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PLANAR MAPPINGS OF SUBEXPONENTIALLY INTEGRABLE DISTORTION: INTEGRABILITY OF DISTORTION INVERSES

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Abstract. We establish the optimal regularity for the distortion of inverses of mappings of finite distortion with logarithm-iterated style subexponentially integrable distortion, which generalizes the Theorem 1 of [7].

1. Introduction

We say that a mapping $f\colon \Omega \to \mathbf{R}^n$ in a domain $\Omega \subset \mathbf{R}^n$ is a mapping of finite distortion, if

(i) $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$,
(ii) the Jacobian determinant $J_f \in L^1_{\mathrm{loc}}(\Omega)$, and
(iii) there is a measurable function $K : \Omega \to [1, +\infty]$ with $K(z) < \infty$ almost everywhere such that

$$|Df(z)|^n \le K(z) J_f(z) \quad \text{for almost all } z \in \Omega, \tag{1.1}$$

where $|Df(z)|$ is the operator norm of the matrix $Df(z)$. For mappings of finite distortion, we define the distortion function by

$$K_f(z) = \begin{cases} \frac{|Df(z)|^n}{J_f(z)}, & \text{if } z \in \{z \in \Omega\colon J_f(z) > 0\}, \\ 1, & \text{if } z \in \{z \in \Omega\colon J_f(z) = 0\}, \end{cases}$$

then the distortion inequality (1.1) becomes

$$|Df(z)|^n = K_f(z) J_f(z) \quad \text{for almost every } z \in \Omega. \tag{1.2}$$

We will limit the discussion in this paper to the planar case, i.e. $n = 2$. In this case, since $|Df(z)| = |f_z| + |f_{\bar z}|$ and $J_f(z) = |f_z|^2 - |f_{\bar z}|^2$, the distortion equality (1.2) is equivalent to the Beltrami equation

$$\frac{\partial f(z)}{\partial \bar z} = \mu(z) \frac{\partial f(z)}{\partial z} \quad \text{for almost every } z \in \Omega \tag{1.3}$$

where $\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)$ and $|\mu(z)| = \frac{K_f(z)-1}{K_f(z)+1}$. For more details about mappings of finite distortion, we refer the reader to [9] and the references therein.

If $\|\mu\|_\infty \le k < 1$, then the classical measurable Riemann mapping theorem tells that the Beltrami equation (1.3) admits a homeomorphic solution and other solutions are represented by composing the homeomorphic solution with holomorphic functions, see [1, 3].

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When $\|\mu\|_\infty = 1$, the Beltrami equation (1.3) becomes degenerate. David dealt with this degenerate Beltrami equation in [4], where he generalized the measurable Riemann mapping theorem for mappings with exponentially integrable distortion K_f, i.e. for mappings with $\exp(pK_f) \in L^1_{\mathrm{loc}}(\Omega)$ for some $p > 0$. David also noted that even if the distortion of f is exponentially integrable the distortion function of f^{-1} might still fail to be exponentially integrable. Later, Hencl and Koskela [8] proved $K_{f^{-1}} \in L^\beta_{\mathrm{loc}}$ where $\beta = c_0 p$ with absolute constant c_0, under the local integrability of $\exp(pK_f)$. Based on Theorem 1.1 of [2], Gill [6] ascertained the sharp inequality $c_0 < 1$. The comprehensive statement is as follows.

Theorem A. [6, Theorem 1] *Suppose that $f\colon \Omega \to \mathbf{C}$ is a homeomorphism of finite distortion which satisfies the Beltrami equation (1.3), with the associated distortion function K_f. If*

$$\exp(pK_f) \in L^1_{\mathrm{loc}}(\Omega) \quad \text{for some } p > 0,$$

then f^{-1} is a mapping of finite distortion and the distortion function $K_{f^{-1}}$ satisfies

$$K_{f^{-1}} \in L^\beta_{\mathrm{loc}}(f(\Omega)) \quad \text{for all } 0 < \beta < p.$$

Moreover, this result is sharp in the sense that for every $p > 0$ there are functions f as above such that $K_{f^{-1}} \notin L^p_{\mathrm{loc}}$.

