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NORMALIZED SOLUTIONS FOR THE CHERN–SIMONS–SCHRÖDINGER EQUATION IN $\mathbf{R}^2$

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Abstract. In this paper, we study the existence and multiplicity of solutions with a prescribed L^2-norm for a class of nonlinear Chern–Simons–Schrödinger equations in $\mathbf{R}^2$

$$-\Delta u - \lambda u + \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s)\,ds\right) u = |u|^{p-2}u,$$

where

$$h(s) = \frac{1}{2}\int_0^s r u^2(r)\,dr.$$

To get such solutions we look for critical points of the energy functional

$$I(u) = \frac{1}{2}\int_{\mathbf{R}^2} |\nabla u|^2 + \frac{1}{2}\int_{\mathbf{R}^2} \frac{|u|^2}{|x|^2}\left(\int_0^{|x|} \frac{s}{2} u^2(s)\,ds\right)^2 - \frac{1}{p}\int_{\mathbf{R}^2} |u|^p$$

on the constraints

$$S_r(c) = \left\{u \in H_r^1(\mathbf{R}^2)\colon \|u\|^2_{L^2(\mathbf{R}^2)} = c\right\}, \quad c > 0.$$

When $p = 4$, we prove a sufficient condition for the nonexistence of constrain critical points of I on $S_r(c)$ for certain c and get infinitely many minimizers of I on $S_r(8\pi)$. For the value $p \in (4, +\infty)$ considered, the functional I is unbounded from below on $S_r(c)$. By using the constrained minimization method on a suitable submanifold of $S_r(c)$, we prove that for certain $c > 0$, I has a critical point on $S_r(c)$. After that, we get an H^1-bifurcation result of our problem. Moreover, by using a minimax procedure, we prove that there are infinitely many critical points of I restricted on $S_r(c)$ for any $c \in \left(0, \frac{4\pi}{\sqrt{p-3}}\right)$.

1. Introduction and main results

In this paper, we study the nonlinear Chern–Simons–Schrödinger equation as follows:

$$-\Delta u - \lambda u + \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s)\,ds\right) u = |u|^{p-2}u, \quad x \in \mathbf{R}^2, \tag{1.1}$$

where

$$h(s) = \frac{1}{2}\int_0^s r u^2(r)\,dr.$$

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Recently, the nonlinear Chern–Simons–Schrödinger equations have been extensively studied, see e.g. [10, 13, 16, 17, 21, 22, 24, 29, 30, 32]. (1.1) is not a pointwise identity as the appearance of the Chern–Simons term

$$\left(\frac{h^2(|x|)}{|x|^2}+\int_{|x|}^{+\infty}\frac{h(s)}{s}u^2(s)\,ds\right)u.$$

Based on such a character, people call it a nonlocal problem and it is quite different from the usual semi-linear Schrödinger equation. The nonlocal term causes some mathematical difficulties that make the study of (1.1) more interesting. As we shall see, (1.1) is also different from the Schrödinger–Poisson equation (see [6, 19, 28]), which is another problem exhibiting the competition between local and nonlocal terms. We point out that (1.1) arises from seeking the standing wave solutions to the following nonlinear Schrödinger equations with the gauge field:

$$\begin{aligned} iD_0+(D_1D_1+D_2D_2)\phi&=-|\phi|^{p-2}\phi, \quad \partial_0A_1-\partial_1A_0=-\mathrm{Im}(\overline{\phi}D_2\phi),\\ \partial_0A_2-\partial_2A_0&=\mathrm{Im}(\overline{\phi}D_1\phi), \quad \partial_1A_2-\partial_2A_1=-\frac{1}{2}|\phi|^2, \end{aligned}\tag{1.2}$$

where i denotes the imaginary unit, $\partial_0=\frac{\partial}{\partial t}$, $\partial_1=\frac{\partial}{\partial x_1}$, $\partial_2=\frac{\partial}{\partial x_2}$ for $(t,x_1,x_2)\in\mathbf{R}^{1+2}$, $\phi\colon\mathbf{R}^{1+2}\to\mathbf{C}$ is the complex scalar field, $A_\mu\colon\mathbf{R}^{1+2}\to\mathbf{R}$ is the gauge field and $D_\mu=\partial_\mu+iA_\mu$ is the covariant derivative for $\mu=0,1,2$. When $p=4$, (1.2) has received much attention, which is related to the following self-dual equations (see [14, 16, 22])

$$\begin{aligned} &D_1\phi+iD_2\phi=0, A_0=\frac{1}{2}|\phi|^2,\\ &\partial_1A_2-\partial_2A_1=-\frac{1}{2}|\phi|^2, \partial_1A_1+\partial_2A_2=0. \end{aligned}\tag{1.3}$$

Indeed, (1.3) provides static solutions to (1.2) if $p=4$. The self-dual equations (1.3) can be transformed into the Liouville equation, whose solutions are known. (1.2) was first proposed in [21, 22, 23]. If we set in (1.2)

$$\begin{aligned} \phi(t,x)&=u(|x|)e^{-i\lambda t}, \quad A_0(x)=A_0(|x|),\\ A_1(t,x)&=\frac{x_2}{|x|^2}h(|x|), \quad A_2(t,x)=-\frac{x_1}{|x|^2}h(|x|), \end{aligned}$$

then u satisfies (1.1). For more details about (1.2) and (1.3), we refer the readers to [10, 11, 14, 16, 24, 29, 30, 32].

Throughout this paper, we denote the norm of $L^p(\mathbf{R}^2)$ by

$$\|u\|_p:=\left(\int_{\mathbf{R}^2}|u|^p\right)^{\frac{1}{p}}$$

for any $1\le p<\infty$. The Hilbert space $H^1(\mathbf{R}^2)$ is defined as

$$H^1(\mathbf{R}^2):=\{u\in L^2(\mathbf{R}^2)\colon \nabla u\in L^2(\mathbf{R}^2)\},$$

with the inner product and norm

$$(u,v):=\int_{\mathbf{R}^2}\nabla u\nabla v+\int_{\mathbf{R}^2}uv,\ \ \|u\|:=(\|\nabla u\|_2^2+\|u\|_2^2)^{\frac{1}{2}},$$

$H^{-1}(\mathbf{R}^2)$ is the dual space of $H^1(\mathbf{R}^2)$ and $H^1_r(\mathbf{R}^2)$ is the subspace of radically symmetric functions in $H^1(\mathbf{R}^2)$ endowed with the usual $H^1(\mathbf{R}^2)$ norm. We use respectively "$\to$" and "$\rightharpoonup$" to denote the strong and weak convergence in the related function spaces. C will denote a positive constant unless specified. Moreover we define, for short, the following quantities

$$A(u) := \|\nabla u\|_2^2 = \int_{\mathbf{R}^2} |\nabla u|^2; \quad B(u) := \int_{\mathbf{R}^2} \frac{|u|^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s)\, ds \right)^2;$$
$$C(u) := \|u\|_p^p = \int_{\mathbf{R}^2} |u|^p; \quad D(u) := \|u\|_2^2 = \int_{\mathbf{R}^2} |u|^2.$$

We say that $u \in H_r^1(\mathbf{R}^2)$ is a weak solution to (1.1) if

$$\int_{\mathbf{R}^2} \nabla u \nabla \varphi - \lambda \int_{\mathbf{R}^2} u\varphi + \int_{\mathbf{R}^2} \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s)\, ds \right) u\varphi - \int_{\mathbf{R}^2} |u|^{p-2} u\varphi = 0,$$

for all $\varphi \in H_r^1(\mathbf{R}^2)$ and $(u_c, \lambda_c) \in H_r^1(\mathbf{R}^2) \times \mathbf{R}$ is a couple of weak solution to (1.1) if u_c is a weak solution to (1.1) with $\lambda = \lambda_c$.

Motivated by the fact that physicists are often interested in normalized solutions, that is, solutions with a prescribed L^2-norm, we consider for each $c > 0$ the following problem:

(P_c) To find a couple $(u_c, \lambda_c) \in H_r^1(\mathbf{R}^2) \times \mathbf{R}$ of weak solution to (1.1) such that $\|u_c\|_2^2 = c$.

Define

$$I(u) = \frac{1}{2} \int_{\mathbf{R}^2} |\nabla u|^2 + \frac{1}{2} \int_{\mathbf{R}^2} \frac{|u|^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s)\, ds \right)^2 - \frac{1}{p} \int_{\mathbf{R}^2} |u|^p \tag{1.4}$$

for $u \in H_r^1(\mathbf{R}^2)$, then $I \in C^1(H_r^1(\mathbf{R}^2), \mathbf{R})$ and a critical point of I restricted on the constraint

$$S_r(c) = \{u \in H_r^1(\mathbf{R}^2) \colon \|u\|_{L^2(\mathbf{R}^2)}^2 = c\}, \quad c > 0 \tag{1.5}$$

corresponds to a couple $(u_c, \lambda_c) \in H_r^1(\mathbf{R}^2) \times \mathbf{R}$ of weak solution to (1.1) such that $\|u_c\|_2^2 = c$ (see Lemma 2.1).

The $\lambda \in \mathbf{R}$ in (1.1) is called a frequency. For fixed λ, [10, 17, 29] obtained weak solutions to (1.1) by looking for critical points of the C^1 functional

$$J(u) = \frac{1}{2} \int_{\mathbf{R}^2} |\nabla u|^2 - \frac{\lambda}{2} \int_{\mathbf{R}^2} |u|^2 + \frac{1}{2} \int_{\mathbf{R}^2} \frac{|u|^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} u^2(s)\, ds \right)^2 - \frac{1}{p} \int_{\mathbf{R}^2} |u|^p$$

defined in $H_r^1(\mathbf{R}^2)$. If $p > 4$, the above functional $J(u)$ has the mountain pass structure when $\lambda < 0$. When applying directly the Mountain Pass Theorem to get a critical point of J in $H_r^1(\mathbf{R}^2)$, it is vital to check whether the Palais–Smale condition holds or not. For the value $p \geq 6$, it is standard to show that the Palais–Smale condition holds for J in $H_r^1(\mathbf{R}^2)$. However, for $p \in (4, 6)$, it seems hard to prove whether or not the Palais–Smale condition holds for J in $H_r^1(\mathbf{R}^2)$. To overcome the difficulty, motivated by [31], [10] considered a minimization problem on a manifold of Pohozaev–Nehari type in $H_r^1(\mathbf{R}^2)$. If $p = 4$, [10] constructed a family of critical points of J in $H_r^1(\mathbf{R}^2)$ only when $\lambda = 0$. In addition, when $p \in (2, 4)$, [10] considered normalized solutions to (1.1) by minimizing $I(u)$ defined by (1.4) on the constraints $S_r(c)$ defined by (1.5). The main result of [10] is that for $p \in (2, 3]$ and any $c > 0$, there exists a positive minimizer of $I(u)$ on $S_r(c)$; for $p \in (3, 4)$, there exists a positive minimizer of $I(u)$ on $S_r(c)$ only for sufficiently small c.

