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A GENERALIZATION OF THE EICHLER TRACE FORMULA FOR MORPHISMS BETWEEN RIEMANN SURFACES

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Abstract. Let T be an automorphism of a closed Riemann surface. The Eichler trace formula asserts that the trace of the pull-back action of T on the space of holomorphic 1-forms can be evaluated in terms of the local behavior of T around the fixed points. In this paper, we will generalize this formula for morphisms between closed Riemann surfaces of possibly different genera.

1. Introduction

Throughout this paper, all of the Riemann surfaces are closed and of genera ≥ 1. We recall the Eichler trace formula. Let T be an automorphism of order q of a Riemann surface X. Suppose that there are t fixed points $p_1, p_2, \cdots, p_t$. We put $\zeta = e^{2\pi i/q}$. Choosing a coordinated neighbourhood around p_j properly, we may write

$$T\colon z \to \zeta^{k_j} z$$

for some k_j, $1 \leq k_j \leq q-1$ in the neighbourhood of p_j. T induces the pull-back action on the space of holomorphic 1-forms on X. Let χ denote the trace of a matrix representation of the action on the space of holomorphic 1-forms. Then the Eichler trace formula is

$$\chi = 1 - \sum_{j=1}^{t} \frac{\zeta^{k_j}}{\zeta^{k_j} - 1}. \tag{1}$$

The Eichler trace formula is closely related to the Lefschetz fixed-point formula. From (1), we can derive

$$t = 2 - \chi - \overline{\chi}.$$

This is the Lefschetz fixed-point formula for closed Riemann surfaces.

Since the trace χ is invariant under base changes, it is characterized by an automorphism. Conversely, it is known that an automorphism is characterized by the trace of induced action on the space of holomorphic 1-forms. A number of articles studying the traces of automorphisms of Riemann surfaces have appeared. We refer the reader to [1] and references therein for further discussion of this point.

Fuertes and González-Diez (see [4]) applied the Lefschetz fixed-point formula to study the number of *coincidences*, that is, the number of points $p \in X$ with $f_1(p) = f_2(p)$ for two distinct morphisms $f_i\colon X \to Y$ $(i = 1, 2)$ between Riemann surfaces. They gave a sharp bound for the number of coincidences of two morphisms.

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Theorem 1.1. (Fuertes and González-Diez) *Let $f_i\colon X \to Y$ be two distinct morphisms of degree d_i $(i = 1, 2)$ between closed Riemann surfaces of genera g and γ, respectively, and let $L(f_1, f_2)$ denote the number of coincidences appropriately counted. We have*

i) $L(f_1, f_2) \leq d_1 + 2\gamma\sqrt{d_1 d_2} + d_2$.

ii) *In case $\gamma \geq 2$, this bound is attained if and only if Y is hyperelliptic and $f_2 = J \circ f_1$, where J denotes the hyperelliptic involution of Y.*

Their results generalize the well known fact about the number of fixed points of automorphisms, namely, the number of fixed points is less than or equal to $2g + 2$ for an arbitrary automorphism of a closed Riemann surface. More recently, Fuertes [3] showed several results concerning the number of coincidences by composing morphisms with meromorphic functions on the target Y.

In [7] the author studied the case where there exists no coincidence of two distinct morphisms $f_i\colon X \to Y$ $(i = 1, 2)$ between closed Riemann surfaces, namely, the case where $L(f_1, f_2) = 0$. Roughly speaking, $\mathrm{trace}(f_1^* \circ f_{2*}|_{H^1_{DR}(X)})$ defines an inner-product on the space of morphisms from X to Y, where $f_*\colon H^1_{DR}(X) \to H^1_{DR}(Y)$ is defined by the property $\int_Y f_* v \wedge w = \int_X v \wedge f^* w$, for any $w \in H^1_{DR}(Y)$. A necessary and sufficient condition for $L(f_1, f_2) = 0$ in terms of the inner-product was given. The holomorphic Lefschetz number for coincidences $L(f_1, f_2, \mathcal{O})$ was defined as

$$L(f_1, f_2, \mathcal{O}) = \sum_{q=0}^{1} (-1)^q \,\mathrm{trace} f_1^* \circ f_{2*}|_{H^{0,q}_{\overline{\partial}}(X)},$$

to be used for the proofs.

In this paper, we investigate the holomorphic Lefschetz number more, and show that the number can be evaluated in terms of the local behavior of two morphisms around the coincidences. As a result, a generalization of the Eichler trace formula for morphisms between Riemann surfaces is given. This is all done in section 4. We exhibit examples for the generalized Eichler trace formula in section 5. In section 6, we examine the possibility of diagonalizing matrix representations of $f_1^* \circ f_{2*}$.