Let $p > 0$, we define

$$\mathcal{A}_{p,n}(x) = \begin{cases} px - p, & \text{if } n = 0, \\ \frac{px}{1+\prod_{k=1}^{n} \log_{(k)}(\exp_{(k-2)}(e)-1+x)} - p, & \text{if } n = 1, 2, \ldots. \end{cases} \tag{1.4}$$

where $\log_{(i)}(x) = \log(\cdots(\log(\log(x)))\cdots)$ and $\exp_{(i)}(x) = \exp(\cdots(\exp(\exp(x)))\cdots)$ are i-iterated logarithm and exponent for $i = 1, 2, \ldots$, $\exp_{(0)}(x) = x$ and $\exp_{(-1)}(x) = 1$. Theorem A tells us that the distortion function of inverse is locally integrable under $\exp[\mathcal{A}_{p,0}(K_f)] \in L^1_{\mathrm{loc}}$. Gill in [7] generalized Theorem 1.1 of [2] to the solution f to the Beltrami equation (1.3) with $\exp[\mathcal{A}_{p,n}(K_f)] \in L^1_{\mathrm{loc}}$ when $n = 1, 2, \ldots$. However, there is no corresponding result analogous to Theorem A. The aim of this article is to present a generalization of Theorem A under the local integrability of $\exp[\mathcal{A}_{p,n}(K_f)]$.

Theorem 1.1. *Suppose that $f\colon \Omega \to \mathbf{C}$ is a homeomorphic mapping of finite distortion, with the associated distortion function K_f. Let $\mathcal{A}_{p,n}(x)$ be (1.4) when $n = 1, 2, \ldots$. If*

$$\exp[\mathcal{A}_{p,n}(K_f)] \in L^1_{\mathrm{loc}}(\Omega) \quad \text{for some } p > 0, \tag{1.5}$$

then f^{-1} is a mapping of finite distortion and the distortion function $K_{f^{-1}}$ satisfies

$$\log_{(n)}(\exp_{(n-1)}(e) + K_{f^{-1}}) \in L^\beta_{\mathrm{loc}}(f(\Omega)) \quad \text{for every } 0 < \beta < p. \tag{1.6}$$

Moreover, for every $p > 0$ there are mappings that satisfy the assumptions of the theorem, yet fail (1.6) for $\beta = p$.

The rest of the paper is organized as follows. In section 2, we recall some basic facts about Legendre Transformation and obtain an inequality of Young type. Section 3 is devoted to the proof of Theorem 1.1.

Notation. By $s \gg 1$ and $t \ll 1$ we mean that s is sufficiently large and t is sufficiently small, respectively. By $f \lesssim g$ we mean that there exists a constant $M > 0$ such that $f(x) \le Mg(x)$ for every x. If $f \lesssim g$ and $g \lesssim f$ we may denote $f \sim g$. By $\mathbf{N}$ we denote the set of positive integers. When concerned only with the convergence of improper integrals, we use notations $\int_*^\infty$ and $\int_0^*$.

2. An inequality of Young type

An Orlicz function is a continuously increasing function $\Phi\colon [0,\infty) \to [0,\infty)$ with

$$\Phi(0) = 0 \text{ and } \lim_{t\to\infty} \Phi(t) = \infty.$$

The conjugate to an Orlicz function Φ is defined by

$$\Phi^*(s) = \sup_{t\geq 0} \{st - \Phi(t)\}, \quad s \geq 0.$$

Directly from definition, we obtain

$$ts \leq \Phi(t) + \Phi^*(s) \text{ for every } t, \quad s \geq 0. \tag{2.1}$$

Inspired by [5], we now impose condition

$$\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty. \tag{2.2}$$

on the Orlicz function Φ to ensure that $\Phi^*(s) \in [0,\infty)$ for all $s \geq 0$. Moreover, there exists $t(s) \in [0,\infty)$ such that the supremum of $st - \Phi(t)$ is attained at $t(s)$.

