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ON A QUESTION OF GUNDERSEN CONCERNING THE GROWTH OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract. We prove that every nontrivial solution of $f'' + A(z)f' + Q(z)f = 0$ is of infinite order, where $A(z)$ is an entire function satisfying $\lambda(A) < \rho(A) < \infty$ and some restrictions, and $Q(z)$ is a non-constant polynomial. This result gives partial solutions to a question posed by Gundersen. Related results are also given.

1. Introduction and main results

We use the standard notations of Nevanlinna theory of meromorphic functions in this paper, such as, $T(r, f)$, $m(r, f)$, $N(r, f)$ and so on; for more detail, see [11, 14, 26]. For a meromorphic function f in the complex plane $\mathbf{C}$, the order of growth and the lower order of growth are defined as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r}, \quad \mu(f) = \liminf_{r\to\infty} \frac{\log^+ T(r, f)}{\log r},$$

respectively. If f is an entire function, then the Nevanlinna characteristic $T(r, f)$ can be replaced with $\log M(r, f)$, where $M(r, f) = \max_{|z|=r} |f(z)|$ is the usual maximum modulus of f, see [26, p. 10]. Let $\lambda(f)$ denote the exponent of convergence of the zeros of f. Obviously, $\lambda(f) \le \rho(f)$ for every meromorphic function f.

The growth of solutions of the following equation

$$f'' + A(z)f' + B(z)f = 0 \tag{1.1}$$

is studied in this paper, where $A(z)$ and $B(z)$ ($\not\equiv 0$) are entire functions. There are many results in the literature that concern the order of growth of solutions of (1.1), see [14] and [15]. For the case of a transcendental entire function $B(z)$, the following result is a summary of results derived from Gundersen [9], Hellerstein, Miles and Rossi [12], and Ozawa [23].

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Theorem 1.1. *Suppose that $A(z)$ and $B(z)$ are entire functions satisfying one of the following conditions:*

(i) $\rho(A) < \rho(B)$;
(ii) $A(z)$ *is a polynomial and* $B(z)$ *is a transcendental entire function;*
(iii) $\rho(B) < \rho(A) \leq \frac{1}{2}$.

Then every nontrivial solution of (1.1) *is of infinite order.*

Motivated by Theorem 1.1, many parallel results written thereafter focus on the case $\rho(A) \geq \rho(B)$ and $B(z)$ is a transcendental entire function; see, for example, [3, 16, 18, 19, 24, 25]. Regarding the case of a polynomial $B(z)$, there are many results concerning the growth of solutions of the following special equation

$$f'' + e^{-z} f' + Q(z) f = 0, \tag{1.2}$$

where $Q(z)$ is a polynomial.

Frei [4] proved that (1.2) has a nontrivial solution of finite order if and only if $Q(z) = -n^2$, where n is a positive integer. The case where $Q(z)$ is non-constant is more difficult to resolve, and the following result is a summary of results derived from Amemiya and Ozawa [1], Gundersen [6], and Ozawa [23].

Theorem 1.2. *Let $Q(z) = b_m z^m + b_p z^p + \cdots + b_0$ be a non-constant polynomial $(b_m \neq 0)$ satisfying one of the following conditions:*

(i) m *is odd;*
(ii) m *is even and* $m \geq 2p + 3$;
(iii) $m = 2$ *and* $b_1 = b_0 = 0$;
(iv) m *is even and* $b_m(-1)^{\frac{m}{2}}$ *is not real and negative.*

Then every nontrivial solution of (1.2) *is of infinite order.*

Motivated by Theorem 1.2, Langley [17] obtained the following general result which completely resolves the growth problem of solutions of (1.2).

Theorem 1.3. *Let $Q(z)$ be a non-constant polynomial. Then all nontrivial solutions of*

$$f'' + Ae^{-z} f' + Q(z) f = 0$$

have infinite order, for any $A \in \mathbf{C} \backslash \{0\}$.

From Theorem 1.3, a natural idea is whether every nontrival solution of (1.2) is of infinite order if e^{-z} is replaced by a more general entire function. Accordingly, Gundersen [10, Question 5.1] posed the following question.

Gundersen's Question. If $A(z)$ is an entire function with $\lambda(A) < \rho(A) < \infty$, and $Q(z) = b_m z^m + \cdots + b_0$ is a non-constant polynomial, then does every nontrivial solution of

$$f'' + A(z) f' + Q(z) f = 0 \tag{1.3}$$

have infinite order?

Here we consider Gundersen's Question and prove the following result.

