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A CHARACTERIZATION OF REAL ANALYTIC FUNCTIONS

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Abstract. We give a characterization of real analytic functions in terms of integral means. The characterization justifies the introduction of a definition of analytic functions on metric measure spaces.

1. Introduction

It is well-known that harmonic functions, i.e., solutions to the Laplace equation $\Delta u = 0$, where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, can be characterized by the mean value property. Namely, a function u continuous on an open set $\Omega \subset \mathbf{R}^n$ is harmonic on Ω if, and only if, for any closed ball $B(x,R) \subset \Omega$ the value of u at the center of the ball is equal to the integral mean of u over the ball. In the previous paper [7] we proved that polyharmonic functions on Ω can be characterized as those continuous functions on Ω for which integral mean over balls of radius R is expressed as an even polynomial of R with coefficients continuous on Ω. Here we extend the above characterization to the case of real analytic functions. Namely we prove

The main theorem. *Let u be a continuous, complex valued function on an open set $\Omega \subset \mathbf{R}^n$. Then u is real analytic on Ω if, and only if, there exist functions $u_{2l} \in C^0(\Omega, \mathbf{C})$ for $l \in \mathbf{N}_0$ and $\epsilon \in C^0(\Omega, \mathbf{R}_+)$ such that*

$$\frac{1}{|B(x,R)|} \int_{B(x,R)} u(y)\,dy = \sum_{l=0}^{\infty} u_{2l}(x) R^{2l} \tag{1}$$

locally uniformly in $\{(x,R)\colon x \in \Omega, 0 \le R < \epsilon(x)\}$.

In fact, the necessity of the expansion (1) for the real analyticity of u is well known, see Theorem 1 below. The novelty of the main theorem is that the sufficient condition for real analyticity of u assumes only continuity of the functions u and u_{2l} in the expansion (1). Thus it justifies the introduction of a definition of analytic functions on metric measure spaces.

2. Preliminary results

Throughout the paper Ω stands for an open set. If $\Omega \subset \mathbf{R}^n$, $x \in \Omega$ and $0 < R < \operatorname{dist}(x, \partial\Omega)$ solid and spherical means of a continuous, complex valued function $u \in C^0(\Omega)$ are defined by

$$M(u;x,R) = \frac{1}{\sigma(n)R^n} \int_{B(x,R)} u(y)\,dy = \frac{1}{\sigma(n)} \int_{B(0,1)} u(x+Rz)\,dz, \tag{2a}$$

$$N(u;x,R) = \frac{1}{n\sigma(n)R^{n-1}} \int_{S(x,R)} u(y)\,dS(y) = \frac{1}{n\sigma(n)} \int_{S(0,1)} u(x+Rz)\,dS(z), \tag{2b}$$

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where $\sigma(n) = \pi^{n/2}/\Gamma(n/2+1)$ (with Γ the Euler Γ-function) is the volume of the unit ball $B(0,1)$ in $\mathbf{R}^n$ and dS denotes the surface measure on the sphere $S(x,R)$. Note that by the second formulas in (2a) and (2b) for a fixed $x \in \Omega$ the functions M and N are defined for $|R| < \operatorname{dist}(x, \partial\Omega)$ and they are even continuous functions of R satisfying $M(u;\, x, 0) = N(u;\, x, 0) = u(x)$. Computing $M(u;\, x, R)$ in spherical coordinates one obtains the following differential relation between the functions $M(u;\, x, R)$ and $N(u;\, x, R)$ (see [5, Lemma 1]),

$$\Big(\frac{R}{n}\frac{\partial}{\partial R} + 1\Big) M(u;\, x, R) = N(u;\, x, R) \tag{3}$$

for any $x \in \Omega$ and $|R| < \operatorname{dist}(x, \partial\Omega)$.

If u is a real analytic function, then its mean value functions $M(u;\, \cdot, \cdot)$ and $N(u;\, \cdot, \cdot)$ are expressed by the so called Pizzetti's serii.

Theorem 1. [8, 6] (Mean-value property) *If u is a real analytic function on $\Omega \subset \mathbf{R}^n$, then for any $x \in \Omega$ and $|R|$ small enough we have*

$$M(u;\, x, R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \big(\frac{n}{2}+1\big)_k k!} \cdot R^{2k} \tag{4}$$

and

$$N(u;\, x, R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \big(\frac{n}{2}\big)_k k!} \cdot R^{2k}, \tag{5}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochhammer symbol. The expansions (4) *and* (5) *are uniform on compact subsets of Ω.*

Conversely real analytic functions are characterized as those smooth ones for which the Pizzetti's serii converge.