Lemma 2.1. *Suppose that Φ is an Orlicz function and satisfies (2.2). If $\Phi(t)$ is differentiable for all $t \geq 0$ and $\Phi(t)$ is twice differentiable with $\Phi''(t) > 0$ for all $t \gg 1$, then*

$$(\Phi^*)'(s) = (\Phi')^{-1}(s) \text{ for all } s \gg 1.$$

Proof. Given $s \gg 1$. From (2.2) and the continuity of Φ, there exists $t(s) \in [0,\infty)$ such that

$$\Phi^*(s) = st(s) - \Phi(t(s)).$$

Then

$$\Phi'(t(s)) = s. \tag{2.3}$$

From (2.2) and $\lim_{t\to\infty} \Phi(t) = \infty$, it follows that $\lim t \to \infty \Phi'(t) = \infty$. Hence, we have $\Phi'(t) \gg 1$ as $t \gg 1$. Since $\Phi''(t) > 0$ for all $t \gg 1$, we know that $(\Phi')^{-1}(s)$ exists for all $s = \Phi'(t) \gg 1$. Applying $(\Phi')^{-1}$ to both sides of (2.3), we have $t(s) = (\Phi')^{-1}(s)$. Consequently, we have

$$(\Phi^*)'(s) = t(s) + (s - \Phi'(t(s)))\frac{d(\Phi')^{-1}(s)}{ds} = (\Phi')^{-1}(s). \qquad \square$$

The $t(s)$ in the proof of Lemma 2.1 is unique. If not, by mean value theorem there is $t_0(s) \gg 1$ such that $\Phi''(t_0(s)) = 0$, which violates the assumption in Lemma 2.1.

Given a strictly convex C^2 function $\Phi(t)$, it is not easy to compute the explicit expression of $\Phi^*(s)$ from the definition. However, by Lemma 2.1, we can obtain the asymptotic behaviour of $\Phi^*(s)$ as $s \gg 1$. The following example, coming from [5], illustrates this.

Example 2.2. Put $\Phi(t) = \exp\left(\frac{t}{\log(e+t)}\right)$. Take some $t_0 \gg 1$ such that $\Phi'(t_0) > 0$. We define $\Phi_1(t)$ by

$$\Phi_1(t) = \begin{cases} \Phi'(t_0)(t - t_0) + t_0\Phi(t_0), & \text{if } 0 \leq t \leq t_0, \\ \Phi(t) + t_0\Phi'(t_0) - \Phi(t_0), & \text{if } t_0 < t. \end{cases}$$

We first compute the asymptotic behaviour of $\Phi_1^*(s)$ as $s \gg 1$. After differentiating and taking the logarithm, we have

$$\log(\Phi_1'(t)) \sim \frac{t}{\log(t)} \text{ as } t \gg 1.$$

Let $\frac{t}{\log(t)} = \log(s)$. Then

$$t \sim \log(s)\log_{(2)}(s) \quad \text{as } s \gg 1. \tag{2.4}$$

Moreover, we can check that $\Phi_1(t)$ satisfies all assumptions in Lemma 2.1. Hence, it follows from Lemma 2.1 that

$$\log(\Phi_1'(t)) \sim \log(s) = \log[\Phi_1'((\Phi_1^*)'(s))].$$

Therefore, by (2.4) and the monotonicity of $\log(\cdot)$ and $\Phi_1'(\cdot)$, we have

$$(\Phi_1^*)'(s) \sim \log(s)\log_{(2)}(s).$$

Hence, by the Newton–Leibniz formula, we show

$$\Phi_1^*(s) \sim s\log(s)\log_{(2)}(s) \quad \text{as } s \gg 1.$$

By the definition of conjugate function, we know that there exist constant C_1 and C_2 with $C_2 \geq C_1$ such that $\Phi^*(s) + C_1 \leq \Phi_1^*(s) \leq \Phi^*(s) + C_2$ for all $s \geq 0$. Therefore, we have

$$\Phi^*(s) \sim s\log(s)\log_{(2)}(s) \quad \text{as } s \gg 1.$$

By the method analogous to Example 2.2, we present an inequality of Young type, which plays a crucial role in the proof of Theorem 1.1.

Lemma 2.3. *Let $p > 0$ and $n \in \mathbf{N}$. Given $\beta > 0$, there exist constants C_1, $C_2 > 0$ such that*

$$ts \leq C_1\Phi(t) + C_2\Psi(s) \quad \textit{for all } t, s \geq 0,$$

where $\Phi(t) = \exp[\mathcal{A}_{p,n}(\exp_{(n)}(t^{\frac{1}{\beta}}))]$ and $\Psi(s) = s\left[\log_{(n+1)}(\exp_{(n)}(e) + s)\right]^{\beta}$.

