mu+\sigma})BV_\mu A(q_{j-\mu})B. \tag{2.19}$$

Then we have $U_n = V_j$. If follows from (2.18) and (2.19) that

$$\begin{aligned} V_{\mu+1} &= \begin{pmatrix} q_{j+\mu+\sigma} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_\mu & b_\mu \\ c_\mu & d_\mu \end{pmatrix} \begin{pmatrix} q_{j-\mu} & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} q_{j-\mu}q_{j+\mu+\sigma}a_\mu + q_{j+\mu+\sigma}b_\mu - q_{j-\mu}c_\mu - d_\mu & -q_{j+\mu+\sigma}a_\mu + c_\mu \\ q_{j-\mu}a_\mu + b_\mu & -a_\mu \end{pmatrix}. \end{aligned} \tag{2.20}$$

Proposition 2.7. *Let* U_n *be as in* (2.6) *and suppose that*

$$q_\nu = q_{n+1-\nu} \neq 0 \text{ for } \nu = 1, \ldots, n. \tag{2.21}$$

Using the notation (2.8) *we then have*

$$\alpha_{n-1} + \beta_n = 0. \tag{2.22}$$

By definition $X_n \cdots X_2 X_1$ is called a *palindrome* if $X_\nu = X_{n-\nu+1}$ for $\nu = 1, \ldots, n$. It follows from (2.21) that U_n is a palindrome with $X_\nu := A(q_\nu)B$.

Proof. We use the bilateral multiplication defined above. We have $U_n = V_j$ and obtain from (2.20) that, for $\mu < j$,

$$b_{\mu+1} + c_{\mu+1} = (q_{\nu-\mu} - q_{j+\mu+\sigma})a_\mu + b_\mu + c_\mu. \tag{2.23}$$

By (2.21) it follows that $b_{\mu+1} + c_{\mu+1} = b_\mu + c_\mu$. Furthermore we see from (2.18) that $b_0 + c_0 = 0$ in both cases. Therefore we obtain by induction that

$$\beta_n + \alpha_{n-1} = b_j + c_j = 0. \qquad \square$$

3. The parabolic subgroup Π_1

3.1. The basic properties. We are going to introduce a subgroup that is entirely generated by parabolic matrices, namely by $A(z)$ and

$$C(z) := \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} = BA(z)B^{-1} = B^{-1}A(z)B. \tag{3.1}$$

Let $W \in \Pi = \Pi[p_1, \ldots, p_m]$ and let

$$\tau(W) := (\text{the number modulo 4 of the symbols } B \text{ in the word } W) \tag{3.2}$$

where $-I = BB$ and $B^{-1} = BBB$. For example we have $\tau(-A(z_1)BA(z_2)B^{-1}) = 2 + 1 + 3 \equiv 2 \bmod 4$. Now we define

$$\Pi_1 = \Pi_1[p_1, \ldots, p_m] := \{W \in \Pi \colon \tau(W) \equiv 0 \bmod 4\}. \tag{3.3}$$

Proposition 3.1. *The set Π_1 is a normal subgroup of Π with index 4.*

Proof. Since Π is a group we see that (3.2) induces a group homorphism $\tau \colon \Pi \to \mathbf{Z}^4$ with kernel Π_1. Hence Π_1 is normal subgroup of Π. Let $\Gamma_k = \{W \in \Pi \colon \tau(W) \equiv k \bmod 4\}$, $k = 0, 1, 2, 3$. Then we have

$$\Pi = \Gamma_0 \mathbin{\dot{\cup}} \Gamma_1 \mathbin{\dot{\cup}} \Gamma_2 \mathbin{\dot{\cup}} \Gamma_3 = \bigcup_{k=0}^{3} B^k \Pi_1 = \bigcup_{k=0}^{3} \Pi_1 B^k$$

so that Π_1 has index 4 in Π. □

Theorem 3.2. *Let $W \in \Pi = \Pi[p_1, \ldots, p_m]$. Then W belongs to Π_1 if and only if it has one of the four forms*