2. The Lefschetz trace formula

In the following, we use the notation of [4] and Chapter 3.4 of [5]. Let X be a Riemann surface and let $T \in \mathrm{Aut}(X)$. We put $\Gamma_T = \{(p, T(p))\} \subset X \times X$ the graph of T. A fixed point of T is corresponds to a point of intersection of the graph Γ_T and the diagonal submanifold $\Delta \subset X \times X$. The *Lefschetz number* of T is defined to be

$$L(T) = \sharp(\Delta \cdot \Gamma_T).$$

By using integral,

$$L(T) = \int_{\Gamma_T} \varphi_\Delta = \int_X (id. \times T)^* \varphi_\Delta,$$

where $\varphi_\Delta \in H^2_{DR}(X \times X)$ is a closed form representing the cohomology class Poincaré dual to the class of Δ. For each q let

$$\{\psi_{q,\mu}\}$$

be a collection of closed q-forms representing a basis for $H^q_{\mathrm{DR}}(X)$, and let

$$\{\psi^*_{2-q,\mu}\}$$

be closed forms representing the dual basis for $H_{\rm DR}^{2-q}(X)$, i.e., such that

$$\int_X \psi_{q,\mu} \wedge \psi^*_{2-q,\nu} = \delta_{\mu,\nu}.$$

Let π_1 and π_2 denote the two projection maps $X \times X \to X$. Then one has

$$\varphi_\Delta = \sum_q (-1)^q \sum_\mu \pi_1^* \psi_{q,\mu} \wedge \pi_2^* \psi^*_{2-q,\mu}.$$

Thus we can evaluate the Lefschetz number by

$$\begin{aligned} L(T) &= \int_{\Gamma_T} \varphi_\Delta = \int_X (id. \times T)^* \varphi_\Delta = \sum_q (-1)^q \sum_\mu \int_X \psi_{q,\mu} \wedge T^* \psi^*_{2-q,\mu} \\ &= \sum_{k=0}^{k=2} (-1)^k \mathrm{trace}\,(T^*|_{H^k_{DR}(X)}), \end{aligned} \tag{2}$$

where $k = 2 - q$. The obtained formula

$$L(T) = \sum_{k=0}^{k=2} (-1)^k \mathrm{trace}\,(T^*|_{H^k_{DR}(X)})$$

is so called the Lefschetz trace formula (for two-dimensional case).

In the rest of this paper, $f_i \colon X \to Y$ will always mean two distinct morphisms of degree d_i $(i = 1, 2)$ between closed Riemann surfaces of genera g and γ, respectively. For two distinct morphisms f_i $(i = 1, 2)$, we define the Lefschetz number as follows.

Definition 2.1. The Lefschetz number of two distinct morphisms $f_i \colon X \to Y$ $(i = 1, 2)$ is defined to be

$$L(f_1, f_2) = \int_X (f_1 \times f_2)^* \varphi_\Delta,$$

where $\varphi_\Delta \in H^2_{DR}(Y \times Y)$ is the Poincaré dual of the diagonal $\Delta \subset Y \times Y$.

Thus letting $\Gamma_{f_1,f_2} = \{p \in X \mid (f_1(p), f_2(p))\} \in \mathrm{H}_2(Y \times Y, \mathbf{Z})$ be the homology class of the image of X via f_1 and f_2, we have

$$L(f_1, f_2) = \int_X (f_1 \times f_2)^* \varphi_\Delta = \int_{\Gamma_{f_1,f_2}} \varphi_\Delta = \sharp(\Delta \cdot \Gamma_{f_1,f_2}).$$

Definition 2.2. Let $f \colon X \to Y$ be a morphism between Riemann surfaces. We define a linear map

$$f_* \colon H^k_{DR}(X) \to H^k_{DR}(Y)$$

by the property

$$\int_Y f_* v \wedge w = \int_X v \wedge f^* w,$$

for any $w \in H^{2-k}_{DR}(Y)$.

Then the analogue to (2) takes the form

$$
\begin{aligned}
L(f_1,f_2) &= \int_X (f_1\times f_2)^*\varphi_\Delta = \sum_q (-1)^q \sum_\mu \int_X f_1^*\psi_{q,\mu}\wedge f_2^*\psi^*_{2-q,\mu} \\
&= \sum_q (-1)^q \sum_\mu \int_Y f_{2*}\circ f_1^*\psi_{q,\mu}\wedge \psi^*_{2-q,\mu} \qquad (3)\\
&= \sum_{q=0}^{q=2}(-1)^q \mathrm{trace}\,(f_{2*}\circ f_1^*|_{H^q_{DR}(Y)}) = \sum_{k=0}^{k=2}(-1)^k \mathrm{trace}\,(f_1^*\circ f_{2*}|_{H^k_{DR}(X)}),
\end{aligned}
$$

where we use the same symbol $\{\psi_{q,\mu}\}$ and $\{\psi^*_{2-q,\mu}\}$ for the basis for $H^q_{\mathrm{DR}}(Y)$ and for the dual basis for $H^{2-q}_{\mathrm{DR}}(Y)$, respectively. The last equality comes from the fact that for any two matrices A and B, the trace of AB and BA agree whenever the two products make sense.

Observing (3), we easily have

$$L(f_1,f_2)=L(f_2,f_1).$$

Fuertes and González-Diez [4] showed the following lemma.

Lemma 2.3. i) $f_1^*\circ f_{2*}\colon H^0(X)\to H^0(X)$ *is multiplication by* d_2.
ii) $f_1^*\circ f_{2*}\colon H^2(X)\to H^2(X)$ *is multiplication by* d_1.