Proof. We divide the proof into two cases.

Case 1. Suppose first that $0 \leq s \leq C_1$ for some $C_1 > 0$. Since $t \leq \Phi(t)$ for all $t \geq 0$ and $\Psi(s) \geq 0$ for all $s \geq 0$, we obtain

$$st \leq C_1\Phi(t) + \Psi(s) \quad \text{for all } t \geq 0 \text{ and } 0 \leq s \leq C_1. \tag{2.5}$$

Case 2. Suppose that $s \gg 1$. Take some $t_0 \gg 1$ such that $\Phi'(t_0) > 0$. We define $\Phi_1(t)$ by

$$\Phi_1(t) = \begin{cases} \Phi'(t_0)(t - t_0) + t_0\Phi(t_0), & \text{if } 0 \leq t \leq t_0, \\ \Phi(t) + t_0\Phi'(t_0) - \Phi(t_0), & \text{if } t_0 < t. \end{cases}$$

We first compute the asymptotic behaviour of $\Phi_1^*(s)$ as $s \gg 1$. Since

$$\log\Phi_1'(t) \sim \frac{\exp_{(n)}(t^{\frac{1}{\beta}})}{\exp_{(n-1)}(t^{\frac{1}{\beta}})\cdots\exp_{(1)}(t^{\frac{1}{\beta}})t^{\frac{1}{\beta}}} = F(t) \quad \text{for all } t \gg 1$$

and for any constant $C > 0$ we have

$$C < \exp_{(n-2)}(t^{\frac{1}{\beta}})\cdots\exp_{(1)}(t^{\frac{1}{\beta}})t^{\frac{1}{\beta}} < C\exp_{(n-1)}(t^{\frac{1}{\beta}}) \quad \text{as } t \gg 1,$$

it follows that

$$\frac{\exp_{(n)}(t^{\frac{1}{\beta}})}{\exp_{(n-1)}(t^{\frac{1}{\beta}})} > \log\Phi_1'(t) > \frac{\exp_{(n)}(t^{\frac{1}{\beta}})}{\left[\exp_{(n-1)}(t^{\frac{1}{\beta}})\right]^2} \quad \text{for all } t \gg 1. \tag{2.6}$$

Next consider the right-hand side of (2.6). Let $b = \exp_{(n)}(t^{\frac{1}{\beta}})$, we consider

$$\frac{b}{[\log(b)]^2} = \log(s), \text{ i.e. } \frac{b^{\frac{1}{2}}}{\log(b^{\frac{1}{2}})} = \sqrt{4\log(s)}. \tag{2.7}$$

By Example 2.2, we have $b^{\frac{1}{2}} \sim \sqrt{\log(s)}\log_{(2)}(s)$ as $s \gg 1$. Taking n successive logarithms, we have

$$t \sim [\log_{(n+1)}(s)]^{\beta} \quad \text{as } s \gg 1. \tag{2.8}$$

Moreover, it is easy to check that $\Phi_1(t)$ satisfies all assumptions in Lemma 2.1. Here, we only check that $\Phi_1''(t) > 0$ for all $t \gg 1$. By chain rule we get $F'(t) > 0$ as $t \gg 1$. Since $\lim_{t\to\infty} \frac{\log \Phi_1'(t)}{F(t)} = p > 0$, by L'Hospital's rule we have $\lim_{t\to\infty} \frac{\Phi_1''(t)}{\Phi_1'(t)F'(t)} = p > 0$. Hence, we have $\Phi_1''(t) > 0$ for all $t \gg 1$. Therefore, it follows from (2.7) and from the right-hand side of (2.6) together with Lemma 2.1 that

$$\log(\Phi_1'(t)) > \log(s) = \log[\Phi_1'((\Phi_1^*)'(s))] \quad \text{for all } t \gg 1.$$

By (2.8) and the monotonicity of $\log(\cdot)$ and $\Phi_1'(\cdot)$, we have

$$(\Phi_1^*)'(s) \lesssim [\log_{(n+1)}(s)]^{\beta} \quad \text{as } s \gg 1. \tag{2.9}$$

