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INVARIANT GRADIENT IN REFINEMENTS OF SCHWARZ AND HARNACK INEQUALITIES

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Abstract. In this paper we prove a refinement of Schwarz's lemma for holomorphic mappings from the unit ball $\mathbf{B}^n \subset \mathbf{C}^n$ to the unit disk $\mathbf{D} \subset \mathbf{C}$ obtained by Kalaj in [3]. We also give some corollaries of this result and a similar result for pluriharmonic functions. In particular, we give an improvement of Schwarz's lemma for non-vanishing holomorphic functions from $\mathbf{B}^n$ to $\mathbf{D}$ that was obtained in a recent paper by Dyakonov [2]. Finally, we give a new and short proof of Marković's theorem on contractivity of harmonic mappings from the upper half-plane $\mathbf{H}$ to the positive reals. The same result does not hold for higher dimensions, as is shown by given counterexamples.

1. Introduction and notation

1.1. Notation. We use terminology and notation from Rudin [7]. Let $\mathbf{B}^n$ denote the unit ball in $\mathbf{C}^n$, $\mathbf{B}_n$ the unit ball in $\mathbf{R}^n$ and let $\mathbf{H}_n$ denote the upper half-space in $\mathbf{R}^n$. Specially, $\mathbf{H}$ is the upper half-plane in $\mathbf{C}$. By $\mathcal{H}(\Omega)$ we denote the space of holomorphic functions on $\Omega \subset \mathbf{C}^n$. The complex scalar product of $z = (z_1, z_2, \ldots, z_n)$, $w = (w_1, w_2, \ldots, w_n) \in \mathbf{C}^n$ is given by

$$\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w_j}.$$

1.2. Bergman and hyperbolic distance. As is well known, the hyperbolic distance on $\mathbf{B}^n$ is given by the expression

$$d(z,w) = \log \frac{1 + |T_w(z)|}{1 - |T_w(z)|},$$

where

$$T_w(z) = \frac{(1-|w|^2)\,(z-w) - |z-w|^2 w}{1 + |z|^2|w|^2 - 2\operatorname{Re}\langle z, w\rangle}.$$

It is easy to check that

$$1 - |T_w(z)|^2 = \frac{(1-|w|^2)\,(1-|z|^2)}{1 + |w|^2|z|^2 - 2\operatorname{Re}\langle z, w\rangle}. \tag{1.1}$$

For more information on the hyperbolic distance and Möbius transformations, one can consult [1].

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We also use hyperbolic distances in some other domains in $\mathbf{C}$ and $\mathbf{R}^n$. Let us recall that the hyperbolic distance on $\mathbf{H}$ is given by

$$d_{\mathbf{H}}(z,w) = 2\tanh^{-1}\left|\frac{z-w}{\overline{z}-w}\right|,$$

and on $\mathbf{R}^+$ by

$$d_{\mathbf{R}^+}(x,y) = \log\frac{y}{x},$$

for $y \geq x > 0$. The hyperbolic distances on $\mathbf{B}_n$ and $\mathbf{H}_n$ are given by

$$d_{\mathbf{B}_n}(x,y) = \cosh^{-1}\left(1+\delta(x,y)\right), \quad \text{where } \delta(x,y) = \frac{2\|x-y\|^2}{(1-\|x\|^2)\,(1-\|y\|^2)},$$

$$d_{\mathbf{H}_n}(x,y) = \cosh^{-1}\left(1+\frac{\|x-y\|^2}{2x_n y_n}\right), \quad \text{where } x_n, y_n > 0.$$

We will use an involutive biholomorphic automorphism $\varphi_w\colon \mathbf{B}^n \to \mathbf{B}^n$ given by

$$\varphi_w(z) = \frac{1}{1-\langle z,w\rangle}\left(w - \frac{\langle z,w\rangle}{|w|^2}w - \left(1-|w|^2\right)^{\frac{1}{2}}\left(z - \frac{\langle z,w\rangle}{|w|^2}w\right)\right).$$

Note that $\varphi_w(0) = w$. An easy verification gives

$$1 - |\varphi_w(z)|^2 = \frac{(1-|w|^2)\,(1-|z|^2)}{|1-\langle z,w\rangle|^2}. \tag{1.2}$$