Theorem 2. [6, Theorem 3.2] (Converse to the mean-value property) *Let $\Omega \subset \mathbf{R}^n$, $u \in C^\infty(\Omega)$ and $\epsilon \in C^0(\Omega, \mathbf{R}_+)$. If the series on the right hand side of* (4) *or* (5) *is locally uniformly convergent in $\{(x,R)\colon x \in \Omega, |R| < \epsilon(x)\}$, then u is real analytic on Ω.*

3. A characterization of real analyticity

In this section we shall prove the main theorem. Lets start with

Lemma 1. *Let Ω be a domain in $\mathbf{R}^n$ and $u \in C^0(\Omega)$. Assume that there exist functions $v_{2l} \in C^0(\Omega)$ for $l \in \mathbf{N}_0$ and $\epsilon \in C^0(\Omega, \mathbf{R}_+)$ such that*

$$N(u;\, x, R) = \sum_{l=0}^{\infty} v_{2l}(x) R^{2l} \tag{6}$$

locally uniformly in $\{(x,R)\colon x \in \Omega, |R| < \epsilon(x)\}$. Then $u \in C^\infty(\Omega)$ and $v_{2l} \in C^\infty(\Omega)$ for $l \in \mathbf{N}_0$.

Proof. Let $\widetilde{\eta}(r)$ be a smooth function on $[0,\infty)$ supported by $[0,1]$ and constant for r close to zero. Assume also that $n\sigma(n)\int_0^1 \widetilde{\eta}(r) r^{n-1}\, dr = 1$. Then $\eta^\varepsilon(y) = \frac{1}{\varepsilon^n}\widetilde{\eta}\big(\frac{|y|}{\varepsilon}\big)$ is a radially symmetric mollifier supported by $\overline{B}(0,\varepsilon)$. Integrating in spherical coordinates we get

$$\int_{B(0,\varepsilon)} \eta^\varepsilon(y)\, dy = \int_0^1 n\sigma(n)\widetilde{\eta}(r) r^{n-1}\, dr = 1.$$

Applying the Laplace operator Δ to η^ε we have

$$\Delta\eta^\varepsilon(y) = \frac{1}{\varepsilon^{n+2}}\widetilde{\eta}''\Big(\frac{|y|}{\varepsilon}\Big) + \frac{1}{\varepsilon^{n+1}}\frac{n-1}{|y|}\widetilde{\eta}'\Big(\frac{|y|}{\varepsilon}\Big) := L_\varepsilon(\widetilde{\eta})(|y|).$$

For $l \in \mathbf{N}_0$ put $m_{2l}(\eta^1) = \int_{B(0,1)} \eta^1(y)y^{2l}\,dy$. For x from a compact subset of Ω and sufficiently small $\varepsilon > 0$ let us compute Δ acting on the convolution $\eta^\varepsilon * u$. Using spherical coordinates, (2b) and (6) we derive

$$\begin{aligned}
\Delta\big(\eta^\varepsilon * u\big)(x) &= \big(\Delta\eta^\varepsilon\big) * u(x)\\
&= \int_0^\varepsilon \bigg(\int_{S(x,r)} u(\zeta)\,dS(\zeta)\bigg) L_\varepsilon(\widetilde{\eta})(r)\,dr\\
&\overset{(2b)}{=} \int_0^\varepsilon n\sigma(n)N(u;x,r)\,L_\varepsilon(\widetilde{\eta})(r)r^{n-1}\,dr\\
&\overset{(6)}{=} \sum_{l=0}^\infty n\sigma(n)v_{2l}(x)\int_0^\varepsilon L_\varepsilon(\widetilde{\eta})(r)r^{2l+n-1}\,dr\\
&= \sum_{l=0}^\infty v_{2l}(x)\cdot n\sigma(n)\varepsilon^{2l-2}\int_0^1 L_1(\widetilde{\eta})(t)t^{2l+n-1}\,dt\\
&= \sum_{l=0}^\infty v_{2l}(x)\cdot n\sigma(n)\varepsilon^{2l-2}\int_{B(0,1)} L_1(\widetilde{\eta})(|y|)|y|^{2l}\,dy\\
&= \sum_{l=1}^\infty v_{2l}(x)\cdot \varepsilon^{2l-2}\int_{B(0,1)} \Delta\eta^1(y)\,y^{2l}\,dy\\
&= \sum_{l=0}^\infty v_{2l+2}(x)\cdot\varepsilon^{2l}m_{2l+2}(\Delta\eta^1)
\end{aligned}$$

