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QUASICONFORMAL HARMONIC MAPPINGS WITH THE CONVEX HOLOMORPHIC PART

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Abstract. Let $F = H + \overline{G}$ be a locally injective and sense-preserving harmonic mapping of the unit disk $\mathbf{D}$ in the complex plane $\mathbf{C}$, where H and G are holomorphic in $\mathbf{D}$ and $G(0) = 0$. In this paper, under the assumption that H maps conformally $\mathbf{D}$ onto a convex domain we obtain modified forms of some results shown for a mapping F such that $F(\mathbf{D})$ is a convex domain in $\mathbf{C}$ in the references [11] and [12].

1. Introduction

Let Ω be a domain in $\mathbf{C}$. A mapping $F\colon \Omega \to \mathbf{C}$ is said to be *Lipschitz* (resp. *co-Lipschitz*) in Ω if there exists a constant $L > 0$ (resp. $C > 0$) such that the following inequality

$$|F(z_1) - F(z_2)| \le L|z_1 - z_2| \quad (\text{resp.}\quad C|z_1 - z_2| \le |F(z_1) - F(z_2)|)$$

holds for all $z_1, z_2 \in \Omega$. We call F bi-Lipschitz in Ω if F is both Lipschitz and co-Lipschitz in Ω. If F is a differentiable mapping at a point $z \in \Omega$, then the differential $\mathrm{d}_z F$ and the Jacobian $\mathrm{J}[F](z)$ of F at z satisfy respectively

$$\mathrm{d}_z F(h) = \partial F(z)h + \bar\partial F(z)\overline{h}\ , \quad h \in \mathbf{C}, \tag{1.1}$$

and

$$\mathrm{J}[F](z) = |\partial F(z)|^2 - |\bar\partial F(z)|^2,$$

where $\partial := \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$ and $\bar\partial := \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$ are the so-called *formal derivatives operators*. Given $a \in \mathbf{C}$ and $r \in \mathbf{R}$ we write $\mathbf{D}(a, r) := \{z \in \mathbf{C}\colon |z - a| < r\}$. Let $\mathbf{D} := \mathbf{D}(0, 1)$ and $\mathbf{T} := \{z \in \mathbf{C}\colon |z| = 1\}$ stand for the unit disk and the unit circle, respectively. From (1.1) it follows that for every $z \in \Omega$ where F is differentiable,

$$\mathrm{d}_z^+ F := \max_{h\in\mathbf{T}} |\,\mathrm{d}_z F(h)| = |\partial F(z)| + |\bar\partial F(z)|$$

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and

$$\mathrm{d}_z^- F := \min_{h\in\mathbf{T}} |\,\mathrm{d}_z F(h)| = \big||\partial F(z)| - |\bar\partial F(z)|\big|\,.$$

A twice continuously differentiable mapping $F\colon \Omega \to \mathbf{C}$ is said to be a harmonic mapping if it satisfies the Laplace equation $\bar\partial\partial F = 0$ in Ω. Lewy's Theorem [7] tells us that a harmonic mapping F in Ω is locally injective in Ω if and only if its Jacobian $\mathrm{J}[F](z) \neq 0$ for every $z \in \Omega$. A harmonic mapping F in Ω is said to be *sense-preserving* if $\mathrm{J}[F](z) > 0$ for every $z \in \Omega$. Suppose that F is a sense-preserving diffeomorphic mapping in Ω. Then for every $K \geq 1$, F is a K-quasiconformal mapping in Ω if and only if

$$\mathrm{K}(F) := \sup_{z\in\Omega} \frac{\mathrm{d}_z^+ F}{\mathrm{d}_z^- F} = \sup_{z\in\Omega} \frac{|\partial F(z)| + |\bar\partial F(z)|}{|\partial F(z)| - |\bar\partial F(z)|} \leq K.$$

Evidently, a sense-preserving diffeomorphic bi-Lipschitz mapping is always quasiconformal, while the converse is not true, in general.

Harmonic quasiconformal mappings are natural generalizations of conformal ones. Recently, many mathematicians have studied such an active topic and obtained many interesting results; cf., e.g., [1, 3, 6, 11, 12, 13, 14, 15]. It is known that every harmonic mapping F defined in $\mathbf{D}$ admits the *canonical representation* $F = H + \overline{G}$, where H and G are holomorphic in $\mathbf{D}$ and are uniquely determined by the condition $G(0) = 0$. In this case we call H (resp. $\overline{G}$) the *holomorphic part* (resp. the *anti-holomorphic part*) of F. One can refer to [3] and the references therein for more details about harmonic mappings.

Given a function $f\colon \mathbf{T} \to \mathbf{C}$ and $z = \mathrm{e}^{\mathrm{i}\theta} \in \mathbf{T}$ we define

$$f'(z) := \lim_{u\to z} \frac{f(u) - f(z)}{u - z}, \tag{1.2}$$

$$\dot f(z) := \lim_{t\to\theta} \frac{f(\mathrm{e}^{\mathrm{i}t}) - f(\mathrm{e}^{\mathrm{i}\theta})}{t - \theta}, \tag{1.3}$$

provided the limits exist as well as $f'(z) := 0$ and $\dot f(z) := 0$ otherwise. Obviously,

$$\dot f(z) = \mathrm{i}z f'(z) \quad\text{and}\quad |\dot f(z)| = |f'(z)|. \tag{1.4}$$

We recall that the *harmonic conjugate operator* A is defined for a function $f\colon \mathbf{T} \to \mathbf{C}$ integrable on $\mathbf{T}$ and $z \in \mathbf{T}$ as follows:

$$\mathrm{A}[f](z) := \frac{1}{2\pi} \lim_{r\to 1^-} \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t}) \,\mathrm{Im}\, \frac{\mathrm{e}^{\mathrm{i}t} + rz}{\mathrm{e}^{\mathrm{i}t} - rz}\, \mathrm{d}t, \tag{1.5}$$

whenever the limit exists and $\mathrm{A}[f](z) := 0$ otherwise. It is known that for a.e. $z \in \mathbf{T}$ the limit exists; cf. [4, Chap. III, Lemma 1.1]. For any Lebesgue measurable function $f\colon \mathbf{T} \to \mathbf{C}$ we set

$$\|f\|_\infty := \operatorname*{ess\,sup}_{z\in\mathbf{T}} |f(z)|.$$

Let $L^\infty(\mathbf{T})$ denote the class of all such functions f with $\|f\|_\infty < +\infty$.

In 2014, the first and second authors of this article improved Kalaj's result [6, Theorem 3.2] and obtained the following theorem; see [12, Theorem 3.4].

Theorem A. *Let $F\colon \mathbf{D} \to \mathbf{C}$ be a sense-preserving injective harmonic mapping such that $F(\mathbf{D})$ is a bounded convex domain in $\mathbf{C}$. Then the following conditions are equivalent to each other:*

(i) *F is a quasiconformal and Lipschitz mapping;*

(ii) *F is a quasiconformal mapping and its boundary limiting valued function f is a Lipschitz mapping;*

(iii) *F is a quasiconformal mapping and the holomorphic part H of F is a bi-Lipschitz mapping;*

(iv) *F is a bi-Lipschitz mapping;*

(v) *F has a continuous extension to the closure $\mathrm{cl}(\mathbf{D})$ of $\mathbf{D}$ and its boundary limiting valued function f is absolutely continuous and satisfies the following condition*

$$0 < d_f, \quad \|\dot{f}\|_\infty < +\infty \quad \textit{and} \quad \|\,\mathrm{A}[\dot{f}]\|_\infty < +\infty,$$

where

$$d_f := \operatorname*{ess\,inf}_{z\in\mathbf{T}} |\dot{f}(z)|.$$

For a sense-preserving harmonic mapping $F = H + \overline{G}$ in $\mathbf{D}$, let

$$\mu_F(z) := \frac{G'(z)}{H'(z)}, \quad z \in \mathbf{D},$$

be the second complex dilatation of F. Then μ_F is a holomorphic mapping of $\mathbf{D}$ and

$$\|\mu_F\|_\infty := \operatorname*{ess\,sup}_{z\in\mathbf{D}} |\mu_F(z)| = \sup_{z\in\mathbf{D}} |\mu_F(z)| \le 1.$$

In 2012 the first and second authors of this article proved the following theorem; see [11, Theorem 3.8].

Theorem B. *Suppose that F is a sense-preserving injective harmonic mapping of $\mathbf{D}$ and that $F(\mathbf{D})$ is a convex domain. Let $F = H + \overline{G}$ be the canonical representation of F. Then the following five conditions are equivalent to each other:*

(i) *F is a quasiconformal mapping;*

(ii) *there exists a constant L_1 such that $1 \le L_1 < 2$ and*

$$|F(z_2) - F(z_1)| \le L_1|H(z_2) - H(z_1)|, \quad z_1, z_2 \in \mathbf{D}; \tag{1.6}$$

(iii) *there exists a constant l_1 such that $0 \le l_1 < 1$ and*

$$|G(z_2) - G(z_1)| \le l_1|H(z_2) - H(z_1)|, \quad z_1, z_2 \in \mathbf{D}; \tag{1.7}$$

(iv) *there exists a constant $L_2 \ge 1$ such that*

$$|H(z_2) - H(z_1)| \le L_2|F(z_2) - F(z_1)|, \quad z_1, z_2 \in \mathbf{D}; \tag{1.8}$$

(v) *$H \circ F^{-1}$ and $F \circ H^{-1}$ are bi-Lipschitz mappings.*

Moreover, the following implications hold: (1.6) $\Rightarrow$ $\|\mu_F\|_\infty \le L_1 - 1$, (1.7) $\Rightarrow$ $\|\mu_F\|_\infty \le l_1$ *and* (1.8) $\Rightarrow$ $\|\mu_F\|_\infty \le 1 - \frac{1}{L_2}$.