Theorem 1.4. *Let $A(z) = \upsilon(z) e^{P(z)}$, where $\upsilon(z)$ $(\not\equiv 0)$ is an entire function and $P(z) = a_n z^n + \cdots + a_0$ is a polynomial of degree n, such that $\rho(\upsilon) < n$. Let $Q(z) = b_m z^m + \cdots + b_0$ be a non-constant polynomial of degree m. Then all nontrivial solutions of* (1.3) *have infinite order if one of the following conditions holds:*

(i) $m + 2 < 2n$;
(ii) $m + 2 > 2n$ *and* $m + 2 \neq 2kn$ *for all integers* k;

(iii) $m+2=2n$ and $\frac{a_n^2}{b_m}$ is not real and negative.

From Theorem 1.4, it follows that Gundersen's Question holds for the case of $m+2<2n$, and we get partial results on Gundersen's Question for the case of $m+2\geq 2n$.

It seems interesting to find a special transcendental entire function $A(z)$ with $\lambda(A)=\rho(A)$ such that every nontrivial solution of (1.3) is of infinite order. To this end, we note that there are many studies in the literature concerning the growth of solutions of the following equation

$$w''+P(z)w=0, \tag{1.4}$$

where $P(z)$ is a non-constant polynomial. Solutions of (1.4) have some striking properties, see Lemma 2.2 below in Section 2. From Lemma 2.2, it is usually the case that $\lambda(A)=\rho(A)$ when $A(z)$ is a nontrivial solution of (1.4). A new idea, in which the properties of solutions of (1.4) are considered, is used to study the growth of solutions of complex differential equations, see, for example, [21, 25]. Here the idea will be used again to study the case of $\lambda(A)=\rho(A)$, and we get the following result.

Theorem 1.5. *Let $A(z)$ be a nontrivial solution of* (1.4), *where $P(z)=a_nz^n+\cdots+a_0$ is a polynomial of degree $n\geq 1$. Let $Q(z)=b_mz^m+\cdots+b_0$ be a polynomial of degree $m\geq 1$. Then all nontrivial solutions of* (1.3) *have infinite order if one of the following conditions holds:*

(i) $m<n$;
(ii) $m>n$ *and* $m+2\neq k(n+2)$ *for all integers* k;
(iii) $m=n$ *and* $\frac{a_n}{b_m}$ *is not real and positive.*

By using the proofs of Theorems 1.4 and 1.5, the conditions of Theorems 1.4 and 1.5 show that, on most rays from the origin, the coefficient $A(z)$ has either fast growth or small growth, so we also find the following conditions to show that all nontrivial solutions of (1.3) have infinite order.

Theorem 1.6. *Let $A(z)$ be a transcendental entire function of finite order with the following property: there exists a set $H\subseteq R$ of linear measure zero, such that for each real $\theta\notin H$, either*

(i) $r^{-N}|A(re^{i\theta})|\to\infty$ *as* $r\to\infty$, *for each* $N>0$; *or*
(ii) *there exists* $n\geq 0$, *possibly depending on* θ, *such that* $n+1<\rho(A)$ *and*

$$A(re^{i\theta})=O(r^n)\quad \text{as}\quad r\to\infty.$$

If $Q(z)=b_mz^m+\cdots+b_0$ is a non-constant polynomial with $m+2<2\rho(A)$, then every nontrivial solution of (1.3) *is of infinite order.*

2. Auxiliary results

In this section, we collect some lemmas which will be used in proving our theorems. The following lemma on logarithmic derivatives is due to Gundersen [8].

Lemma 2.1. *Let f be a transcendental meromorphic function of finite order $\rho(f)$. Let $\varepsilon>0$ be a given real constant, and let k and j be two integers such that $k>j\geq 0$. Then there exists a set $E\subset[0,2\pi)$ that has linear measure zero, such that if $\psi_0\in[0,2\pi)-E$, then there is a constant $R_0=R_0(\psi_0)>1$ such that for all z*

satisfying $\arg z = \psi_0$ *and* $|z| \geq R_0$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho(f)-1+\varepsilon)}.$$

In order to prove the Theorem 1.5, an auxiliary result is also needed, in which the properties of solutions of $w'' + P(z)w = 0$ are described. To this end, some notations are stated. Let $\alpha < \beta$ be such that $\beta - \alpha < 2\pi$, and let $r > 0$. Denote

$$S(\alpha, \beta) = \{z\colon \alpha < \arg z < \beta\},$$
$$S(\alpha, \beta, r) = \{z\colon \alpha < \arg z < \beta\} \cap \{z\colon |z| < r\}.$$

Let $\overline{F}$ denote the closure of F. Let A be an entire function of order $\rho(A) \in (0, \infty)$. For simplicity, set $\rho = \rho(A)$ and $S = S(\alpha, \beta)$. We say that A blows up exponentially in $\overline{S}$ if for any $\theta \in (\alpha, \beta)$,

$$\lim_{r\to\infty} \frac{\log\log|A(re^{i\theta})|}{\log r} = \rho$$

holds. We also say that A decays to zero exponentially in $\overline{S}$ if for any $\theta \in (\alpha, \beta)$,

$$\lim_{r\to\infty} \frac{\log\log|A(re^{i\theta})|^{-1}}{\log r} = \rho$$

holds.

