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A NOTE ON A MULTILINEAR LOCAL *Tb* THEOREM FOR CALDERÓN–ZYGMUND OPERATORS

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Abstract. In this short note, we extend a local Tb theorem that was proved in [4] to a full multilinear local Tb theorem.

1. Introduction

In [4], we proved (in collaboration with Grau de la Herran) a multilinear local Tb theorem for square functions, and applied it to prove a local Tb theorem for singular integrals. The local Tb theorem for singular integrals had "local Tb type" testing conditions for the operator T on pseudo-accretive collections $\mathbf{b} = \{b_Q^i\}_{1\le i\le m}$, but tested the adjoints of T on the constant function 1; see [4] for more details on this.

There has been interest recently in multilinear local Tb theorems for square function and singular integrals, for example in the works [6], [1], [5] and [7]. In particular, in [7], the authors are interested in multilinear local Tb theorems that test all adjoints of a multilinear operator T on pseudo-accretive systems, rather than only on the operator itself; two examples are in [7] and the authors cite these as a feature of their multilinear local Tb result for a class of n-linear forms known as perfect Calderón–Zygmund operators. In this note, we show that our result from [4] can be easily extended to a local Tb theorem for Calderón–Zygmund operators where T and all of its adjoints are tested on pseudo-accretive systems.

The following local Tb theorem for multilinear singular integral operators, which is an extension of Theorem 1.2 from [4], is the main result of the article.

Theorem 1.1. *Let T be a continuous m linear operator from $\mathscr{S}\times\cdots\times\mathscr{S}$ into $\mathscr{S}'$ with a standard Calderón–Zygmund kernel K. Suppose that $T\in WBP$ and for each $j=0,1,\dots,m$ there exist $2\le q<\infty$ and $1<q_{i,j}<\infty$ with $\frac{1}{q_j}=\sum_{i=1}^m\frac{1}{q_{i,j}}$ and an m-compatible collection of pseudo-accretive systems $\mathbf{b}_j=\{b_Q^{i,j}\}_{1\le i\le m}$ indexed by dyadic cubes Q such that*

$$(1)\qquad \int_Q\left(\int_0^{\ell(Q)}|Q_tT^{*j}(P_tb_Q^{1,j},\dots,P_tb_Q^{m,j})(x)|^2\frac{dt}{t}\right)^{\frac{q_j}{2}}dx\lesssim|Q|.$$

Then T is bounded from $L^{p_1}\times\cdots\times L^{p_m}$ into L^p for all $1<p_i<\infty$ such that $\frac{1}{p}=\frac{1}{p_1}+\cdot+\frac{1}{p_m}$. Here we assume that this estimate holds for any approximation to

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the identity P_t with smooth compactly supported kernels and any Littlewood–Paley–Stein projection operators Q_t whose kernels also are smooth compactly supported function.

The precise meaning of (1) is the following: For any $\varphi, \psi \in C_0^\infty$ such that $\widehat{\varphi}(0) = 1$ and $\widehat{\psi}(0) = 0$, (1) holds for $P_t f = \varphi_t * f$, $Q_t f = \psi_t * f$, $\varphi_t(x) = \frac{1}{t^n}\varphi(\frac{x}{t})$, and $\psi_t(x) = \frac{1}{t^n}\psi(\frac{x}{t})$, where the constant is independent of the dyadic cube Q, but may depend on φ and ψ.

We prove this theorem by applying the square function estimates that were also proved in [4], but with a few minor modifications to allow for the extension to Theorem 1.1. This note is intended to be an addendum to the article [4]. So the reader should refer to that article for definitions, discussion, and history related to these results.

2. A few definitions and results from [4]

Define the family of multilinear of operators $\{\Theta_t\}_{t>0}$

$$\Theta_t(f_1, \dots, f_m)(x) = \int_{\mathbf{R}^{mn}} \theta_t(x, y_1, \dots, y_m) \prod_{i=1}^m f_i(y_i)\, dy_i \tag{2}$$

where $\theta_t \colon \mathbf{R}^{(m+1)n} \to \mathbf{C}$ and the square function

$$S(f_1, \dots, f_m)(x) = \left(\int_0^\infty |\Theta_t(f_1, \dots, f_m)(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \tag{3}$$

associated to $\{\Theta_t\}_{t>0}$, where f_i for $i = 1, \dots, m$ are initially functions in $C_0^\infty(\mathbf{R}^n)$. Also assume that θ_t satisfies for all $x, y_1, \dots, y_m, x', y_1', \dots, y_m' \in \mathbf{R}^n$