In this paper we consider the case where F is a sense-preserving harmonic mapping of $\mathbf{D}$ such that the holomorphic part H of F is convex, that is, H is injective and $H(\mathbf{D})$ is a convex domain. Then we show Theorem 4.2 and Theorem 3.3, which give modified forms of Theorem A and Theorem B, respectively, under the assumption that H is convex, instead of the assumption that $F(\mathbf{D})$ is a convex domain. This is done in Sections 4 and 3, respectively, where also some quite essential relevant results are presented. In Section 2 we study the mappings of the form (2.1), which are very useful in the context of harmonic mappings with the injective holomorphic part; cf. Remark 2.1. As a result we infer Theorem 2.8 and Corollary 2.9, much relative to [11, Theorems 3.2 and 3.4]. It is worth pointing out that some results are proved in a more general case, where the convexity property is replaced by the rectifiably arcwise connectivity one; cf. [8]. For example Corollary 3.2 gives a variant of the classical result by Clunie and Sheil-Small; cf. [1, Corollary 5.8]. In Section 5 we give examples which illustrate our considerations.

2. Anti-holomorphic distortion of the identity mapping

Given a nonempty set $\Omega \subset \mathbf{C}$ and a mapping $\phi\colon \Omega \to \mathbf{C}$ we define the following mapping

$$\Omega \ni z \mapsto \mathrm{I}[\phi](z) := z + \overline{\phi(z)}. \tag{2.1}$$

Remark 2.1. The operator I is useful while studying harmonic mappings $F = H + \overline{G}$ in $\mathbf{D}$ in the case where the holomorphic part H of F is injective. Then the obvious decomposition holds

$$F(z) = \mathrm{I}[G \circ H^{-1}] \circ H(z), \quad z \in \mathbf{D}. \tag{2.2}$$

The mapping $\mathrm{I}[G \circ H^{-1}]$ is harmonic in $H(\mathbf{D})$. Its holomorphic part is the identity mapping and the anti-holomorphic one coincides with $\overline{G \circ H^{-1}}$.

By $\mathrm{L}^+(\phi)$ and $\mathrm{L}^-(\phi)$ we denote the *Lipschitz constant of* ϕ and the *co-Lipschitz constant of* ϕ, respectively, i.e.,

$$\mathrm{L}^+(\phi) := \sup\left\{\Big|\frac{\phi(z)-\phi(w)}{z-w}\Big| : z, w \in \Omega,\ z \neq w\right\}$$

$$\mathrm{L}^-(\phi) := \inf\left\{\Big|\frac{\phi(z)-\phi(w)}{z-w}\Big| : z, w \in \Omega,\ z \neq w\right\}.$$

Note that ϕ is a Lipschitz (resp. co-Lipschitz) mapping if and only if $\mathrm{L}^+(\phi) < +\infty$ (resp. $\mathrm{L}^-(\phi) > 0$), and so ϕ is a bi-Lipschitz mapping if and only if $\mathrm{L}^+(\phi) < +\infty$ and $\mathrm{L}^-(\phi) > 0$.

Lemma 2.2. *If ϕ is a Lipschitz mapping in Ω, then $\mathrm{I}[\phi]$ is Lipschitz with*

$$\mathrm{L}^+(\mathrm{I}[\phi]) \le 1 + \mathrm{L}^+(\phi). \tag{2.3}$$

If additionally $\mathrm{L}^+(\phi) < 1$, then $\mathrm{I}[\phi]$ is co-Lipschitz with

$$\mathrm{L}^-(\mathrm{I}[\phi]) \ge 1 - \mathrm{L}^+(\phi); \tag{2.4}$$

in particular, $\mathrm{I}[\phi]$ is injective in Ω.

Proof. For any Lipschitz mapping $\phi\colon \Omega \to \mathbf{C}$ and $z, w \in \Omega$ we see by (2.1) that

$$|\,\mathrm{I}[\phi](z) - \mathrm{I}[\phi](w)| \le |z-w| + |\overline{\phi(z)} - \overline{\phi(w)}| \le (1 + \mathrm{L}^+(\phi))|z-w|,$$

which yields (2.3). If additionally $\mathrm{L}^+(\phi) < 1$, then

$$|\,\mathrm{I}[\phi](z) - \mathrm{I}[\phi](w)| \ge |z-w| - |\overline{\phi(z)} - \overline{\phi(w)}| \ge (1 - \mathrm{L}^+(\phi))|z-w|.$$

Hence the inequality (2.4) holds and $|\,\mathrm{I}[\phi](z) - \mathrm{I}[\phi](w)| > 0$ provided $z \neq w$, and so $\mathrm{I}[\phi]$ is injective in Ω. $\square$

For a C^1-mapping ϕ in a domain $\Omega \subset \mathbf{C}$ we define

$$\mathrm{D}^+(\phi) := \sup_{z\in\Omega} \mathrm{d}_z^+\phi = \sup_{z\in\Omega}\big(|\partial\phi(z)| + |\bar\partial\phi(z)|\big) \tag{2.5}$$

as well as

$$\mathrm{D}^-(\phi) := \inf_{z\in\Omega} \mathrm{d}_z^-\phi = \inf_{z\in\Omega}\big||\partial\phi(z)| - |\bar\partial\phi(z)|\big|. \tag{2.6}$$

Lemma 2.3. *If ϕ is a C^1-mapping in a domain $\Omega \subset \mathbf{C}$, then*

$$\mathrm{D}^+(\phi) \le \mathrm{L}^+(\phi) \quad \textit{and} \quad \mathrm{D}^-(\phi) \ge \mathrm{L}^-(\phi). \tag{2.7}$$

In particular, if ϕ is a holomorphic mapping in Ω, then

$$\mathrm{D}^+(\phi) \le \mathrm{L}^+(\mathrm{I}[\phi]) - 1 \tag{2.8}$$

as well as

$$\mathrm{D}^+(\phi) \le 1 \Rightarrow \mathrm{D}^+(\phi) \le 1 - \mathrm{L}^-(\mathrm{I}[\phi]). \tag{2.9}$$

Proof. Fix $z \in \Omega$. Then $\mathbf{D}(z, r_z) \subset \Omega$ for some $r_z > 0$ and

$$\phi(w) - \phi(z) = \partial\phi(z)(w-z) + \bar\partial\phi(z)\overline{(w-z)} + o(w-z), \quad w \in \mathbf{D}(z, r_z), \tag{2.10}$$

where $o\colon \mathbf{C} \to \mathbf{C}$ is a function such that $o(w)/w \to 0$ as $w \to 0$. Since $\partial\phi(z) = |\partial\phi(z)|\mathrm{e}^{\mathrm{i}\alpha}$ and $\bar\partial\phi(z) = |\bar\partial\phi(z)|\mathrm{e}^{\mathrm{i}\beta}$ for certain $\alpha, \beta \in \mathbf{R}$, we conclude from (2.10) that for all $r \in (0; r_z)$ and $\theta \in \mathbf{R}$,

$$\Bigl|\frac{\phi(z + r\mathrm{e}^{\mathrm{i}\theta}) - \phi(z)}{r}\Bigr| = \Bigl||\partial\phi(z)|\mathrm{e}^{\mathrm{i}(\alpha+\theta)} + |\bar\partial\phi(z)|\mathrm{e}^{\mathrm{i}(\beta-\theta)} + \frac{o(r\mathrm{e}^{\mathrm{i}\theta})}{r}\Bigr|.$$

Passage to the limit with $r \to 0^+$ leads to

$$\mathrm{L}^+(\phi) \ge |\partial\phi(z)| + |\bar\partial\phi(z)| = \mathrm{d}_z^+\phi \tag{2.11}$$

for $\theta := (\beta - \alpha)/2$, and to

$$\mathrm{L}^-(\phi) \le \bigl||\partial\phi(z)| - |\bar\partial\phi(z)|\bigr| = \mathrm{d}_z^-\phi \tag{2.12}$$

for $\theta := (\pi + \beta - \alpha)/2$. Therefore the inequalities (2.7) follow directly from the ones (2.11) and (2.12), respectively.

Assume now that ϕ is a holomorphic mapping in Ω. Then

$$\mathrm{d}_z^+\,\mathrm{I}[\phi] = 1 + |\phi'(z)| = 1 + \mathrm{d}_z^+\phi, \quad z \in \Omega.$$

Applying now the first inequality in (2.7) with ϕ replaced by $\mathrm{I}[\phi]$ we derive the inequality (2.8). If $\mathrm{D}^+(\phi) \le 1$, then

$$\mathrm{d}_z^-(\mathrm{I}[\phi]) = \bigl|1 - |\phi'(z)|\bigr| = 1 - |\phi'(z)| = 1 - \mathrm{d}_z^+\phi, \quad z \in \Omega.$$

Applying this time the second inequality in (2.7) with ϕ replaced by $\mathrm{I}[\phi]$ we obtain the implication (2.9), which completes the proof. □

According to [8], for any $M \ge 1$ a domain Ω in $\mathbf{C}$ is said to be *rectifiably M-arcwise connected* if for all $z, w \in \Omega$ there exists an arc γ joining the points z and w in Ω with the length $|\gamma|_1 \le M|w - z|$. Note that Ω is a convex domain if and only if Ω is a rectifiably 1-arcwise connected domain.