In [29], by studying the global behavior of the functional $J(u)$, Pomponio and Ruiz proved the existence and nonexistence of positive solutions to (1.1) for different value of λ when $p \in (2,4)$. Precisely, they showed that J is bounded from below if and only if

$$\lambda \leq \lambda_0 := \frac{p-4}{p+2} 3^{\frac{p-2}{2(4-p)}} 2^{\frac{2}{4-p}} \left(\frac{m^2(2+p)}{p-2} \right)^{-\frac{p-2}{2(4-p)}},$$

where $m = \int_{-\infty}^{+\infty} \omega_1^2(r)\, dr$ and $\omega_1(r) = \left(\frac{2}{p} \cos h^2 \left(\frac{(p-2)r}{2}\right)\right)^{-\frac{2}{p-2}}$ is the unique positive even solution of the problem $-\omega'' + \omega = \omega^p$ in $\mathbf{R}$. Furthermore, regarding the existence of solutions to (1.1), they obtained that there exists $\overline{\lambda} < \overline{\overline{\lambda}} < \lambda_0$ such that (1.1) has no nontrivial solutions if $\lambda < \overline{\lambda}$; (1.1) admits at least two positive solutions, one is a global minimizer for J and the other is a mountain pass solution if $\lambda \in (\overline{\overline{\lambda}}, \lambda_0)$; (1.1) admits a positive solution for almost every $\lambda \in (\lambda_0, 0)$. They also studied in [30] the bounded domain case for $p \in (2,4)$. By using singular perturbation arguments based on a Lyapunov–Schmidt reduction, they obtained some results on boundary concentration of solutions.

Recently, normalized solutions to elliptic PDEs and systems attract much attention of researchers, see e.g. [2, 3, 4, 6, 7, 8, 18, 19, 20, 27, 28, 36]. In [18], Jeanjean considered the following semi-linear Schrödinger equation:

$$-\Delta u - \lambda u = g(u), \quad \lambda \in \mathbf{R}, \quad x \in \mathbf{R}^N, \tag{1.6}$$

where $N \geq 1$ and g satisfies

(H_1) $g\colon \mathbf{R} \to \mathbf{R}$ is continuous and odd;

(H_2) there exists $(\alpha, \beta) \in \mathbf{R} \times \mathbf{R}$ satisfying

$$\begin{cases} \frac{2N+4}{N} < \alpha \leq \beta < \frac{2N}{N-2}, & N \geq 3, \\ \frac{2N+4}{N} < \alpha \leq \beta, & N = 2, \end{cases}$$

such that

$$\alpha G(s) \leq g(s)s \leq \beta G(s), \quad G(s) = \int_0^s g(z)\, dz;$$

(H_3) let $\widetilde{G}\colon \mathbf{R} \to \mathbf{R}$, $\widetilde{G}(s) = g(s)s - 2G(s)$. Then $\widetilde{G}'$ exists and

$$\widetilde{G}'(s)s > \frac{2N+4}{N} \widetilde{G}(s).$$

Under assumptions (H_1) and (H_2) for $N \geq 2$ or (H_1)–(H_3) for $N \geq 1$, by using a minimax procedure, [18] proved that for each $c > 0$, there is a couple $(u_c, \lambda_c) \in H^1(\mathbf{R}^N) \times \mathbf{R}^-$ of weak solution to (1.6) with $\|u_c\|_2^2 = c$. After that, an $H^1(\mathbf{R}^N)$-bifurcation result associated with (1.6), i.e. a dependence of $\|\nabla u_c\|_2$ and λ_c on the value of c was proved (see Corollary 3.1 and Theorem 3.2 in [18]).

In [6], the following Schrödinger–Poisson equation was considered:

$$-\Delta u - \lambda u + (|x|^{-1} * u^2)u = |u|^{p-2}u, \quad \lambda \in \mathbf{R}, \quad x \in \mathbf{R}^3. \tag{1.7}$$

By using a mountain pass argument on

$$S(c) = \{u \in H^1(\mathbf{R}^N) \colon \|u\|^2_{L^2(\mathbf{R}^N)} = c\}, \quad c > 0, \tag{1.8}$$

[6] proved that for $p \in (\frac{10}{3}, 6)$, there exists $c_0 > 0$ such that for any $c \in (0, c_0)$ there exists a couple $(u_c, \lambda_c) \in H^1(\mathbf{R}^N) \times \mathbf{R}^-$ of weak solution to (1.7) with $\|u_c\|_2^2 = c$.

In [4], Bartsch and De Valeriola considered the semi-linear Schrödinger equation (1.6) above. Under assumptions (H_1) and (H_2) with $G(s) > 0$, [4] proved a multiplicity result for normalized solutions to equation (1.6). Luo in [28] then generalized the main result in [4] concerning (1.6) to the Schrödinger–Poisson equation (1.7) above. Luo proved that when $p \in (\frac{10}{3}, 6)$, there exists $c_0 > 0$ such that for any $c \in (0, c_0)$, (1.7) admits an unbounded sequence of couple of weak solutions $\{(\pm u_n, \lambda_n)\} \subseteq H_r^1(\mathbf{R}^N) \times \mathbf{R}^-$ with $\|u_n\|_2^2 = c$ for each $n \in \mathbf{N}^+$.

In [27], Li and Ye considered the following semilinear Choquard equation:

$$-\Delta u - \lambda u = (I_\alpha * G(u))G'(u), \quad x \in \mathbf{R}^N, \quad \lambda \in \mathbf{R},$$

where $N \geq 3$, $\alpha \in (0, N)$, $I_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{N}{2})\pi^{\frac{N}{2}}2^\alpha} \frac{1}{|x|^{N-\alpha}}$. Under certain assumptions on $G(u)$, by using a minimax procedure inspired by [6], the authors of [27] proved that for any $c > 0$, there is at least a couple $(u_c, \lambda_c) \in H^1(\mathbf{R}^N) \times \mathbf{R}^-$ of weak solution to the equation above with $\|u_c\|_2 = c$.

In [20], Jeanjean et al. considered the following quasi-linear Schrödinger equation:

$$-\Delta u - u\Delta(u^2) - \lambda u = |u|^{p-1}u, \quad \text{in } \mathbf{R}^N,$$

where $p \in (1, \frac{3N+2}{N-2})$ if $N \geq 3$ and $p \in (1, +\infty)$ if $N = 1, 2$. By a perturbation method, they prove the existence of two normalized solutions for the above problem. One is a mountain pass solution on a constraint and the other is a minimum either local or global.

Recently, Bartsch et al. considered normalized solutions to the nonlinear Schrödinger systems in [2, 3]. In [3], the following coupled cubic Schrödinger systems was considered:

$$\begin{cases} -\Delta u - \lambda_1 u = \mu_1 u^3 + \beta u v^2, \\ -\Delta v - \lambda_2 v = \mu_1 v^3 + \beta u^2 v, \end{cases} \quad \text{in } \mathbf{R}^3.$$

By using different constrain minimization methods, for different ranges of the coupling parameter $\beta > 0$, they proved the existence of positive solutions satisfying the additional condition

$$\int_{\mathbf{R}^3} |u|^2 = a_1 > 0 \quad \text{and} \quad \int_{\mathbf{R}^3} |v|^2 = a_2 > 0$$

to the above systems.

In this paper, we discuss the existence, $H^1(\mathbf{R}^2)$-bifurcation and multiplicity of normalized solutions to the nonlocal problem (1.1). For any $c > 0$, we set

$$\gamma(c) := \inf_{u \in S_r(c)} I(u).$$

It is standard that the minimizers of $\gamma(c)$ are critical points of $I\big|_{S_r(c)}$ as well as normalized solutions to (1.1). Letting $u^t(x) = tu(tx)$, $t > 0$, it is easy to know that $p = 4$ is L^2-critical or mass-critical exponent for our minimizing problem in the sense that for any $c > 0$, $\gamma(c) > -\infty$ if $p \in (2, 4]$ and $\gamma(c) = -\infty$ if $p \in (4, +\infty)$. In the mass-subcritical case $p \in (2, 4)$, $I(u)$ is bounded from below and coercive on $S_r(c)$. As mentioned above, [10] proved that when $p \in (2, 4)$, under certain condition on c, $I(u)$ has a minimum point on $S_r(c)$ (see Proposition 4.3 in [10]). To the best knowledge of ours, in the mass-critical case where $p = 4$ and mass-supercritical case where $p \in (4, +\infty)$, the existence of critical points of $I(u)$ restricted on $S_r(c)$ are still unknown. In this paper, we consider normalized solutions to (1.1) in the mass-critical case where $p = 4$ and mass-supercritical case where $p \in (4, +\infty)$.

Our main results are as follows:

Theorem 1.1. *Let $p = 4$. Then*

(i) $\gamma(c) = 0$ *for all* $c > 0$;

(ii) $\gamma(c)$ *has no minimizer if* $0 < c < 2\|W_4\|_2^2 = (1.86225\cdots) \times (4\pi)$;

(iii) I *has no constraint critical point on* $S_r(c)$ *if* $0 < c < 2\|W_4\|_2^2$;

(iv) $\gamma(8\pi) = 0$ *has a family of minimizers:*

$$\left\{ u(l,x) = \frac{\sqrt{8}l}{1+|lx|^2} \in H_r^1(\mathbf{R}^2) \mid l \in (0,+\infty) \right\};$$

(v) $(u(l,x),0)$ *is a couple of weak solution to* (1.1) *for any* $l \in (0,+\infty)$,

where $\gamma(c) := \inf_{u\in S_r(c)} I(u)$ *and* W_4 *is the unique ground state solution of*

$$-\Delta W + W = W^3, \quad x \in \mathbf{R}^2.$$

Theorem 1.2. *Let* $p \in (4,+\infty)$. *Then there exists* $c^* > 0$ *such that for any* $c \in (0, c^*]$ *there exists a couple of weak solution* $(u_c, \lambda_c) \in H_r^1(\mathbf{R}^2) \times (\mathbf{R}^- \cup \{0\})$ $(\lambda_c \in \mathbf{R}^-$ *if* $c < \frac{4\pi}{\sqrt{p-3}})$ *to* (1.1) *with* $\|u_c\|_2^2 = c$ *and* u_c *is nonnegative. Furthermore,*

$$\begin{cases} \|\nabla u_c\|_2 \to +\infty, \\ \lambda_c \to -\infty, \\ I(u_c) \to +\infty, \end{cases}$$

as $c \to 0$.

Theorem 1.3. *Let* $p \in (4,+\infty)$ *and* $c \in (0, \frac{4\pi}{\sqrt{p-3}})$. *Then* (1.1) *has a sequence of couples of weak solutions* $\{(v_n, \widetilde{\lambda}_n)\} \subseteq H_r^1(\mathbf{R}^2) \times \mathbf{R}^-$ *with* $\|v_n\|_2^2 = c$ *and* $\|v_n\|_{H^1(\mathbf{R}^2)}^2 \to +\infty$ *as* $n \to +\infty$.

Remark 1.4. To the best of our knowledge, the main results in this paper are new. Theorem 1.1 and Theorem 1.2 generalize the result of Proposition 4.3 in [10] to the mass-critical case $p = 4$ and mass-supercritical case $p > 4$. Theorem 1.1 also extends partially the main results in [19] considering the Schrödinger–Poisson equation (1.7) to the Chern–Simons–Schrödinger equation (1.1). Notice that, for mass-critical Schrödinger–Poisson equation i.e. $p = \frac{10}{3}$ in (1.7), there is no result related to the existence of normalized solutions to (1.7). Theorem 1.2 also extends the results of Theorem 2.1 and Corollary 3.1 in [18] which considered the semi-linear Schrödinger equation (1.6) to the Chern–Simons–Schrödinger equation (1.1). Theorem 1.3 generalizes the results of Theorem 1.1 in [4] and Theorem 1.1 in [28] concerning the Schrödinger equation (1.6) and Schrödinger–Poisson equation (1.7) to the Chern–Simons–Schrödinger equation (1.1). Notice that, although [10] obtained a positive solution $u \in H_r^1(\mathbf{R}^2)$ to (1.1) when $p \in (4,6)$ and [17] obtained infinitely many solutions to (1.1) when $p > 6$, there is no information about the L^2-norm of the solutions. So Theorem 1.2 and Theorem 1.3 in this paper can also be viewed as a complement of the main results in [10, 17].