since by the Green formula $\int_{B(0,1)}\Delta\eta^1(y)\,dy = \int_{S(0,1)}\frac{\partial\eta^1}{\partial n}(y)\,dS(y) = 0$. Analogously for $k \in \mathbf{N}_0$ we obtain

$$\Delta^k\big(\eta^\varepsilon * u\big)(x) = \sum_{l=0}^\infty v_{2l+2k}(x)\cdot\varepsilon^{2l}m_{2l+2k}(\Delta^k\eta^1). \tag{7}$$

Since $\Delta^k(\eta^\varepsilon * u)$ is convergent in the distributional sense as $\varepsilon \to 0$ we get

$$m_{2k}(\Delta^k\eta^1)v_{2k} = \lim_{\varepsilon\to 0}\Delta^k(\eta^\varepsilon * u) = \Delta^k\big(\lim_{\varepsilon\to 0}\eta^\varepsilon * u\big) = \Delta^k v_0 \in D'(\Omega).$$

Since $v_{2k} \in C(\Omega)$ applying the Weyl lemma [9, Lemma 2] we conclude that $u = v_0 \in C^{2k}(\Omega)$. Note that for $0 \le l \le k$ we have $\Delta^k v_0 = \Delta^{k-l}(\Delta^l v_0) = m_{2l}(\Delta^l\eta^1)\cdot\Delta^{k-l}v_{2l} \in C^0(\Omega)$. So $v_{2l} \in C^{2k-2l}(\Omega)$ since by (8) below, $m_{2l}(\Delta^l\eta^1) = 4^l l!\big(\frac{n}{2}\big)_l m_0(\eta^1)$ and $m_0(\eta^1) = 1$. Since k is arbitrary big we conclude that $v_{2l} \in C^\infty(\Omega)$ for $l \in \mathbf{N}_0$. □

Now the main theorem is a consequence of Theorem 1 and the following.

Theorem 3. *Let Ω be a domain in $\mathbf{R}^n$, $\epsilon \in C^0(\Omega, \mathbf{R}_+)$ and $u \in C^0(\Omega)$. If there exist functions $u_k \in C^0(\Omega)$ for $k \in \mathbf{N}_0$ and $\epsilon \in C^0(\Omega, \mathbf{R}_+)$ such that*

$$M(u;\,x,R) = \sum_{k=0}^\infty u_k(x)R^k$$

locally uniformly in $\{(x,R)\colon x \in \Omega, |R| < \epsilon(x)\}$, then u is real analytic on Ω, $u_k = 0$ if k is odd and $u_k = \big[4^l\big(\frac{n}{2}+1\big)_l l!\big]^{-1}\Delta^l u$ if $k = 2l$ with $l \in \mathbf{N}_0$.

Proof. Since, by the second formula in (2a), $M(u;\, x, R)$ is an even function of R we have $u_k = 0$ if k is odd. Next applying (3) we get for $x \in \Omega$ and $|R| < \epsilon(x)$,

$$N(u;\, x, R) = \Big(\frac{R}{n}\frac{\partial}{\partial R} + 1\Big)\Big(\sum_{l=0}^{\infty} u_{2l}(x)R^{2l}\Big) = \sum_{l=0}^{\infty}\Big(\frac{2l}{n} + 1\Big)u_{2l}(x)R^{2l}.$$