Lemma 2.4. *Given $M \ge 1$ let ϕ be a C^1-mapping in a rectifiably M-arcwise connected domain Ω. Then the following implications hold:*

(i) *If $\mathrm{D}^+(\phi) < +\infty$, then ϕ and $\mathrm{I}[\phi]$ are Lipschitz with*

$$\mathrm{L}^+(\phi) \le M\,\mathrm{D}^+(\phi) \quad \textit{and} \quad \mathrm{L}^+(\mathrm{I}[\phi]) \le 1 + M\,\mathrm{D}^+(\phi); \tag{2.13}$$

(ii) *If $\mathrm{D}^+(\phi) < 1/M$, then $\mathrm{I}[\phi]$ is co-Lipschitz with*

$$\mathrm{L}^-(\mathrm{I}[\phi]) \ge 1 - M\,\mathrm{D}^+(\phi). \tag{2.14}$$

Proof. Fix $M \geq 1$ and assume that Ω is a rectifiably M-arcwise connected domain and ϕ is a C^1-mapping in Ω. Fix arbitrarily chosen distinct points $z, w \in \Omega$. Then there exists a path $\gamma\colon [0;1] \to \Omega$ connecting the points $\gamma(0) = z$ and $\gamma(1) = w$ such that $|\gamma|_1 \leq M|w - z|$. Since γ is a uniformly continuous mapping, the image $\gamma([0;1])$ is a compact subset of the domain Ω, and consequently there exist $r > 0$ and $n \in \mathbf{N}$ such that

$$\gamma([t - 1/n; t + 1/n] \cap [0;1]) \subset \mathbf{D}(\gamma(t), r) \subset \Omega, \quad t \in [0;1]. \tag{2.15}$$

Write $\mathbf{Z}_{p,q} := \{k \in \mathbf{Z}\colon p \leq k \leq q\}$ for $p, q \in \mathbf{Z}$. Setting $\mathbf{Z}_{0,n} \ni k \mapsto t_k := k/n$ and

$$[0;1] \ni t \mapsto \gamma_k(t) := t\gamma(t_k) + (1 - t)\gamma(t_{k-1}), \quad k \in \mathbf{Z}_{1,n},$$

we conclude from (2.15) that for each $k \in \mathbf{Z}_{1,n}$, $\gamma_k([0;1]) \subset \Omega$. Therefore

$$\begin{aligned}
|\phi(w) - \phi(z)| &= \Big|\sum_{k=1}^{n}(\phi(\gamma(t_k)) - \phi(\gamma(t_{k-1}))\Big| \\
&\leq \sum_{k=1}^{n} |\phi(\gamma_k(1)) - \phi(\gamma_k(0))| = \sum_{k=1}^{n}\Big|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(\phi \circ \gamma_k)(t)\,\mathrm{d}t\Big| \\
&\leq \sum_{k=1}^{n} \int_0^1 |\partial\phi(\gamma_k(t))\gamma_k'(t) + \bar\partial\phi(\gamma_k(t))\overline{\gamma_k'(t)}|\,\mathrm{d}t \\
&\leq \sum_{k=1}^{n} \int_0^1 (|\partial\phi(\gamma_k(t))| + |\bar\partial\phi(\gamma_k(t))|)|\gamma_k'(t)|\,\mathrm{d}t \\
&= \sum_{k=1}^{n} \int_0^1 \mathrm{D}^+(\phi)(\gamma_k(t))|\gamma_k'(t)|\,\mathrm{d}t \leq \mathrm{D}^+(\phi)\sum_{k=1}^{n} \int_0^1 |\gamma_k'(t)|\,\mathrm{d}t.
\end{aligned}$$

Since

$$\sum_{k=1}^{n} \int_0^1 |\gamma_k'(t)|\,\mathrm{d}t = \sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})| \leq |\gamma|_1 \leq M|w - z|,$$

we see that

$$|\phi(w) - \phi(z)| \leq M\,\mathrm{D}^+(\phi)|w - z|.$$

Thus ϕ is Lipschitz with $\mathrm{L}^+(\phi) \leq M\,\mathrm{D}^+(\phi)$, and the implications (i) and (ii) follow directly from Lemma 2.2, which completes the proof. □

Lemma 2.5. *Given $M \geq 1$ let ϕ be a C^1-mapping in a domain Ω such that $\phi(\Omega)$ is a rectifiably M-arcwise connected domain. If ϕ is injective in Ω and $\mathrm{D}^-(\phi) > 0$, then ϕ is co-Lipschitz with*

$$\mathrm{L}^-(\phi) \geq \frac{\mathrm{D}^-(\phi)}{M}. \tag{2.16}$$

Proof. Under the assumptions of the lemma there exists the inverse mapping $f\colon \phi(\Omega) \to \Omega$ to ϕ and

$$\big||\partial\phi(z)| - |\bar\partial\phi(z)|\big| = \mathrm{d}_z^-\phi \geq \mathrm{D}^-(\phi) > 0, \quad z \in \Omega. \tag{2.17}$$

Hence

$$\begin{aligned}
|\,\mathrm{J}[\phi](z)| &= \big||\partial\phi(z)|^2 - |\bar\partial\phi(z)|^2\big| \\
&= \big(|\partial\phi(z)| + |\bar\partial\phi(z)|\big) \cdot \big||\partial\phi(z)| - |\bar\partial\phi(z)|\big| \\
&\geq \big||\partial\phi(z)| - |\bar\partial\phi(z)|\big|^2 \geq \mathrm{D}^-(\phi)^2 > 0, \quad z \in \Omega.
\end{aligned} \tag{2.18}$$

Since ϕ is a C^1-mapping in Ω, we conclude from (2.18) that f is differentiable in $\phi(\Omega)$ and

$$\partial f \circ \phi(z) = \frac{\overline{\partial\phi(z)}}{\mathrm{J}[\phi](z)} \quad \text{and} \quad \bar{\partial} f \circ \phi(z) = -\frac{\bar{\partial}\phi(z)}{\mathrm{J}[\phi](z)}, \quad z \in \Omega.$$

Combining this with (2.17) and (2.18) we see that for every $z \in \Omega$,

$$|\partial f \circ \phi(z)| + |\bar{\partial} f \circ \phi(z)| = \frac{|\partial\phi(z)| + |\bar{\partial}\phi(z)|}{|\,\mathrm{J}[\phi](z)|} = \frac{1}{\big||\partial\phi(z)| - |\bar{\partial}\phi(z)|\big|} = \frac{1}{\mathrm{d}_z^-\phi} \le \frac{1}{\mathrm{D}^-(\phi)}.$$

Therefore $\mathrm{D}^+(f) \le 1/\,\mathrm{D}^-(\phi) < +\infty$, and by Lemma 2.4, f is a Lipschitz mapping in $\phi(\Omega)$ with $\mathrm{L}^+(f) \le M\,\mathrm{D}^+(f) \le M/\,\mathrm{D}^-(\phi) < +\infty$. Since $\mathrm{L}^-(\phi) = 1/\,\mathrm{L}^+(f)$, we see that ϕ is co-Lipschitz and the inequality (2.16) holds, which proves the lemma. □

Remark 2.6. Note that if Ω is a convex domain, then from Lemma 2.3 and Lemma 2.4 we deduce that the equalities hold in place of the first inequality in (2.7), the inequality in (2.8) and the second inequality in (2.9). Suppose now that Ω is an arbitrary simply connected domain in $\mathbf{C}$. If ϕ is an injective C^1-mapping in Ω and $\phi(\Omega)$ is a convex domain, then combining the second inequality in (2.7) with the inequality (2.16) we obtain $\mathrm{D}^-(\phi) = \mathrm{L}^-(\phi)$, and so the equality holds in place of the second inequality in (2.7).

Corollary 2.7. *Given $M \ge 1$ and $R > 0$ let H be a conformal mapping in $\mathbf{D}$ such that $H(\mathbf{D})$ is a rectifiably M-arcwise connected domain and $\mathbf{D}(H(0), R) \subset H(\mathbf{D})$. Then H is co-Lipschitz with*

$$\mathrm{L}^-(H) \ge \frac{\mathrm{D}^-(H)}{M} \ge \frac{R}{4M}. \tag{2.19}$$

Proof. Under the assumption of the corollary we see that the mapping $\mathbf{D} \ni z \mapsto \tilde{H}(z) := H(z) - H(0)$ maps $\mathbf{D}$ onto a rectifiably M-arcwise connected domain and $\mathbf{D}(0, R) \subset \tilde{H}(\mathbf{D})$. Since $\tilde{H}(0) = 0$ we conclude from [10, Corollary 3.1] (see also [5, Theorem 2.5]) that

$$|H'(z)| = |\tilde{H}'(z)| \ge \frac{R}{4}, \quad z \in \mathbf{D}.$$

Therefore $\mathrm{D}^-(H) \ge R/4$, and so the second inequality in (2.19) holds. Lemma 2.5 now shows that H is a co-Lipschitz mapping and the first inequality in (2.19) holds, which is our claim. □

Let us consider the following deformations of a harmonic mapping $F = H + \overline{G}$ in $\mathbf{D}$,

$$\mathbf{D} \ni z \mapsto F_\varepsilon(z) := H(z) + \varepsilon\overline{G(z)}, \quad \varepsilon \in \mathbf{C}. \tag{2.20}$$

Using now the decomposition (2.2) we derive the following theorem.