Now, we give the main idea of the proof of our main results. The key points of proving Theorem 1.1 are some established inequalities in Lemma 2.2–2.4. Here, we shall see the difference between (1.1) and the Schrödinger–Poisson equation (see [19]). In the mass-supercritical case $p \in (4,+\infty)$, the functional $I(u)$ is no longer bounded from below on $S(c)$ (Lemma 2.5), the minimization method on $S_r(c)$ used in [10] does not work. Motivated by minimization method on Nehari manifold and some recent works of [15, 26, 31], we try to construct a submanifold of $S_r(c)$, on which $I(u)$ is bounded from below and coercive, and then we look for minimizers on such a submanifold. The idea of constructing such a suitable submanifold is in the following. We notice that, if u is a critical point of $I\,\big|_{S_r(c)}$, then $I'(u) - \lambda u = 0$ in $H_r^{-1}(\mathbf{R}^2)$ for some $\lambda \in \mathbf{R}$. Hence u satisfies the following Pohozaev identity (Lemma 2.8):

$$P_\lambda(u) := \lambda \int_{\mathbf{R}^2} |u|^2 - 2\int_{\mathbf{R}^2} \frac{|u|^2}{|x|^2}\left(\int_0^{|x|} \frac{s}{2}u^2(s)\,ds\right)^2 + \frac{2}{p}\int_{\mathbf{R}^2} |u|^p = 0. \tag{1.9}$$

Combining the Pohozaev functional $P_\lambda(u)$ with the Nehari functional $N_\lambda(u) = \langle I'(u) -\lambda u, u\rangle$, we introduce another auxiliary functional

$$\begin{aligned} Q(u) &:= N_\lambda(u) + P_\lambda(u) \\ &= \int_{\mathbf{R}^2} |\nabla u|^2 + \int_{\mathbf{R}^2} \frac{|u|^2}{|x|^2}\left(\int_0^{|x|} \frac{s}{2}u^2(s)\,ds\right)^2 - \frac{p-2}{p}\int_{\mathbf{R}^2} |u|^p \end{aligned} \tag{1.10}$$

and construct a submanifold $V(c)$ as follows:

$$V(c) := \{u \in S_r(c) \colon Q(u) = 0\}. \tag{1.11}$$

If u is a critical point of I with $\|u\|_2^2 = c$, then $u \in V(c)$. By considering the following minimization problem

$$m(c) := \inf_{u\in V(c)} I(u), \tag{1.12}$$

we find a critical point of I restricted to $V(c)$ and prove that it is indeed a critical point of I restricted to $S_r(c)$. Notice that we have two restrictions in $V(c)$, which is different from the situation in [15, 26, 31]. In order to use Lagrange Theorem, we need to prove that $Q'(u)$ and $D'(u)$ are linearly independent if u is a critical point of I restricted to $V(c)$ (see Lemma 2.13 for details). The main difficulty in proving the existence of a minimizer for $m(c)$ is due to the lack of compactness of the embedding $H_r^1(\mathbf{R}^2) \hookrightarrow L^2(\mathbf{R}^2)$. To overcome this difficulty, we need the monotonicity of the function $c \to m(c)$. We would like to mention that the two methods used in [6] (see Theorem 1.2) and in [25] (see Lemma 2.9) seem difficult to be used here due to the existence of the Chern–Simons term in (1.1). Motivated by [5], after getting an equality related to $m(c)$ (see Lemma 2.14), we succeed in proving the monotonicity property of $m(c)$ by a scaling argument. Then we can obtain the L^2 compactness of a minimizing sequence and a minimizer of $m(c)$ for certain c. Let us denote the set of minimizers of $I(u)$ on $V(c)$ as

$$\mathcal{M}_c := \{u \in V(c) \colon I(u) = \inf_{v\in V(c)} I(v)\}. \tag{1.13}$$

Then we prove the first part of Theorem 1.1 by showing a simple property of $\mathcal{M}_c$ (see Proposition 2.19).

The idea of proving the dependence of $\|\nabla u_c\|_2$ and λ_c on the value of c comes from [18, 32, 33]. The fact that u_c is a minimizer of $I(u)$ restricted on $V(c)$ and $Q(u_c) = 0$ are crucial. Due to the nonlocal property of our problem, we need some improvements of the method used in [18]. We proved an important inequality for $u \in H_r^1(\mathbf{R}^2)$ (see Lemma 2.3)

$$\int_{\mathbf{R}^2} \frac{|u|^2}{|x|^2}\left(\int_0^{|x|} \frac{s}{2}u^2(s)\,ds\right)^2 \le \frac{1}{16\pi^2}\,\|\nabla u\|_2^2\,\|u\|_2^4, \tag{1.14}$$

which is vital for proving $\lambda_c \to -\infty$ as $c \to 0$.

Next, we give the main idea of the proof of Theorem 1.3. Since I is unbounded from below on $S_r(c)$ if $p \in (4, +\infty)$, the genus of the sublevel set

$$I^d := \{u \in S_r(c) \colon I(u) \le d\}$$

is always infinite. Thus, to obtain the existence of infinitely many solutions, classical argument based on the Kranoselski genus (see [33]) does not work. We use the argument in [4] to present a new type of linking geometry which is inspired by the Fountain theorem for the functional I restricted on $S_r(c)$. Then a min-max scheme is set up to construct an unbounded sequence $\{\gamma_n(c)\}$ of critical values for I on $S_r(c)$. At each level $\gamma_n(c)$, by using Lemma 2.3 in [18] in $H_r^1(\mathbf{R}^2)$, we get a Palais–Smale sequence $\{v_k^n\}_{k=1}^{+\infty}$ with an additional condition $Q(v_k^n) \to 0$ as $k \to +\infty$ (see Proposition 2.25), where $Q(u)$ is given in (1.10). This extra condition is crucial in proving the boundness and non-vanishing of $\{v_k^n\}$ (see the proof of Proposition 2.27). By working in the radially symmetric Soblev space $H_r^1(\mathbf{R}^2)$, which embeds compactly in $L^q(\mathbf{R}^2)$ for $2 < q < +\infty$, we could recover the compactness of our Palais–Smale sequence. Here we need the fact that the associated Lagrange multiplier is strictly negative. Therefore, we get a critical point v_n at each level $\gamma_n(c)$. By using the corresponding Pohozaev identity, we prove that each critical point v_n of I restricted on $S_r(c)$ satisfies $Q(v_n) = 0$, which is useful in proving that the critical point sequence $\{v_n\}$ is unbounded in $H_r^1(\mathbf{R}^2)$.

The paper is organized as follows. In Section 2, we present some preliminary results. In Section 3, we will prove our main results Theorem 1.1 Theorem 1.2 and Theorem 1.3.

2. Preliminary results

In this section, we give some preliminary results.

Lemma 2.1. [10, Proposition 2.2] *Let $p > 2$. Then $I \in C^1(H_r^1(\mathbf{R}^2))$ and a critical point of I on $S_r(c)$ is a weak solution of* (1.1).

Lemma 2.2. [10, Proposition 2.4] *For $u \in H_r^1(\mathbf{R}^2)$, the following inequality holds*

$$\int_{\mathbf{R}^2} |u|^4 \le 4\left(\int_{\mathbf{R}^2} |\nabla u|^2\right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^2} \frac{|u|^2}{|x|^2}\left(\int_0^{|x|} \frac{s}{2} u^2(s)\, ds\right)^2\right)^{\frac{1}{2}}.$$

Furthermore, the equality is attained by a continuum of functions

$$\left\{ u(l,x) = \frac{\sqrt{8}l}{1 + |lx|^2} \in H_r^1(\mathbf{R}^2) \mid l \in (0, +\infty) \right\}$$

and

$$\frac{1}{4}\int_{\mathbf{R}^2} |u(l,x)|^4 = \int_{\mathbf{R}^2} |\nabla u(l,x)|^2 = \int_{\mathbf{R}^2} \frac{|u(l,x)|^2}{|x|^2}\left(\int_0^{|x|} \frac{s}{2} u(l,s)^2\, ds\right)^2 = \frac{16}{3}\pi l^2.$$

Lemma 2.3. *For $u \in H_r^1(\mathbf{R}^2)$, the following inequality holds*

$$\int_{\mathbf{R}^2} \frac{|u|^2}{|x|^2}\left(\int_0^{|x|} \frac{s}{2} u^2(s)\, ds\right)^2 \le \frac{1}{16\pi^2} \|\nabla u\|_2^2 \|u\|_2^4.$$

Proof. By Hölder's inequality, we have

$$\begin{aligned}(2.1)\qquad \int_0^{|x|}\frac{s}{2}u^2(s)\,ds &= \frac{1}{4\pi}\int_{B(0,|x|)}u^2 \le \frac{1}{4\pi}\left(\int_{B(0,|x|)}u^4\right)^{\frac12}\cdot\left(\int_{B(0,|x|)}1^2\right)^{\frac12}\\ &\le \frac{1}{4\pi}\left(\int_{\mathbf{R}^2}u^4\right)^{\frac12}\cdot(\pi x^2)^{\frac12} = \frac{1}{4\sqrt{\pi}}\,|x|\cdot\left(\int_{\mathbf{R}^2}u^4\right)^{\frac12}.\end{aligned}$$

Thus, by Lemma 2.2 we have

$$\begin{aligned}(2.2)\qquad B(u) &= \int_{\mathbf{R}^2}\frac{|u|^2}{|x|^2}\left(\int_0^{|x|}\frac{s}{2}u^2(s)\,ds\right)^2 \le \frac{1}{16\pi}\,\|u\|_4^4\,\|u\|_2^2\\ &\le \frac{1}{4\pi}A(u)^{\frac12}B(u)^{\frac12}D(u),\end{aligned}$$

which concludes

$$(2.3)\qquad B(u) \le \frac{1}{16\pi^2}A(u)D(u)^2. \qquad \square$$

Lemma 2.4. [35, Gagliardo–Nirenberg inequality] *Let* $p \ge 2$ *and* $u \in H^1(\mathbf{R}^2)$. *Then*

$$\|u\|_p \le \left(\frac{p}{2\,\|W_p\|_2^{p-2}}\right)^{\frac1p}\|\nabla u\|_2^{\frac{p-2}{p}}\,\|u\|_2^{\frac2p}$$

with equality holds only for $u = W_p$, *where up to translations,* W_p *is the unique ground state solution of*

$$-\frac{p-2}{2}\Delta W + W = |W|^{p-2}W, \quad x \in \mathbf{R}^2.$$

Furthermore, when $p = 4$,

$$\frac12 C(W_4) = A(W_4) = D(W_4) := \|W_4\|_2^2 = (1.86225\cdots)\times(2\pi).$$

Then, we introduce the Cazenave rescaling [12], for $u \in S_r(c)$, set $u^t(x) = tu(tx)$, $t > 0$, then

$$A(u^t) = t^2A(u),\ B(u^t) = t^2B(u),\ C(u^t) = t^{p-2}C(u),\ D(u^t) = D(u)$$

and

$$(2.4)\qquad I(u^t) = \frac12 t^2(A(u)+B(u)) - \frac1p t^{p-2}C(u).$$

Lemma 2.5. *Let* $p \in (4,+\infty)$. *Then for any* $u \in S_r(c)$, $u^t \in S_r(c)$, $A(u^t) \to +\infty$ *and* $I(u^t) \to -\infty$ *as* $t \to \infty$.