Bergman distance on $\mathbf{B}^n$ is given by the expression

$$\rho(z,w) = \log\frac{|1-\langle z,w\rangle| + \sqrt{|z-w|^2 + |\langle z,w\rangle|^2 - |z|^2|w|^2}}{|1-\langle z,w\rangle| - \sqrt{|z-w|^2 + |\langle z,w\rangle|^2 - |z|^2|w|^2}}; \tag{1.3}$$

it can be also written in the following form, using $\varphi_w(z)$:

$$\rho(z,w) = \log\frac{1+|\varphi_w(z)|}{1-|\varphi_w(z)|}. \tag{1.4}$$

Some general information and theorems on Bergman distance can be found in [4].

1.3. $\mathcal{M}$-invariant gradient. For a function $f \in C^1(\mathbf{B}^n)$, we define $\mathcal{M}$-invariant gradient by expression

$$\widetilde{D}f(z) = D\left(f\circ\varphi_z\right)(0),$$

where D denotes the complex gradient

$$Df(z) = \left\{\frac{\partial f}{\partial z_j}\right\}_{j=1}^{n}, \quad z_j = x_j + iy_j, \quad \frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + \frac{1}{i}\frac{\partial f}{\partial y_j}\right)$$

and φ_z is aforementioned automorphism of the unit ball $\mathbf{B}^n$. For $f \in \mathcal{H}\left(\mathbf{B}^n\right)$,

$$\left|\widetilde{D}f(0)\right| = |Df(0)| = |f'(0)|. \tag{1.5}$$

The main property of $\mathcal{M}$-invariant gradient is the following invariance property

$$\left|\widetilde{D}\left(f\circ\varphi_z\right)(0)\right| = \left|\widetilde{D}f(z)\right|. \tag{1.6}$$

Basic properties about this notion can be found in [7] and [6]. We will use the following identity

$$\left|\widetilde{D}f(z)\right|^2 = \left(1-|z|^2\right)\left(|Df(z)|^2 - |\langle Df(z),\overline{z}\rangle|^2\right), \quad f \in C^1(\mathbf{B}^n), \tag{1.7}$$

see [6] for details. On the other hand, the real gradient ∇ is defined by

$$\nabla f(z) = \left\{ \frac{\partial f}{\partial x_j}, \frac{\partial f}{\partial y_j} \right\}, \quad j = 1, 2 \ldots, n, \quad z_j = x_j + iy_j.$$

For a function $f \in C^1(\mathbf{B}^n)$ we also define $\mathcal{M}$-invariant real gradient

$$\widetilde{\nabla} f(z) = \nabla \left(f \circ \varphi_z \right)(0).$$

Analogous formulae hold for $\widetilde{\nabla}$ and $\widetilde{D}$, except for (1.5) where we have only

$$\left| \widetilde{\nabla} f(0) \right| = \left| \nabla f(0) \right|,$$

for $f \in C^1(\mathbf{B}^n)$.

1.4. Results. Schwarz's lemma is a fundamental result which states that for a holomorphic mapping $f \colon \mathbf{D} \to \mathbf{D}$, with $f(0) = 0$, we have

$$\left| f(z) \right| \le \left| z \right|.$$

This leads to another inequality which estimates the pseudohyperbolic distance between images of two points in terms of the pseudohyperbolic distance of points:

$$\left| \frac{f(z) - f(w)}{1 - \overline{f(z)} f(w)} \right| \le \left| \frac{z - w}{1 - \overline{z} w} \right|,$$

and also the magnitude of the derivative at a point z in terms of moduli of z and its image $f(z)$:

$$\left| f'(z) \right| \le \frac{1 - \left| f(z) \right|^2}{1 - \left| z \right|^2}.$$

These basic results have been extended in numerous ways. The one that is of special interest for us is the following theorem.