The following lemma, originally due to Hille [13, Chapter 7.4], see also [7, 20, 22], plays an important role in proving Theorem 1.5. The method used in proving the lemma is typically referred to as the method of asymptotic integration.

Lemma 2.2. *Let* A *be a nontrivial solution of* $w'' + P(z)w = 0$, *where* $P(z) = a_n z^n + \cdots + a_0$, $a_n \neq 0$. *Set* $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ *and* $S_j = S(\theta_j, \theta_{j+1})$, *where* $j = 0, 1, 2, \ldots, n+1$ *and* $\theta_{n+2} = \theta_0 + 2\pi$. *Then* A *has the following properties.*

(1) *In each sector* S_j, A *either blows up or decays to zero exponentially.*
(2) *If, for some* j, A *decays to zero in* S_j, *then it must blow up in* S_{j-1} *and* S_{j+1}. *However, it is possible for* A *to blow up in many adjacent sectors.*
(3) *If* A *decays to zero in* S_j, *then* A *has at most finitely many zeros in any closed sub-sector within* $S_{j-1} \cup \overline{S_j} \cup S_{j+1}$.
(4) *If* A *blows up in* S_{j-1} *and* S_j, *then for each* $\varepsilon > 0$, A *has infinitely many zeros in each sector* $\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon)$, *and furthermore, as* $r \to \infty$,

$$n(\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r), 0, A) = (1 + o(1))\frac{2\sqrt{|a_n|}}{\pi(n+2)} r^{\frac{n+2}{2}},$$

where $n(\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r), 0, A)$ *is the number of zeros of* A *in the region* $\overline{S}(\theta_j - \varepsilon, \theta_j + \varepsilon, r)$.

The next two lemmas show the asymptotic properties of solutions of $w'' + Q(z)w = 0$ when the coefficient $Q(z)$ has small growth in some domain.

Lemma 2.3. [17] *Let* S *be the strip*

$$z = x + iy, \quad x \geq x_0, \quad |y| \leq 4.$$

Suppose that $Q(z)$ *is analytic in* S *such that*

$$Q(z) = b_m z^m + O(|z|^{m-2}), \quad z \in S,$$

where m is a positive integer and $b_m > 0$. Then there exists a path Γ tending to infinity in S such that all solutions of

$$w'' + Q(z)w = 0$$

tend to zero on Γ.

Lemma 2.4. [2] *Suppose that $Q(z)$ is analytic in a sector containing the ray $\gamma\colon re^{i\theta}$ and that as $r \to \infty$,*

$$|Q(re^{i\theta})| = O(r^m)$$

for some $m \geq 0$. Then all solutions of

$$g'' + Q(z)g = 0$$

satisfy

$$\log^+ |g(re^{i\theta})| = O(r^{\frac{m+2}{2}})$$

on γ.

We introduce the definitions of critical rays of $\exp(P(z))$ and $Q(z)$ which are needed in the proof of our results, where $P(z)$ and $Q(z)$ are polynomials.

Definition 2.5. Let $P(z) = a_n z^n + \cdots + a_0$ be a polynomial, where $a_n = \alpha + i\beta \neq 0$. Set $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. A ray $\arg z = \theta$ from the origin is said to be a critical ray of $e^{P(z)}$ if $\delta(P, \theta) = 0$.

Obviously, $e^{P(z)}$ has $2n$ critical rays, which divide the whole plane into $2n$ sectors, where the length of each sector is equal to $\frac{\pi}{n}$, say $\arg z = \theta_j$, $\arg z = \varphi_j$, $\theta_1 < \varphi_1 < \theta_2 < \varphi_2 < \cdots < \theta_n < \varphi_n < \theta_{n+1} = \theta_1 + 2\pi$, $j = 1, 2, \ldots, n$. It is not hard to see that the function $e^{P(z)}$ has the property that there are n disjoint sectors satisfying $\delta(P, \theta) > 0$, and n other disjoint sectors satisfying $\delta(P, \theta) < 0$, see [19]. Without loss of generality, set

$$S_j^+(\theta_j, \varphi_j) = \{z\colon \delta(P, \theta) > 0, \theta_j < \arg z = \theta < \varphi_j\}, \quad j = 1, 2, \ldots, n,$$

and

$$S_j^-(\varphi_j, \theta_{j+1}) = \{z\colon \delta(P, \theta) < 0, \varphi_j < \arg z = \theta < \theta_{j+1}\}, \quad j = 1, 2, \ldots, n.$$