Proof. For any $u \in S_r(c)$, since $D(u^t) = D(u)$, $u^t(x) \in S_r(c)$. By (2.4), $A(u^t) \to +\infty$ and $I(u^t) \to -\infty$ as $t \to \infty$ follow from the fact that $p > 4$. $\square$

Lemma 2.6. *Let* $p \in (4,+\infty)$. *Then for any* $u \in S_r(c)$, $c > 0$, *there exists a unique* $t_0 > 0$ *such that*

$$I(u^{t_0}) = \max_{t>0} I(u^t) = \frac{(p-4)p^{\frac{2}{p-4}}}{2(p-2)^{\frac{p-2}{p-4}}}\frac{[A(u)+B(u)]^{\frac{p-2}{p-4}}}{[C(u)]^{\frac{2}{p-4}}}$$

and $u^{t_0} \in V(c)$. *In particular,*

(i) $t_0 < 1 \iff Q(u) < 0$;
(ii) $t_0 = 1 \iff Q(u) = 0$;

where $V(c)$ is given in (1.11) *and $Q(u)$ is given in* (1.10).

Proof. Define $\tau(t) := I(u^t) = \frac{1}{2}t^2(A(u)+B(u)) - \frac{1}{p}t^{p-2}C(u)$. By Lemma 2.4 and an elementary analysis, we know that $\tau(t)$ has a unique critical point $t_0 > 0$ corresponding to its maximum on $(0,+\infty)$. Hence $I(u^{t_0}) = \max_{t>0} I(u^t)$ and $\tau'(t_0) = t_0(A(u)+B(u)) - \frac{p-2}{p}t_0^{p-3}C(u) = 0$, thus $Q(u^{t_0}) = t_0^2(A(u)+B(u)) - \frac{p-2}{p}t_0^{p-2}C(u) = 0$, i.e. $u^{t_0} \in V(c)$ and

$$I(u^{t_0}) = \frac{(p-4)p^{\frac{2}{p-4}}}{2(p-2)^{\frac{p-2}{p-4}}} \frac{[A(u)+B(u)]^{\frac{p-2}{p-4}}}{[C(u)]^{\frac{2}{p-4}}}.$$

Moreover,

$$Q(u) = A(u)+B(u) - \frac{p-2}{p}C(u) = (A(u)+B(u))(1-t_0^{4-p}),$$

which concludes (i) and (ii). □

Recall that a functional $F\colon X \to \mathbf{R}$ on a Banach space X is called coercive if, for every sequence $\{u_k\} \subset X$ with $\|u_k\| \to +\infty$ implies $F(u_k) \to +\infty$ (see Definition 1.5.5 in [1]).

Lemma 2.7. *Let $p \in (4,+\infty)$. Then $I(u)$ is bounded from below and coercive on $V(c)$. Moreover, there exists a constant $C_0 > 0$ such that $I(u) \geq C_0$ for all $u \in V(c)$.*

Proof. For any $u \in V(c)$, $Q(u) = A(u) + B(u) - \frac{p-2}{p}C(u) = 0$, then $C(u) = \frac{p}{p-2}(A(u)+B(u))$. We have

$$I(u) = \frac{1}{2}(A(u)+B(u)) - \frac{1}{p}C(u) = \left(\frac{1}{2} - \frac{1}{p-2}\right)(A(u)+B(u)) \geq 0,$$

and I is coercive on $V(c)$. Furthermore, by Lemma 2.4,

$$A(u)+B(u) = \frac{p-2}{p}C(u) \leq C(p)\left\|\nabla u\right\|_2^{p-2}\left\|u\right\|_2^2 = C(p,c)A(u)^{\frac{p-2}{2}}.$$

Since $p \in (4,+\infty)$, there exists a constant $\widetilde{C_0} > 0$ such that $A(u) \geq \widetilde{C_0} > 0$. Then there exists $C_0 = \left(\frac{1}{2} - \frac{1}{p-2}\right)\widetilde{C_0} > 0$ such that $I(u) \geq \left(\frac{1}{2} - \frac{1}{p-2}\right)A(u) \geq C_0$. □

Lemma 2.8. [10, Proposition 2.3] *Let $p > 2$, $\overline{b}, \overline{c}, \overline{d} \in \mathbf{R}$ and $u \in H_r^1(\mathbf{R}^2)$ be a weak solution of the equation:*

$$\Delta u + \overline{b}u + \overline{c}\left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s}u^2(s)\,ds\right)u + \overline{d}|u|^{p-2}u = 0 \quad \text{in } \mathbf{R}^2,$$

where $h(s) = \frac{1}{2}\int_0^s ru^2(r)\,dr$. Then there holds the following Pohozaev identity

$$\overline{b}\int_{\mathbf{R}^2} |u|^2 + 2\overline{c}\int_{\mathbf{R}^2} \frac{|u|^2}{|x|^2}\left(\int_0^{|x|} \frac{s}{2}u^2(s)\,ds\right)^2 + \frac{2\overline{d}}{p}\int_{\mathbf{R}^2} |u|^p = 0.$$

Lemma 2.9. *For any $p > 2$ and $\lambda > 0$, there exists no positive solution to* (1.1) *in $H_r^1(\mathbf{R}^2)$.*

Proof. The proof mainly comes from [10] with some modifications. Just suppose that there exists a positive solution $u \in H_r^1(\mathbf{R}^2)$ to (1.1). Denote

$$a_0(x) := \lambda - \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s)\, ds\right) + |u|^{p-2},$$

then u satisfies $-\Delta u = a_0(x)u$. By the Strauss inequality,

$$|u(x)| \le C\frac{\|u\|}{|x|^{\frac{1}{2}}}, \quad |x| > 0.$$

By Hölder inequality, we have

$$h(|x|) = \frac{1}{2}\int_0^{|x|} su^2(s)\, ds = \frac{1}{4\pi}\int_{B(0,|x|)} u^2(|y|)\, dy \le \frac{\|u\|_2^2}{4\pi},$$

and by (2.1),

$$\begin{aligned}
\int_0^{+\infty} \frac{h(s)}{s} u^2(s)\, ds &= \int_0^1 \frac{h(s)}{s} u^2(s)\, ds + \int_1^{+\infty} \frac{h(s)}{s} u^2(s)\, ds \\
&\le \left(\int_0^1 (h(s)s^{-\frac{4}{3}})^{\frac{3}{2}}\, ds\right)^{\frac{2}{3}} \left(\int_0^1 (s^{\frac{1}{3}}u^2(s))^3\, ds\right)^{\frac{1}{3}} + \int_1^{+\infty} \frac{h(s)}{s} u^2(s) ds \\
&\le C\,\|u\|_4^2 \left(\int_0^1 s^{-\frac{1}{2}}\, ds\right)^{\frac{2}{3}} \left(\int_0^1 su^6(s)\, ds\right)^{\frac{1}{3}} + C\,\|u\|_4^2 \int_1^{+\infty} su^2(s)\, ds \\
&\le C\,\|u\|_4^2\,(\|u\|_6^2 + \|u\|_2^2)
\end{aligned}$$

Then, we can choose an R_0 sufficiently large such that$\inf_{|x|>R_0} a_0(x) := a_0 > 0$. For $R_1 > R_0$, we consider the following eigenvalue problem:

$$\begin{cases} -\Delta\phi = \mu\phi & \text{on } A(R_0, R_1), \\ \phi = 0 & \text{on } \partial A(R_0, R_1), \end{cases} \tag{2.5}$$

where $A(R_0, R_1) = \{x \in \mathbf{R}^2\colon R_0 < |x| < R_1\}$. Let $\mu_1 = \mu_1(R_0, R_1)$ be the first eigenvalue of the problem (2.5) and ϕ_1 is a corresponding positive eigenfunction. Then we have

$$0 = \int_{A(R_0,R_1)} (\Delta u + a_0(x)u)\phi_1 = -\int_{\partial A(R_0,R_1)} u\frac{\partial\phi_1}{\partial n} + \int_{A(R_0,R_1)} (-\mu_1 + a_0(x))u\phi_1,$$

where $\frac{\partial\phi_1}{\partial n}$ denotes the outer normal derivative of ϕ_1. We note that $\int_{\partial A(R_0,R_1)} u\frac{\partial\phi_1}{\partial n} \le 0$ and that $\mu_1 \to 0$ as $R_1 \to +\infty$. Thus, taking large $R_1 > 0$ large enough such that $\mu_1 \le \frac{a_0}{2}$, which is a contradiction. The proof is completed. □

Lemma 2.10. *Let $p \in (3, +\infty)$. If $v \in H_r^1(\mathbf{R}^2)$ is a weak solution to (1.1), then $Q(v) = 0$. Moreover, $v = 0$ if $\lambda \ge 0$ and $D(v) = \|v\|_2^2 < \frac{4\pi}{\sqrt{p-3}}$.*

Proof. Let $v \in H_r^1(\mathbf{R}^2)$ be a weak solution to (1.1). By Lemma 2.8, the following Pohozaev identity

$$\lambda\int_{\mathbf{R}^2} |v|^2 - 2\int_{\mathbf{R}^2} \frac{|v|^2}{|x|^2}\left(\int_0^{|x|} \frac{s}{2}v^2(s)\, ds\right)^2 + \frac{2}{p}\int_{\mathbf{R}^2} |v|^p = 0$$

holds. Multiplying (1.1) by v and integrating we derive a second identity,

$$\int_{\mathbf{R}^2} |\nabla v|^2 - \lambda \int_{\mathbf{R}^2} |v|^2 + 3\int_{\mathbf{R}^2} \frac{|v|^2}{|x|^2}\left(\int_0^{|x|} \frac{s}{2} v^2(s)\, ds\right)^2 - \int_{\mathbf{R}^2} |v|^p = 0.$$

Thus we have immediately

$$Q(v) = \int_{\mathbf{R}^2} |\nabla v|^2 + \int_{\mathbf{R}^2} \frac{|v|^2}{|x|^2}\left(\int_0^{|x|} \frac{s}{2} v^2(s)\, ds\right)^2 - \frac{p-2}{p}\int_{\mathbf{R}^2} |v|^p = 0.$$

Also with simple calculations, we obtain

$$\lambda D(v) = \frac{2}{2-p} A(v) + \frac{2p-6}{p-2} B(v). \tag{2.6}$$

By Lemma 2.3,

$$\begin{aligned} \lambda D(v) &= \frac{2}{2-p}A(v) + \frac{2p-6}{p-2}B(v) \le \frac{2}{2-p}A(v) + \frac{p-3}{8\pi^2(p-2)}A(v)D(v)^2 \\ &= \left(\frac{2}{2-p} + \frac{p-3}{8\pi^2(p-2)}D(v)^2\right)A(v) \le 0, \end{aligned} \tag{2.7}$$

if $D(v) = \|v\|_2^2 < \frac{4\pi}{\sqrt{p-3}}$. Thus, the proof is completed. □

Lemma 2.11. *Let $p \in (3, +\infty)$. If u_c is a critical point of $I\big|_{S_r(c)}$, then there exists $\lambda_c \in \mathbf{R}$ such that (u_c, λ_c) is a couple of solution to (1.1). Furthermore, $\lambda_c < 0$ if $c < \frac{4\pi}{\sqrt{p-3}}$.*