Thus the Lefschetz trace formula is written as

$$L(f_1,f_2)=\sum_{k=0}^{k=2}(-1)^k\mathrm{trace}\,(f_1^*\circ f_{2*}|_{H^k_{DR}(X)}) = d_1 - \mathrm{trace}\, f_1^*\circ f_{2*}|_{H^1_{DR}(X)} + d_2. \qquad (4)$$

Definition 2.4. Let $p\in X$ be a coincidence of f_1 and f_2, and let

$$f_1(z)-f_2(z)=c_k z^k + c_{k+1}z^{k+1}+\dots,\quad c_k\neq 0$$

be the Taylor expansion of f_1-f_2 with respect to small parametric discs D around p and D' around $f_i(p)$. We define the multiplicity of f_1 and f_2 at p to be

$$m_p(f_1,f_2)=k.$$

By the definition, $m_p(f_1,f_2)$ is always positive. Furthermore, one can show that (see [4])

$$L(f_1,f_2)=\sum_{\{p\in X;\, f_1(p)=f_2(p)\}} m_p(f_1,f_2). \qquad (5)$$

Thus $L(f_1,f_2)$ is always greater than or equal to the actual number of coincidences.

3. The holomorphic Lefschetz number

A morphism $f\colon X\to Y$ acts not only on the de Rham cohomology groups but on the Dolbeault cohomology groups. Let M be a compact Kähler manifold. By the Hodge decomposition,

$$H^r(M,\mathbf{C})\cong\bigoplus_{p+q=r} H^{p,q}_{\bar\partial}(M),$$
$$H^{p,q}_{\bar\partial}(M)=\overline{H^{q,p}_{\bar\partial}(M)}.$$

Thus, for a Riemann surface X,

$$H^1(X,\mathbf{C})\cong H^{1,0}_{\bar\partial}(X)\bigoplus H^{0,1}_{\bar\partial}(X)$$

holds, where we may identify $H^{1,0}_{\overline{\partial}}(X)$ with the space of holomorphic 1-forms and $H^{0,1}_{\overline{\partial}}(X)$ being the complex conjugate of $H^{1,0}_{\overline{\partial}}(X)$. H^0 and H^2 are trivial in this case. Now the Lefschetz number $L(f_1, f_2)$ is written as

$$L(f_1, f_2) = \int_X (f_1 \times f_2)^* \varphi_\Delta = \sum_{p,q} (-1)^{p+q} \mathrm{trace}\,(f_1^* \circ f_{2*}|_{H^{p,q}_{\overline{\partial}}(X)}).$$

Let π_1 and π_2 denote the two projection maps $Y \times Y \to Y$. For each p and q let

$$\{\psi_{p,q,\mu}\}$$

be a collection of $\overline{\partial}$-closed (p,q)-forms representing a basis for $H^{p,q}_{\overline{\partial}}(Y)$, and let

$$\{\psi^*_{1-p,1-q,\mu}\}$$

be $\overline{\partial}$-closed forms representing the dual basis for $H^{1-p,1-q}_{\overline{\partial}}(Y)$ under the pairing

$$H^{p,q}_{\overline{\partial}}(Y) \otimes H^{1-p,1-q}_{\overline{\partial}}(Y) \to \mathbf{C}$$

given by

$$\psi \otimes \varphi \mapsto \int_Y \psi \wedge \varphi.$$

A basis for $H^{1,1}_{\overline{\partial}}(Y \times Y)$ is represented by the forms

$$\{\varphi_{p,q,\mu,\nu} = \pi_1^* \psi_{p,q,\mu} \wedge \pi_2^* \psi^*_{1-p,1-q,\nu}\},$$

and the dual basis for $H^{1,1}_{\overline{\partial}}(Y \times Y)$ is represented by

$$\{\varphi^*_{1-p,1-q,\mu,\nu} = \pi_1^* \psi^*_{1-p,1-q,\mu} \wedge \pi_2^* \psi_{p,q,\nu}\}.$$

The Dolbeault class of the diagonal is represented by the form

$$\varphi_\Delta = \sum_{p,q,\mu} (-1)^{p+q} \varphi_{p,q,\mu,\mu}.$$

Set

$$\varphi^0_\Delta = \sum_{q,\mu} (-1)^q \varphi_{0,q,\mu,\mu} = \varphi_{0,0} - \sum_\mu \varphi_{0,1,\mu,\mu}.$$

We put

$$\eta^0_\Delta(\Gamma_{f_1,f_2}) = \int_{\Gamma_{f_1,f_2}} \varphi^0_\Delta$$

integration of φ^0_Δ over Γ_{f_1,f_2}. We have

$$\begin{aligned}
\eta^0_\Delta(\Gamma_{f_1,f_2}) &= \int_{\Gamma_{f_1,f_2}} \varphi^0_\Delta = \int_X (f_1 \times f_2)^* \varphi^0_\Delta = \int_X \sum_{q,\mu} (-1)^q f_1^* \psi_{0,q,\mu} \wedge f_2^* \psi^*_{1,1-q,\mu} \\
&= \int_Y \sum_{q,\mu} (-1)^q f_{2*} \circ f_1^* \psi_{0,q,\mu} \wedge \psi^*_{1,1-q,\mu} = \sum_{q=0}^{1} (-1)^q \,\mathrm{trace} f_{2*} \circ f_1^*|_{H^{0,q}_{\overline{\partial}}(Y)} \\
&= \sum_{q=0}^{1} (-1)^q \,\mathrm{trace} f_1^* \circ f_{2*}|_{H^{0,q}_{\overline{\partial}}(X)}.
\end{aligned}$$

The last equality comes from the fact that for any two matrices A and B, the trace of AB and BA agree whenever the two products make sense.