In order to state later, the following notations are needed. Set

$$S^+ = \bigcup_{j=1}^{n} S_j^+(\theta_j, \varphi_j), \quad S^- = \bigcup_{j=1}^{n} S_j^-(\varphi_j, \theta_{j+1}),$$

and

$$E^+ = \bigcup_{j=1}^{n} (\theta_j, \varphi_j), \quad E_- = \bigcup_{j=1}^{n} (\varphi_j, \theta_{j+1}).$$

Definition 2.6. Let $Q(z) = b_m z^m + \cdots + b_0$ be a polynomial with $b_m \neq 0$. A ray $\arg z = \theta$ from the origin is said to be a critical ray of $Q(z)$ if $\arg b_m + (m+2)\theta = 0 \pmod{2\pi}$.

The next lemma gives the asymptotic properties on most rays of the function $\upsilon(z)\exp(P(z))$ in Theorem 1.4.

Lemma 2.7. [2] *Let $P(z)$ be a polynomial of degree $n \geq 1$, and let $\varepsilon > 0$ be a given constant. Let $\upsilon(z)$ ($\not\equiv 0$) be analytic for all z of sufficiently large modulus, and of order less than n. Consider the function $A(z) = \upsilon(z)\exp(P(z))$ on a ray $\arg z = \theta$. Then there exists a set $E \subset [0, 2\pi)$ with linear measure zero, such that*

(i) *If* $\theta \in E^+ - E$, *there exists a* $R(\theta) > 1$ *such that for* $r > R(\theta)$,

$$|A(re^{i\theta})| \geq \exp((1-\varepsilon)\delta(P,\theta)r^n).$$

(ii) *If* $\theta \in E^- - E$, *there exists a* $R(\theta) > 1$ *such that for* $r > R(\theta)$,

$$|A(re^{i\theta})| \leq \exp((1-\varepsilon)\delta(P,\theta)r^n).$$

Lemma 2.8. *Let* $A(z)$ *be an entire function satisfying* $\lim_{r\to\infty} |A(re^{i\phi})| = 0$ *for* $\phi \in (\theta, \varphi)$, *where* θ, φ *are two real constants such that* $0 < \varphi - \theta < 2\pi$. *Then for any* $\eta \in (0, \frac{\varphi-\theta}{4})$, *there exists a positive constant* M *such that the following three statements hold.*

(i) $A(z)$ *is uniformly continuous in* $S^1 = S(\theta+2\eta, \varphi-2\eta) \cap \{z\colon |z| \geq 1\}$;

(ii) $|A'(re^{i\nu})| \leq \frac{M}{\sin\eta}$ *for* $\nu \in (\theta+2\eta, \varphi-2\eta)$ *and* $r > 2$;

(iii) $|\int_{z_0}^{re^{i\nu}} A(t)\,dt| < Mr + Mr_0(2\pi-1)$ *for* $\nu \in (\theta+2\eta, \varphi-2\eta)$ *and* $r > 2$, *where* $z_0 = r_0e^{i\theta_0}$ *is a given point in* $S(\theta+\eta, \varphi-\eta)$.

Proof. Obviously, there exists a positive constant M such that $|A(z)| \leq M$ for $z \in S(\theta+\eta, \varphi-\eta)$. Note that

$$d(z, \partial S(\theta,\varphi)) \geq \sin\eta, \quad z \in S^1,$$

where $d(z, \partial S(\theta,\varphi))$ denotes the distance between z and the boundary of $S(\theta,\varphi)$.

(i) For any given $\varepsilon \in (0, M)$, there exists a $\delta < \frac{\varepsilon\sin\eta}{4M}$, such that for any z_1, $z_2 \in S^1$ with $|z_1 - z_2| < \delta$, we get

$$|\zeta - z_2| \geq |\zeta - z_1| - |z_1 - z_2| \geq \frac{\sin\eta}{2} - \frac{\sin\eta}{4} = \frac{\sin\eta}{4},$$

where we denote by γ the circle $|\zeta - z_1| = \frac{\sin\eta}{2}$. Then it follows from Cauchy's formula that

$$\begin{aligned}|A(z_1) - A(z_2)| &= \left|\frac{1}{2\pi i}\int_\gamma \frac{A(\zeta)}{\zeta - z_1}\,d\zeta - \frac{1}{2\pi i}\int_\gamma \frac{A(\zeta)}{\zeta - z_2}\,d\zeta\right| \leq \frac{1}{2\pi}\int_\gamma \frac{|A(\zeta)||z_1 - z_2|}{|\zeta - z_1||\zeta - z_2|}|d\zeta| \\ &\leq \frac{1}{2\pi}\cdot M|z_1 - z_2|\cdot\frac{1}{\frac{\sin\eta}{2}\cdot\frac{\sin\eta}{4}}\cdot 2\pi\cdot\frac{\sin\eta}{2} \leq \frac{4M}{\sin\eta}|z_1 - z_2| < \varepsilon.\end{aligned}$$