Hence the assumptions of Lemma 1 are satisfied with $v_{2l} = \big(\frac{2l}{n} + 1\big)u_{2l}$ and so $u_{2l} \in C^\infty(\Omega)$ for $l \in \mathbf{N}_0$. By the Green formula we get for $l \in \mathbf{N}_0$,

$$\begin{aligned} m_{2l}(\Delta\eta^1) &= \int_{B(0,1)} \Delta\eta^1(y)y^{2l}\,dy = \int_{B(0,1)} \eta^1(y)\Delta\big(y^{2l}\big)\,dy \\ &= 2l(n+2l-2)\int_{B(0,1)} \eta^1(y)y^{2l-2}\,dy \\ &= \begin{cases} 0 & \text{if } l = 0, \\ 2l(n+2l-2)m_{2l-2}(\eta^1) & \text{if } l \geq 1. \end{cases} \end{aligned}$$

So for $k \in \mathbf{N}_0$ we obtain

$$(8) \qquad m_{2l}(\Delta^k\eta^1) = \begin{cases} 0 & \text{if } l = 0, \dots, k-1, \\ 4^k(l-k+1)_k(l-k+\frac{n}{2})_k\, m_{2l-2k}\,(\eta^1) & \text{if } l \geq k. \end{cases}$$

By (7) for $k \in \mathbf{N}_0$ we get

$$\begin{aligned} \Delta^k\big(\eta^\varepsilon * u\big)(x) &= \sum_{l=0}^{\infty} v_{2l+2k}(x)\cdot\varepsilon^{2l}m_{2l+2k}(\Delta^k\eta^1) \\ &= \sum_{l=0}^{\infty} \big(\tfrac{2l+2k}{n} + 1\big)u_{2l+2k}(x)\cdot\varepsilon^{2l}4^k(l+1)_k(l+\tfrac{n}{2})_k\, m_{2l}\,(\eta^1). \end{aligned}$$

Taking the limit as $\varepsilon \to 0$ we get

$$\Delta^k u_0(x) = \lim_{\varepsilon\to 0}\Delta^k(\eta^\varepsilon * u)(x) = \big(\tfrac{2k}{n} + 1\big)u_{2k}(x)\cdot 4^k k!(\tfrac{n}{2})_k\, m_0\,(\eta^1).$$

Since $u_0 = u$ and $m_0\,(\eta^1) = 1$ we conclude that $4^k k!\big(\frac{n}{2} + 1\big)_k u_{2k} = \Delta^k u$ for $k \in \mathbf{N}_0$. Hence

$$M(u;\, x, R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k\big(\frac{n}{2} + 1\big)_k k!}R^{2k}$$

locally uniformly in $\{(x, R)\colon x \in \Omega, |R| < \epsilon(x)\}$ and Theorem 2 implies analyticity of u in Ω. □

Remark 1. Analogues of Lemma 1 and Theorem 3 hold true if one only assumes that $u, u_l, v_{2l} \in L^1_{\text{loc}}(\Omega)$ for $l \in \mathbf{N}_0$. Also the condition of continuity of ϵ can be replaced by its uniform positivity on compact subsets of Ω.

4. Analytic functions on metric measure spaces

It is well known that the mean value characterization of harmonic functions can be used to define harmonic functions on metric measure spaces (MMS), see [3, Definition 2.1]. If the measure is continuous with respect to the metric, then harmonic functions on MMS satisfy the maximum principle, the Harnack type inequality and the Weierstrass and Montel convergence theorem, see [3]. On the other hand Alabern, Mateu and Verdera obtained in [1] a characterization of Sobolev spaces on $\mathbf{R}^n$ only in terms of the Euclidean metric and the Lebesgue measure which allowed them to define higher order Sobolev spaces on MMS.

Here we propose a definition of analytic functions on MMS.

Definition 1. Let (X, ρ, μ) be a metric measure space with a metric ρ and a regular Borel measure μ which is positive on open sets and finite on bounded sets. Let Ω be an open subset of X. For any $x \in \Omega$ and $0 < R < \operatorname{dist}(x, \partial\Omega)$ define the *solid mean* of a continuous, complex valued function $u \in C^0(\Omega)$ by

$$M_X(u;\, x, R) = \frac{1}{\mu(B_\rho(x, R))} \int_{B_\rho(x,R)} u(y)\, d\mu(y),$$

where $B_\rho(x, R)$ is a ball with respect to the metric ρ of center at x and radius R. We also put $M_X(u;\, x, 0) := \lim_{R\to 0} M_X(u;\, x, R) = u(x)$.