Definition 3.1. We define the holomorphic Lefschetz number of (f_1, f_2) to be

$$L(f_1, f_2, \mathcal{O}) = \sum_{q=0}^{1} (-1)^q \operatorname{trace} f_1^* \circ f_{2*}|_{H_{\bar\partial}^{0,q}(X)}.$$

By Lemma 2.3 i), we see

$$L(f_1, f_2, \mathcal{O}) = d_2 - \operatorname{trace} f_1^* \circ f_{2*}|_{H_{\bar\partial}^{0,1}(X)}. \tag{6}$$

We also have

$$L(f_2, f_1, \mathcal{O}) = d_1 - \operatorname{trace} f_1^* \circ f_{2*}|_{H_{\bar\partial}^{1,0}(X)} \tag{7}$$

since

$$\begin{aligned}\int_X (f_2 \times f_1)^* \varphi_\Delta^0 &= \int_X \sum_{q,\mu} (-1)^q f_2^* \psi_{0,q,\mu} \wedge f_1^* \psi_{1,1-q,\mu}^* \\ &= \int_Y \sum_{q,\mu} (-1)^q \psi_{0,q,\mu} \wedge f_{2*} \circ f_1^* \psi_{1,1-q,\mu}^* \\ &= \sum_{q=0}^{1} (-1)^{1-q} \operatorname{trace} f_{2*} \circ f_1^*|_{H_{\bar\partial}^{1,q}(Y)} \\ &= \sum_{q=0}^{1} (-1)^{1-q} \operatorname{trace} f_1^* \circ f_{2*}|_{H_{\bar\partial}^{1,q}(X)}.\end{aligned}$$

Summing (6) and (7), we have

$$L(f_1, f_2) = L(f_1, f_2, \mathcal{O}) + L(f_2, f_1, \mathcal{O}).$$

In [7], the holomorphic Lefschetz number was used to show a theorem giving a condition for $L(f_1, f_2) = 0$. As a consequence of the theorem, the following corollary was deduced.

Corollary 3.2. *Let $f_i \colon X \to Y$ be two distinct morphisms of degree d_i $(i = 1, 2)$ between closed Riemann surfaces of genera g and $\gamma = 1$, respectively. The following two conditions are equivalent.*

1) $L(f_1, f_2) = 0$.
2) *The difference between f_1 and f_2 is only a translation on the torus Y.*

For details, we refer to [7].

4. The Eichler trace formula

According to the method in [5], we can evaluate the number $\eta_\Delta^0(\Gamma_{f_1,f_2})$ in terms of the local behavior of f_1 and f_2 around their coincidences.

Let $A^{p,q}(Y)$ denote the space of differential forms of type (p, q), and let

$$A^{(p_1,q_1),(p_2,q_2)}(Y \times Y)$$

denote the space of differential forms of bitype $(p_1, q_1), (p_2, q_2)$ where (p_1, q_1) and (p_2, q_2) come from the first and the second factor of the product $Y \times Y$, respectively. We have the decomposition of forms on $Y \times Y$ into bitype

$$A^{p,q}(Y \times Y) = \bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} A^{(p_1,q_1),(p_2,q_2)}(Y \times Y).$$

We denote by T_Δ the current of the diagonal $\Delta \subset Y \times Y$. Let T_Δ^0 be the component of T_Δ of bitype $(0,*)$, $(1,1-*)$, where $* = 0,1$, that is, the current defined by the linear function

$$T_\Delta^0(\psi) = \int_\Delta \sum_{q=0}^{1} \psi^{(1,1-q),(0,q)}$$

on test forms ψ. Then T_Δ^0 is $\overline{\partial}$-closed and represents the Dolbeault cohomology class η_Δ^0. We will smooth the current, namely, solve the equation of currents

$$T_\Delta^0 = \varphi + \overline{\partial} k$$

with k any $(1,0)$-current and φ a smooth form. Then we will have

$$\eta_\Delta^0(\Gamma_{f_1,f_2}) = \int_{\Gamma_{f_1,f_2}} \varphi.$$

The Bochner–Martinelli kernel on $\mathbf{C} \times \mathbf{C}$ is given by

$$k(z,\zeta) = C_1 \frac{d\zeta}{z-\zeta},$$

where

$$C_1 = \frac{-1}{2\pi i}.$$

Apparently, $\overline{\partial} k(z,\zeta) = 0$ on $\mathbf{C} \times \mathbf{C} - \Delta$ and therefore the current defined by $k(z,\zeta)$ has distributional derivative $\overline{\partial} k$ supported on Δ. Further, it is known that $\overline{\partial} k = T_\Delta^0$ in the sense of distributions. This gives the smoothing of T_Δ^0 in $\mathbf{C} \times \mathbf{C}$.