We now turn to the left-hand side of (2.6). By the similar arguments used to deduce (2.9), we obtain

$$[\log_{(n+1)}(s)]^{\beta} \lesssim (\Phi_1^*)'(s) \quad \text{as } s \gg 1. \tag{2.10}$$

Combining (2.9) and (2.10), we obtain $(\Phi_1^*)'(s) \sim [\log_{(n+1)}(s)]^{\beta}$ as $s \gg 1$. Hence, by the Newton–Leibniz formula, we get

$$\Phi_1^*(s) \sim s[\log_{(n+1)}(s)]^{\beta} \quad \text{as } s \gg 1.$$

By the definition of conjugate function, we know that there exist constant C_1 and C_2 with $C_2 \geq C_1$ such that $\Phi^*(s) + C_1 \leq \Phi_1^*(s) \leq \Phi^*(s) + C_2$ for all $s \geq 0$. Therefore, we have

$$\Phi^*(s) \sim s[\log_{(n+1)}(s)]^{\beta} < \Psi(s) \quad \text{as } s \gg 1. \tag{2.11}$$

It follows from (2.1) and (2.11) that there exists constant $C_2 > 0$ such that

$$ts \leq \Phi(t) + C_2\Psi(s) \quad \text{for all } t \geq 0 \text{ and all } s \gg 1. \tag{2.12}$$

Combining (2.5) and (2.12), we complete the proof. □

3. Proof of Theorem 1.1

We begin with four lemmas.

Lemma 3.1. [11, Theorem 1.1] *Suppose that Ψ is a strictly increasing, differentiable function and satisfies*

$$\int_1^{\infty} \frac{\Psi'(t)}{t}\, dt = \infty, \tag{C–1}$$

$$\lim_{t\to\infty} t\Psi'(t) = \infty. \tag{C–2}$$

Let $f\colon \Omega \to \mathbf{R}^n$ be a mapping of finite distortion and the distortion function K_f satisfies $\exp(\Psi(K_f)) \in L^1_{\mathrm{loc}}(\Omega)$. Then f satisfies the Lusin's condition (N), i.e. $f(E)$ has Lebesgue measure zero if E has Lebesgue measure zero.

Given a mapping $f\colon \Omega \to \mathbf{R}^n$, we denote $N(f,\Omega,y)$ by the number of preimages of point y in Ω under f. We say f has essentially bounded multiplicity, if $N(f,\Omega,y)$ is bounded for a.e. $y \in \mathbf{R}^n$.

From the proof of Theorem 1.2 in [12], we know the assertion of Theorem 1.2 in [12] remains valid if both the mapping and its distortion function lie in local Sobolev spaces. So, we have the following result:

Lemma 3.2. *Let $f\colon \Omega \to \mathbf{R}^2$ be a mapping of finite distortion and the distortion function K_f satisfies $K_f \in L^1_{\mathrm{loc}}(\Omega)$. If f has essentially bounded multiplicity and f is not a constant, then $J_f > 0$ almost everywhere in Ω.*

Suppose that a function $\mathcal{A}$ has the properties:

(A–1) $\mathcal{A}\colon [1,\infty) \to [0,\infty)$ is a smooth increasing function with $\mathcal{A}(1) = 0$.

$$\int_1^\infty \frac{\mathcal{A}(t)}{t^2}\,\mathrm{d}t = \infty. \tag{A–2}$$

The associated function of $\mathcal{A}$ is denoted by

$$P(t) = \begin{cases} t^2, & 0 \le t \le 1, \\ \frac{t^2}{\mathcal{A}^{-1}(\log t^2)}, & t \ge 1. \end{cases} \tag{3.1}$$

Let us recall the notation

$$W^{1,P}_{\mathrm{loc}}(\Omega) = \left\{ f \in W^{1,1}_{\mathrm{loc}}(\Omega) \colon P(|Df|) \in L^1_{\mathrm{loc}}(\Omega) \right\}.$$

Lemma 3.3. [3, Theorem 20.5.1] *Given a function $\mathcal{A}$ satisfying (A–1) and (A–2) and the associated function P which is defined by (3.1). Let $f\colon \Omega \to \mathbf{R}^2$ be a mapping of finite distortion such that the distortion function K_f satisfies* $\exp[\mathcal{A}(K_f)] \in L^1_{\mathrm{loc}}(\Omega)$, *then*

$$f \in W^{1,P}_{\mathrm{loc}}(\Omega).$$

Obviously, $\mathcal{A}_{p,n}$ satisfies (A–1) and (A–2). We denote the associated function of $\mathcal{A}_{p,n}$ by P_n. Next we present a lemma essentially due to Gill [7].