So $A(z)$ is uniformly continuous in S^1.

(ii) Suppose that $\nu \in (\theta+2\eta, \varphi-2\eta)$ and $r > 2$, we denote by γ_1 the circle $|\zeta - re^{i\nu}| = \sin\eta$. By the Cauchy's formula we deduce that

$$|A'(re^{i\nu})| = \left|\frac{1}{2\pi i}\int_{\gamma_1} \frac{A(\zeta)}{(\zeta - re^{i\nu})^2}\,d\zeta\right| \leq \frac{M}{2\pi}\cdot\frac{1}{(\sin\eta)^2}\cdot 2\pi\sin\eta = \frac{M}{\sin\eta}.$$

(iii) Let $z_0 = r_0e^{i\theta_0} \in S(\theta+\eta, \varphi-\eta)$ be a given point. For any $\nu \in (\theta+2\eta, \varphi-2\eta)$, and $r > 2$, we get

$$\begin{aligned}\left|\int_{z_0}^{re^{i\nu}} A(t)\,dt\right| &\leq \left|\int_{\theta_0}^{\nu} A(r_0e^{i\theta})ir_0e^{i\theta}d\theta\right| + \left|\int_{r_0}^{r} A(xe^{i\nu})e^{i\nu}dx\right| \\ &\leq Mr_0|\nu - \theta_0| + M(r - r_0) \leq Mr_0(2\pi - 1) + Mr.\end{aligned}$$

□

3. Proof of Theorem 1.4

Suppose that $\arg z = \theta_j, \varphi_j$ are $2n$ critical rays of $e^{P(z)}$, $j = 1, 2, \ldots, n$, $\theta_{n+1} = \theta_1 + 2\pi$. Note that by a suitable translation, for instance, $z = \xi - \frac{b_{m-1}}{mb_m}$, the equation (1.3) is turned into the following form

$$f'' + A_1(\xi)f' + Q_1(\xi)f = 0,$$

where $A_1(\xi)$ satisfies $\lambda(A_1) < \rho(A_1) < \infty$, and

$$Q_1(\xi) = b_m\xi^m + c_{m-2}\xi^{m-2} + \cdots, \quad m \geq 2,$$

or

$$Q_1(\xi) = b_m\xi, \quad m = 1.$$

Hence we assume directly that $Q(z)$ in (1.3) has the following form

$$Q(z) = b_m z^m + b_{m-2}z^{m-2} + \cdots + b_0.$$

Now suppose on the contrary to the assertion that there exists a nontrivial solution f of (1.3) with $\rho(f) < \infty$. By (1.3), we get

$$\frac{f''}{f} + A(re^{i\theta})\frac{f'}{f} + Q(re^{i\theta}) = 0. \tag{3.1}$$

By using Lemma 2.1, there exists a set $E_1 \subset [0, 2\pi)$ with linear measure zero and a positive number K, such that for all sufficiently large r and $\theta \notin E_1$, we have

$$\left|\frac{f''(re^{i\theta})}{f(re^{i\theta})}\right| \leq r^K, \quad \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| \leq r^K. \tag{3.2}$$

By using Lemma 2.7, for any $\theta \in ([0, 2\pi) \cap E^+) - (E_1 \cup E_2)$ and positive number ε, there exists a positive number $R_1(\theta)$, such that

$$|A(re^{i\theta})| \geq \exp((1-\varepsilon)\delta(P, \theta)r^n) \tag{3.3}$$

holds for $r > R_1(\theta)$, where $E_2 \subset [0, 2\pi)$ with linear measure zero. Combining (3.1), (3.2) and (3.3), for any $\theta \in ([0, 2\pi) \cap E^+) - (E_1 \cup E_2)$, we get

$$\frac{f'(re^{i\theta})}{f(re^{i\theta})} = o(\frac{1}{r^2})$$

as $r \to \infty$. By the Phragmén–Lindelöf principle, $f(re^{i\theta})$ tends to a nonzero finite constant for $\theta \in E^+$, without loss of generality, say