Let $f_i \colon X \to Y$ be two distinct morphisms of degree d_i $(i = 1,2)$ between closed Riemann surfaces and let $\{p_\alpha\}$ be their coincidences possibly empty set. Put the image of coincidences $q_\alpha = f_i(p_\alpha)$. We denote by $B_\epsilon(q_\alpha, q_\alpha)$ the ball of radius ϵ around (q_α, q_α) in $Y \times Y$, and let ρ_α be a bump function with $\rho_\alpha \equiv 1$ in $B_\epsilon(q_\alpha, q_\alpha)$ and $\rho_\alpha \equiv 0$ in $Y \times Y - B_{2\epsilon}(q_\alpha, q_\alpha)$. Let k_ϵ be the current on $Y \times Y$ given by

$$k_\epsilon = \sum_\alpha \rho_\alpha \cdot k(z_\alpha, \zeta_\alpha),$$

where $k(z_\alpha, \zeta_\alpha)$ is the Bochner–Martinelli kernel.

In $B_\epsilon(q_\alpha, q_\alpha)$

$$\overline{\partial} k_\epsilon = \overline{\partial} k(z_\alpha, \zeta_\alpha) = T_\Delta^0,$$

and if we put

$$\varphi = T_\Delta^0 - \overline{\partial} k_\epsilon,$$

φ is a $\overline{\partial}$-closed current representing η_Δ^0, smooth in an open set containing Γ_{f_1,f_2} and equal to $-\overline{\partial} k_\epsilon$ away from Δ. Let ξ_α be a local coordinate around p_α. We evaluate

$$\begin{aligned}\eta_\Delta^0(\Gamma_{f_1,f_2}) &= \int_{\Gamma_{f_1,f_2}} \varphi = -\int_{\Gamma_{f_1,f_2} - \cup B_\epsilon(q_\alpha,q_\alpha)} \overline{\partial} k_\epsilon = \sum_\alpha \int_{\partial(\Gamma_{f_1,f_2} \cap B_\epsilon(q_\alpha,q_\alpha))} k_\epsilon \\ &= \sum_\alpha \int_{|\xi_\alpha|=\epsilon} k(f_1(\xi_\alpha),\, f_2(\xi_\alpha)) = \sum_\alpha C_1 \int_{|\xi_\alpha|=\epsilon} \frac{f_2'(\xi_\alpha)}{f_1(\xi_\alpha) - f_2(\xi_\alpha)}\, d\xi_\alpha.\end{aligned}$$

If $\{p_\alpha\}$ is the empty set, it is easy to see that the third integral is 0. Then

$$C_1 \int_{|\xi_\alpha|=\epsilon} \frac{f_2'(\xi_\alpha)}{f_1(\xi_\alpha) - f_2(\xi_\alpha)}\, d\xi_\alpha = \operatorname{Res}\left(\frac{f_2'(\xi_\alpha)}{f_2(\xi_\alpha) - f_1(\xi_\alpha)};\, p_\alpha \right),$$

where

$$\mathrm{Res}\left(\frac{f_2'(\xi_\alpha)}{f_2(\xi_\alpha)-f_1(\xi_\alpha)};\, p_\alpha\right)$$

denotes the residue of the function at p_α, which is independent of the choice of ϵ. Thus

$$\eta^0_\Delta(\Gamma_{f_1,f_2}) = \sum_\alpha \mathrm{Res}\left(\frac{f_2'(\xi_\alpha)}{f_2(\xi_\alpha)-f_1(\xi_\alpha)};\, p_\alpha\right),$$

and we have established

Theorem 4.1. *Let $f_i\colon X \to Y$ be two distinct morphisms of degree d_i $(i=1,2)$ between closed Riemann surfaces and let $\{p_\alpha\}$ be their coincidences possibly empty set. The holomorphic Lefschetz number of (f_1, f_2) is given by*

$$L(f_1, f_2, \mathcal{O}) = \sum_\alpha \mathrm{Res}\left(\frac{f_2'(\xi_\alpha)}{f_2(\xi_\alpha)-f_1(\xi_\alpha)};\, p_\alpha\right).$$

If $\{p_\alpha\}$ is empty, then we take the right hand side to be 0*.*

By the definition of $L(f_1, f_2, \mathcal{O})$ and the fact that

$$\mathrm{trace} f_1^* \circ f_{2*}|_{H^{0,1}_{\overline{\partial}}(X)} = \overline{\mathrm{trace} f_1^* \circ f_{2*}|_{H^{1,0}_{\overline{\partial}}(X)}},$$

we easily derive

Corollary 4.2. (The Eichler trace formula for coincidences) *With the same notation as in Theorem 4.1, we have*

$$\mathrm{trace} f_1^* \circ f_{2*}|_{H^{1,0}_{\overline{\partial}}(X)} = d_2 - \sum_\alpha \overline{\mathrm{Res}\left(\frac{f_2'(\xi_\alpha)}{f_2(\xi_\alpha)-f_1(\xi_\alpha)};\, p_\alpha\right)}.$$