$$\begin{aligned} A(q_{2\nu})C(q_{2\nu-1}) \cdots A(q_2)C(q_1) &= (-1)^\nu U_{2\nu}, \\ C(q_{2\nu})A(q_{2\nu-1}) \cdots C(q_2)A(q_1) &= (-1)^\nu BU_{2\nu}B^{-1}, \\ A(q_{2\nu+1})C(q_{2\nu}) \cdots C(q_2)A(q_1) &= (-1)^{\nu+1} U_{2\nu+1}B, \\ C(q_{2\nu+1})A(q_{2\nu}) \cdots A(q_2)C(q_1) &= (-1)^{\nu+1} BU_{2\nu+1} \end{aligned} \tag{3.4}$$

with $\nu \in \mathbf{N}_0$ where q_k is defined in (2.7) and U_n in (2.6).

We write the elements of the matrices $W \in \Pi_1$ in the same order as in (3.4). It follows from Theorem 3.2 and Corollary 2.3 that

$$\begin{aligned} &A \ldots C = (-1)^\nu \begin{pmatrix} \alpha_{2\nu} & \beta_{2\nu} \\ \alpha_{2\nu-1} & \beta_{2\nu-1} \end{pmatrix}, \quad C \ldots A = (-1)^\nu \begin{pmatrix} \beta_{2\nu-1} & -\alpha_{2\nu-1} \\ -\beta_{2\nu} & \alpha_{2\nu} \end{pmatrix}, \\ &A \ldots A = (-1)^\nu \begin{pmatrix} -\beta_{2\nu+1} & \alpha_{2\nu+1} \\ -\beta_{2\nu} & \alpha_{2\nu} \end{pmatrix}, \quad C \ldots C = (-1)^\nu \begin{pmatrix} \alpha_{2\nu} & \beta_{2\nu} \\ -\alpha_{2\nu+1} & -\beta_{2\nu+1} \end{pmatrix}. \end{aligned} \tag{3.5}$$

Proof. (a) First we verify the four equations (3.4) using (2.6). As an abbreviation we write $A_k := A(q_k)$, $C_k := C(q_k)$. In this representation of W the symbol B occurs $\kappa + n + \lambda$ times. Therefore we obtain from the definition (3.2) that $\kappa \equiv -n - \lambda \bmod 4$. It follows from (3.1) that

$$A_k B = BC_k, \quad A_k B A_{k+1} B = A_k BBC_{k+1} = -A_k C_{k+1} = B^2 A_k C_{k+1}. \tag{3.6}$$

First let $n = 2\nu$, $\lambda = 0$ and therefore $\kappa \equiv -2\nu \bmod 4$. Then we obtain from (3.1), (2.6) and (3.6) that

$$\begin{aligned} W = (-1)^\nu U_{2\nu} &= B^{-2\nu} \cdot (A_{2\nu} B A_{2\nu-1} B) \cdots (A_2 B A_1 B) \\ &= B^{-2\nu} (B^2 A_{2\nu} C_{2\nu-1}) \cdots (B^2 A_2 C_1) = (A_{2\nu} C_{2\nu-1}) \cdots (A_2 C_1). \end{aligned}$$

Now let $n = 2\nu$, $\lambda = 1$ and therefore $\kappa \equiv -2\nu - 1$. It follows that

$$\begin{aligned} W &= (-1)^{\nu+1} B U_{2\nu} B = B^{-2\nu-1} A_{2\nu} B \cdot (A_{2\nu-1} B A_{2\nu-2} B) \cdots (A_3 B A_2 B) \cdot A_1 \\ &= B^{-2\nu-1} B C_{2\nu} \cdot (B^2 A_{2\nu-1} C_{2\nu-2}) \cdots (B^2 A_3 C_2) \cdot A_1. \end{aligned}$$

Let $n = 2\nu + 1$, $\lambda = 1$ and therefore $\kappa \equiv -2\nu - 2$. Then we have

$$\begin{aligned} W &= (-1)^{\nu+1} U_{2\nu+1} B = B^{-2\nu-2} (A_{2\nu+1} B A_{2\nu} B) \cdots (A_3 B A_2 B) \cdot A_1 B^2 \\ &= B^{-2\nu} (B^2 A_{2\nu+1} C_{2\nu}) \cdots (B^2 A_3 C_2) \cdot A_1. \end{aligned}$$