Theorem 2.8. *Let $F = H + \overline{G}$ be a sense-preserving harmonic mapping in $\mathbf{D}$. Suppose that H is injective, $H(\mathbf{D})$ is a rectifiably M-arcwise connected domain with a given $M \ge 1$ and that F is not a conformal mapping. Then for every $\varepsilon \in \mathbf{D}(1/M\|\mu_F\|_\infty)$, F_ε is a quasiconformal harmonic and co-Lipschitz mapping.*

Proof. Fix $\varepsilon \in \mathbf{D}(1/M\|\mu_F\|_\infty)$. By setting $H(\mathbf{D}) \ni z \mapsto \phi(z) := \overline{\varepsilon} G \circ H^{-1}(z)$ we see that for every $z \in H(\mathbf{D})$,

$$|\phi'(z)| = \left|\overline{\varepsilon}\frac{G'(H^{-1}(z))}{H'(H^{-1}(z))}\right| = |\varepsilon||\mu_F(H^{-1}(z))| \le |\varepsilon|\|\mu_F\|_\infty. \tag{2.21}$$

Hence $M\,\mathrm{D}^+(\phi) \le M|\varepsilon|\|\mu_F\|_\infty < 1$. From Lemma 2.4 it follows that $\mathrm{I}[\phi]$ is bi-Lipschitz, and so $\mathrm{I}[\phi]$ is quasiconformal. Since $F_\varepsilon = \mathrm{I}[\phi] \circ H$, F_ε is a quasiconformal mapping as a composition of quasiconformal ones. By the conformality of H, $\mathbf{D}(H(0), R) \subset H(\mathbf{D})$ for a certain positive number R. Then by Corollary 2.7 we see that H is a co-Lipschitz mapping. Therefore F_ε is a co-Lipschitz mapping as a composition of co-Lipschitz ones, which proves the theorem. □

Let us recall that a holomorphic function $H\colon \mathbf{D} \to \mathbf{C}$ is said to be *convex* if H is injective and $H(\mathbf{D})$ is a convex domain, i.e., H maps conformally $\mathbf{D}$ onto a convex domain.

Corollary 2.9. *Let $F = H + \overline{G}$ be a sense-preserving harmonic mapping in $\mathbf{D}$. Suppose that H is convex and that F is not a conformal mapping. Then for every $\varepsilon \in \mathbf{D}(1/\|\mu_F\|_\infty)$, F_ε is a quasiconformal harmonic and co-Lipschitz mapping. In particular, if additionally F is quasiconformal, then F is a co-Lipschitz mapping.*

Proof. Fix $\varepsilon \in \mathbf{D}(1/\|\mu_F\|_\infty)$. Since $H(\mathbf{D})$ is convex, we see that $H(\mathbf{D})$ is rectifiably 1-arcwise connected. From Theorem 2.8 it follows that F_ε is a quasiconformal harmonic and co-Lipschitz mapping. If F is additionally quasiconformal, then $1 \in \mathbf{D}(1/\|\mu_F\|_\infty)$, and hence F is a co-Lipschitz mapping since $F = F_1$. This completes the proof. □

3. A counterpart of Theorem B

Lemma 3.1. *Given $M \ge 1$ suppose that $F = H + \overline{G}$ is a sense-preserving harmonic mapping in $\mathbf{D}$, H is injective in $\mathbf{D}$ and $H(\mathbf{D})$ is a rectifiably M-arcwise connected domain. Then the following estimations hold:*

$$|G(z_2) - G(z_1)| \le M\|\mu_F\|_\infty |H(z_2) - H(z_1)|, \quad z_1, z_2 \in \mathbf{D}, \tag{3.1}$$

as well as

$$\begin{aligned}(1 - M\|\mu_F\|_\infty)|H(z_2) - H(z_1)| &\le |F(z_2) - F(z_1)| \\ &\le (1 + M\|\mu_F\|_\infty)|H(z_2) - H(z_1)|, \quad z_1, z_2 \in \mathbf{D}.\end{aligned} \tag{3.2}$$

If additionally $M\|\mu_F\|_\infty < 1$, then $F \circ H^{-1}$ is a bi-Lipschitz mapping with

$$\mathrm{L}^+(F \circ H^{-1}) \le 1 + M\|\mu_F\|_\infty \quad \text{and} \quad \mathrm{L}^-(F \circ H^{-1}) \ge 1 - M\|\mu_F\|_\infty. \tag{3.3}$$

Proof. Under the assumption of the lemma we define the mapping $\phi := G \circ H^{-1}$. Then $\mathrm{D}^+(\phi) = \|\mu_F\|_\infty \le 1$, and by Lemma 2.4, $\mathrm{L}^+(\phi) \le M\,\mathrm{D}^+(\phi) = M\|\mu_F\|_\infty$. This yields the estimation (3.1). Using the formula (2.2) we have $F = \mathrm{I}[\phi] \circ H$. From Lemma 2.4 it follows that $\mathrm{L}^+(\mathrm{I}[\phi]) \le 1 + M\,\mathrm{D}^+(\phi) = 1 + M\|\mu_F\|_\infty$, which implies the second estimation in (3.2). If $M\|\mu_F\|_\infty \ge 1$, then the first estimation in (3.2) is obvious. Otherwise, we have $M\|\mu_F\|_\infty < 1$. Applying Lemma 2.4 once more we see that $\mathrm{L}^-(\mathrm{I}[\phi]) \ge 1 - M\,\mathrm{D}^+(\phi) = 1 - M\|\mu_F\|_\infty$, and so the first estimation in (3.2) holds in both cases. Since $F \circ H^{-1} = \mathrm{I}[\phi]$ we obtain the inequalities (3.3), which completes the proof. □

Corollary 3.2. *Given $M \ge 1$ suppose that $F = H + \overline{G}$ is a sense-preserving harmonic mapping in $\mathbf{D}$, H is injective in $\mathbf{D}$ and $H(\mathbf{D})$ is a rectifiably M-arcwise connected domain. Then the following inequality*

$$|G(z_2) - G(z_1)| < \max(1, M\|\mu_F\|_\infty)|H(z_2) - H(z_1)| \tag{3.4}$$

holds for all $z_1, z_2 \in \mathbf{D}$ such that $z_1 \ne z_2$. If additionally $M\|\mu_F\|_\infty \le 1$, then F is an injective mapping.

Proof. Fix $z_1, z_2 \in \mathbf{D}$ such that $z_1 \neq z_2$. Under the assumption of the corollary suppose that $M\|\mu_F\|_\infty \geq 1$ and

$$|G(z_2) - G(z_1)| = M\|\mu_F\|_\infty |H(z_2) - H(z_1)|. \tag{3.5}$$

Since H is injective, we can consider the following holomorphic function

$$\mathbf{D} \setminus \{z_1\} \ni z \mapsto \Phi(z) := \frac{G(z) - G(z_1)}{H(z) - H(z_1)}.$$

From Lemma 3.1 it follows that $|\Phi(z)| \leq M\|\mu_F\|_\infty$ for $z \in \mathbf{D} \setminus \{z_1\}$. Hence and by (3.5) we see that the function $|\Phi|$ attains the maximum value at the point z_2. Then by the maximum principle for holomorphic functions we conclude that Φ is a constant function. Since $\Phi(z) \to G'(z_1)/H'(z_1)$ as $z \to z_1$ and F is sense-preserving, we conclude from (3.5) that

$$M\|\mu_F\|_\infty = |\Phi(z_2)| = \left|\frac{G'(z_1)}{H'(z_1)}\right| = |\mu_F(z_1)| < 1,$$

which contradicts the assumption $M\|\mu_F\|_\infty \geq 1$. Therefore, if $M\|\mu_F\|_\infty \geq 1$, then the equality (3.5) does not hold, and so

$$|G(z_2) - G(z_1)| < M\|\mu_F\|_\infty |H(z_2) - H(z_1)|. \tag{3.6}$$

If $M\|\mu_F\|_\infty < 1$, then by Lemma 3.1,

$$|G(z_2) - G(z_1)| \leq M\|\mu_F\|_\infty |H(z_2) - H(z_1)| < |H(z_2) - H(z_1)|.$$

Combining this with (3.6) we get the inequality (3.4). Assume now that $M\|\mu_F\|_\infty \leq 1$. From the inequality (3.4) it follows that for all $z_1, z_2 \in \mathbf{D}$ such that $z_1 \neq z_2$,

$$\begin{aligned}|F(z_2) - F(z_1)| &= |H(z_2) - H(z_1) + \overline{G(z_2) - G(z_1)}| \\ &\geq |H(z_2) - H(z_1)| - |G(z_2) - G(z_1)| > 0.\end{aligned}$$

Therefore, F is an injective mapping, which completes the proof. □

Theorem 3.3. *Let $F = H + \overline{G}$ be a sense-preserving harmonic mapping in $\mathbf{D}$ such that H is convex. Then F is injective and the following five conditions are equivalent to each other:*

(i) *F is a quasiconformal mapping;*

(ii) *there exists a constant L_1 such that $1 \leq L_1 < 2$ and*

$$|F(z_2) - F(z_1)| \leq L_1 |H(z_2) - H(z_1)|, \quad z_1, z_2 \in \mathbf{D}; \tag{3.7}$$

(iii) *there exists a constant l_1 such that $0 \leq l_1 < 1$ and*

$$|G(z_2) - G(z_1)| \leq l_1 |H(z_2) - H(z_1)|, \quad z_1, z_2 \in \mathbf{D}; \tag{3.8}$$

(iv) *there exists a constant $L_2 \geq 1$ such that*

$$|H(z_2) - H(z_1)| \leq L_2 |F(z_2) - F(z_1)|, \quad z_1, z_2 \in \mathbf{D}; \tag{3.9}$$

(v) *$H \circ F^{-1}$ and $F \circ H^{-1}$ are bi-Lipschitz mappings.*

Proof. Under the assumption of the theorem we see that $H(\mathbf{D})$ is a convex domain. Therefore $H(\mathbf{D})$ is a rectifiably M-arcwise connected domain with $M := 1$. Hence $M\|\mu_F\|_\infty \leq 1$, and by Corollary 3.2, F is an injective mapping. Applying Lemma 3.1 we see that the condition (i) implies the remaining conditions (ii)–(v). Thus we need to show the inverse implications. Setting $\phi := G \circ H^{-1}$ we see that $F = \mathrm{I}[\phi] \circ H$ and $G = \phi \circ H$.