Definition 2. We say that a function $u \in C^0(\Omega)$ is (X, ρ, μ)-*analytic* on Ω if there exist functions $u_k \in C^0(\Omega)$ for $k \in \mathbf{N}_0$ and $\epsilon \in C^0(\Omega, \mathbf{R}_+)$ such that

$$M_X(u;\, x, R) = \sum_{k=0}^{\infty} u_k(x) R^k$$

locally uniformly in $\{(x, R)\colon x \in \Omega, 0 \le R < \epsilon(x)\}$.

Since an (X, ρ, μ)-analytic function u on Ω uniquely determines $M_X(u;\, x, R)$ and vice versa the topology on the space $\mathcal{A}_X(\Omega, \rho, \mu)$ of (X, ρ, μ)-analytic functions on Ω can be defined by

$$\mathcal{A}_X(\Omega, \rho, \mu) = \operatorname{projlim}_{K \Subset \Omega} \operatorname{indlim}_{\epsilon > 0} \mathcal{E}(K, \epsilon),$$

where

$$\mathcal{E}(K, \epsilon) = \left\{ F \in C^0(K; C^\infty(-\epsilon, \epsilon)) : \|F\|_{K,\epsilon} = \sup_{k \in \mathbf{N}_0, x \in K} \frac{\left| \frac{\partial^k}{\partial R^k} F(x, R)_{|R=0} \right| \epsilon^k}{k!} < \infty \right\}.$$

By Theorem 3 and the Pringsheim type theorem [4, Theorem] we get

Corollary 1. *Let $X = \mathbf{R}^n$ with the Euclidean metric ρ and the Lebesgue measure λ. Let $\Omega \subset X$. Then $\mathcal{A}_X(\Omega, \rho, \lambda) = \mathcal{A}(\Omega)$. In other words a function u continuous on Ω is (X, ρ, λ)-analytic on Ω if, and only if, it is real analytic on Ω.*

The following definition gives an important class of MMS for which metric-measure theoretic properties of analytic functions are the same as the ones of real analytic functions on $\mathbf{R}^n$.

Definition 3. The metric measure space (X, ρ, μ) is called *analytizable* if for any $x \in X$ there exist open sets $U \subset \mathbf{R}^n$, $x \in \Omega \subset X$ and a homeomorphism $\Phi\colon U \to \Omega$ such that $\Phi\big(B(\Phi^{-1}(y), R)\big) = B_\rho(y, R)$ for $y \in \Omega$ and R small enough and $\mu(A) = |\Phi^{-1}(A)|$ for Borel sets $A \subset \Omega$.

Theorem 4. *Under the notations of Definition 3 let $u\colon \Omega \to \mathbf{C}$ be a continuous function. Then u is (X, ρ, μ)-analytic on Ω if, and only if, $u \circ \Phi$ is real analytic on $\Phi^{-1}(\Omega)$.*

Proof. For $x \in \Omega$ and R small enough we have

$$\begin{aligned} M_X(u;\, x, R) &= \frac{1}{\mu(B_\rho(x,R))} \int_{B_\rho(x,R)} u(y)\, d\mu(y) \\ &= \frac{1}{|\Phi^{-1}(B_\rho(x,R))|} \int_{\Phi^{-1}(B_\rho(x,R))} u(\Phi(z))\, dz \\ &= \frac{1}{|B(\Phi^{-1}(x),R)|} \int_{B(\Phi^{-1}(x),R)} u(\Phi(z))\, dz \\ &= M(u \circ \Phi;\, \Phi^{-1}(x), R), \end{aligned}$$

which proves the theorem. □

5. L_p-mean value expansions

The following theorem establishes the mean value expansion of a real analytic function with respect to the L_p metric on $\mathbf{R}^n$ and the Lebesgue measure λ.

Theorem 5. (Mean value property) *Let $1 \le p \le \infty$, $X = (\mathbf{R}^n, \|\cdot\|_p, \lambda)$, $\Omega \subset \mathbf{R}^n$ and $u \in \mathcal{A}(\Omega)$. Then for $x \in \Omega$ and R small enough we have*

$$M_X(u;\, x, R) = \sum_{k=0}^{\infty} D(n,p,k) \sum_{\kappa \in \mathbf{N}_0^n, |\kappa|=k} D(p,\kappa) \frac{\partial^{2k} u(x)}{\partial x^{2\kappa}} R^{2k} \tag{9}$$

where

$$D(n,p,k) = \frac{\Gamma\big(\frac{n}{p}+1\big)}{\Gamma^n\big(\frac{1}{p}+1\big)\Gamma\big(\frac{n+2k}{p}+1\big)}, \quad D(p,\kappa) = \frac{\Gamma\big(\frac{2\kappa_1+1+p}{p}\big)\cdots\Gamma\big(\frac{2\kappa_n+1+p}{p}\big)}{(2\kappa_1+1)!\cdots(2\kappa_n+1)!}. \tag{10}$$