Finally let $n = 2\nu + 1$, $\lambda = 0$ and therefore $\kappa \equiv -2\nu - 1$. It follows that

$$\begin{aligned} W &= (-1)^{\nu+1} B U_{2\nu+1} = B^{-2\nu-1} A_{2\nu+1} B \cdot (A_{2\nu} B A_{2\nu-1} B) \cdots (A_2 B A_1 B) \\ &= B^{-2\nu} C_{2\nu+1} \cdot (B^2 A_{2\nu} C_{2\nu-1}) \cdots (B^2 A_2 C_1). \end{aligned}$$

(b) We have seen in (a) that the four words at the left side of (3.4) are all the possibilities for Π_1. Conversely, the words on the right side of (3.4) belong to Π_1 by definition (3.3) and because $\tau(A(q_k)) = \tau(C(q_k)) = 0$ by (3.1). □

3.2. Two-bridge knots. The groups of the knots [BZ85, HTT02] have the abstract presentation $\langle x_1, x_2 \colon x_1 w = w x_2 \rangle$ where w has a certain symmetry, see for instance [HTT02], [PT11b, Th.4.3]. An example is the "figure-eight knot" [MR03, p. 60]. The relation is $w = x_1 x_2^{-1} x_1^{-1} x_2$ and the matrix group is generated by $A(1), C(\omega)$, $\omega = e^{2\pi i/3}$ which is a subgroup of $\Pi_1[1, \omega]$.

The following proposition generalizes a well-known method to deal with the single relation w, see e.g. [MR03, p. 140]. To apply it to two-bridge knots, we assign $x_1 \mapsto C(\zeta_1)$, $x_2 \mapsto A(\zeta_2)$.

Proposition 3.3. *We assume that* (2.14) *holds and that* $q_\nu = q_{n-\nu+1} \neq 0$ *for* $\nu = 1, \ldots, n$, n *even. Let*

$$W := A(q_n) C(q_{n-1}) \cdots A(q_2) C(q_1). \tag{3.7}$$

Then there exists $\zeta \in \mathbf{C}$ *such that, for all* $z \in \mathbf{C}$,

$$c = -b, \ \deg(a) = n, \quad W(\zeta) A(z) = C(z) W(\zeta). \tag{3.8}$$

Proof. We write $W = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. By (3.7) and (3.1) we have

$$W = U_n = A(q_n) B \cdot A(q_{n-1}) B \cdots A(q_2) B \cdot A(q_1) B$$

where U_n is defined in (2.6). Since $q_\nu = q_{n-\nu+1} \neq 0$ for $\nu = 1, \ldots, n$, this is a palindrome in $X_\nu := A(q_\nu) B$, see Section 2.3. It follows from Proposition 2.7 that $b + c = 0$ and from Proposition 2.4 that $\deg(a) = n > 0$. Hence there exists ζ such that $a(\zeta) = 0$. Then we have

$$\begin{aligned} W(\zeta) A(z) &= \begin{pmatrix} 0 & b(\zeta) \\ -b(\zeta) & d(\zeta) \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & b(\zeta) \\ -b(\zeta) & -z b(\zeta) + d(\zeta) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} \begin{pmatrix} 0 & b(\zeta) \\ -b(\zeta) & d(\zeta) \end{pmatrix} = C(z) W(\zeta). \end{aligned}$$

□

3.3. Three-bridge knots. The theory of three-bridge knots is much more difficult [Ril82, HMTT12]. The knot group has the abstract presentation

$$\langle\, x_1, x_2, x_3 \colon x_1 w_1 = w_1 x_2, \ x_2 w_2 = w_2 x_3, \ x_3 w_3 = w_3 x_1 \,\rangle. \tag{3.9}$$

We will not use the fact [BZ85] that any relation is a consequence of the other two. The following proposition is, in part, an adaptation of Riley [Ril82, p. 119].