Proof. Since u_c is a critical point of $I\big|_{S_r(c)}$, there exists $\lambda_c \in \mathbf{R}$ such that $I'(u_c) - \lambda_c u_c = 0$ in $H_r^{-1}(\mathbf{R}^2)$. Thus u_c satisfies (1.1) with $\lambda = \lambda_c$. By Lemma 2.10, we conclude that $\lambda_c < 0$ if $c < \frac{4\pi}{\sqrt{p-3}}$. □

Lemma 2.12. [12, Corollary 4.1.2] *Let X be a real Banach space, $U \subset X$ be an open set. Suppose that $f, g_1, \cdots, g_m \colon U \to \mathbf{R}^1$ are C^1 functions and $x_0 \in M$ is such that $f(x_0) = \inf_{x\in M} f(x)$ with*

$$M = \{x \in U \mid g_i(x) = 0,\ i = 1, 2, \cdots, m\}.$$

If $\{g_i'(x_0)\}_{i=1}^m$ is linearly independent, then there exists $\lambda_1, \cdots, \lambda_m \in \mathbf{R}$ such that

$$f'(x_0) + \sum_{i=1}^m \lambda_i g_i'(x_0) = 0.$$

Lemma 2.13. *Let $p \in (4, +\infty)$ and $c > 0$. Then each critical point of $I\big|_{V(c)}$ is a critical point of $I\big|_{S_r(c)}$.*

Proof. Suppose that u is a critical point of $I\big|_{V(c)}$, then by Lemma 2.12, either (i) $Q'(u)$ and $D'(u)$ are linearly dependent, or (ii) there exists $\lambda_1, \lambda_2 \in \mathbf{R}$ such that

$$I'(u) - \lambda_1 Q'(u) - \lambda_2 u = 0 \quad \text{in } H^{-1}(\mathbf{R}^2). \tag{2.8}$$

If (i) holds, then u satisfies

$$-\Delta u - \lambda^* u + \left(\frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{+\infty} \frac{h(s)}{s} u^2(s)\, ds\right) u = \frac{p-2}{2}|u|^{p-2} u$$

for some $\lambda^* \in \mathbf{R}$. Multiplying the above equation by u and integrating, we get

$$A(u) + 3B(u) - \frac{p-2}{2}C(u) - \lambda^* D(u) = 0.$$

By Pohozaev identity, we derive

$$\lambda^* D(u) - 2B(u) + \frac{p-2}{p}C(u) = 0.$$

Hence

$$A(u) + B(u) - \frac{(p-2)^2}{2p}C(u) = 0.$$

Notice that $Q(u) = 0$ and $p > 4$, then we have, immediately, that $C(u) = 0$, a contradiction. This implies that (i) does not occur and (ii) is true. It is enough to show that $\lambda_1 = 0$. By (2.8) we have

$$\begin{aligned}&\left\langle I'(u) - \lambda_1 Q'(u) - \lambda_2 u, u\right\rangle \\ &= (1-2\lambda_1)(A(u)+3B(u)) - [1-\lambda_1(p-2)]C(u) - \lambda_2 D(u) = 0.\end{aligned} \tag{2.9}$$

By Pohozaev identity (Lemma 2.8),

$$\frac{\lambda_2}{1-2\lambda_1}D(u) - 2B(u) + \frac{2-2(p-2)\lambda_1}{p(1-2\lambda_1)}C(u) = 0. \tag{2.10}$$

Combining (2.9) with (2.10) we have

$$(1-2\lambda_1)A(u) + (1-2\lambda_1)B(u) - \frac{p-2}{p}[1-(p-2)\lambda_1]C(u) = 0. \tag{2.11}$$

Since $u \in V(c)$, $A(u) + B(u) = \frac{p-2}{p}C(u)$, then by (2.11) we have

$$\frac{(p-4)(p-2)}{p}\lambda_1 C(u) = 0.$$

Hence $\lambda_1 = 0$, for $p > 4$. $\square$

Lemma 2.14. *Let* $p \in (4, +\infty)$, *then* $\inf_{u\in V(c)} I(u) = \inf_{u\in S_r(c)} \max_{t>0} I(u^t)$, *where* $u^t(x) = tu(tx)$.

Proof. For any $u \in V(c)$, $Q(u) = 0$. By Lemma 2.6,

$$I(u) = \max_{t>0} I(u^t) \geq \inf_{u\in S_r(c)} \max_{t>0} I(u^t),$$

then $\inf_{u\in V(c)} I(u) \geq \inf_{u\in S_r(c)} \max_{t>0} I(u^t)$. On the other hand, by Lemma 2.6, for any $u \in S_r(c)$, there exists a unique $t_0 > 0$ such that $u^{t_0} \in V(c)$ and

$$\max_{t>0} I(u^t) = I(u^{t_0}) \geq \inf_{u\in V(c)} I(u).$$

Thus,

$$\inf_{u\in S_r(c)} \max_{t>0} I(u^t) \geq \inf_{u\in V(c)} I(u).$$

We end the proof. $\square$

Lemma 2.15. *Let* $p \in (4, +\infty)$. *Define* $m(c) := \inf_{u\in V(c)} I(u)$, *then there exists a* $c^* > 0$ *such that the function* $c \to m(c)$ *is strictly decreasing on* $(0, c^*]$, *where* $V(c)$ *is given in* (1.10).

Proof. By Lemma 2.7, $m(c) \geq C_0 > 0$ is well defined. For any $0 < c_1 < c_2 < +\infty$, by Lemma 2.13, there exists $u_1 \in S_r(c_1)$ such that

$$\max_{t>0} I(u_1^t) < \left(\frac{c_2}{c_1}\right)^{\frac{1}{p-4}} m(c_1).$$

Set

$$u_2(x) = \left(\sqrt{\frac{c_2}{c_1}}\right)^p u_1\left(\left(\sqrt{\frac{c_2}{c_1}}\right)^{p-1} x\right),$$

then

$$A(u_2) = \left(\frac{c_2}{c_1}\right)^p A(u_1), \quad B(u_2) = \left(\frac{c_2}{c_1}\right)^{p+2} B(u_1),$$

$$C(u_2) = \left(\frac{c_2}{c_1}\right)^{\frac{p^2-2p+2}{2}} C(u_1), \quad D(u_2) = \frac{c_2}{c_1} D(u_1) = c_2.$$

We claim that there exists a $c^* > 0$ such that

$$\left(\frac{c_2}{c_1}\right)^p A(u_1) + \left(\frac{c_2}{c_1}\right)^{p+2} B(u_1) \leq \left(\frac{c_2}{c_1}\right)^{p+\frac{1}{p-2}} A(u_1) + \left(\frac{c_2}{c_1}\right)^{p+\frac{1}{p-2}} B(u_1)$$

for $0 < c_1 < c_2 \leq c^*$. Indeed, by a simple calculation, we have

$$\frac{c_2^{\frac{1}{p-2}}}{16\pi^2}\left(c_2^2 - c_1^2\right) \leq c_2^{\frac{1}{p-2}} - c_1^{\frac{1}{p-2}},$$

if $0 < c_1 < c_2 \leq [2(p-2)]^{\frac{2-p}{2p-5}}$. Let $0 < c_1 < c_2 \leq [2(p-2)]^{\frac{2-p}{2p-5}}$ afterwards. Then we get that

$$\frac{1}{16\pi^2}\left(c_2^2 c_1^{\frac{1}{p-2}} - c_1^2 c_2^{\frac{1}{p-2}}\right) \leq \frac{c_2^{\frac{1}{p-2}}}{16\pi^2}\left(c_2^2 - c_1^2\right) \leq c_2^{\frac{1}{p-2}} - c_1^{\frac{1}{p-2}}.$$

This implies that

$$\frac{c_1^2}{16\pi^2}\left[\left(\frac{c_2}{c_1}\right)^2 - \left(\frac{c_2}{c_1}\right)^{\frac{1}{p-2}}\right] A(u_1) \leq \left[\left(\frac{c_2}{c_1}\right)^{\frac{1}{p-2}} - 1\right] A(u_1).$$

By Lemma 2.3, we have that

$$B(u_1) \leq \frac{1}{16\pi^2} A(u_1) c_1^2,$$

which implies that

$$\left[\left(\frac{c_2}{c_1}\right)^2 - \left(\frac{c_2}{c_1}\right)^{\frac{1}{p-2}}\right] B(u_1) \leq \frac{c_1^2}{16\pi^2}\left[\left(\frac{c_2}{c_1}\right)^2 - \left(\frac{c_2}{c_1}\right)^{\frac{1}{p-2}}\right] A(u_1) \leq \left[\left(\frac{c_2}{c_1}\right)^{\frac{1}{p-2}} - 1\right] A(u_1).$$

As a consequence,

$$A(u_1) + \left(\frac{c_2}{c_1}\right)^2 B(u_1) \leq \left(\frac{c_2}{c_1}\right)^{\frac{1}{p-2}} [A(u_1) + B(u_1)],$$

which is equivalent to our claim for $0 < c_1 < c_2 \le c^* := [2(p-2)]^{\frac{2-p}{2p-5}}$. Thus, by Lemma 2.6,

$$
\begin{aligned}
m(c_2) &\le \max_{t>0} I(u_2^t) = \frac{(p-4)p^{\frac{2}{p-4}}}{2(p-2)^{\frac{p-2}{p-4}}} \frac{[A(u_2)+B(u_2)]^{\frac{p-2}{p-4}}}{[C(u_2)]^{\frac{2}{p-4}}} \\
&= \frac{(p-4)p^{\frac{2}{p-4}}}{2(p-2)^{\frac{p-2}{p-4}}} \frac{[(\frac{c_2}{c_1})^p A(u_1) + (\frac{c_2}{c_1})^{p+2} B(u_1)]^{\frac{p-2}{p-4}}}{(\frac{c_2}{c_1})^{\frac{p^2-2p+2}{p-4}} [C(u_1)]^{\frac{2}{p-4}}} \\
&\le \frac{(p-4)p^{\frac{2}{p-4}}}{2(p-2)^{\frac{p-2}{p-4}}} \left(\frac{c_2}{c_1}\right)^{\frac{-1}{p-4}} \frac{[A(u_1)+B(u_1)]^{\frac{p-2}{p-4}}}{[C(u_1)]^{\frac{2}{p-4}}} \\
&= \left(\frac{c_2}{c_1}\right)^{\frac{-1}{p-4}} \max_{t>0} I(u_1^t) < \left(\frac{c_2}{c_1}\right)^{\frac{-1}{p-4}} \left(\frac{c_2}{c_1}\right)^{\frac{1}{p-4}} m(c_1) = m(c_1)
\end{aligned}
\tag{2.12}
$$

holds for $0 < c_1 < c_2 \le c^*$. Thus, we complete the proof. $\square$

Lemma 2.16. [10, Proposition 2.2 and Lemma 3.2] *Let $p > 2$. Then $B \in C^1(H_r^1(\mathbf{R}^2))$. Furthermore, if $u_n \rightharpoonup u$ in $H_r^1(\mathbf{R}^2)$, as $n \to +\infty$, then*

$$
\lim_{n\to+\infty} B(u_n) = B(u), \quad \lim_{n\to+\infty} B'(u_n)u_n = B'(u)u \quad \text{and} \quad \lim_{n\to+\infty} B'(u_n)\varphi = B'(u)\varphi,
$$

for any $\varphi \in H_r^1(\mathbf{R}^2)$.