For automorphisms, namely if $X = Y$ and f_2 is the identity map on X and $f_1 \neq \mathrm{id}.$ then a coincidence is a fixed point of f_1 and

$$\mathrm{Res}\left(\frac{f_2'(\xi_\alpha)}{f_2(\xi_\alpha)-f_1(\xi_\alpha)};\, p_\alpha\right) = \frac{1}{1-f_1'(\xi_\alpha(p_\alpha))}.$$

Substituting this into Corollary 4.2, we obtain the Eichler trace formula

$$\mathrm{trace} f_1^*|_{H^{1,0}_{\overline{\partial}}(X)} = 1 - \sum_\alpha \overline{\frac{1}{1-f_1'(\xi_\alpha(p_\alpha))}} = 1 - \sum_\alpha \frac{f_1'(\xi_\alpha(p_\alpha))}{f_1'(\xi_\alpha(p_\alpha))-1},$$

since $|f_1'(\xi_\alpha(p_\alpha))| = 1$.

Remark. For another proof of the Eichler trace formula, we refer to [2] in which for an automorphism T of a Riemann surface X, $\mathrm{tr}\, T$ denotes $\mathrm{trace}\,(T^{-1})^*|_{H^{1,0}_{\overline{\partial}}(X)}$ in our notation. Also in [2], the genera of Riemann surfaces are assumed to be > 1 although we do not exclude tori.

We return to Theorem 4.1. Changing the order of f_1 and f_2, we have

$$L(f_2, f_1, \mathcal{O}) = \sum_\alpha \mathrm{Res}\left(\frac{f_1'(\xi_\alpha)}{f_1(\xi_\alpha)-f_2(\xi_\alpha)};\, p_\alpha\right).$$

Thus

$$\begin{aligned} &L(f_1, f_2, \mathcal{O}) + L(f_2, f_1, \mathcal{O}) \\ &= \sum_{\alpha} \operatorname{Res}\left(\frac{f_2'(\xi_\alpha)}{f_2(\xi_\alpha) - f_1(\xi_\alpha)}; p_\alpha\right) + \sum_{\alpha} \operatorname{Res}\left(\frac{f_1'(\xi_\alpha)}{f_1(\xi_\alpha) - f_2(\xi_\alpha)}; p_\alpha\right) \\ &= \sum_{\alpha} \operatorname{Res}\left(\frac{f_2'(\xi_\alpha) - f_1'(\xi_\alpha)}{f_2(\xi_\alpha) - f_1(\xi_\alpha)}; p_\alpha\right). \end{aligned}$$

Recalling (5), we have again obtained

$$L(f_1, f_2) = L(f_1, f_2, \mathcal{O}) + L(f_2, f_1, \mathcal{O}).$$

If the multiplicity of f_1 and f_2 at p_α is 1, then

$$\operatorname{Res}\left(\frac{f_2'(\xi_\alpha)}{f_2(\xi_\alpha) - f_1(\xi_\alpha)}; p_\alpha\right) = \frac{f_2'(\xi_\alpha(p_\alpha))}{f_2'(\xi_\alpha(p_\alpha)) - f_1'(\xi_\alpha(p_\alpha))}.$$

In this case, we have

Corollary 4.3. *Let the conditions of Theorem 4.1 hold, and suppose that every coincidence has multiplicity one. Then the difference in degree between f_1 and f_2 is given by*

$$d_2 - d_1 = \sum_{\alpha} \frac{|f_2'(\xi_\alpha(p_\alpha))|^2 - |f_1'(\xi_\alpha(p_\alpha))|^2}{|f_2'(\xi_\alpha(p_\alpha)) - f_1'(\xi_\alpha(p_\alpha))|^2}.$$

Proof. Recalling that

$$\begin{aligned} L(f_1, f_2, \mathcal{O}) &= d_2 - \operatorname{trace} f_1^* \circ f_{2*}|_{H_{\bar\partial}^{0,1}(X)}, \\ L(f_2, f_1, \mathcal{O}) &= d_1 - \operatorname{trace} f_1^* \circ f_{2*}|_{H_{\bar\partial}^{1,0}(X)}, \end{aligned}$$

and

$$\operatorname{trace} f_1^* \circ f_{2*}|_{H_{\bar\partial}^{0,1}(X)} = \overline{\operatorname{trace} f_1^* \circ f_{2*}|_{H_{\bar\partial}^{1,0}(X)}},$$

we easily derive the conclusion from Theorem 4.1. □

5. Examples

In this section, we exhibit examples for Corollary 4.2.