Theorem 1. (Kalaj, [3]) *If f is a holomorphic mapping of the unit ball $\mathbf{B}^n \subset \mathbf{C}^n$ into $\mathbf{B}^m \subset \mathbf{C}^m$, then for $m \ge 2$*

$$|f'(z)| \le \frac{\sqrt{1 - |f(z)|^2}}{1 - |z|^2}, \quad z \in \mathbf{B}^n,$$

and for $m = 1$ we have that

$$\left| f'(z) \right| \le \frac{1 - \left| f(z) \right|^2}{1 - \left| z \right|^2}, \quad z \in \mathbf{B}^n.$$

Here $|f'(z)|$ denotes the norm of the Frechet derivative of the mapping f. Let us recall that Frechet derivative of a holomorphic function $f \colon A \subset \mathbf{C}^n \to \mathbf{C}^m$ is the unique linear mapping $L = f'(z) \colon \mathbf{C}^n \to \mathbf{C}^m$ such that $f(z+h) - f(z) = f'(z)h + O(|h|^2)$.

Corollary 2. (Kalaj) *Every holomorphic function $f \colon \mathbf{B}^n \to \mathbf{D}$ is a contraction with respect to the hyperbolic metric on both $\mathbf{B}^n$ and $\mathbf{D}$.*

In section 2, we prove a refined version of Kalaj's theorem, using the so-called $\mathcal{M}$-invariant gradient $\widetilde{D}$.

Theorem 1.1. *For each holomorphic function $f \colon \mathbf{B}^n \to \mathbf{D}$ we have*

$$\left| \widetilde{D} f(z) \right| \le 1 - \left| f(z) \right|^2, \quad z \in \mathbf{B}^n,$$

while for each holomorphic mapping $f\colon \mathbf{B}^n \to \mathbf{B}^m$, $m \geq 2$, *we have*

$$\left|\widetilde{D}f(z)\right| \leq \sqrt{1-|f(z)|^2}, \quad z \in \mathbf{B}^n.$$

Since $|\widetilde{D}f(z)| \geq (1-|z|^2)\,|Df(z)|$ (see section 2) this indeed improves Kalaj's theorem. In the case of the unit disc $\mathbf{D}$ we give a new technique, while for $n \geq 2$ we only refine the Kalaj's result.

Theorem 1.2. *Every holomorphic function* $f\colon \mathbf{B}^n \to \mathbf{D}$ *is a contraction with respect to Bergman metric on* $\mathbf{B}^n$ *and* $\mathbf{D}$.

As a direct consequence we will get

Corollary 1.3. (Schwarz–Pick inequality for several variables) *For each holomorphic function* $f\colon \mathbf{B}^n \to \mathbf{D}$ *we have*

$$\left|\frac{f(z)-f(w)}{1-\overline{f(z)}f(w)}\right| \leq \frac{\sqrt{|z-w|^2+|\langle z,w\rangle|^2-|z|^2|w|^2}}{|1-\langle z,w\rangle|}. \tag{1.8}$$

Note that for $n=1$, this is the classical Schwarz–Pick inequality.

The following result on plurisubharmonic functions from $\mathbf{B}^n$ to $(-1,1)$ can be found in[3]:

Theorem 3. *Let* f *be a pluriharmonic function from the unit ball* $\mathbf{B}^n \subset \mathbf{C}^n$ *to* $(-1,1)$. *Then the following sharp inequality holds*

$$|\nabla f(z)| \leq \frac{4}{\pi}\frac{1-|f(z)|^2}{1-|z|^2}, \quad z \in \mathbf{B}^n.$$

Using the $\mathcal{M}$-invariant gradient, we get the following refinement:

Theorem 1.4. *For each pluriharmonic function* $f\colon \mathbf{B}^n \to (-1,1)$ *there holds the following inequality*

$$\left|\widetilde{\nabla} f(z)\right| \leq \frac{4}{\pi}\left(1-|f(z)|^2\right), \quad z \in \mathbf{B}^n.$$

We also deal with Harnack's inequalities. It is well known that for positive harmonic functions $u : \mathbf{D} \to \mathbf{R}^+$ we have

$$\frac{1-|z|}{1+|z|} \leq \frac{u(z)}{u(0)} \leq \frac{1+|z|}{1-|z|} \quad \text{and} \quad |\nabla u(z)| \leq \frac{2u(z)}{1-|z|}.$$

Dyakonov [2] proved the following lemma:

Lemma 4. *Suppose* $f \in \mathcal{H}(\mathbf{B}^n)$ *is a function satisfying* $0 < |f(z)| \leq 1$ *for all* $z \in \mathbf{B}^n$. *Then*

$$|Df(z)| \leq \frac{2\,|f(z)|}{1-|z|^2}\log\frac{1}{|f(z)|}.$$

Using the $\mathcal{M}$-invariant gradient form of Harnack's inequality, we prove the following refinement of Dyakonov's lemma:

Theorem 1.5. *For each holomorphic function* $f\colon \mathbf{B}^n \to \mathbf{D}$ *without zeros in* $\mathbf{B}^n$ *we have*

$$\left|\widetilde{D}f(z)\right| \leq 2\,|f(z)|\log\frac{1}{|f(z)|}, \quad z \in \mathbf{B}^n.$$

In section 3, we present a different proof, based only on Harnack's inequality, of the following Marković's theorem.

Theorem 5. (Marković, [5]) *Every harmonic function* $f\colon \mathbf{H} \to \mathbf{R}^+$ *acts as a contraction with respect to hyperbolic metric considered on both* $\mathbf{H}$ *and* $\mathbf{R}^+$.

We investigate the same problem in the upper half-space $\mathbf{H}_n \subset \mathbf{R}^n$ and the unit ball $\mathbf{B}_n \subset \mathbf{R}^n$: Are positive harmonic functions contractions if we consider hyperbolic metric on $\mathbf{H}^n$ or $\mathbf{B}^n$ and $\mathbf{R}^+$? The answer is "no", see Section 3.

2. Variations of the Schwarz lemma on the unit ball

Using the Cauchy–Schwarz inequality we obtain:

$$\begin{aligned}\left|\widetilde{D}f(z)\right|^2 &= \left(1-|z|^2\right)\left(|Df(z)|^2 - |\langle Df(z),\overline{z}\rangle|^2\right)\\ &\geq \left(1-|z|^2\right)\left(|Df(z)|^2 - |z|^2\,|Df(z)|^2\right)\\ &= \left(1-|z|^2\right)^2 |Df(z)|^2,\end{aligned}$$

and we have proved

$$\left|\widetilde{D}f(z)\right| \geq \left(1-|z|^2\right)|Df(z)|.$$

Proof of Theorem 1.1. (Case $m=1$.) Let us prove Theorem 1.1 in case $z=0$, i.e.

$$\left|\widetilde{D}f(0)\right| = |Df(0)| \leq 1-|f(0)|^2. \tag{2.1}$$

Let us fix $\zeta = (\zeta_1,\zeta_2,\dots,\zeta_n) \in \partial\mathbf{B}^n$ and define a one-variable complex function

$$g_\zeta(z) = f\left(\zeta_1 z, \zeta_2 z, \dots, \zeta_n z\right), \quad z \in \mathbf{D}.$$

Note that g_ζ maps $\mathbf{D}$ to $\mathbf{D}$ so we can apply Schwarz–Pick lemma to get

$$\left|g'_\zeta(0)\right| \leq 1-|g_\zeta(0)|^2.$$

Since

$$g'_\zeta(z) = \zeta_1\frac{\partial f}{\partial z_1}(z) + \zeta_2\frac{\partial f}{\partial z_2}(z) + \cdots + \zeta_n\frac{\partial f}{\partial z_n}(z),$$

by choosing

$$\zeta_i = \frac{1}{|Df(0)|}\overline{\frac{\partial f}{\partial z_i}(0)},$$

we obtain the needed estimate

$$|Df(0)| \leq 1-|g(0)|^2 = 1-|f(0)|^2.$$

Applying (2.1) to $f\circ\varphi_z$ we get

$$\left|\widetilde{D}\left(f\circ\varphi_z\right)(0)\right| \leq 1-\left|\left(f\circ\varphi_z\right)(0)\right|^2.$$

The left-hand side, by (1.6), is equal to $\left|\widetilde{D}f(z)\right|$, so we have

$$\left|\widetilde{D}f(z)\right| \leq 1-|f(z)|^2.$$

(Case $m \geq 2$.) According to [3], we have

$$\left|\widetilde{D}f(0)\right| = |Df(0)| \leq \sqrt{1-|f(0)|^2}.$$

Applying this inequality to the function $f\circ\varphi_z$, we get:

$$\left|\widetilde{D}f(z)\right| = \left|D\left(f\circ\varphi_z\right)(0)\right| \leq \sqrt{1-\left|f\circ\varphi_z(0)\right|^2} = \sqrt{1-|f(z)|^2}. \qquad \square$$