Proposition 2.17. *Let $p \in (4, +\infty)$ and $c \in (0, c^*]$. Then $m(c) := \inf_{u \in V(c)} I(u)$ is attained, where $V(c)$ and c^* are given in Lemma 2.15.*

Proof. Let $\{u_n\}$ be a minimizing sequence for $m(c)$. By Lemma 2.7, $\{u_n\}$ is bounded in $H_r^1(\mathbf{R}^2)$, then there exists $u \ne 0$ in $H_r^1(\mathbf{R}^2)$ such that

$$
\begin{cases}
u_n \rightharpoonup u & \text{in } H_r^1(\mathbf{R}^2), \\
u_n \to u & \text{in } L^q(\mathbf{R}^2), \\
u_n \to u & \text{a.e. in } \mathbf{R}^2,
\end{cases}
$$

for $2 < q < +\infty$. Otherwise, $u_n \to 0$ in $L^p(\mathbf{R}^2)$. Since

$$
Q(u_n) = A(u_n) + B(u_n) - \frac{p-2}{p} C(u_n) = 0,
$$

we have $A(u_n) \to 0$ and $B(u_n) \to 0$. Therefore, $I(u_n) \to 0$ and $m(c) = 0$, which contradicts to the fact that $m(c) > 0$. Next, we shall prove that $\|u\|_2^2 = c$. Just suppose that $\|u\|_2^2 = \overline{c} \in (0, c)$, then by Lemma 2.15, $m(\overline{c}) > m(c)$. Since $u_n \rightharpoonup u$ in $H_r^1(\mathbf{R}^2)$, $Q(u) \le \lim_{n\to\infty} Q(u_n) = 0$. By Lemma 2.6, there exists $t_0 \in (0, 1]$ such that $u^{t_0} \in V(\overline{c})$. Then

$$
\begin{aligned}
m(\overline{c}) &\le I(u^{t_0}) = I(u^{t_0}) - \frac{1}{p-2} Q(u^{t_0}) = \frac{p-4}{2(p-2)} \left[A(u^{t_0}) + B(u^{t_0})\right] \\
&= \frac{p-4}{2(p-2)} t_0^2 \left[A(u) + B(u)\right] \le \frac{p-4}{2(p-2)} \left[A(u) + B(u)\right] \\
&\le \lim_{n\to\infty} \left\{ \frac{p-4}{2(p-2)} \left[A(u_n) + B(u_n)\right] \right\} = \lim_{n\to\infty} \left[I(u_n) - \frac{1}{p-2} Q(u_n) \right] \\
&= m(c),
\end{aligned}
\tag{2.13}
$$

which is a contradiction. So $t_0 = 1$, $c = \bar{c}$, i.e. $\|u\|_2^2 = c$ and $I(u) = m(c)$. Then by (2.13) we have $A(u_n - u) = o(1)$, $u_n \to u$ in $H_r^1(\mathbf{R}^2)$ and u is a minimizer for $m(c)$. □

Proposition 2.18. *Assume that $p \in (4, +\infty)$, $c > 0$ and $\mathcal{M}_c$ is defined by (1.13). Then $|u_c| \in \mathcal{M}_c$ if $u_c \in \mathcal{M}_c$.*

Proof. Let $u_c \in H_r^1(\mathbf{R}^2)$ with $u_c \in V(c)$. Since $B(|u_c|) = B(u_c)$, $A(|u_c|) \le A(u_c)$, we have that $I(|u_c|) \le I(u_c)$ and $Q(|u_c|) \le Q(u_c) = 0$. In addition, by Lemma 2.6, there exists $t_0 \in (0, 1]$ such that $Q(|u_c|^{t_0}) = 0$. We claim that $I(|u_c|^{t_0}) \le t_0^2 \cdot I(u_c)$. Indeed, for $u_c \in V(c)$, $|u_c|^{t_0} \in V(c)$, by the relationship between $I(u)$ and $Q(u)$ in (2.13), we have $I(u_c)$, $I(|u_c|^{t_0}) > 0$. Thus,

$$
\begin{aligned}
I(|u_c|^{t_0}) &= I\left(|u_c|^{t_0}\right) - \frac{1}{p-2} Q\left(|u_c|^{t_0}\right) = \frac{p-4}{2(p-2)}\left(A\left(|u_c|^{t_0}\right) + B\left(|u_c|^{t_0}\right)\right) \\
&= \frac{p-4}{2(p-2)} {t_0}^2 \left(A\left(|u_c|\right) + B\left(|u_c|\right)\right) \le \frac{p-4}{2(p-2)} {t_0}^2 \left(A\left(u_c\right) + B\left(u_c\right)\right) \\
&= t_0^2 \left(I\left(u_c\right) - \frac{1}{p-2} Q\left(u_c\right)\right) = t_0^2 I(u_c).
\end{aligned} \tag{2.14}
$$

Therefore, if $u_c \in H_r^1(\mathbf{R}^2)$ is a minimizer of $I(u)$ on $V(c)$ we have

$$
I(u_c) = \inf_{u \in V(c)} I(u) \le I\left(|u_c|^{t_0}\right) \le t_0^2 I(u_c),
$$

which implies $t_0 = 1$. Then $Q(|u_c|) = 0$ and we conclude that $A(|u_c|) = A(u_c)$ and $I(|u_c|) = I(u_c)$, thus the proof is completed. □

Let $\{V_n\} \subset H_r^1(\mathbf{R}^2)$ be a strictly increasing sequence of finite-dimensional linear subspaces in $H_r^1(\mathbf{R}^2)$ such that $\bigcup_n V_n$ is dense in $H_r^1(\mathbf{R}^2)$. We denote by $V_n^\perp$ the orthogonal space of V_n in $H_r^1(\mathbf{R}^2)$.

Lemma 2.19. [4, Lemma 2.1] *Let $p \in (2, +\infty)$. Then there holds*

$$
\mu_n := \inf_{u \in V_{n-1}^\perp} \frac{\int_{\mathbf{R}^2}\left(|\nabla u|^2 + |u|^2\right)}{\left(\int_{\mathbf{R}^2} |u|^p\right)^{2/p}} = \inf_{u \in V_{n-1}^\perp} \frac{\|u\|^2}{\|u\|_p^2} \to \infty \quad \text{as } n \to \infty.
$$

Now for $c > 0$ fixed and for each $n \in \mathbf{N}^+$ and $n \ge 2$, we define $S_r(c)$ by (1.5),

$$
\begin{aligned}
&\rho_n := L^{-\frac{2}{p-2}} \cdot {\mu_n}^{\frac{2}{p-2}} \quad \text{with} \quad L = \max_{x>0} \frac{(x^2+c)^{p/2}}{x^p + c^{p/2}}, \\
&B_n := \left\{u \in V_{n-1}^\perp \cap S_r(c) \colon \|\nabla u\|_2^2 = \rho_n\right\},
\end{aligned} \tag{2.15}
$$

and

$$
b_n := \inf_{u \in B_n} I(u). \tag{2.16}
$$

Then we have:

Lemma 2.20. *Let $p \in (2, +\infty)$, then $b_n \to \infty$ as $n \to \infty$.*

Proof. For any $u \in B_n$, we have that

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbf{R}^2} |\nabla u|^2 + \frac{1}{2}\int_{\mathbf{R}^2} \frac{|u|^2}{|x|^2}\left(\int_0^{|x|} \frac{s}{2}u^2(s)\,ds\right)^2 - \frac{1}{p}\int_{\mathbf{R}^2} |u|^p \\ &\geq \frac{1}{2}\int_{\mathbf{R}^2} |\nabla u|^2 - \frac{1}{p\mu_n}\left(\|\nabla u\|_2^2 + c\right)^{p/2} \\ &\geq \frac{1}{2}\int_{\mathbf{R}^2} |\nabla u|^2 - \frac{L}{p\mu_n}\left(\|\nabla u\|_2^p + c^{p/2}\right) = \left(\frac{1}{2} - \frac{1}{p}\right)\rho_n - \frac{L}{p\mu_n}c^{p/2}. \end{aligned} \tag{2.17}$$

From this estimate and Lemma 2.19, it follows since $p > 2$, that $b_n \to \infty$ as $n \to \infty$. □

Now we begin to set up our min-max procedure. First we introduce the map

$$\begin{aligned} &\kappa\colon H_r^1(\mathbf{R}^2) \times \mathbf{R} \to H_r^1(\mathbf{R}^2), \\ &\qquad (u,\theta) \to \kappa(u,\theta) := e^{\theta}u\left(e^{\theta}x\right). \end{aligned} \tag{2.18}$$

Observe that for any given $u \in S_r(c)$, we have $\kappa(u,\theta) \in S_r(c)$ for all $\theta \in \mathbf{R}$. Also we know from Lemma 2.5 that

$$\begin{cases} A(\kappa(u,\theta)) \to 0, & I(\kappa(u,\theta)) \to 0, \qquad \theta \to -\infty, \\ A(\kappa(u,\theta)) \to +\infty, & I(\kappa(u,\theta)) \to -\infty, \quad \theta \to +\infty. \end{cases} \tag{2.19}$$

Thus, we deduce that for each $n \in \mathbf{N}$, there exists $\theta_n > 0$, such that

$$\overline{g}_n\colon [0,1] \times (S_r(c) \cap V_n) \to S_r(c), \quad \overline{g}_n(t,u) \to \kappa(u,(2t-1)\theta_n) \tag{2.20}$$

satisfying

$$\begin{cases} A(\overline{g}_n(0,u)) < \rho_n, & A(\overline{g}_n(1,u)) > \rho_n, \\ I(\overline{g}_n(0,u)) < b_n, & I(\overline{g}_n(1,u)) < b_n. \end{cases} \tag{2.21}$$

Now we define

$$\begin{aligned} \Gamma_n := \{g\colon [0,1] \times (S_r(c) \cap V_n) \to S_r(c) \mid g \text{ is continuous, odd in } u \\ \text{and such that } \forall u\colon g(0,u) = \overline{g}_n(0,u),\ g(1,u) = \overline{g}_n(1,u)\}. \end{aligned} \tag{2.22}$$

Clearly $\overline{g}_n \in \Gamma_n$. Before proving the key intersection result, we need the following linking property:

Lemma 2.21. [4, Lemma 2.3] *For each* $g \in \Gamma_n$*, there exists* $(t,u) \in [0,1] \times (S_r(c) \cap V_n)$ *such that* $g(t,u) \in B_n$ *with* B_n *defined in* (2.15).

Lemma 2.22. *For each* $n \in \mathbf{N}^+$,

$$\gamma_n(c) := \inf_{g \in \Gamma_n} \max_{0 \leq t \leq 1, u \in S_r(c) \cap V_n} I(g(t,u)) \geq b_n.$$

Proof. It follows from Lemma 2.21 immediately. □

Next, we shall prove that the sequence $\{\gamma_n(c)\}$ is indeed a sequence of critical values for I restricted to $S_r(c)$. To this end, we first show that there exists a bounded Palais–Smale sequence at each level $\gamma_n(c)$. From now on we fix an arbitrary $n \in \mathbf{N}^+$. To find such a Palais–Smale sequence, we apply the approach developed by Jeanjean [18], already applied in [4]. First, we introduce the auxiliary functional

$$\widetilde{I}\colon S_r(c) \times \mathbf{R} \to \mathbf{R}, \quad (u,\theta) \to I(\kappa(u,\theta)),$$

where $\kappa(u,\theta)$ is given in (2.21), and the set

$$\widetilde{\Gamma}_n := \{\widetilde{g}\colon [0,1]\times(S_r(c)\cap V_n)\to S_r(c)\times\mathbf{R} \mid \widetilde{g} \text{ is continuous, odd in } u, \text{ and such that } \kappa\circ\widetilde{g}\in\Gamma_n\}. \tag{2.23}$$

Clearly, for any $g\in\Gamma_n$, $\widetilde{g}:=(g,0)\in\widetilde{\Gamma}_n$.