$$|\theta_t(x, y_1, \dots, y_m)| \lesssim \frac{t^{-mn}}{\prod_{i=1}^m (1 + t^{-1}|x - y_i|)^{N+\gamma}}, \tag{4}$$

$$|\theta_t(x, y_1, \dots, y_m) - \theta_t(x, y_1, \dots, y_i', \dots, y_m)| \lesssim t^{-mn}(t^{-1}|y_i - y_i'|)^\gamma, \tag{5}$$

$$|\theta_t(x, y_1, \dots, y_m) - \theta_t(x', y_1, \dots, y_m)| \lesssim t^{-mn}(t^{-1}|x - x'|)^\gamma \tag{6}$$

for some $N > n$ and $0 < \gamma \le 1$. The following results were proved in [4].

Theorem. *Let Θ_t and S be defined as in (2) and (3) where θ_t satisfies (4)–(6). Suppose there exist $q_i, q > 1$ for $i = 1, \dots, m$ with $\frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i}$ and functions $\mathbf{b} = \{b_Q^i\}_{1 \le i \le m}$ indexed by dyadic cubes $Q \subset \mathbf{R}^n$ such that for every dyadic cube Q*

$$\int_{\mathbf{R}^n} |b_Q^i|^{q_i} \le B_1 |Q|, \tag{7}$$

$$\frac{1}{B_2} \le \left| \frac{1}{|Q|} \int_Q \prod_{i=1}^m b_Q^i(x)\, dx \right|, \tag{8}$$

$$\left| \frac{1}{|R|} \int_R \prod_{i=1}^m b_Q^i(x)\, dx \right| \le B_3 \prod_{i=1}^m \left| \frac{1}{|R|} \int_R b_Q^i(x)\, dx \right| \tag{9}$$

for all dyadic subcubes $R \subset Q$,

$$\int_Q \left(\int_0^{\ell(Q)} |\Theta_t(b_Q^1, \dots, b_Q^m)(x)|^2 \frac{dt}{t} \right)^{\frac{q}{2}} dx \le B_4 |Q|. \tag{10}$$

Then S satisfies

$$\|S(f_1,\dots,f_m)\|_{L^p} \lesssim \prod_{i=1}^m \|f\|_{L^{p_i}} \tag{11}$$

for all $1<p_i<\infty$ and $2\le p<\infty$ satisfying $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$.

If $\{b_Q\}$ satisfies (7) and (8), we say that $\{b_Q\}$ is a pseudo-accretive system. We say that $\mathbf{b}=\{b_Q^i\}_{1\le i\le m}$ is an m-compatible, or just compatible, collection of pseudo-accretive systems if they satisfy (7)–(9).

In Theorem 1.1, we only impose that $\mathbf{b}_j=\{b_Q^{i,j}\}_{1\le i\le m}$ is an m-compatible pseudo-accretive system for each $j=0,1,\dots,m$. In particular, $\mathbf{b}_j$ must satisfy (7)–(9) for each $j=0,1,\dots,m$, but there is no dependence between pseudo-accretive systems $\mathbf{b}_j$ and $\mathbf{b}_k$ for $j\ne k$.

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Let P_t be a smooth approximation to identity operators with smooth compactly supported kernels. Then it follows that $P_t^2 f\to f$ as $t\to 0^+$ and $P_t^2 f\to 0$ as $t\to\infty$ in $\mathscr{S}$ for $f\in\mathscr{S}_0$. Here $\mathscr{S}_0$ is the subspace of Schwartz functions satisfying $|\widehat{f}(\xi)|\le C_M|\xi|^M$ for all $M\in\mathbf{N}$. There exist Littlewood–Paley–Stein projection operators $Q_t^{(i)}$ for $i=1,2$ with smooth compactly supported kernels such that $t\frac{d}{dt}P_t^2=Q_t^{(2)}Q_t^{(1)}=Q_t$. Using these operators, we decompose T for $f_i\in\mathscr{S}_0$, $i=0,\dots,m$