Example 1. Consider the torus

$$T = \mathbf{C}/(i, 1)\mathbf{Z}^2.$$

For any positive integer n, define the multiplication by n on T

$$n_T \colon T \to T$$

by

$$x \mapsto nx.$$

Let n_1 and n_2 be positive integers distinct from each other. We consider the coincidences of $f_1 = n_{1T}$ and $f_2 = n_{2T}$. The number of coincidences is $(n_1 - n_2)^2$. Now, we recall that the push-forward of a holomorphic 1-form $f_*\omega$ is expressed by the local behavior of the morphism f and the local value of the form ω as below (we refer to [3, p. 344]). Let $f \colon X \to Y$ be a morphism between Riemann surfaces of degree d and ω be a holomorphic 1-form on X. Let U' be an open set of Y with the property that

$f^{-1}(U')$ is the disjoint union of open sets U_i $(i = 1, 2, \cdots, d)$ such that the restriction of f to each of them is an isomorphism. We assign to each such open set U' the form

$$\omega'|_{U'} = \sum_{i=1}^{d} ((f|_{U_i})^{-1})^* \omega.$$

Then we can obtain a globally well defined holomorphic form ω' and this is $f_*\omega$.

Observing this expression, we calculate the left hand side of the equation in Corollary 4.2 to be $n_1 n_2$. The right hand side is $n_2^2 - (n_1 - n_2)^2 \cdot \frac{n_2}{n_2 - n_1}$ which is equal to $n_1 n_2$.

Fuertes [3] computed the multiplicities of coincidences of morphisms given below.

Example 2. Let X and Y be hyperelliptic Riemann surfaces defined by the equations $y^2 = x^{2n} - 1$ and $y^2 = x^n - 1$ with n even, respectively. Then X is of genus $n - 1$ and Y is of genus $(n - 2)/2$. Let $f_j \colon X \to Y$ $(0 \le j \le n - 1)$ be morphisms defined by

$$f_j(x, y) = (\zeta^j x^2, y),$$

where $\zeta = e^{2\pi i/n}$. Then $\deg f_j = 2$ for any j. We put

$$p_1 = (x, y) = (0, +i), \quad p_2 = (x, y) = (0, -i)$$

on X. Then p_1 and p_2 are coincidences of f_0 and f_k for any $k \neq 0$. It is seen that the multiplicity of each of these coincidences is 2 as indicated in [3]. Taking the variable x of defining equation for X as a local coordinate around p_i $(i = 1, 2)$, we compute

$$\operatorname{Res}\left(\frac{f_0'(x)}{f_0(x) - f_k(x)}; p_i\right) = \frac{1}{2\pi i} \int_{|x|=\epsilon} \frac{2x}{x^2 - \zeta^k x^2}\, dx = \frac{2}{1 - \zeta^k}.$$

We denote by ∞_1 and ∞_2 the two points at infinity of X. These points are the rest of the coincidences of f_0 and f_k. We take $s = x^{-1}$ as a local coordinate around ∞_1. Then the local expression of f_0 around ∞_1 is

$$s \mapsto \left(\frac{1}{s}, \frac{\sqrt{1 - s^{2n}}}{s^n}\right) \mapsto \left(\frac{1}{s^2}, \frac{\sqrt{1 - s^{2n}}}{s^n}\right) \mapsto s^2,$$

and similarly the local expression of f_k around ∞_1 is

$$s \mapsto \frac{s^2}{\zeta^k}.$$

Thus we compute

$$\operatorname{Res}\left(\frac{f_0'(s)}{f_0(s) - f_k(s)}; \infty_1\right) = \frac{1}{2\pi i} \int_{|s|=\epsilon} \frac{2s}{s^2 - \zeta^{-k} s^2}\, ds = \frac{2}{1 - \zeta^{-k}}.$$

Similarly, we obtain

$$\operatorname{Res}\left(\frac{f_0'(s)}{f_0(s) - f_k(s)}; \infty_2\right) = \frac{2}{1 - \zeta^{-k}}.$$

It is known that

$$\frac{dx}{y}, \frac{xdx}{y}, \cdots, \frac{x^{n-2}\,dx}{y}$$

form a basis for $H^{1,0}_{\bar\partial}(X)$. The eigenvectors of $f_k^* \circ f_{0*}$ are pull-back forms via f_0

$$\frac{xdx}{y}, \frac{x^3dx}{y}, \cdots, \frac{x^{n-3}dx}{y}$$

corresponding eigenvalues being

$$2\zeta^{2k}, 2\zeta^{4k}, \cdots, 2\zeta^{(n-2)k},$$

where the factor 2 comes from $\deg f_0$. Then the left hand side of the equation in Corollary 4.2 is

$$\operatorname{trace} f_k^* \circ f_{0*}|_{H_{\overline{\partial}}^{1,0}(X)} = \sum_{l=1}^{\frac{n-2}{2}} 2\zeta^{2kl} = 2\zeta^{2k}\,\frac{1-\zeta^{2k(\frac{n}{2}-1)}}{1-\zeta^{2k}} = -2,$$

and the right hand side is

$$\deg f_0 - \sum_{\alpha} \overline{\operatorname{Res}\left(\frac{f_0'(s)}{f_0(s)-f_k(s)}; p_\alpha\right)} = 2 - 2\,\overline{\frac{2}{1-\zeta^k}} - 2\,\overline{\frac{2}{1-\zeta^{-k}}} = -2.$$