Lemma 3.4. *Let $p > 0$ and $n \in \mathbf{N}$. Given a Beltrami equation (1.3) with compactly supported $\mu(z)$, and $|\mu(z)| < 1$ almost everywhere with* $\exp\left[\mathcal{A}_{p,n}\left(\frac{1+|\mu(z)|}{1-|\mu(z)|}\right)\right] \in L^1_{\mathrm{loc}}(\mathbf{C})$. *Then any solution $f \in W^{1,P_n}_{\mathrm{loc}}(\Omega)$ to this Beltrami equation in a domain $\Omega \subset \mathbf{C}$ admits*

$$J_f \left[\log_{(n+1)}(\exp_{(n)}(e) + J_f)\right]^\beta \in L^1_{\mathrm{loc}}(\Omega) \quad \textit{for all } 0 < \beta < p.$$

Proof of Theorem 1.1. Since

$$\frac{1}{\log_{(1)}(x)\log_{(2)}(x)\cdots\log_{(n)}(x)} \lesssim \mathcal{A}'_{p,n}(x) \quad \text{as } x \gg 1,$$

we know $\mathcal{A}_{p,n}(x)$ satisfies (C–1) and (C–2). It follows from Lemma 3.1 that f satisfies the Lusin's condition (N).

Since

$$x \lesssim \exp(\mathcal{A}_{p,n}(x)) \quad \text{for all } x \ge 1,$$

it follows from (1.5) that

$$K_f \in L^1_{\mathrm{loc}}(\Omega). \tag{3.2}$$

So, Lemma 3.2 tells us $J_f > 0$ almost everywhere in Ω.

Given compact set $\widetilde{M} \subset f(\Omega)$, we have $M = f^{-1}(\widetilde{M}) \subset \Omega$ is a compact set. By Corollary 3.3.3 in [3], we obtain that f is differentiable almost everywhere in Ω. So, we can divide the set M into two subsets M' and M'', where M' is the subset in which f is differentiable and $J_f(z) > 0$ and $M'' = M \setminus M'$ has Lebesgue measure zero. For any $z \in M'$, by Lemma A.29 of [9], we have

$$Df^{-1}(f(z)) = (Df(z))^{-1}.$$

Hence, by Cramer's rule we have $|Df^{-1}(f(z))|^2 J_f(z) = K_f(z)$ and $K_{f^{-1}}(f(z)) = K_f(z)$ for all $z \in M'$. So, it follows from Corollary A.36 (c) of [9] and the Lusin's condition (N) of f that

$$\int_{\widetilde{M}} |Df^{-1}(w)|^2 \,\mathrm{d}w = \int_M K_f(z) \,\mathrm{d}z \tag{3.3}$$

and

$$\int_{\widetilde{M}} \left[\log_{(n)}(\exp_{(n-1)}(e) + K_{f^{-1}})\right]^\beta \,\mathrm{d}w = \int_M \left[\log_{(n)}(\exp_{(n-1)}(e) + K_f)\right]^\beta J_f \,\mathrm{d}z. \tag{3.4}$$

By (3.2) and $J_{f^{-1}} \le |Df^{-1}|^2$, it follows from (3.3) that $J_{f^{-1}} \in L^1_{\mathrm{loc}}(f(\Omega))$. Therefore, by [8, Theorem 3.3] f^{-1} is a mapping of finite distortion.