The expansion (9) *is locally uniform with respect to $x \in \Omega$.*

Proof. Expending u into Taylor series we compute

$$\begin{aligned} M_X(u;x,R) &= \frac{1}{|B_p(x,R)|} \int_{B_p(x,R)} u(y)\, dy \\ &= \frac{1}{|B_p(x,R)|} \int_{B_p(x,R)} \sum_{\ell \in \mathbf{N}_0^n} \frac{1}{\ell!} \frac{\partial^l u(x)}{\partial x^\ell} (y-x)^\ell\, dy \\ &= \frac{1}{|B_p(0,1)|} \int_{B_p(0,1)} \sum_{\ell \in \mathbf{N}_0^n} \frac{1}{\ell!} \frac{\partial^l u(x)}{\partial x^\ell} (Rz)^\ell\, dz. \end{aligned}$$

Note that if at least one of the exponents $\ell_1, \ldots, \ell_n$ is odd, then the integral of $z^\ell = z_1^{\ell_1} \cdots z_n^{\ell_n}$ over $B_p(0,1)$ vanishes. Next using [2, formula 676, 5)] for $k \in \mathbf{N}_0$ and $\kappa \in \mathbf{N}_0^n$ with $|\kappa| = k$ we have

$$\begin{aligned} \int_{B_p(0,1)} z^{2\kappa}\, dz &= \frac{2^n}{p^n} \frac{\Gamma\big(\frac{2\kappa_1+1}{p}\big)\cdots\Gamma\big(\frac{2\kappa_n+1}{p}\big)}{\Gamma\big(\frac{2k+n}{p}+1\big)} \\ &= \frac{2^n}{\Gamma\big(\frac{2k+n}{p}+1\big)} \frac{\Gamma\big(\frac{2\kappa_1+1}{p}+1\big)\cdots\Gamma\big(\frac{2\kappa_n+1}{p}+1\big)}{(2\kappa_1+1)\cdots(2\kappa_n+1)}. \end{aligned}$$

In particular,

$$|B_p(0,1)| = \frac{2^n}{p^n} \frac{\Gamma\big(\frac{1}{p}\big)\cdots\Gamma\big(\frac{1}{p}\big)}{\Gamma\big(\frac{n}{p}+1\big)} = 2^n \frac{\Gamma^n\big(\frac{1}{p}+1\big)}{\Gamma\big(\frac{n}{p}+1\big)}.$$

Hence

$$M_X(u;x,R) = \frac{1}{|B_p(0,1)|}\sum_{\kappa\in\mathbf{N}_0^n}\frac{1}{(2\kappa)!}\frac{\partial^{2k}u(x)}{\partial x^{2\kappa}}R^{2k}\int_{B_p(0,1)} z^{2\kappa}\,dz$$
$$= \sum_{k=0}^{\infty}\frac{\Gamma\big(\frac{n}{p}+1\big)}{\Gamma^n\big(\frac{1}{p}\big)\Gamma\big(\frac{n+2k}{p}+1\big)}\sum_{|\kappa|=k}\frac{\Gamma\big(\frac{2\kappa_1+1}{p}\big)\cdots\Gamma\big(\frac{2\kappa_n+1}{p}\big)}{(2\kappa)!}\frac{\partial^{2k}u(x)}{\partial x^{2\kappa}}R^{2k},$$

which implies (9).