Observe the definition

$$\widetilde{\gamma}_n(c) := \inf_{\widetilde{g}\in\widetilde{\Gamma}_n}\max_{0\le t\le 1, u\in S_r(c)\cap V_n}\widetilde{I}(\widetilde{g}(t,u)),$$

we have that $\widetilde{\gamma}_n(c)=\gamma_n(c)$. Indeed, by the definition of $\widetilde{\gamma}_n(c)$ and $\gamma_n(c)$, this identity follows immediately from the fact that the maps

$$\varphi\colon \Gamma_n\to\widetilde{\Gamma}_n,\quad g\to\varphi(g):=(g,0),$$

and

$$\psi\colon \widetilde{\Gamma}_n\to\Gamma_n,\quad \widetilde{g}\to\psi(\widetilde{g}):=\kappa\circ\widetilde{g},$$

satisfy

$$\widetilde{I}(\varphi(g))=I(g)\quad\text{and}\quad I(\psi(\widetilde{g}))=\widetilde{I}(\widetilde{g}).$$

For $r\in\mathbf{R}$, We define $|r|_{\mathbf{R}}=r$. Then we denote by E the space $H_r^1(\mathbf{R}^2)\times\mathbf{R}$ endowed with the norm $\|\cdot\|_E^2=\|\cdot\|^2+|\cdot|_{\mathbf{R}}^2$, and by E^* its dual space and give an useful result, which was proved by using Ekeland's variational principle.

Lemma 2.23. *Let $\varepsilon>0$. Suppose that $\widetilde{g}_0\in\widetilde{\Gamma}_n$ satisfies*

$$\max_{0\le t\le 1, u\in S_r(c)\cap V_n}\widetilde{I}(\widetilde{g}_0(t,u))\le\widetilde{\gamma}_n(c)+\varepsilon.$$

Then there exists a pair of $(u_0,\theta_0)\in S_r(c)\times\mathbf{R}$ such that:

(1) $\widetilde{I}(u_0,\theta_0)\in[\widetilde{\gamma}_n(c)-\varepsilon,\widetilde{\gamma}_n(c)+\varepsilon]$;

(2) $\displaystyle\min_{0\le t\le 1, u\in S_r(c)\cap V_n}\|(u_0,\theta_0)-\widetilde{g}_0(t,u)\|_E\le\sqrt{\varepsilon}$;

(3) $\displaystyle\left\|\widetilde{I}'\big|_{S_r(c)\times\mathbf{R}}(u_0,\theta_0)\right\|_{E^*}\le 2\sqrt{\varepsilon}$, *i.e.* $\displaystyle\left|\left\langle\widetilde{I}'(u_0,\theta_0),z\right\rangle_{E^*\times E}\right|\le 2\sqrt{\varepsilon}\|z\|_E$ *holds, for all*

$$z\in\widetilde{T}_{(u_0,\theta_0)}:=\{(z_1,z_2)\in E, \langle u_0,z_1\rangle_{L^2}=0\}.$$

Proof. The proof is the same as the proof of Lemma 2.3 in [18], so we omit it here. □

Proposition 2.24. *Let $p\in(4,+\infty)$. Then for any fixed $c>0$ and $n\in N^+$, there exists a sequence $\{v_k^n\}\subset S_r(c)$ satisfying as $k\to\infty$,*

$$\begin{cases} I(v_k^n)\to\gamma_n(c),\\ I'\big|_{S_r(c)}(v_k^n)\to 0,\\ Q(v_k^n)\to 0.\end{cases}\tag{2.24}$$

In particular $\{v_k^n\}\subset S_r(c)$ is bounded in $H_r^1(\mathbf{R}^2)$.

Proof. From the definition of $\gamma_n(c)$, we know that for each $k\in\mathbf{N}^+$, there exists a $g_k\in\Gamma_n$ such that

$$\max_{0\le t\le 1, u\in S_r(c)\cap V_n} I(g_k(t,u))\le\gamma_n(c)+\frac{1}{k}.$$

Since $\widetilde{\gamma}_n(c) = \gamma_n(c)$, $\widetilde{g}_k = (g_k, 0) \in \widetilde{\Gamma}_n$ satisfies

$$\max_{0\le t\le 1, u\in S_r(c)\cap V_n} \widetilde{I}(\widetilde{g}_k(t,u)) \le \widetilde{\gamma}_n(c) + \frac{1}{k}.$$

Thus applying Lemma 2.23, we obtain a sequence $\{(u_k^n, \theta_k^n)\} \subset S_r(c) \times \mathbf{R}$ such that

(i) $\widetilde{I}(u_k^n, \theta_k^n) \in \left[\gamma_n(c) - \frac{1}{k}, \gamma_n(c) + \frac{1}{k}\right]$;

(ii) $\displaystyle\min_{0\le t\le 1, u\in S_r(c)\cap V_n} \|(u_k^n, \theta_k^n) - (g_k(t,u), 0)\|_E \le \sqrt{\tfrac{1}{k}}$;

(iii) $\left\|\widetilde{I}'\Big|_{S_r(c)\times\mathbf{R}}(u_k^n, \theta_k^n)\right\|_{E^*} \le 2\sqrt{\frac{1}{k}}$, i.e. $\left|\left\langle \widetilde{I}'(u_k^n, \theta_k^n), z\right\rangle_{E^*\times E}\right| \le 2\sqrt{\frac{1}{k}}\|z\|_E$ holds for all

$$z \in \widetilde{T}_{(u_k^n, \theta_k^n)} := \{(z_1, z_2) \in E,\ \langle u_k^n, z_1\rangle_{L^2} = 0\}.$$

For each $k \in \mathbf{N}^+$, let $v_k^n = \kappa(u_k^n, \theta_k^n)$. We shall prove that $\{v_k^n\} \subset S_r(c)$ satisfies (2.24). First from (i) we have that $I(v_k^n) \to \gamma_n(c)$ as $k \to \infty$, since $I(v_k^n) = I(\kappa(u_k^n, \theta_k^n)) = \widetilde{I}(u_k^n, \theta_k^n)$. Secondly, note that

$$\begin{aligned}
\left\langle \widetilde{I}'(u,\theta), (\phi, r)\right\rangle &= re^{2\theta}\left(\int_{\mathbf{R}^2}|\nabla u|^2 + \int_{\mathbf{R}^2}\frac{|u|^2}{|x|^2}\left(\int_0^{|x|}\frac{s}{2}u^2(s)\,ds\right)^2\right)\\
&\quad + e^{2\theta}\left[\int_{\mathbf{R}^2}\nabla u\nabla\phi + \int_{\mathbf{R}^2}\frac{u\phi}{|x|^2}\left(\int_0^{|x|}\frac{s}{2}u^2(s)\,ds\right)^2\right]\\
&\quad + \int_{\mathbf{R}^2}\frac{|u|^2}{|x|^2}\left(\int_0^{|x|}\frac{s}{2}u^2(s)\,ds\right)\cdot\left(\int_0^{|x|}su(s)\phi(s)\,ds\right)\\
&\quad - \frac{(p-2)r}{p}e^{(p-2)\theta}\int_{\mathbf{R}^2}|u|^p - e^{(p-2)\theta}\int_{\mathbf{R}^2}|u|^{p-2}u\phi,
\end{aligned} \tag{2.25}$$

then we obtain

$$\begin{aligned}
Q(v_k^n) &= A(v_k^n) + B(v_k^n) - \frac{p-2}{p}C(v_k^n)\\
&= e^{2\theta_k^n}(A(u_k^n) + B(u_k^n)) - \frac{p-2}{p}e^{\theta_k^n(p-2)}C(u_k^n) = \left\langle \widetilde{I}'(u_k^n, \theta_k^n), (0,1)\right\rangle.
\end{aligned} \tag{2.26}$$

Thus (iii) yields $Q(v_k^n) \to 0$ as $k \to \infty$, for $(0,1) \in \widetilde{T}_{(u_k^n, \theta_k^n)}$. Finally, we prove that

$$I'\Big|_{S_r(c)}(v_k^n) \to 0 \ \text{ as } \ k \to \infty.$$

We claim that for $k \in \mathbf{N}$ sufficiently large,

$$\left|\left\langle I'(v_k^n), \omega\right\rangle\right| \le \frac{2\sqrt{2}}{\sqrt{k}}\|\omega\| \ \text{ holds for all } \ \omega \in T_{v_k^n},$$

where $T_{v_k^n} = \{\omega \in H_r^1(\mathbf{R}^2),\ \langle v_k^n, \omega\rangle_{L^2} = 0\}$. Indeed, for $\omega \in T_{v_k^n}$, setting $\widetilde{\omega} = \kappa(\omega, -\theta_k)$, one has

$$\begin{aligned}\left\langle I'(v_k^n), \omega\right\rangle &= \int_{\mathbf{R}^2} \nabla v_k^n \nabla \omega + \int_{\mathbf{R}^2} \frac{v_k^n \omega}{|x|^2} \int_0^{|x|} \frac{s}{2}\big(v_k^n(s)\big)^2\, ds\big)^2 \\ &\quad + \int_{\mathbf{R}^2} \frac{|v_k^n|^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2}(v_k^n(s))^2\, ds\right) \cdot \left(\int_0^{|x|} s v_k^n(s)\omega(s)\, ds\right) \\ &\quad - \int_{\mathbf{R}^2} |v_k^n|^{p-2} v_k^n \omega = \left\langle \widetilde{I}'(u_k^n, \theta_k^n), (\widetilde{\omega}, 0)\right\rangle.\end{aligned} \tag{2.27}$$

Since $\int_{\mathbf{R}^2} u_k^n \widetilde{\omega} = \int_{\mathbf{R}^2} v_k^n \omega$, we obtain $(\widetilde{\omega}, 0) \in \widetilde{T}_{(u_k^n, \theta_k^n)} \Leftrightarrow \omega \in T_{v_k^n}$. From (ii) it follows that

$$|\theta_k^n| = |\theta_k^n - 0| \le \min_{0\le t\le 1, u \in S_r(c)\cap V_n} \|(u_k^n, \theta_k^n) - (g_k(t,u), 0)\|_E \le \frac{1}{\sqrt{k}},$$

by which we deduce that, for k large enough,

$$\|(\widetilde{\omega}, 0)\|_E^2 = \|\widetilde{\omega}\|^2 = \int_{\mathbf{R}^2} |\omega|^2 + e^{-2\theta_k^n s} \int_{\mathbf{R}^2} |\nabla \omega|^2 \le 2\|\omega\|^2.$$

Thus, by (iii) we have,

$$\left|\left\langle I'(v_k^n), \omega\right\rangle\right| = \left\langle \widetilde{I}'(u_k^n, \theta_k^n), (\widetilde{\omega}, 0)\right\rangle \le \frac{2}{\sqrt{k}} \|(\widetilde{\omega}, 0)\|_E \le \frac{2\sqrt{2}}{\sqrt{k}}\|\omega\|.$$

As a consequence,

$$\left\| I'\Big|_{S_r(c)}(v_k^n)\right\| = \sup_{\omega \in T_{v_k^n}, \|\omega\|\le 1} \left|\left\langle I'(v_k^n), \omega\right\rangle\right| \le \frac{2\sqrt{2}}{\sqrt{k}} \to 0, \quad k \to \infty.$$

To end the proof of the proposition, it remains to show that $\{v_k^n\} \subset S_r(c)$ is bounded in $H_r^1(\mathbf{R}^2)$. But since $p \in (4, +\infty)$, this follows from the relationship between $I(u)$ and $Q(u)$,

$$I(u) - \frac{1}{p-2} Q(u) = \frac{p-4}{2(p-2)}(A(u) + B(u)). \tag{2.28}$$ □

Next, we show the compactness of our Palai–Smale sequence $\{v_k^n\}$ obtained in Proposition 2.24. First, we give a useful lemma.