(ii) ⇒ (iii): From the condition (3.7) it follows that $\mathrm{L}^+(\mathrm{I}[\phi]) \le L_1$. Then Lemma 2.3 shows that $\mathrm{D}^+(\phi) \le \mathrm{L}^+(\mathrm{I}[\phi]) - 1 \le l_1 := L_1 - 1 < 1$. Applying now Lemma 2.4 with $M := 1$ we see that $\mathrm{L}^+(\phi) \le l_1$, which leads to the estimation (3.8). Thus (ii) implies (iii).

(iii) ⇒ (iv): From the condition (3.8) it follows that $\mathrm{L}^+(\phi) \le l_1 < 1$. Then Lemma 2.2 shows that $\mathrm{L}^-(\mathrm{I}[\phi]) \ge 1-\mathrm{L}^+(\phi) \ge 1-l_1 > 0$. Setting now $L_2 := 1/(1-l_1)$ we see that $L_2 \ge 1$ and the the estimation (3.9) holds. Thus (iii) implies (iv).

(iv) ⇒ (v): From the condition (3.9) we have $\mathrm{L}^-(\mathrm{I}[\phi]) \ge 1/L_2$. Applying now Lemma 2.4 with $M := 1$ we see that $\mathrm{L}^+(\mathrm{I}[\phi]) \le 1 + \mathrm{D}^+(\phi) \le 2$. Therefore $\mathrm{I}[\phi]$ is a bi-Lipschitz mapping. Since $F \circ H^{-1} = \mathrm{I}[\phi]$, $F \circ H^{-1}$ is a bi-Lipschitz mapping, and so is the inverse mapping $H \circ F^{-1}$. Thus (iv) implies (v).

(v) ⇒ (i): Since the mapping F is sense-preserving, the condition (v) implies that $F \circ H^{-1}$ is a quasiconformal mapping. Thus F is a qusiconformal mapping as a composition of a qusiconformal one with the conformal mapping H. Thus (v) implies (i), which completes the proof. □

Remark 3.4. Let $F = H + \overline{G}$ be a sense-preserving harmonic mapping in $\mathbf{D}$ such that H is injective. Then the conditions (i)–(v) in Theorem 3.3 are involved by the implications: (iii) ⇒ (ii), (iii) ⇒ (iv) and (iv) ⇒ (i). Moreover, the following implications hold: (3.7) ⇒ $\|\mu_F\|_\infty \le L_1 - 1$, (3.8) ⇒ $\|\mu_F\|_\infty \le l_1$ and (3.9) ⇒ $\|\mu_F\|_\infty \le 1 - \frac{1}{L_2}$. If additionally F is injective, then the implication (ii)⇒(i) holds. Note that we do not assume here that $\Omega := H(\mathbf{D})$ is a convex domain.

For the proof let us consider the mapping $\phi := G \circ H^{-1}$. Then $F = \mathrm{I}[\phi] \circ H$ and $G = \phi \circ H$. Since F is a sense-preserving harmonic mapping in $\mathbf{D}$, the mapping $\phi := G \circ H^{-1}$ satisfies the condition

$$(3.10)\qquad |\phi'(z)|^2 = \left|\frac{G'(H^{-1}(z))}{H'(H^{-1}(z))}\right|^2 = 1 - \frac{\mathrm{J}[F](H^{-1}(z))}{|H'(H^{-1}(z))|^2} < 1, \quad z \in \Omega,$$

and so $\mathrm{d}_z^+\phi < 1$ for $z \in \Omega$. Since H is a conformal mapping and $F = \mathrm{I}[\phi] \circ H$ we see that

$$(3.11)\qquad \|\mu_F\|_\infty = \mathrm{D}^+(\phi) \le 1.$$

Assume that the condition (3.7) holds. Then $\mathrm{L}^+(\mathrm{I}[\phi]) \le L_1$, and by Lemma 2.3 we get $\mathrm{D}^+(\phi) \le \mathrm{L}^+(\mathrm{I}[\phi]) - 1 \le L_1 - 1$. Combining this with (3.11) we see that $\|\mu_F\|_\infty \le L_1 - 1$. This yields the implication (3.7) ⇒ $\|\mu_F\|_\infty \le L_1 - 1$. Thus the condition (ii) implies $\|\mu_F\|_\infty < 1$, and so F is a quasiregular mapping. Hence the implication (ii)⇒(i) holds provided F is injective.

Assume now that the condition (3.8) holds. Then $\mathrm{L}^+(\phi) \le l_1 < 1$. By Lemmas 2.2 and 2.3 we see that $\mathrm{D}^+(\phi) \le \mathrm{L}^+(\mathrm{I}[\phi]) - 1 \le \mathrm{L}^+(\phi) \le l_1 < 1$. Combining this with (3.11) we obtain $\|\mu_F\|_\infty \le l_1$, which yields the implication (3.8) ⇒ $\|\mu_F\|_\infty \le l_1$.

Assume finally that the condition (3.9) holds. Then $\mathrm{L}^-(\mathrm{I}[\phi]) \ge 1/L_2$. The condition (3.9) implies, by the injectivity of H, that F is injective. Since $\mathrm{D}^+(\phi) \le 1$, we conclude from Lemma 2.3 that $\mathrm{D}^+(\phi) \le 1 - \mathrm{L}^-(\mathrm{I}[\phi]) \le 1 - 1/L_2$. Combining this with (3.11) we have $\|\mu_F\|_\infty \le 1 - 1/L_2$. This yields the implication (3.9) ⇒ $\|\mu_F\|_\infty \le 1 - 1/L_2$, which leads to the implication (iv) ⇒ (i).

The implications (iii) ⇒ (ii) and (iii) ⇒ (iv) follow directly from Lemma 2.2 and the inequality $\mathrm{L}^+(\phi) \le l_1$.

4. A counterpart of Theorem A

Let $\mathrm{P}[f]$ stand for the Poisson integral of an integrable function $f\colon \mathbf{T}\to\mathbf{C}$, i.e.,

$$\mathbf{D}\ni z\mapsto \mathrm{P}[f](z):=\frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t})\frac{1-|z|^2}{|\mathrm{e}^{\mathrm{i}t}-z|^2}\,\mathrm{d}t.$$

If f is continuous, then the Poisson integral $\mathrm{P}[f]$ provides the unique solution to the Dirichlet problem for the unit disk $\mathbf{D}$ with the boundary function f. This means that $\mathrm{P}[f]$ is a harmonic mapping in $\mathbf{D}$ which has a continuous extension to the closed disk $\mathrm{cl}(\mathbf{D})$ and its boundary limiting valued function coincides with f. Given an absolutely continuous function $f\colon \mathbf{T}\to\mathbf{C}$ we know that $\dot{f}$ is integrable on $\mathbf{T}$, and therefore we can define

$$d_f^* := \operatorname*{ess\,inf}_{z\in\mathbf{T}} \operatorname{Im}\Big(\dot{f}(z)\overline{\mathrm{A}[\dot{f}](z)}\Big). \tag{4.1}$$

Lemma 4.1. *Let $K\geq 1$ and let F be a K-quasiconformal and Lipschitz harmonic mapping in $\mathbf{D}$. Then F has a continuous extension $\tilde{F}$ to the closure $\mathrm{cl}(\mathbf{D})$ of $\mathbf{D}$ and the restriction f of $\tilde{F}$ to $\mathbf{T}$ is absolutely continuous on $\mathbf{T}$ and satisfies the following conditions*

$$\|\,\mathrm{A}[\dot{f}]\|_\infty \leq \sqrt{2}\,\mathrm{L}^+(F) \quad \textit{and} \quad \|\dot{f}\|_\infty \leq \mathrm{L}^+(F) \tag{4.2}$$

as well as

$$\frac{K}{K^2+1}\big(|\,\mathrm{A}[\dot{f}](z)|^2+|\dot{f}(z)|^2\big) \leq \operatorname{Im}\Big(\dot{f}(z)\overline{\mathrm{A}[\dot{f}](z)}\Big) \quad \textit{for a.e. } z\in\mathbf{T}; \tag{4.3}$$

in particular

$$d_f^* \geq \frac{K}{K^2+1}\big(\operatorname*{ess\,inf}_{z\in\mathbf{T}}|\,\mathrm{A}[\dot{f}](z)|^2+d_f^2\big). \tag{4.4}$$

If additionally F is a co-Lipschitz mapping, then

$$d_f^* \geq \mathrm{L}^-(F)^2\ ,\ d_f \geq \frac{\mathrm{L}^-(F)^2}{\sqrt{2}\,\mathrm{L}^+(F)} \quad \textit{and} \quad \operatorname*{ess\,inf}_{z\in\mathbf{T}}|\,\mathrm{A}[\dot{f}](z)| \geq \frac{\mathrm{L}^-(F)^2}{\mathrm{L}^+(F)}. \tag{4.5}$$

Proof. Let F be a Lipschitz harmonic mapping in $\mathbf{D}$. Then $\mathrm{L}^+(F)<+\infty$. From [12, Theorem 3.2] we know that F has a continuous extension $\tilde{F}$ to the closure $\mathrm{cl}(\mathbf{D})$ and that the restriction f of $\tilde{F}$ to $\mathbf{T}$ is absolutely continuous and satisfies the inequalities (4.2).