To prove the uniform convergence of the series (9) on a compact subset $K \Subset \Omega$ note that by the Cauchy inequalities we can find $(R_1,\ldots,R_n)\in\mathbf{R}_+^n$ such that for any $x\in K$ and any $\ell=(l_1,\ldots,l_n)\in\mathbf{N}_0^n$,

$$\Big|\frac{\partial^{|\ell|}u(x)}{\partial x^{\ell}}\Big| \le \frac{C}{R_1^{l_1}\cdots R_n^{l_n}}\; l_1!\cdots l_n!.$$

Since for any $\varepsilon>0$ one can find C_ε such that

$$\left(\frac{s}{e}\right)^s \le \Gamma(s+1) \le C_\varepsilon\left(\frac{(1+\varepsilon)s}{e}\right)^s \quad \text{for } s\ge 0 \tag{11}$$

we derive that the series (9) converges for $|R|<\min(R_1,\ldots,R_n)$. □

Theorem 6. (Converse to the mean-value property) *Let* $1\le p\le\infty$, $X=(\mathbf{R}^n,\|\cdot\|_p,\lambda)$, $\Omega\subset\mathbf{R}^n$, $\epsilon\in C^0(\Omega,\mathbf{R}_+)$ *and* $u\in C^\infty(\Omega)$. *If the series*

$$\widetilde{M}_X(x,R) = \sum_{k=0}^{\infty}D(n,p,k)\sum_{\kappa\in\mathbf{N}_0^n,|\kappa|=k}D(p,\kappa)\frac{\partial^{2k}u(x)}{\partial x^{2\kappa}}R^{2k},$$

where $D(n,p,k)$ *and* $D(n,\kappa)$ *are given by* (10), *is locally uniformly convergent in* $\{(x,R)\colon x\in\Omega, |R|<\epsilon(x)\}$, *then* $u\in\mathcal{A}(\Omega)$ *and* $M_X(u;x,R)=\widetilde{M}_X(x,R)$ *for* $x\in\Omega$ *and* $0<R<\min\big(\epsilon(x),\operatorname{dist}(x,\partial\Omega)\big)$.

Proof. Fix a compact set $K\Subset\Omega$ and set $\epsilon=\inf_{x\in K}\epsilon(x)>0$. Then the assumption implies that

$$\frac{1}{(2k)!}\sum_{\kappa\in\mathbf{N}_0^n,|\kappa|=k}\binom{2k}{2\kappa}\frac{\Gamma\big(\frac{2\kappa_1+1}{p}+1\big)\cdots\Gamma\big(\frac{2\kappa_n+1}{p}+1\big)}{(2\kappa_1+1)\cdots(2\kappa_n+1)\Gamma\big(\frac{n+2k}{p}+1\big)}\frac{\partial^{2k}u(x)}{\partial x^{2\kappa}}R^{2k}\to 0$$

as $k\to\infty$ uniformly on $K\times\{|R|\le\epsilon_1\}$ with any $\epsilon_1<\epsilon$. Hence for any $\epsilon_1<\epsilon$ there exists a constant $C(\epsilon_1)<\infty$ such that for $k\in\mathbf{N}_0$,

$$\sup_{x\in K}\sum_{\kappa\in\mathbf{N}_0^n,|\kappa|=k}\binom{2k}{2\kappa}\frac{\Gamma\big(\frac{2\kappa_1+1}{p}+1\big)\cdots\Gamma\big(\frac{2\kappa_n+1}{p}+1\big)}{\Gamma\big(\frac{n+2k}{p}+1\big)}\frac{\partial^{2k}u(x)}{\partial x^{2\kappa}}\le\frac{C(\epsilon_1)}{\epsilon_1^{2k}}(2k)!.$$

Since by (11) the coefficient of $\frac{\partial^{2k}u(x)}{\partial x^{2\kappa}}$ in the above expression is not less then $\delta^k\binom{k}{\kappa}$ with some $\delta>0$ we conclude that for any compact set $K\Subset\Omega$ one can find $C<\infty$ and $L<\infty$ such that

$$\sup_{x\in K}|\Delta^k u(x)|\le CL^{2k}(2k)! \quad\text{for}\quad k\in\mathbf{N}_0.$$

But by [4, Theorem] this inequality implies that $u\in\mathcal{A}(\Omega)$. Finally, by Theorem 5 we get $\widetilde{M}_X(x,R)=M_X(u;x,R)$. □

Finally let us remark that Theorems 5 and 6 remain valid also for $0 < p < 1$. In fact Definitions 1 and 2 make sense if ρ is only a (non necessary symmetric) pseudometric.

Added in proof. The uniqueness property for the analytic functions on MMS will be studied in a subsequent paper.

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