Proposition 3.4. *Let* $D(z) := A(1)C(z)A(-1) = \left(\begin{smallmatrix} 1+z & -z \\ z & 1-z \end{smallmatrix}\right)$. *We assume that* $\zeta_1\zeta_2\zeta_3 \neq 0$ *and assign*

$$x_1 \mapsto A(\zeta_1), \quad x_2 \mapsto C(\zeta_2), \quad x_3 \mapsto D(\zeta_3), \quad w_\nu \mapsto W_\nu = \left(\begin{smallmatrix} a_\nu & b_\nu \\ c_\nu & d_\nu \end{smallmatrix}\right)$$

where the W_ν *are products of the matrices* A, C, D *that depend on the specific knot. Then the relations in* (3.9) *hold if and only if*

$$\begin{aligned} &d_1 = 0, \quad \zeta_1 c_1 + \zeta_2 b_1 = 0, \\ &a_2 + b_2 = 0, \quad \zeta_2 a_2 + \zeta_3(c_2 + d_2) = 0, \\ &a_3 - c_3 = 0, \quad \zeta_1 c_3 + \zeta_3(d_3 - b_3) = 0. \end{aligned} \tag{3.10}$$

All these matrices V belong to the group generated by $A(\zeta_1), A(\zeta_2), A(\zeta_3), A(1)$ and B. Since the sum of the exponents of B is $\sigma(B) = 0$ it follows from (3.3) that $V \in \Pi_1[\zeta_1, \zeta_2, \zeta_3, 1]$. The algebraic integers $\zeta_1, \zeta_2, \zeta_3$ depend on the knot.

Proof. We restrict ourselves to show that $x_1 w_1 = w_1 x_2$ holds if and only if the first line in (3.10) holds. We must have

$$A(\zeta_1)W_1 = \begin{pmatrix} a_1 + \zeta_1 c_1 & b_1 + \zeta_1 d_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} a_1 - b_1\zeta_2 & b_1 \\ c_1 - d_1\zeta_2 & d_1 \end{pmatrix} = W_1 C(\zeta_2).$$

The first elements of the matrices are equal if and only if $\zeta_1 c_1 = -b_1\zeta_2$. The second elements are equal if and only if $d_1 = 0$. The same is true for the third elements, and the last elements are the same. □

Note that $a_\nu, b_\nu, c_\nu, d_\nu$ depend on $\zeta_1, \zeta_2, \zeta_3$. Thus we have six equations for the three unknowns $\zeta_1, \zeta_2, \zeta_3$. We do not know whether these equations always have non-trivial solutions. If they exist it may turn out that they are of no use for knot theory. Moreover, following Riley, we choose three parabolic generators but it might perhaps be necessary to use non-parabolic generators. These are important questions on which we are currently working.

4. Extensions of Π by other matrices

4.1. The matrices Q and R. Now we study the connection of the matrices

$$Q := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = R^2, \quad R := \begin{pmatrix} \frac{1+i}{\sqrt{2}} & 0 \\ 0 & \frac{1-i}{\sqrt{2}} \end{pmatrix}, \tag{4.1}$$

with the group Π. They satisfy

$$Q\begin{pmatrix} a & b \\ c & d \end{pmatrix} Q^{-1} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}, \quad R\begin{pmatrix} a & b \\ c & d \end{pmatrix} R^{-1} = \begin{pmatrix} a & ib \\ -ic & d \end{pmatrix}. \tag{4.2}$$

Theorem 4.1. *Let* $\Pi = \Pi[p_1, \ldots, p_m]$ *and let*

$$\Gamma = \langle A(p_1), \ldots, A(p_m), B, Q \rangle. \tag{4.3}$$

If $Q \in \Pi$ *then* $\Gamma = \Pi$, *if* $Q \notin \Pi$ *then* $\Gamma = \Pi \,\dot\cup\, (Q\Pi)$.

The assumption that $Q \notin \Pi$ is for instance satisfied if $p_\mu = r_\mu \xi$ with $r_\mu \in \mathbf{C}$, $r_\mu \neq 0$, $(\mu = 1, \ldots, m)$ because then all words W of Π except $\pm I, \pm B$ are non-constant polynomials.