Next we prove (1.6). Because of (3.4), it suffices to prove

$$\int_M \left[\log_{(n)}(\exp_{(n-1)}(e) + K_f(z))\right]^\beta J_f(z) \,\mathrm{d}z < \infty \tag{3.5}$$

for any compact set $M \subset \Omega$. Let

$$s = J_f(z) \quad \text{and} \quad t = \left[\log_{(n)}(\exp_{(n-1)}(e) + K_f(z))\right]^\beta.$$

Since

$$\mathcal{A}_{p,n}\big(\exp_{(n)}\big(t^{\frac{1}{\beta}}\big)\big) \le \mathcal{A}_{p,n}(K_f(z)) + p(\exp_{(n-1)}(e) - 1),$$

it follows from Lemma 2.3 that there exist constants C' and C'' such that

$$ts \le C' \exp[\mathcal{A}_{p,n}(K_f)] + C'' J_f \left[\log_{(n+1)}(\exp_{(n)}(e) + J_f)\right]^\beta. \tag{3.6}$$

Note that $\mathcal{A}_{p,n}(x)$ satisfies (A–1) and (A–2) conditions, and thus Lemma 3.3 implies

$$f \in W^{1,P_n}_{\mathrm{loc}}(\Omega),$$

where P_n is the associated function of $\mathcal{A}_{p,n}$. So, it follows from Lemma 3.4 that

$$J_f \left[\log_{(n+1)}(\exp_{(n)}(e) + J_f)\right]^\beta \in L^1_{\mathrm{loc}}(\Omega). \tag{3.7}$$

Hence, according to (3.6), (1.5) and (3.7), (3.5) is proved.

To show Theorem 1.1 is sharp, as in Theorem 4 of [7], we consider Kovalev-type function h in $\Omega = \mathbf{D}$ as

$$h(z) = \frac{z}{|z|}\rho(|z|) \tag{3.8}$$

where $\rho(t) = \left[\log_{(n+1)}(\exp_{(n+1)}(e) + \frac{1}{t})\right]^{-\frac{p}{2}} \left[\log_{(n+2)}(\exp_{(n+1)}(e) + \frac{1}{t})\right]^{-\frac{1}{2}}$, $p > 0$ and $n \in \mathbf{N}$. For the reader's convenience, we carry out the main computation. By (3.4), it is enough to check

$$J_h \left[\log_{(n)}(\exp_{(n-1)}(e) + K_h)\right]^p \notin L^1_{\mathrm{loc}}(\mathbf{D}). \tag{3.9}$$

From the definition of h, it is sufficient to consider h in the small enough neighbourhood of 0. So with the formulas in section 6.5.1 of [10], when $|z| \ll 1$, we have

$$J_h(z) \sim \frac{1}{|z|^2} \frac{1}{\log_{(1)}(\frac{1}{|z|})} \cdots \frac{1}{\log_{(n)}(\frac{1}{|z|})} \left[\log_{(n+1)}(\frac{1}{|z|})\right]^{-p-1} \left[\log_{(n+2)}(\frac{1}{|z|})\right]^{-1} \tag{3.10}$$

and

$$K_h(z) = \frac{\rho(|z|)}{|z|\rho'(|z|)} \sim \log_{(1)}\left(\frac{1}{|z|}\right) \log_{(2)}\left(\frac{1}{|z|}\right) \cdots \log_{(n+1)}\left(\frac{1}{|z|}\right).$$

Since

$$\log(\exp_{(n-1)}(e) + K_h(z)) \sim \log(K_h(z)) \sim \log_{(2)}\left(\frac{1}{|z|}\right) \quad \text{as } |z| \ll 1,$$

we get

$$\left[\log_{(n)}(\exp_{(n-1)}(e) + K_h)\right]^p \sim \left[\log_{(n+1)}(\frac{1}{|z|})\right]^p \quad \text{as } |z| \ll 1. \tag{3.11}$$

Combining (3.10) and (3.11), we obtain

$$J_h \left[\log_{(n)}(\exp_{(n-1)}(e) + K_h)\right]^p \sim \frac{1}{|z|^2} \frac{1}{\log_{(1)}(\frac{1}{|z|})} \cdots \frac{1}{\log_{(n+2)}(\frac{1}{|z|})}$$

Now, (3.9) is obtained from

$$\int_0^* \frac{1}{t} \frac{1}{\log_{(1)}(\frac{1}{t})} \cdots \frac{1}{\log_{(n+2)}(\frac{1}{t})}\, \mathrm{d}t = \int_*^{+\infty} \frac{1}{s} \frac{1}{\log_{(1)}(s)} \cdots \frac{1}{\log_{(n+2)}(s)}\, \mathrm{d}s$$
$$= \cdots = \int_*^{+\infty} \frac{1}{\log(x)}\, \mathrm{d}x = \infty.$$

The proof is complete. □

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