Assume now that F is additionally K-quasiconformal with a given $K\geq 1$. Then $\|\mu[F]\|_\infty \leq (K-1)/(K+1)$, and so

$$\frac{|\partial F(z)|^2+|\bar\partial F(z)|^2}{|\partial F(z)|^2-|\bar\partial F(z)|^2} = \frac{1+|\mu[F](z)|^2}{1-|\mu[F](z)|^2} \leq \frac{K^2+1}{2K}, \quad z\in\mathbf{D}. \tag{4.6}$$

Since $F=\mathrm{P}[f]$, we conclude from [12, Corollary 1.3 and (1.18)] that for a.e. $z\in\mathbf{T}$,

$$\begin{aligned}\lim_{r\to 1^-}\partial F(rz) &= \frac{\overline{z}}{2}(\mathrm{A}[\dot{f}](z)+zf'(z)),\\ \lim_{r\to 1^-}\bar\partial F(rz) &= \frac{z}{2}(\mathrm{A}[\dot{f}](z)-zf'(z)).\end{aligned} \tag{4.7}$$

Hence and by (1.4) we obtain for a.e. $z \in \mathbf{T}$,

$$\text{(4.8)} \qquad \lim_{r\to 1^-}\big(|\partial F(rz)|^2 + |\bar\partial F(rz)|^2\big) = \frac{1}{4}|\operatorname{A}[\dot f](z) - \mathrm{i}\dot f(z)|^2 + \frac{1}{4}|\operatorname{A}[\dot f](z) + \mathrm{i}\dot f(z)|^2 = \frac{1}{2}\big(|\operatorname{A}[\dot f](z)|^2 + |\dot f(z)|^2\big)$$

as well as

$$\text{(4.9)} \qquad \lim_{r\to 1^-}\big(|\partial F(rz)|^2 - |\bar\partial F(rz)|^2\big) = \frac{1}{4}|\operatorname{A}[\dot f](z) - \mathrm{i}\dot f(z)|^2 - \frac{1}{4}|\operatorname{A}[\dot f](z) + \mathrm{i}\dot f(z)|^2 = \operatorname{Im}\Big(\dot f(z)\overline{\operatorname{A}[\dot f](z)}\Big).$$

Combining (4.6) with (4.8) and (4.9) we obtain the condition (4.3), which implies the inequality (4.4).

Assume finally that F is co-Lipschitz, i.e., $\mathrm{L}^-(F) > 0$. Since

$$|\partial F(z)| - |\bar\partial F(z)| \geq \mathrm{L}^-(F), \quad z \in \mathbf{D},$$

we see that for every $z \in \mathbf{D}$,

$$|\partial F(z)|^2 - |\bar\partial F(z)|^2 = (|\partial F(z)| - |\bar\partial F(z)|)(|\partial F(z)| + |\bar\partial F(z)|) \geq \mathrm{L}^-(F)^2.$$

Combining this with (4.9) and (4.1) yields $d_f^* \geq \mathrm{L}^-(F)^2 > 0$. Since for a.e. $z \in \mathbf{T}$,

$$\operatorname{Im}\Big(\dot f(z)\overline{\operatorname{A}[\dot f](z)}\Big) \leq |\dot f(z)| \cdot |\operatorname{A}[\dot f](z)| \leq \|\dot f\|_\infty |\operatorname{A}[\dot f](z)|,$$

we conclude from (4.1) that for a.e. $z \in \mathbf{T}$,

$$\mathrm{L}^-(F)^2 \leq d_f^* \leq \|\dot f\|_\infty |\operatorname{A}[\dot f](z)|,$$

which implies

$$\operatorname*{ess\,inf}_{z\in\mathbf{T}} |\operatorname{A}[\dot f](z)| \geq \frac{d_f^*}{\|\dot f\|_\infty} \geq \frac{\mathrm{L}^-(F)^2}{\mathrm{L}^+(F)}.$$

Likewise, we get

$$d_f \geq \frac{d_f^*}{\|\operatorname{A}[\dot f]\|_\infty} \geq \frac{\mathrm{L}^-(F)^2}{\sqrt{2}\,\mathrm{L}^+(F)}.$$

Thus the inequalities (4.5) hold, which completes the proof. □

Theorem 4.2. *Let $F = H + \overline{G}$ be a sense-preserving injective harmonic mapping in $\mathbf{D}$ such that the holomorphic part H of F is convex. Then the following five conditions are equivalent to each other:*

(i) *F is a quasiconformal and Lipschitz mapping;*

(ii) *F is a quasiconformal mapping and identical with the Poisson integral $\mathrm{P}[f]$ of a Lipschitz function $f\colon \mathbf{T} \to \mathbf{C}$;*

(iii) *F is a quasiconformal mapping and H is a bi-Lipschitz mapping;*

(iv) *F is a bi-Lipschitz mapping;*

(v) *F has a continuous extension $\tilde F$ to the closure $\mathrm{cl}(\mathbf{D})$ of $\mathbf{D}$ and the restriction f of $\tilde F$ to $\mathbf{T}$ is absolutely continuous on $\mathbf{T}$ and satisfies the following conditions*

$$0 < d_f^*\,, \quad \|\dot f\|_\infty < +\infty \quad \text{and} \quad \|\operatorname{A}[\dot f]\|_\infty < +\infty.$$

Proof. We will prove the following cycle of five implications.

(i) ⇒ (ii): Suppose that the condition (i) holds. Then by [12, Theorem 3.2] F has the continuous extension $\tilde F$ to the closure $\mathrm{cl}(\mathbf{D})$ and the restriction f of $\tilde F$ to $\mathbf{T}$ is a Lipschitz mapping. Since $F = \mathrm{P}[f]$, we see that (i) implies (ii).

(ii) ⇒ (iii): Suppose that the condition (ii) holds. Since the Lipschitz function $f\colon \mathbf{T} \to \mathbf{C}$ is continuous in $\mathbf{T}$, $F = \mathrm{P}[f]$ has a continuous extension $\tilde{F}$ to the closure $\mathrm{cl}(\mathbf{D})$ and the restriction of $\tilde{F}$ to $\mathbf{T}$ is identical with f. Moreover, F is a K-quasiconformal mapping for some $K \geq 1$. By Theorem 3.3, $F \circ H^{-1}$ is a bi-Lipschitz mapping. By the conformality of H, $\mathbf{D}(H(0), R) \subset H(\mathbf{D})$ for some positive number R. Then by Corollary 2.7 we see that H is a co-Lipschitz mapping. Therefore F is a co-Lipschitz mapping as a composition of co-Lipschitz ones. Hence $\tilde{F}$ is continuous and co-Lipschitz in $\mathrm{cl}(\mathbf{D})$, and consequently $F(\mathbf{D})$ is a bounded Jordan domain. Applying now [9, Lemma 2.3] we have

$$\sup_{z\in\mathbf{D}} |H'(z)| = \sup_{z\in\mathbf{D}} |\partial F(z)| \leq \frac{K+1}{2}\,\mathrm{L}^+(f) < +\infty,$$

and thus, by Lemma 2.4, H is a Lipschitz mapping. Therefore H is a bi-Lipschitz mapping, and so (ii) implies (iii).

(iii) ⇒ (iv): Suppose that the condition (iii) holds. Then Theorem 3.3 shows that $F \circ H^{-1}$ is a bi-Lipschitz mapping, and so is F. This shows that (iii) implies (iv).

(iv) ⇒ (v): Suppose that the condition (iv) holds. Then $\mathrm{L}^+(F) < +\infty$ and $\mathrm{L}^-(F) > 0$. Lemma 4.1 now shows that the condition (v) holds, and so (iv) implies (v).

(v) ⇒ (i): Suppose that the condition (v) holds. Since F is a sense-preserving injective harmonic mapping, we conclude from Lewy's theorem that the Jacobian $\mathrm{J}[F]$ is positive in $\mathbf{D}$; cf. [7]. Therefore

$$|\partial F(z)|^2 - |\bar{\partial} F(z)|^2 = \mathrm{J}[F](z) > 0, \quad z \in \mathbf{D}, \tag{4.10}$$

and so the second dilatation

$$\mathbf{D} \ni z \mapsto \omega(z) := \frac{\overline{\bar{\partial} F(z)}}{\partial F(z)} = \frac{G'(z)}{H'(z)} \tag{4.11}$$

of F is well defined as well as

$$|\omega(z)| < 1, \quad z \in \mathbf{D}. \tag{4.12}$$

Applying now [2, Theorem 1.3] we see that there exists a function $\omega^*\colon \mathbf{T} \to \mathbf{C}$ such that

$$\omega^*(z) = \lim_{r\to 1^-} \omega(rz) \quad \text{for a.e. } z \in \mathbf{T}. \tag{4.13}$$

From (4.7) and (1.4) it follows that for a.e. $z \in \mathbf{T}$,

$$4|\lim_{r\to 1^-} \partial F(rz)|^2 = \left|\mathrm{A}[\dot{f}](z) - \mathrm{i}\dot{f}(z)\right|^2 \leq 2\left|\mathrm{A}[\dot{f}](z)\right|^2 + 2\left|\dot{f}(z)\right|^2.$$

Therefore

$$|\lim_{r\to 1^-} \partial F(rz)|^2 \leq M_f := \frac{1}{2}\left\|\mathrm{A}[\dot{f}]\right\|_\infty^2 + \frac{1}{2}\left\|\dot{f}\right\|_\infty^2 < +\infty \quad \text{for a.e. } z \in \mathbf{T}. \tag{4.14}$$

Using now (4.9) we have

$$\lim_{r\to 1^-} \mathrm{J}[F](rz) = \mathrm{Im}\Big(\dot{f}(z)\overline{\mathrm{A}[\dot{f}](z)}\Big) \geq d_f^* > 0 \quad \text{for a.e. } z \in \mathbf{T}.$$

Combining this with (4.10), (4.11), (4.13) and (4.14) we see that for a.e. $z \in \mathbf{T}$,

$$M_f(1 - |\omega^*(z)|^2) \geq \lim_{r\to 1^-} |\partial F(rz)|^2(1 - |\omega(rz)|^2) = \lim_{r\to 1^-} \mathrm{J}[F](rz) \geq d_f^* > 0.$$

Hence $M_f > 0$ and

$$\|\omega^*\|_\infty^2 \le 1 - \frac{d_f^*}{M_f} < 1.$$

This implies, by (4.12) and (4.13), that

$$\left|\frac{\bar\partial F(z)}{\partial F(z)}\right| = |\omega(z)| = |\operatorname{P}[\omega^*](z)| \le \|\omega^*\|_\infty \le \sqrt{1 - \frac{d_f^*}{M_f}} < 1, \quad z \in \mathbf{D}. \tag{4.15}$$

Therefore F is a quasiconformal mapping. By [12, Theorem 3.2] F is also a Lipschitz mapping. Thus the condition (i) holds. This leads to the implication (v) ⇒ (i), and the proof is complete. □

Remark 4.3. Note that the implications (iv) ⇒ (v) and (v) ⇒ (i) in Theorem 4.2 hold without the assumption that H is convex.