Proof of Theorem 1.2. We have to prove $\rho\left(f(z), f(w)\right) \leq \rho(z, w)$ for a holomorphic $f : \mathbf{B}^n \to \mathbf{D}$. By Corollary 2, we have

$$d\left(f(z), f(0)\right) \leq d(z, 0).$$

Since

$$\rho(z, 0) = \log \frac{1+|z|}{1-|z|} = d(z, 0), \quad z \in \mathbf{B}^n,$$

and the Bergman and hyperbolic metrics coincide on $\mathbf{D}$, i.e.

$$\rho\left(f(z), f(0)\right) = d\left(f(z), f(0)\right),$$

we conclude

$$\rho\left(f(z), f(0)\right) \leq \rho(z, 0). \tag{2.2}$$

For prescribed $z, w \in \mathbf{B}^n$ there exist an automorphism φ of the unit ball $\mathbf{B}^n$ such that $\varphi(0) = z$, $\varphi(a) = w$. Since φ is an isometry in the Bergman metric, we have

$$\rho(z, w) = \rho\left(\varphi(0), \varphi(a)\right) = \rho(0, a).$$

Using (2.2) with $f \circ \varphi$ in place of f, we obtain

$$\rho\left((f \circ \varphi)(0), (f \circ \varphi)(a)\right) \leq \rho(0, a),$$

which gives

$$\rho\left(f(z), f(w)\right) \leq \rho(0, a) = \rho(z, w). \qquad \square$$

From (1.1) and (1.2) it easily follows that $|T_w(z)| \geq |\varphi_w(z)|$, which leads to $\rho(z, w) \leq d(z, w)$. We see that Theorem 1.2 indeed refines Corollary 2.

Proof of Corollary 1.3. By Theorem 1.2 and (1.3) we have

$$\log \frac{1 + \left|\frac{f(z)-f(w)}{1-\overline{f(z)}f(w)}\right|}{1 - \left|\frac{f(z)-f(w)}{1-\overline{f(z)}f(w)}\right|} \leq \log \frac{|1-\langle z, w\rangle| + \sqrt{|z-w|^2 + |\langle z, w\rangle|^2 - |z|^2|w|^2}}{|1-\langle z, w\rangle| - \sqrt{|z-w|^2 + |\langle z, w\rangle|^2 - |z|^2|w|^2}}$$

which implies

$$\left|\frac{f(z)-f(w)}{1-\overline{f(z)}f(w)}\right| \leq \frac{\sqrt{|z-w|^2 + |\langle z, w\rangle|^2 - |z|^2|w|^2}}{|1-\langle z, w\rangle|}. \qquad \square$$

Proof of the Theorem 1.4. The proof is similar to that of Theorem 1.1. For $z = 0$ we have

$$\left|\widetilde{\nabla} f(0)\right| = |\nabla f(0)| \leq \frac{4}{\pi}\left(1 - |f(0)|^2\right).$$

Now, using this inequality for $f \circ \varphi_z$ we have, by $\widetilde{\nabla}$-version of (1.6):

$$\left|\widetilde{\nabla} f(z)\right| = \left|\widetilde{\nabla}\left(f \circ \varphi_z\right)(0)\right| \leq \frac{4}{\pi}\left(1 - |f \circ \varphi_z(0)|^2\right) = \frac{4}{\pi}\left(1 - |f(z)|^2\right). \qquad \square$$

A consequence of the Harnack's inequality for a positive harmonic function u in $\mathbf{D}$ is

$$|\nabla u(z)| \leq \frac{2u(z)}{1-|z|^2}. \tag{2.3}$$

Using Poisson representation we prove this for $z = 0$ and extend the result to any $z \in \mathbf{D}$ using automorphisms φ_z.