$$f(re^{i\theta}) \to 1 \tag{3.4}$$

as $r \to \infty$ with $\theta \in E^+$. Meanwhile, for any $\theta \in ([0, 2\pi) \cap E^-) - (E_1 \cup E_2)$ and ε given above, by using Lemma 2.7 again, there exists a positive number $R_2(\theta)$, such that

$$|A(re^{i\theta})| \leq \exp((1-\varepsilon)\delta(P, \theta)r^n) \tag{3.5}$$

holds for $r > R_2(\theta)$. This implies that for $\theta \in ([0, 2\pi) \cap E^-) - (E_1 \cup E_2)$,

$$\lim_{r\to\infty} |A(re^{i\theta})| = 0.$$

Let $0 < \eta \leq \min\{\frac{\pi}{5\rho(f)}, \frac{\theta_{i+1}-\varphi_i}{5}\}$. By Lemma 2.8, we know that both $|A(re^{i\theta})|$ and $|A'(re^{i\theta})|$ are bounded in every sector $S_i^-(\varphi_i + 2\eta, \theta_{i+1} - 2\eta)$, $i = 1, 2, \ldots, n$.

Set $f(z) = g(z)\exp(-\frac{1}{2}\int^z A(t)dt)$. Then $g(z)$ satisfies the following equation

$$g'' + (Q(z) - \frac{A^2(z)}{4} - \frac{A'(z)}{2})g = 0. \tag{3.6}$$

Hence, for $\nu \in (\varphi_i + 2\eta, \theta_{i+1} - 2\eta)$, we obtain that

$$\left|Q(re^{i\nu}) - \frac{A^2(re^{i\nu})}{4} - \frac{A'(re^{i\nu})}{2}\right| = O(r^m)$$

as $r \to \infty$. It is not hard to see from Lemma 2.4 that for every nontrivial solution $g(z)$ of (3.6),

$$\log^+ |g(re^{i\nu})| = O(r^{(m+2)/2})$$

as $r \to \infty$ with $\nu \in (\varphi_i + 2\eta, \theta_{i+1} - 2\eta)$. By (iii) of Lemma 2.8, for $\nu \in \bigcup_{i=1}^n (\varphi_i + 2\eta, \theta_{i+1} - 2\eta)$,

$$\log^+ |f(re^{i\nu})| \le \log^+ |g(re^{i\nu})| + \frac{Mr(1 + o(1))}{2} = O(r^{(m+2)/2})$$

as $r \to \infty$. Combining (3.4) and the Phragmén–Lindelöf principle, for any $\theta \in [0, 2\pi)$, we get

$$\log^+ |f(re^{i\theta})| = O(r^{(m+2)/2})$$

as $r \to \infty$. That implies that

$$\rho(f) \le \frac{m+2}{2}. \tag{3.7}$$

(1) If the condition (i) holds, then $\rho(f) < n = \rho(A)$. This is a contradiction with equation (1.3). Therefore, we know that every nontrivial solution of (1.3) is of infinite order.

(2) If the condition (ii) holds, then $\frac{\pi}{n} \neq \frac{2k\pi}{m+2}$ for any integers k. Therefore, for any given i, $i = 1, 2, \dots, n$, there can exist at most one ray among the two rays $\arg z = \varphi_i$, $\arg z = \theta_{i+1}$ that could be a critical ray of $Q(z)$. For the sector $S_i^-(\varphi_i, \theta_{i+1})$, without loss of generality, we suppose that $\arg z = \theta_{i+1}$ is not a critical ray of $Q(z)$. Then there must exist a critical ray of $Q(z)$, say $\arg z = \phi_0$, such that $\theta_{i+1} - \phi_0 < \frac{2\pi}{m+2}$. By using similar reasoning as in [17], set $z = xe^{i\phi_0}$, and then (3.6) becomes to

$$\frac{d^2g}{dx^2} + \left[Q_2(x) + e^{2i\phi_0}(-\frac{A^2(xe^{i\phi_0})}{4} - e^{i\phi_0}\frac{A'(xe^{i\phi_0})}{2})\right] g = 0,$$

where $Q_2(x) = \alpha_1 x^m + O(x^{m-2})$ and $\alpha_1 > 0$.

By Lemma 2.8 and Lemma 2.3, there exists a path Γ_{ϕ_0} tending to infinity, such that $\arg z \to \phi_0$ on Γ_{ϕ_0} and $g(z) \to 0$ on Γ_{ϕ_0}. Therefore, there exists a sector $S(\phi_0-\varepsilon, \theta_{i+1}+2\varepsilon)$, such that $f(z) \to 0$ on Γ_{ϕ_0} and $f(z) \to 1$ on the ray $\arg z = \theta_{i+1}+\varepsilon$. Combining $\theta_{i+1}-\phi_0+3\varepsilon < \frac{2\pi}{m+2}$ and the Phragmén–Lindelöf principle (see [5, p. 104]), we get that $\rho(f) > \frac{m+2}{2}$, and this contradicts with (3.7).