Corollary 4.4. *Let $F = H + \overline{G}$ be a sense-preserving injective harmonic mapping in $\mathbf{D}$ such that the holomorphic part H of F is convex. If F is a quasiconformal and Lipschitz mapping, then F has a continuous extension $\tilde F$ to the closure $\operatorname{cl}(\mathbf{D})$ of $\mathbf{D}$ and the restriction f of $\tilde F$ to $\mathbf{T}$ is absolutely continuous on $\mathbf{T}$ and satisfies*

$$0 < d_f, \quad \|\dot f\|_\infty < +\infty, \quad \|\operatorname{A}[\dot f]\|_\infty < +\infty \quad \text{and} \quad 0 < \operatorname*{ess\,inf}_{z\in\mathbf{T}} |\operatorname{A}[\dot f](z)|. \tag{4.16}$$

Proof. By Theorem 4.2 we see that F is a bi-Lipschitz mapping. This means, that $\operatorname{L}^+(F) < +\infty$ and $\operatorname{L}^-(F) > 0$. Then the condition (4.16) follows from Lemma 4.1. □

5. Examples

In this section we collect a few examples which complete our considerations.

Example 5.1. Given a constant k satisfying $0 \le k < 1$, consider the mapping $\mathbf{C} \ni z \mapsto \phi(z) := kz$. Then $F := \operatorname{I}[\phi] \circ H$ is a K-quasiconformal harmonic mapping in $\mathbf{D}$ with $K := \frac{1+k}{1-k}$ for every conformal mapping H in $\mathbf{D}$. Evidently, $F(\mathbf{D})$ is a convex domain if and only if $H(\mathbf{D})$ is so. Suppose that H is convex. From Corollary 2.9 it follows that F ($\varepsilon := 1$) and H ($\varepsilon := 0$) are co-Lipschitz mappings. Moreover, from Theorem 4.2 we see that F is a bi-Lipschitz mapping if and only if H is so. In particular, the mapping

$$\mathbf{D} \ni z \mapsto F(z) := \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) + \frac{k}{2}\overline{\log\left(\frac{1+z}{1-z}\right)}$$

is co-Lipschitz but not Lipschitz, because the mapping

$$\mathbf{D} \ni z \mapsto H(z) := \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

maps the unit disk $\mathbf{D}$ onto the strip $\{z \in \mathbf{C} \colon |\operatorname{Im} z| < \pi/4\}$, which is an unbounded convex domain. Since

$$\operatorname{d}_z^- F = \left|\left|\frac{1}{1-z^2}\right| - \left|\frac{k}{1-\overline{z}^2}\right|\right| = \frac{1-k}{|1-z^2|} > \frac{1-k}{2}, \quad z \in \mathbf{D},$$

we conclude from Lemma 2.5 that

$$\frac{1-k}{2}|z_1 - z_2| \le |F(z_1) - F(z_2)|, \quad z_1, z_2 \in \mathbf{D}.$$

Example 5.2. Similarly as in Example 5.1 we see that for any constants α and k satisfying $0 < \alpha \le 1$ and $0 \le k < 1$ the mapping

$$\mathbf{D} \ni z \mapsto F(z) := (1-z)^\alpha + k(1-\bar{z})^\alpha$$

is harmonic co-Lipschitz and K-quasiconformal with $K := \frac{1+k}{1-k}$, because the mapping

$$\mathbf{D} \ni z \mapsto H(z) := (1-z)^\alpha$$

maps the unit disk $\mathbf{D}$ onto a convex domain. However, this time the image $H(\mathbf{D})$ is bounded. Since

$$\mathrm{d}_z^+ F = \left|\frac{-\alpha}{(1-z)^{1-\alpha}}\right| + \left|\frac{-k\alpha}{(1-\bar{z})^{1-\alpha}}\right| = \frac{(1+k)\alpha}{|1-z|^{1-\alpha}}, \quad z \in \mathbf{D},$$

we see that $\mathrm{D}^+(F) = +\infty$ provided $0 < \alpha < 1$. Lemma 2.3 now shows that $\mathrm{L}^+(F) = +\infty$, and so F is not a Lipschitz mapping for $\alpha \in (0;1)$. Therefore F is bi-Lipschitz if and only if $\alpha = 1$. Moreover, if $\alpha = 1$, then

$$|F(z_1) - F(z_2)| \le (1+k)|z_1 - z_2|, \quad z_1, z_2 \in \mathbf{D}.$$

Since

$$\mathrm{d}_z^- F = \left|\left|\frac{-\alpha}{(1-z)^{1-\alpha}}\right| - \left|\frac{-k\alpha}{(1-\bar{z})^{1-\alpha}}\right|\right| = \frac{(1-k)\alpha}{|1-z|^{1-\alpha}} > \frac{(1-k)\alpha}{2^{1-\alpha}}, \quad z \in \mathbf{D},$$

we conclude from Lemma 2.5 that

$$\frac{(1-k)\alpha}{2^{1-\alpha}}|z_1 - z_2| \le |F(z_1) - F(z_2)|, \quad z_1, z_2 \in \mathbf{D}.$$

Example 5.3. Given $\alpha \in (0;1]$ we define the functions

$$\mathbf{D} \ni z \mapsto H(z) := \frac{1-(1-z)^\alpha}{\alpha} \quad \text{and} \quad \mathbf{D} \ni z \mapsto G(z) := \int_0^z \zeta H'(\zeta)\,\mathrm{d}\zeta.$$

The holomorphic part H of the harmonic mapping $F := H + \overline{G}$ in $\mathbf{D}$ is convex and $H(0) = 0$ and $H'(0) = 1$. Integrating by substitution we can determine explicitly the anti-holomorphic part G of F as follows

$$G(z) = -\int_1^{1-z} (1-\eta)\eta^{\alpha-1}\,\mathrm{d}\eta = \frac{1-(1-z)^\alpha(1+\alpha z)}{\alpha(\alpha+1)}, \quad z \in \mathbf{D}.$$

In particular $F(z) = z + \overline{z^2}/2$ for $z \in \mathbf{D}$ provided $\alpha = 1$. Since $G'(z) = zH'(z)$ for $z \in \mathbf{D}$ we see that $\|\mu_{F_\varepsilon}\|_\infty = |\varepsilon|$ for a fixed $\varepsilon \in \mathbf{C}$, where F_ε is the mapping given by the formula (2.20). From Theorem 3.3 it follows that F_ε is an injective harmonic mapping if and only if $|\varepsilon| \le 1$. Therefore, F_ε is quasiconformal if and only if $|\varepsilon| < 1$. Moreover, by [1, Theorem 5.17] we know that F_ε is a close-to-convex mapping provided $|\varepsilon| \le 1$. Suppose that $0 < \alpha < 1$. Since $H'(z) = (1-z)^{\alpha-1}$ for $z \in \mathbf{D}$, we deduce from Lemma 2.3 that $\mathrm{L}^+(H) \ge \mathrm{D}^+(H) = +\infty$, and so H is not a Lipschitz mapping. Hence and by Theorem 4.2, F_ε is not a Lipschitz mapping for $\varepsilon \in \mathbf{D}$.

Example 5.4. Given $\varepsilon \in \mathbf{C}$, $|\varepsilon| \le 1$, we determine the holomorphic mappings H and G in $\mathbf{D}$ satisfying the following conditions

$$H(z) - G(z) = z \quad \text{and} \quad \frac{G'(z)}{H'(z)} = \varepsilon^2 z^2, \quad z \in \mathbf{D}, \tag{5.1}$$

as well as $H(0) = 0 = G(0)$. Hence

$$H'(z) = \frac{1}{1-\varepsilon^2 z^2}, \quad z \in \mathbf{D}, \tag{5.2}$$

and so

$$H(z)=\frac{1}{2\varepsilon}\log\frac{1+\varepsilon z}{1-\varepsilon z}\quad\text{and}\quad G(z)=-z+\frac{1}{2\varepsilon}\log\frac{1+\varepsilon z}{1-\varepsilon z},\quad z\in\mathbf{D},$$

provided $\varepsilon\neq 0$, and otherwise $H(z)=z$ and $G(z)=0$ for $z\in\mathbf{D}$. Then the holomorphic part H of the harmonic mapping $F:=H+\overline{G}$ is convex. By (5.1), the Jacobian $\mathrm{J}[F]$ is positive in $\mathbf{D}$ and $\|\mu_F\|_\infty=|\varepsilon|^2$. Then Theorem 3.3 shows that F is an injective harmonic mapping. Therefore, F is quasiconformal if and only if $|\varepsilon|<1$. By Lemmas 2.4 and 2.5 we have

$$\mathrm{L}^+(H)\le\mathrm{D}^+(H)\le\frac{1}{1-|\varepsilon|^2}\quad\text{and}\quad\mathrm{L}^-(H)\ge\mathrm{D}^-(H)\ge\frac{1}{1+|\varepsilon|^2},\quad\varepsilon\in\mathbf{D},\tag{5.3}$$

and so H is a bi-Lipschitz mapping for $\varepsilon\in\mathbf{D}$. Hence and by Theorem 4.2, F is a bi-Lipschitz mapping for $\varepsilon\in\mathbf{D}$. Moreover, from Lemma 3.1 we deduce that for every $\varepsilon\in\mathbf{D}$,

$$(1-|\varepsilon|^2)|H(z_2)-H(z_1)|\le|F(z_2)-F(z_1)|\le(1+|\varepsilon|^2)|H(z_2)-H(z_1)|,\quad z_1,z_2\in\mathbf{D},$$

which combined with (5.3) leads to

$$\frac{1-|\varepsilon|^2}{1+|\varepsilon|^2}|z_2-z_1|\le|F(z_2)-F(z_1)|\le\frac{1+|\varepsilon|^2}{1-|\varepsilon|^2}|z_2-z_1|,\quad z_1,z_2\in\mathbf{D}.$$