Proof of Theorem 1.5. Let us fix $\zeta \in \partial\mathbf{B}^n$ and apply (2.3) to positive harmonic function

$$u(z) = \log \frac{1}{|F_\zeta(z)|},$$

where $F_\zeta(z) = f\left(\zeta_1 z, \zeta_2 z, \ldots, \zeta_n z\right),\ z \in \mathbf{D}$ for point $z = 0$:

$$\left|F'_\zeta(0)\right| \leq 2\left|F_\zeta(0)\right| \log \frac{1}{|F_\zeta(0)|} = 2|f(0)| \log \frac{1}{|f(0)|}.$$

Next

$$F'_\zeta(0) = \sum_{i=1}^{n} \zeta_i \frac{\partial f}{\partial z_i}(0),$$

by choosing

$$\zeta_i = \frac{1}{|Df(0)|} \frac{\partial f}{\partial z_i}(0),$$

where $|Df(0)| = \sqrt{\sum_{i=1}^n \left|\frac{\partial f}{\partial z_i}(0)\right|^2}$, we obtain

$$|Df(0)| \leq 2\left|f(0)\right| \log \frac{1}{|f(0)|}.$$

Because of (1.5) we get

$$\left|\widetilde{D}f(0)\right| \leq 2\left|f(0)\right| \log \frac{1}{|f(0)|}. \tag{2.4}$$

Now, using inequality (2.4) for $f \circ \varphi_z$, $z \in \mathbf{B}^n$ in place of f, we obtain

$$\left|\widetilde{D}f(z)\right| = \left|\widetilde{D}\left(f \circ \varphi_z\right)(0)\right| \leq 2\left|\left(f \circ \varphi_z\right)(0)\right| \log \frac{1}{\left|\left(f \circ \varphi_z\right)(0)\right|} = 2\left|f(z)\right| \log \frac{1}{|f(z)|},$$

and Theorem 1.5 is proved. □

3. Marković's theorem and counterexamples

Here we give a new proof of Marković's theorem on positive harmonic functions on the upper half-plane and obtain counterexamples for higher-dimensional analogues.

We will use common Harnack inequality, which estimates the ratio of values of positive harmonic functions at an arbitrary point z and at the zero:

$$\frac{1-|z|}{1+|z|} \leq \frac{v(z)}{v(0)} \leq \frac{1+|z|}{1-|z|}. \tag{3.1}$$

3.1. Marković's theorem. For every harmonic function $v\colon \mathbf{D} \to \mathbf{R}^+$ we can define harmonic function $u\colon \mathbf{H} \to \mathbf{R}^+$ by

$$u(z) = v(\varphi(z)), \quad \text{where} \quad \varphi(z) = \frac{z-i}{z+i},$$

is a conformal mapping from $\mathbf{H}$ to $\mathbf{D}$. Conversely, for every harmonic $u\colon \mathbf{H} \to \mathbf{R}^+$ there is a harmonic $v\colon \mathbf{D} \to \mathbf{R}^+$ given by $v(z) = u(\varphi^{-1}(z))$, for which we have $u = v \circ \varphi$.

So, for $\zeta \in \mathbf{H}$, $\frac{\zeta-i}{\zeta+i} \in \mathbf{D}$, using (3.1) we obtain

$$\frac{v\left(\frac{\zeta-i}{\zeta+i}\right)}{v(0)} \leq \frac{1+\left|\frac{\zeta-i}{\zeta+i}\right|}{1-\left|\frac{\zeta-i}{\zeta+i}\right|} = \frac{1+\left|\frac{\zeta-i}{\overline{\zeta}-i}\right|}{1-\left|\frac{\zeta-i}{\overline{\zeta}-i}\right|}, \quad \text{i.e.} \quad \frac{u(\zeta)}{u(i)} \leq \frac{1+\left|\frac{\zeta-i}{\overline{\zeta}-i}\right|}{1-\left|\frac{\zeta-i}{\overline{\zeta}-i}\right|},$$

and after taking logarithms of both sides

$$d\left(u(\zeta), u(i)\right) \leq d(\zeta, i). \tag{3.2}$$

Let ψ be a conformal automorphism of the upper half-plane which sends ζ and i to z and w, respectively. We have

$$d(\zeta, i) = d(\psi(\zeta), \psi(i)) = d(z, w)$$

and

$$d(u \circ \psi(\zeta), u \circ \psi(i)) = d(u(z), u(w)),$$

so (3.2) with $u \circ \psi$ instead of u gives

$$d(u(z), u(w)) \leq d(z, w).$$

So, Theorem 5 is proved.