(3) Suppose that the condition (iii) holds. Then we assert that $e^{P(z)}$ and $Q(z)$ cannot have a common critical ray. Otherwise, there exists a real number ϕ_1 such that $\delta(P, \phi_1) = 0$ and $\arg b_m + (m + 2)\phi_1 = 0 \pmod{2\pi}$. This implies that

$$\alpha \cos n\phi_1 - \beta \sin n\phi_1 = 0,$$

where $a_n = \alpha + i\beta$. Thus

$$\sqrt{\alpha^2+\beta^2}\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\cos n\phi_1 - \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\sin n\phi_1\right) = 0.$$

Set $\varphi = \arg a_n$, then

$$\sqrt{\alpha^2+\beta^2}(\cos\varphi\cos n\phi_1 - \sin\varphi\sin n\phi_1) = 0.$$

So,

$$\phi_1 = \frac{k_1\pi + \pi/2 - \arg a_n}{n},$$

where k_1 is integer. On the other hand, $\arg b_m + (m+2)\phi_1 = 2k_2\pi$, where k_2 is integer. Thus

$$\phi_1 = \frac{2k_2\pi - \arg b_m}{m+2} = \frac{2k_2\pi - \arg b_m}{2n}.$$

Combining the two lines above, we get

$$(2k_1 + 1 - 2k_2)\pi = 2\arg a_n - \arg b_m.$$

Therefore,

$$(2k_1 + 1 - 2k_2 + 2k_3)\pi = \arg\frac{a_n^2}{b_m},$$

where k_3 is integer. This contradicts with the fact that $\frac{a_n^2}{b_m}$ is not real and negative. Thus $Q(z)$ has just one critical ray in every sector $S_i^-(\varphi_i, \theta_{i+1})$, $i = 1, 2, \ldots, n$. Therefore we suppose that $\arg z = \phi_1$ is a critical ray of $Q(z)$ in $S_1^-(\varphi_1, \theta_2)$. Then

$$\theta_2 - \phi_1 < \frac{\pi}{n} = \frac{2\pi}{m+2}, \quad \phi_1 - \varphi_1 < \frac{\pi}{n} = \frac{2\pi}{m+2}.$$

By using similar reasoning as in the proof of (ii), we get $\rho(f) > \frac{m+2}{2}$, which contradicts with (3.7). This completes the proof. □

4. Proof of Theorems 1.5 and 1.6

We prove Theorems 1.5 and 1.6 by using similar reasoning as in the proof of Theorem 1.4. Thus we point out only the important steps. Suppose that there exists a nontrivial solution f of (1.3) with $\rho(f) < \infty$.

Proof of Theorem 1.5. We divide the proof into two cases.

Case 1. Suppose that $A(z)$ blows up exponentially in every sector S_j, $j = 0, 1, \ldots n+1$, where S_j is defined in Lemma 2.2. We deduce as in the proof of Theorem 1.4 that

$$f(re^{i\theta}) \to 1$$

as $z \to \infty$, for every θ in each S_j. Then by the Phragmén–Lindelöf principle and Liouville's theorem, $f(z)$ must be a constant, and this contradicts with equation (1.3).

Case 2. Suppose that $A(z)$ decays to zero exponentially in some sector S_{j_0}. By using similar reasoning as in the proof of Theorem 1.4, we also obtain (3.7).

If the condition (i) holds, then $\rho(f) \le \frac{m+2}{2} < \frac{n+2}{2} = \rho(A)$. This contradicts with (1.3).