Example 5.5. Fix $k\in\mathbf{D}$ and define $\mathbf{C}\ni z\mapsto\phi(z):=kz$. Since the mapping $\mathbf{D}\ni z\mapsto H(z):=(1+z)^2$ is injective, as in Example 5.1 we see that $F:=\mathrm{I}[\phi]\circ H$ is a quasiconformal harmonic mapping in $\mathbf{D}$. It is clear that F is a Lipschitz mapping, but $H(\mathbf{D})$ is not a convex domain. Since

$$\mathrm{L}^-(F)\le|\partial F(z)|-|\bar\partial F(z)|=2(1-|k|)|1+z|,\quad z\in\mathbf{D},$$

we see that $\mathrm{L}^-(F)=0$, and so F is not co-Lipschitz. Moreover,

$$d_f^*=\operatorname*{ess\,inf}_{z\in\mathbf{T}}\lim_{r\to1^-}\mathrm{J}[F](rz)=4(1-|k|^2)\operatorname*{ess\,inf}_{z\in\mathbf{T}}|1+z|^2=0,$$

where f is the boundary limiting valued function of F. Thus the mapping F satisfies the property (i), but does not satisfy the properties (iv) and (v) in Theorem 4.2.

Example 5.6. Let $\mathbf{D}\ni z\mapsto F(z):=\mathrm{I}[\phi](z)$ and $\mathbf{T}\ni z\mapsto f(z):=\mathrm{I}[\phi](z)$, where

$$\mathbf{C}\ni z\mapsto\phi(z):=-\mathrm{e}^{z-1}+\mathrm{e}^{-1}.\tag{5.4}$$

Since ϕ is a holomorphic function in $\mathbf{C}$, f is absolutely continuous in $\mathbf{T}$ and $F=\mathrm{P}[f]$. By the formula (5.4),

$$\mathrm{J}[F](z)=|\partial F(z)|^2-|\bar\partial F(z)|^2=1-\left|\mathrm{e}^{z-1}\right|^2=1-\mathrm{e}^{2\operatorname{Re}(z-1)}>0,\quad z\in\mathbf{D}.$$

Thus F is a sense-preserving harmonic mapping in $\mathbf{D}$ and its holomorphic part is convex as the identity mapping. From Theorem 3.3 it follows that F is injective. By Lemma 2.4 we see that F is a Lipschitz mapping. However, F is not quasiconformal, because

$$\|\mu_F\|_\infty=\sup_{z\in\mathbf{D}}\left|\mathrm{e}^{z-1}\right|=\sup_{z\in\mathbf{D}}\mathrm{e}^{\operatorname{Re}(z-1)}=1.$$

By the formula (1.4) we have

$$\dot f(z)=\mathrm{i}(z+\overline{z\mathrm{e}^{z-1}}),\quad z\in\mathbf{T},\tag{5.5}$$

and therefore $\dot f$ is continuous in $\mathbf{T}$. Hence

$$d_f=\operatorname*{ess\,inf}_{z\in\mathbf{T}}|\dot f(z)|=\min_{z\in\mathbf{T}}|z+\overline{z\mathrm{e}^{z-1}}|=|z_0+\overline{z_0\mathrm{e}^{z_0-1}}|\tag{5.6}$$

for some $z_0 \in \mathbf{T}$. If $z_0 + \overline{z_0 e^{z_0 - 1}} = 0$, then $1 = |z_0| = |z_0 e^{z_0-1}| = e^{\operatorname{Re}(z_0-1)}$, and thereby $z_0 = 1$, which is impossible. Thus $d_f > 0$. From (5.8) we deduce that the function $\mathrm{A}[\dot{f}]$ is continuous in $\mathbf{T}$, and so $\|\dot{f}\|_\infty < +\infty$ and $\|\mathrm{A}[\dot{f}]\|_\infty < +\infty$. Therefore the inequality $d_f^* > 0$ in condition (v) of Theorem 4.2 can not be replaced by $d_f > 0$.

In order to determine the function $\mathrm{A}[\dot{f}]$ assume that $\varphi\colon \mathbf{T} \to \mathbf{C}$ is a function which has a holomorphic extension Φ to a domain containing $\operatorname{cl}(\mathbf{D})$. From the formula (1.5) it follows that for every $z \in \mathbf{T}$,

$$\mathrm{A}[\operatorname{Re}\varphi](z) = \operatorname{Im}\Phi(z) - \operatorname{Im}\Phi(0) \quad \text{and} \quad \mathrm{A}[\operatorname{Im}\varphi](z) = -\operatorname{Re}\Phi(z) + \operatorname{Re}\Phi(0),$$

which leads to

$$\begin{aligned}\mathrm{A}[\varphi](z) &= \mathrm{A}[\operatorname{Re}\varphi + \mathrm{i}\operatorname{Im}\varphi](z) = \mathrm{A}[\operatorname{Re}\varphi](z) + \mathrm{i}\,\mathrm{A}[\operatorname{Im}\varphi](z)\\ &= \operatorname{Im}\Phi(z) - \mathrm{i}\operatorname{Re}\Phi(z) - [\operatorname{Im}\Phi(0) - \mathrm{i}\operatorname{Re}\Phi(0)].\end{aligned}$$

Therefore,

$$\mathrm{A}[\varphi](z) = \frac{1}{\mathrm{i}}[\Phi(z) - \Phi(0)], \quad z \in \mathbf{T}. \tag{5.7}$$

Applying now (5.7) we deduce from (5.5) that

$$\mathrm{A}[\dot{f}](z) = \frac{\mathrm{i}z}{\mathrm{i}} + \overline{\left(\frac{-\mathrm{i}z e^{z-1}}{\mathrm{i}}\right)} = z - \overline{z e^{z-1}}, \quad z \in \mathbf{T}, \tag{5.8}$$

which gives $\mathrm{A}[\dot{f}](1) = 0$. Hence $\operatorname{ess\,inf}_{z\in\mathbf{T}} |\mathrm{A}[\dot{f}](z)| = 0$ which means that Corollary 4.4 does not hold without the quasiconformality assumption on F. Moreover, by (4.1), (4.9) and (5.5) we have $d_f^* = 0$. Therefore, the inequality $d_f^* > 0$ is not valid in general provided $\mathrm{P}[f]$ is a sense-preserving harmonic mapping in $\mathbf{D}$ and its holomorphic part is convex.

References

[1] Clunie, J., and T. Sheil-Small: Harmonic univalent functions. - Ann. Acad. Sci. Fenn. Ser. A I Math. 9, 1984, 3–25.

[2] Duren, P.: Theory of H^p-spaces. - Dover Publications, Inc., Mineola, New York, 2000.

[3] Duren, P.: Harmonic mappings in the plane. - Cambridge Tracts in Math. 156, Cambridge Univ. Press, Cambridge, 2004.

[4] Garnett, J. B.: Bounded analytic functions. - Academic Press, New York, 1981.

[5] Kalaj, D.: On harmonic diffeomorphisms of the unit disc onto a covex domain. - Complex Variables 48:2, 2003, 175–187.

[6] Kalaj, D.: Quasiconformal harmonic functions between convex domains. - Publ. Inst. Math. (Beograd) (N.S.) 90, 2004, 3–20.

[7] Lewy, H.: On the non-vanishing of the Jacobian in certain one-to-one mappings. - Bull. Amer. Math. Soc. 42, 1936, 689–692.

[8] Näkki, R., and B. Palka: Lipschitz conditions, b-arcwise connectedness and conformal mappings. - J. Anal. Math. 42, 1982/83, 38–50.

[9] Partyka, D., and K. Sakan: On bi-Lipschitz type inequalities for quasiconformal harmonic mappings. - Ann. Acad. Sci. Fenn. Math. 32, 2007, 579–594.

[10] Partyka, D., and K. Sakan: On a variant of Heinz's inequality for harmonic mappings of the unit disk onto bounded convex domains. - Bull. Soc. Sci. Lett. Łódź Sér. Rech. Déform. 59:2, 2009, 25–36.

[11] Partyka, D., and K. Sakan: A simple deformation of quasiconformal harmonic mappings in the unit disk. - Ann. Acad. Sci. Fenn. Ser. Math. 37, 2012, 539–556.

[12] Partyka, D., and K. Sakan: Quasiconformal and Lipschitz harmonic mappings of the unit disk onto bounded convex domains. - Ann. Acad. Sci. Fenn. Ser. Math. 39, 2014, 811–830.

[13] Zhu, J. F.: Some estimates for harmonic mappings with given boundary function. - J. Math. Anal. Appl. 411, 2014, 631–638.

[14] Zhu, J. F.: Landau theorem for planar harmonic mappings. - Complex Anal. Oper. Theory 9, 2015, 1819–1826.

[15] Zhu, J. F.: Coefficients estimate for harmonic v-Bloch mappings and harmonic K-quasiconformal mappings. - Bull. Malays. Math. Sci. Soc. (2) 39, 2016, 349–358.

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