The situation is different for the groups $\Pi(\zeta)$, see (2.5). It is possible that $Q \in \Pi(\zeta)$ so that $Q\Pi = \Pi$. An example is given by $Q \in \Pi[1, i]$ as we will see in part (a) of the proof of Corollary 4.3. On the other hand, we will later see that $Q \notin \Pi[1+i, 1-i]$.

Proof. From the definition (2.3) of Π and from (4.3) we obtain that $Q \in \Pi$ implies $\Gamma = \Pi$. Now let $Q \notin \Pi$. The words of Γ consist of blocks of powers of $A(p_\mu)$ as in (2.4), divided by blocks of powers of B and Q. We have $B^k = \pm I, \pm B$ and $Q^k = \pm Q, \pm I$. Since $-I$ commutes with every matrix we may therefore assume that W consists of blocks of powers of $A(p_\mu)$ that are divided by single symbols B and Q.

It follows from (4.2) that $BQ = -QB$ and $A(p_\mu)Q = QA(-p_\mu)$. Therefore we can move all Q to the left. Finally we obtain $W = Q^kV$ with $k = 0, 1$ and $V \in \Pi$. If $k = 0$ then $W \in \Pi$, if $k = 1$ then $W \in Q\Pi$. $\square$

Theorem 4.2. *Let m be even and let $\Pi = \Pi[p_1, \dots, p_m]$ where the parameters come in pairs p_μ, ip_μ for $\mu = 1, \dots, m/2$. If $R \notin \Pi$ then $\Gamma := \langle\, A(p_1), \dots, A(p_m), R\,\rangle$ satisfies*

$$\begin{aligned} &\Gamma = \Pi \mathbin{\dot\cup} (R\Pi) \mathbin{\dot\cup} R^2\Pi \mathbin{\dot\cup} (R^3\Pi) \quad \textit{if} \;\; Q \notin \Pi, \\ &\Gamma = \Pi \mathbin{\dot\cup} (R\Pi) \quad \textit{if} \;\; Q \in \Pi. \end{aligned} \tag{4.4}$$

Proof. By (2.7) the assumption about the parameters implies that $iq_\nu = q'_\nu = k'_1p_1 + \dots + k'_mp_m$ with $k'_\mu \in \mathbf{Z}$. Hence we obtain from (4.2) that $A(q_\nu)R = RA(q'_\nu)$. Using this identity repeatedly we can move all R to the left as in the proof of Theorem 4.1. Finally we obtain the union (4.4) but possibly without the disjunctions.

Suppose that $R^k\Pi \cap R^l\Pi \neq \emptyset$ where $0 \leq k < l \leq 3$. First let $Q \notin \Pi$. Then we have $R^j = R^{l-k} = \Pi$ with $1 \leq j \leq 3$. But $R \in \Pi$ implies $R^2 = Q \in \Pi$ and $R^3 \in \Pi$ implies $R^6 = Q \in \Pi$, which is false by assumption. Hence the four sets in (4.4) are disjoint. Now let $Q \in \Pi$. As $R \notin \Pi$ we see that the second line of (4.4) is true. $\square$

4.2. The euclidean Bianchi groups. Let $d \in \mathbf{N}$ and let $\mathcal{O}_d$ be the ring of the integers of the quadratic field $\mathbf{Q}(\sqrt{-d})$. We only consider the euclidean Bianchi groups, that is, the five d for which $\mathcal{O}_d$ allows a euclidian algorithm. The following theorem is based on the work of Swan [Swa71] and is not new.

Corollary 4.3. *If $d = 1, 2, 3, 7, 11$, then*

$$\mathrm{SL}(2, \mathcal{O}_d) = \Pi[1, \omega] \tag{4.5}$$

where, in the same order, $\omega = i,\ i\sqrt{2},\ \frac{1}{2}(-1 + i\sqrt{3}),\ \frac{1}{2}(1 + i\sqrt{7}),\ \frac{1}{2}(1 + i\sqrt{11})$.

Swan has proved (4.5) for $d = 2, 7, 11$ using the method of fundamental domains. For $d = 1, 3$, this method gives also Q as a generator. Almost all articles that we know quote this additional generator [FN87, YC00], [MR03, p. 59]. But Cohn [Coh68] has shown that Q is superfluous as a generator.