$$\begin{aligned}\langle T(f_1,\dots,f_m),f_0\rangle &= \int_0^\infty t\frac{d}{dt}\left\langle T(P_t^2f_1,\dots,P_t^2f_m),P_t^2f_0\right\rangle\frac{dt}{t}\\ &=\sum_{i=0}^m\left\langle\int_0^\infty Q_tT^{*i}(P_t^2f_1,\dots,P_tf_{i-1},P_tf_0,P_tf_{i+1}\dots,P_t^2f_m)\frac{dt}{t},f_i\right\rangle\\ &=\sum_{i=0}^m\langle T_i(f_1,\dots,f_{i-1},f_0,f_{i+1},\dots,f_m),f_i\rangle\,,\end{aligned}\tag{12}$$

where we take the last line in (12) as the definition of T_i for $i=0,1,\dots,m$. It follows that T_i is a multilinear singular integral operator with standard kernel

$$K_i(x,y_1,\dots,y_m)=\int_0^\infty\left\langle T^{*i}(\varphi_t^{y_1},\dots,\varphi_t^{y_m}),\psi_t^x\right\rangle\frac{dt}{t},$$

Also let Θ_t^i be the multilinear operator associated to

$$\theta_t^i(x,y_1,\dots,y_m)=Q_t^{(1)}T^{*i}(\varphi_t^{y_1},\dots,\varphi_t^{y_m})(x),$$

and let S_i be the square function associated to Θ_t^i. Note that $\theta_t^i(x,y_1,\dots,y_m)\ne\langle T^{*i}(\varphi_t^{y_1},\dots,\varphi_t^{y_m}),\psi_t^x\rangle$, and that T_i is not actually the integral of Θ_t^i (one has Q_t and the other has $Q_t^{(1)}$). Furthermore, by the hypotheses on T^{*i} and by the local Tb theorem for square functions from [4], it follows that S_i is bounded from $L^{2m}\times\cdots\times$

L^{2m} into L^2. Therefore we have

$$\begin{aligned}\langle T_i(f_1,\dots,f_m),f_0\rangle &= \int_0^\infty \int_{\mathbf{R}^n} Q_t T^{*i}(P_t^2 f_1,\dots,P_t^2 f_m)(x) f_0(x)dx\frac{dt}{t}\\ &= \int_0^\infty \int_{\mathbf{R}^n} Q_t^{(1)} T^{*i}(P_t^2 f_1,\dots,P_t^2 f_m)(x) Q_t^{(2)\,*} f_0(x)dx\frac{dt}{t}\\ &\le \|S_i(f_1,\dots,f_m)\|_{L^2}\left\|\left(\int_0^\infty |Q_t^{(2)\,*}f_0|^2\frac{dt}{t}\right)^{\frac12}\right\|_{L^2}\\ &\lesssim \|f_1\|_{L^{2m}}\cdots\|f_m\|_{L^{2m}}\|f_0\|_{L^2}.\end{aligned}$$

Hence T_i is bounded from $L^{2m}\times\cdots\times L^{2m}$ into L^2, and by the multilinear Calderón–Zygmund theory developed by Grafakos and Torres in [2, 3], it follows that T_i also bounded from $L^{p_1}\times\cdots\times L^{p_m}$ into L^p for all $1<p_1,\dots,p_m<\infty$ with $\frac1p=\frac1{p_1}+\cdots+\frac1{p_m}$ for each $i=0,1,\dots,m$. In particular, it follows that T_i is bounded from $L^{m+1}\times\cdots\times L^{m+1}$ into $L^{\frac{m+1}{m}}$ for each $i=0,1,\dots,m$. Then continuing from (12), we have

$$\begin{aligned}|\langle T(f_1,\dots,f_m),f_0\rangle| &\le \sum_{i=0}^m |\langle T_i(f_1,\dots,f_{i-1},f_0,f_{i+1},\dots,f_m),f_i\rangle|\\ &\le \sum_{i=0}^m \|T_i(f_1,\dots,f_{i-1},f_0,f_{i+1},\dots,f_m)\|_{L^{\frac{m+1}{m}}}\|f_i\|_{L^{m+1}}\\ &\lesssim \prod_{j=0}^m \|f_j\|_{L^{m+1}}.\end{aligned}$$

Therefore T is bounded from $L^{m+1}\times\cdots L^{m+1}$ into $L^{\frac{m+1}{m}}$. Then it again follows from the multilinear Calderón–Zygmund theory in [2, 3] that T is bounded from $L^{p_1}\times\cdots\times L^{p_m}$ into L^p for all $1<p_1,\dots,p_m<\infty$ such that $\frac1p=\frac1{p_1}+\cdots+\frac1{p_m}$. □

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