Using the method of the above proof, one can show that any positive harmonic function from $\mathbf{D}$ to $(0, +\infty)$ is a contraction with respect to hyperbolic metrics on $\mathbf{D}$ and $\mathbf{R}^+$.

In the next subsection we provide counterexamples which show that these results do not extend to higher dimensions.

3.2. Counterexamples for $n \geq 3$.

Example 1. (The unit ball $\mathbf{B}_n$) The following Harnack's inequality for positive harmonic functions in the unit ball $\mathbf{B}_n$

$$\frac{1 - |x|}{(1 + |x|)^{n-1}} \leq \frac{v(x)}{v(0)} \leq \frac{1 + |x|}{(1 - |x|)^{n-1}}$$

gives us a clue for counterexamples in higher dimensions.

For $n \geq 3$, the hyperbolic metric in the unit ball $\mathbf{B}_n$ in $\mathbf{R}^n$ is given by

$$d(x, y) = \cosh^{-1}(1 + \delta(x, y)) = \log\left(1 + \delta(x, y) + \sqrt{(1 + \delta(x, y))^2 - 1}\right),$$

where

$$\delta(x, y) = \frac{2\|x - y\|^2}{(1 - \|x\|^2)(1 - \|y\|)^2}$$

and $\|\cdot\|$ denotes the usual Euclidean norm.

So, for $y = 0$ we have

$$\delta(x, 0) = \frac{2\|x\|^2}{1 - \|x\|^2},$$

$$d(x, 0) = \log\left(1 + \frac{2\|x\|^2}{1 - \|x\|^2} + \sqrt{\left(\frac{2\|x\|^2}{1 - \|x\|^2}\right)^2 - 1}\right) = \log\frac{1 + \|x\|}{1 - \|x\|}.$$

If $u\colon \mathbf{B}_n \to \mathbf{R}^+$ is harmonic, then

$$d(u(x), u(0)) \leq d(x, 0) \quad \text{if and only if} \quad \frac{u(x)}{u(0)} \leq \frac{1 + \|x\|}{1 - \|x\|}. \tag{3.3}$$

But, for $\mathbf{1} = (1, 0, \ldots, 0) \in \mathbf{R}^n$ and $n \geq 3$, the function

$$u(x) = \frac{1 - \|x\|^2}{\|\mathbf{1} - x\|^n}$$

is positive harmonic, and setting $x = (t, 0, \dots, 0)$, $0 < t < 1$, inequality (3.3) gives

$$\frac{1-t^2}{(1-t)^n} \leq \frac{1+t}{1-t},$$

that is $(1-t)^2 \leq (1-t)^n$ which cannot hold for $n \geq 3$.

Example 2. (The upper half-space) In the case of the upper half-space $\mathbf{H}_n \subset \mathbf{R}^n$ for $n \geq 3$, the hyperbolic metric is given by

$$d(x,y) = \cosh^{-1}\left(1 + \frac{\|x-y\|^2}{2x_n y_n}\right),$$

where $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$ and $x_n, y_n > 0$.

For $x = (0, 0, \dots, t)$, $t > 1$ and $y = (0, 0, \dots, 1)$, we have

$$\delta(x,y) = 1 + \frac{\|x-y\|^2}{2x_n y_n} = 1 + \frac{(t-1)^2}{2t} = \frac{t^2+1}{2t}$$

and

$$d(x,y) = \log\left(\delta(x,y) + \sqrt{\delta(x,y)^2 - 1}\right) = \log\left(\frac{t^2+1}{2t} + \sqrt{\left(\frac{t^2+1}{2t}\right)^2 - 1}\right) = \log t,$$

so $d(u(x), u(y)) \leq d(x,y)$ is equivalent to

$$\frac{u(y)}{u(x)} \leq t. \tag{3.4}$$

Choosing positive harmonic function $u(x)$ to be

$$u(x) = \frac{x_n}{\|x\|^n},$$

we obtain

$$\frac{u(y)}{u(x)} = \frac{1}{t^{1-n}} = t^{n-1},$$

which cannot be smaller than t for $n \geq 3$, for $t > 1$.

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