Proof. (a) Let $d = 1$, the case of the Picard group $\mathrm{SL}(2, \mathbf{Z}[i])$. According to [Swa71, p. 39] the generators are $A(1), A(i), B$ and furthermore Q. Using (2.11) with $\alpha_3 = i, \alpha_2 = \beta_3 = 0, \beta_2 = -i$ we construct the following matrix

$$Q = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -i & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -i & -1 \\ 1 & 0 \end{pmatrix} \in \Pi[1, i].$$

Hence Q is superfluous as a generator.

(b) Let $d = 3$ and $\omega = \frac{1}{2}(-1 + i\sqrt{3})$. According to [Swa71, p. 41] the generators are $A(1), A(\omega), B$ and furthermore $G := \left(\begin{smallmatrix} \omega & 0 \\ 0 & \bar\omega \end{smallmatrix}\right)$. Using again (2.11) we obtain

$$G = \begin{pmatrix} \omega & 0 \\ 0 & \bar\omega \end{pmatrix} = \begin{pmatrix} \omega & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\bar\omega & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \omega & -1 \\ 1 & 0 \end{pmatrix} \in \Pi[1, \omega].$$

Hence G is superfluous as a generator. $\square$

4.3. The Schild group. In the standard form of discrete relativity theory the space-time vector (t,x,y,z) lies in the grid $\mathbf{Z}^4$. By definition, the Schild group $\mathbf{S} \subset \mathrm{SL}(2,\mathbf{C})$ leaves the 2x2-matrix form of $\mathbf{Z}^4$ invariant. See e.g. [Sch49, LK99, JP16].

The Schild group $\mathbf{S}$ is the disjoint union of the four rather different sets $\mathbf{G}$, $\mathbf{H}$, $\mathbf{V}$ and $\mathbf{W}$, see e.g. [JP16]. The basic group [JP17, Th. 2.1] is

$$\begin{aligned}\mathbf{G} &= \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \{\mathrm{SL}(2,\mathbf{Z}[i])\colon |a|^2+|b|^2+|c|^2+|d|^2 \in 2\mathbf{Z}\} \\ &= \langle A(1+i), A(1-i), B, Q\rangle\,.\end{aligned}$$

The only subgroups of $\mathbf{S}$ between $\mathbf{G}$ and $\mathbf{S}$ are the two groups [JP17, Th. 3.2, Sect. 4.2] given by

$$\mathbf{G} \cup \mathbf{V} = \langle A(1+i), A(1-i), B, R\rangle\,, \tag{4.6}$$

$$\mathbf{G} \cup \mathbf{H} = \langle A(1+i), A(1-i), B, P\rangle\,, \quad P := \begin{pmatrix} (1-i)/2 & (1-i)/2 \\ -(1+i)/2 & (1+i)/2 \end{pmatrix}. \tag{4.7}$$

Furthermore it is shown [JP17, Th. 6.2] that $\mathbf{G}$ is a subgroup of index 3 of the Picard group $\Pi[1,i]$ considered in Corollary 4.3.

Now we derive a result proved by [JP17] in a different form and by a different method. Let $\Pi = \Pi[1+i, 1-i]$, the group considered in Section 2.3. It follows from Proposition 2.6 that $Q \notin \Pi$, furthermore we have $R \notin \Pi$ because $R^2 = Q$ by (4.1). Therefore we obtain from Theorems 4.1 and 4.2 that

$$\mathbf{G} = \Pi \,\dot{\cup}\, (Q\Pi), \quad \mathbf{G} \cup \mathbf{V} = \Pi \,\dot{\cup}\, (R\Pi) \,\dot{\cup}\, R^2\Pi \,\dot{\cup}\, (R^3\Pi). \tag{4.8}$$

The situation is different for the group (4.7). We do not know any analogue of Theorem 4.1 with Q replaced by the technically more difficult P. The cosets of $\mathbf{G}$ in $\mathbf{G} \cup \mathbf{H}$ are described in [JP17, Th. 3.2].

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