If the condition (ii) holds, then $\frac{2\pi}{n+2} \ne \frac{2k\pi}{m+2}$ for any integer k. Then there can exist at most one ray among the two rays $\arg z = \theta_{j_0}$, $\arg z = \theta_{j_0+1}$ that could be a critical ray of $Q(z)$. Without loss of generality, we suppose that $\arg z = \theta_{j_0+1}$ is not a critical ray of $Q(z)$. Then there must exist a critical ray of $Q(z)$, say $\arg z = \phi_0$, such that $\theta_{j_0+1} - \phi_0 < \frac{2\pi}{m+2}$. By using similar reasoning as in the proof of Theorem 1.4, there exists a path Γ_{ϕ_0} in S_{j_0} tending to infinity, such that $\arg z \to \phi_0$ on Γ_{ϕ_0}, while $f(z) \to 0$ on Γ_{ϕ_0}. On the other hand, by (ii) of Lemma 2.2, we know that $A(z)$ must blow up exponentially in S_{j_0+1}. For any given $\varepsilon \in (0, \frac{2\pi/(m+2)-\theta_{j_0+1}+\phi_0}{3})$, there exists a sector $S(\phi_0 - \varepsilon, \theta_{j_0+1} + 2\varepsilon)$, such that $f(z) \to 1$ on the ray $\arg z = \theta_{j_0+1} + \varepsilon$ and $f(z) \to 0$ on Γ_{ϕ_0}. Noting that $\theta_{j_0+1} - \phi_0 + 3\varepsilon < \frac{2\pi}{m+2}$ and using the Phragmén–Lindelöf principle, we get that $\rho(f) > \frac{m+2}{2}$, and this contradicts with (3.7).

If the condition (iii) holds, then we assert that $\arg z = \theta_j$ are not critical rays of $Q(z)$, $j = 0, 1, \ldots, n+1$. Otherwise, suppose that there exists at least one ray among the $n+2$ rays $\arg z = \theta_j$ that is a critical ray of $Q(z)$, say $\arg z = \theta_{j'}$, $j' \in \{0, 1, 2, \ldots, n+1\}$. Then $\arg b_m + (m+2)\theta_{j'} = 2k_1\pi$, where k_1 is integer. Thus

$$\theta_{j'} = \frac{2k_1\pi - \arg b_m}{m+2} = \frac{2k_1\pi - \arg b_m}{n+2}.$$

On the other hand,

$$\theta_{j'} = \frac{2j'\pi - \arg a_n}{n+2}.$$

Therefore,

$$(2j' - 2k_1)\pi = \arg a_n - \arg b_m.$$

This implies

$$(2j' - 2k_1 + 2k_2)\pi = \arg \frac{a_n}{b_m},$$

where k_2 is integer. This contradicts with the fact that $\frac{a_n}{b_m}$ is not real and positive. Thus $Q(z)$ has just one critical ray in S_{j_0}. If $\arg z = \phi_1$ is the critical ray of $Q(z)$ in S_{j_0}, then

$$\theta_{j_0+1} - \phi_1 < \frac{2\pi}{n+2} = \frac{2\pi}{m+2}, \quad \phi_1 - \theta_{j_0} < \frac{2\pi}{n+2} = \frac{2\pi}{m+2}.$$

By using a similar discussion as in (ii) above, we get $\rho(f) > \frac{m+2}{2}$, and this contradicts with (3.7). This completes the proof. □

Proof of Theorem 1.6. Let $\theta \in [0, 2\pi) - H$. Suppose that θ satisfies the condition (i). By using similar reasoning as in the proof of Theorem 1.4, we get

$$f(re^{i\theta}) \to 1 \quad \text{as} \quad r \to \infty. \tag{4.1}$$

Suppose that θ satisfies the condition (ii). If we set

$$f(z) = g(z) \exp\left(-\frac{1}{2}\int^z A(t)\, dt\right),$$

then $g(z)$ satisfies the equation

$$g'' + (Q(z) - \frac{A^2(z)}{4} - \frac{A'(z)}{2})g = 0. \tag{4.2}$$

Since $\theta \in [0, 2\pi) - H$ and θ satisfies the condition (ii), then

$$\left|Q(re^{i\theta}) - \frac{A^2(re^{i\theta})}{4} - \frac{A'(re^{i\theta})}{2}\right| = O(r^K) \quad \text{as} \quad r \to \infty,$$

where $K = \max\{2n, m\}$.

It is not hard to see from Lemma 2.4 that for every nontrivial solution g of (4.2),

$$\log^+ |g(re^{i\theta})| = O(r^{(K+2)/2}) \quad \text{as} \quad r \to \infty.$$

Furthermore, we have

$$\text{(4.3)}\qquad \begin{aligned}\log^+|f(re^{i\theta})| &\le \log^+|g(re^{i\theta})| + \left|\int^{re^{i\theta}} A(t)\,dt\right| \\ &\le O\left(r^{(K+2)/2} + \int^{r} |t|^n|\,dt|\right) \le O(r^{(K+2)/2}).\end{aligned}$$

Then it follows from (4.1), (4.3) and the Phragmén–Lindelöf principle that

$$\log^+|f(re^{i\theta})| = O(r^{(K+2)/2})$$

as $r \to \infty$ with $\theta \in [0, 2\pi)$. That implies that $\rho(f) \le \frac{K+2}{2} < \rho(A)$, which contradicts with (1.3). This completes the proof. □

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