Lemma 2.25. *Let F be a C^1 functional on $H^1(\mathbf{R}^2)$, if $\{x_k\} \subset S(c)$ is bounded in $H^1(\mathbf{R}^2)$, then*

$$F'\Big|_{S(c)}(x_k) \to 0 \text{ in } H^{-1}(\mathbf{R}^2) \Longleftrightarrow F'(x_k) - \langle F'(x_k), x_k\rangle x_k \to 0 \text{ in } H^{-1}(\mathbf{R}^2)$$

as $k \to \infty$.

Proof. The proof is the same as the proof of Lemma 3 in [9], so we omit it here. □

Proposition 2.26. *Let $p \in (4, +\infty)$, $c > 0$ and $\{v_k\} \subset S_r(c)$ be a sequence satisfying as $k \to \infty$,*

$$\begin{cases} I(v_k) \to \rho(c) \in \mathbf{R} \setminus \{0\}, \\ I'\big|_{S_r(c)}(v_k) \to 0, \\ Q(v_k) \to 0. \end{cases} \tag{2.29}$$

Then there exists $v \in H_r^1(\mathbf{R}^2)$ and $\{\lambda_k\} \subset \mathbf{R}$ such that up to a subsequence, as $k \to +\infty$,

(i) $v_k \rightharpoonup v \neq 0$ *in* $H_r^1(\mathbf{R}^2)$;
(ii) $\lambda_k \to \widetilde{\lambda}$ *in* $\mathbf{R}$;
(iii) $I'(v_k) - \lambda_k v_k \to 0$ *in* $H_r^{-1}(\mathbf{R}^2)$;
(iv) $I'(v) - \widetilde{\lambda} v = 0$ *in* $H_r^{-1}(\mathbf{R}^2)$.

Moreover, if $\widetilde{\lambda} < 0$, then we have $v_k \to v$ in $H_r^1(\mathbf{R}^2)$ as $k \to \infty$.

Proof. Since by (2.28) and (2.29), $\{v_k\} \subset S_r(c)$ is bounded, up to a subsequence, there exists $v \in H_r^1(\mathbf{R}^2)$ such that

$$\begin{cases} v_k \rightharpoonup v & \text{in } H_r^1(\mathbf{R}^2), \\ v_k \to v & \text{in } L^p(\mathbf{R}^2), \\ v_k \to v & \text{a.e. in } \mathbf{R}^2. \end{cases}$$

If $v = 0$, we have $C(v_k) = o(1)$. Thus we obtain $A(v_k) = o(1)$ and $B(v_k) = o(1)$ for $Q(v_k) = o(1)$. As a consequence, $I(v_k) = o(1)$, which contradicts with $\rho(c) \neq 0$. Thus (i) is obtained. By Lemma 2.25 above,

$$I'\Big|_{S(c)}(v_k) \to 0 \text{ in } H^{-1}(\mathbf{R}^2) \Longleftrightarrow I'(v_k) - \langle I'(v_k), v_k\rangle v_k \to 0 \text{ in } H^{-1}(\mathbf{R}^2) \text{ as } k \to \infty.$$

Since for any $\omega \in H^1(\mathbf{R}^2)$,

$$\begin{aligned} &\left\langle I'(v_k) - \left\langle I'(v_k), v_k \right\rangle v_k, \omega \right\rangle \\ &= \int_{\mathbf{R}^2} \nabla v_k \nabla \omega + \int_{\mathbf{R}^2} \frac{v_k \omega}{|x|^2} \int_0^{|x|} \frac{s}{2} \big(v_k(s)\big)^2 \, ds\big)^2 \\ &\quad + \int_{\mathbf{R}^2} \frac{|v_k|^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2} (v_k(s))^2 \, ds \right) \cdot \left(\int_0^{|x|} s v_k(s) \omega(s) \, ds \right) \\ &\quad - \int_{\mathbf{R}^2} |v_k|^{p-2} v_k \omega - \lambda_k \int_{\mathbf{R}^2} v_k \omega, \end{aligned} \tag{2.30}$$

where

$$\lambda_k = \langle I'(v_k), v_k \rangle = A(v_k) + 3B(v_k) - C(v_k). \tag{2.31}$$

Thus (iii) is proved. Since each term in the right hand of (2.31) is bounded, there exists $\widetilde{\lambda} \in \mathbf{R}$ such that $\lambda_k \to \widetilde{\lambda}$ as $k \to +\infty$ up to a subsequence. Thus (ii) is proved and (iv) follows from (iii). By (ii),(iii) and (iv) we have

$$\left\langle I'(v_k) - \widetilde{\lambda} v_k, v_k - v \right\rangle = o(1) \quad \text{and} \quad \left\langle I'(v) - \widetilde{\lambda} v, v_k - v \right\rangle = 0. \tag{2.32}$$

By Lemma 2.16, we get

$$A(v_k - v) - \widetilde{\lambda} D(v_k - v) = o(1).$$

If $\widetilde{\lambda} < 0$, we have $A(v_k - v) = o(1)$ and $D(v_k - v) = o(1)$, thus $v_k \to v$ in $H_r^1(\mathbf{R}^2)$ as $k \to \infty$. ☐

3. Proof of main results

At this point we can prove our main results.

Proof of Theorem 1.1. Let $u \in S_r(c)$, set $u^t(x) = tu(tx)$, $t > 0$. Then $u^t \in S_r(c)$ and

$$I(u^t) = \frac{1}{2}t^2(A(u) + B(u)) - \frac{1}{p}t^2 C(u) \to 0 \quad \text{as } t \to 0.$$

Thus $\gamma(c) \le 0$ for all $c > 0$. On the other hand, by Lemma 2.2, for any $u \in S_r(c)$,

$$\begin{aligned} I(u) &= \frac{1}{2}A(u) + \frac{1}{2}B(u) - \frac{1}{4}\|u\|_4^4 \ge \frac{1}{2}A(u) + \frac{1}{2}B(u) - A(u)^{\frac{1}{2}}B(u)^{\frac{1}{2}} \\ &= \frac{1}{2}\Big(A(u)^{\frac{1}{2}} - B(u)^{\frac{1}{2}}\Big)^2 \ge 0. \end{aligned} \tag{3.1}$$

Then $\gamma(c) \ge 0$. (i) is proved. Just suppose $\gamma(c)$ has a minimizer u if $0 < c < 2\|W_4\|_2^2$, by Lemma 2.2,

$$\begin{aligned} 0 = I(u) &= \frac{1}{2}A(u) + \frac{1}{2}B(u) - \frac{1}{4}\|u\|_4^4 \ge \frac{1}{2}A(u) + \frac{1}{2}B(u) - A(u)^{\frac{1}{2}}B(u)^{\frac{1}{2}} \\ &= \frac{1}{2}\Big(A(u)^{\frac{1}{2}} - B(u)^{\frac{1}{2}}\Big)^2 \ge 0. \end{aligned} \tag{3.2}$$

Then, by Lemma 2.4,

$$A(u) = B(u) = \frac{1}{4}\|u\|_4^4 \le \frac{c}{2\|W_4\|_2^2}A(u).$$

Thus $A(u) = 0$, $u = 0$, a contradiction. (ii) is proved. To prove (iii), we suppose that I has a constraint critical point v on $S_r(c)$ if $c < 2\|W_4\|_2^2$. By Lemma 2.10, $Q(v) = 0$, then $I(v) = \frac{1}{2}Q(v) = 0$, we can get a contradiction as in proving (ii). To prove (iv), set

$$u(l, x) = \frac{\sqrt{8}l}{1 + |lx|^2} \in H_r^1(\mathbf{R}^2), \quad l \in (0, +\infty),$$

then $D(u(l, x)) = 8\pi$. By Lemma 2.2, $I(u(l, x)) = 0$ and $u(l, x)$ is a minimizer for $\gamma(8\pi)$ for any $l \in (0, +\infty)$. Thus, there exists $\lambda(l) \in \mathbf{R}$ such that $(u(l, x), \lambda(l))$ satisfies (1.1). Then we have

$$A(u(l, x)) + 3B(u(l, x)) - C(u(l, x)) - \lambda(l)D(u(l, x)) = 0.$$

By Lemma 2.2 and the fact that $D(u(l, x)) = 8\pi$, $\lambda(l) = 0$. We end the proof. □

Proof of Theorem 1.2. The first part follows from Lemma 2.11, Lemma 2.13, Proposition 2.17 Proposition 2.18 and Lemma 2.9. By Lemma 2.10, $Q(u_c) = A(u_c) + B(u_c) - \frac{p-2}{p}C(u_c) = 0$, then, by Lemma 2.4,

$$A(u_c) + B(u_c) = \frac{p-2}{p}C(u_c) \le \frac{p-2}{2\|W\|_2^{p-2}}A(u_c)^{\frac{p-2}{2}} \cdot c.$$

Then,

$$A(u_c)^{\frac{4-p}{2}} \le \frac{p-2}{2\|W\|_2^{p-2}} \cdot c \to 0$$

as $c \to 0^+$, i.e. $A(u_c) \to +\infty$ as $c \to 0^+$. Moreover,

$$m(c) = I(u_c) = \frac{p-4}{2(p-2)}(A(u_c) + B(u_c)) \to +\infty$$

as $c \to 0^+$. From (1.1), we have $A(u_c) + 3B(u_c) - C(u_c) - \lambda_c D(u_c) = 0$, then, by Lemma 2.3,

$$\begin{aligned} \lambda_c &= \frac{1}{c}[A(u_c) + 3B(u_c) - C(u_c)] = \frac{1}{c} \cdot [\frac{2}{2-p}A(u_c) + \frac{2p-6}{p-2}A(u_c)] \\ &\le \frac{1}{c} \cdot [\frac{2}{2-p}A(u_c) + \frac{p-3}{8\pi^2(p-2)}A(u_c)D(u_c)^2] \qquad (3.3) \\ &= \frac{1}{c} \cdot [\frac{2}{2-p} + \frac{p-3}{8\pi^2(p-2)}D(u_c)^2]A(u_c) \le \frac{1}{c} \cdot \frac{1}{2-p}A(u_c) \to -\infty, \end{aligned}$$

as $c \to 0^+$, for $p \in (4, +\infty)$. Thus the proof is completed. □

Proof of Theorem 1.3. By Proposition 2.24 and Proposition 2.26, there exists $(v_n, \widetilde{\lambda}_n) \in H^1_r(\mathbf{R}^2)\backslash\{0\} \times \mathbf{R}$ which satisfies (1.1) and it is enough to prove that $\widetilde{\lambda}_n < 0$, for each $n \in \mathbf{N}^+$. However, this point has been proved in Lemma 2.11. Since

$$I(v_n) - \frac{1}{p-2}Q(v_n) = \frac{p-4}{2(p-2)}(A(v_n) + B(v_n)) = \gamma_n(c),$$

for $Q(v_n) = 0$, then by Lemma 2.3, we get that $\{v_n\}$ is unbounded in $H^1_r(\mathbf{R}^2)$ from the fact in Lemma 2.20 and Lemma 2.22 that $\gamma_n(c) \ge b_n \to \infty$ as $n \to \infty$. Thus the proof is completed. □

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