6. Diagonalization of $f_1^* \circ f_{2*}$

Our purpose in this section is to examine the possibility of diagonalizing matrix representations of $f_1^* \circ f_{2*}$. We recall that the vector space $H_{\overline{\partial}}^{1,0}(X)$ carries a hermitian inner-product defined by

$$\langle v, w\rangle_X = i\int_X v \wedge \overline{w}.$$

We denote by

$$\langle v, w\rangle_Y$$

the hermitian inner-product on $H_{\overline{\partial}}^{1,0}(Y)$ as well. Then for any $v \in H_{\overline{\partial}}^{1,0}(X)$ and $w \in H_{\overline{\partial}}^{1,0}(Y)$,

$$\langle f_{i*}v, w\rangle_Y = i\int_Y f_{i*}v \wedge \overline{w} = i\int_X v \wedge \overline{f_i^* w} = \langle v, f_i^* w\rangle_X. \tag{8}$$

Let

$$\Omega = \{\omega_1, \cdots, \omega_g\}$$

be an orthonormal basis for $H_{\overline{\partial}}^{1,0}(X)$ and let

$$\Lambda = \{\lambda_1, \cdots, \lambda_\gamma\}$$

be an orthonormal basis for $H_{\overline{\partial}}^{1,0}(Y)$. We denote by A the matrix representation of f_{i*} with respect to Ω and Λ. Then (8) means that

$$A^* = \overline{{}^tA}$$

is the matrix representation of f_i^* with respect to Λ and Ω. This observation gives

Lemma 6.1. *Let A be the matrix representation of $f_{i*}\colon H_{\overline{\partial}}^{1,0}(X) \to H_{\overline{\partial}}^{1,0}(Y)$ with respect to orthonormal bases. Then the matrix representation of f_i^* is A^*.*

Now we will show

Theorem 6.2. *Let $f_i\colon X \to Y$ $(i = 1, 2)$ be distinct morphisms between closed Riemann surfaces of genera greater than one. Let $\Omega = \{\omega_1, \cdots, \omega_g\}$ be an orthonormal basis for $H_{\overline{\partial}}^{1,0}(X)$. A matrix representation of $f_1^* \circ f_{2*}$ with respect to Ω is diagonalizable by a unitary matrix if and only if there is an automorphism h of Y with $f_1 = h \circ f_2$.*

Proof. A matrix is diagonalizable by a unitary matrix if and only if it is a normal matrix. Thus we will show that there is an automorphism h of Y with $f_1 = h \circ f_2$ if and only if the matrix representation of $f_1^* \circ f_{2*}$ is normal.

Suppose that there is an automorphism h of Y with $f_1 = h \circ f_2$. Let $\Lambda = \{\lambda_1, \cdots, \lambda_\gamma\}$ be an orthonormal basis for $H_{\overline{\partial}}^{1,0}(Y)$. Let $\mathcal{F}_i\,(i = 1, 2)$ and $\mathcal{H}$ denote the matrix representations of f_{i*} and h_* respectively with respect to Ω and Λ. It suffices to show that $\mathcal{F}_1^*\mathcal{F}_2$ is a normal matrix. Using Lemma 6.1, we compute

$$(\mathcal{F}_1^*\mathcal{F}_2)^*\,(\mathcal{F}_1^*\mathcal{F}_2) = (\mathcal{F}_2^*\mathcal{H}^*\mathcal{F}_2)^*\,(\mathcal{F}_2^*\mathcal{H}^*\mathcal{F}_2) = \mathcal{F}_2^*\mathcal{H}\mathcal{F}_2\,\mathcal{F}_2^*\mathcal{H}^*\mathcal{F}_2 = d_2\,\mathcal{F}_2^*\mathcal{F}_2.$$

On the other hand,

$$(\mathcal{F}_1^*\mathcal{F}_2)\,(\mathcal{F}_1^*\mathcal{F}_2)^* = (\mathcal{F}_2^*\mathcal{H}^*\mathcal{F}_2)\,(\mathcal{F}_2^*\mathcal{H}^*\mathcal{F}_2)^* = \mathcal{F}_2^*\mathcal{H}^*\mathcal{F}_2\,\mathcal{F}_2^*\mathcal{H}\mathcal{F}_2 = d_2\,\mathcal{F}_2^*\mathcal{F}_2.$$

Thus we see that a matrix representation of $\mathcal{F}_1^*\mathcal{F}_2$ is normal.

Conversely, we assume that the matrix representation of $f_1^* \circ f_{2*}$ is normal. Then $(\mathcal{F}_1^*\mathcal{F}_2)^*\,(\mathcal{F}_1^*\mathcal{F}_2) = (\mathcal{F}_1^*\mathcal{F}_2)\,(\mathcal{F}_1^*\mathcal{F}_2)^*$. We compute

$$(\mathcal{F}_1^*\mathcal{F}_2)^*\,(\mathcal{F}_1^*\mathcal{F}_2) = \mathcal{F}_2^*\,\mathcal{F}_1\,\mathcal{F}_1^*\mathcal{F}_2 = d_1\mathcal{F}_2^*\,\mathcal{F}_2,$$

and

$$(\mathcal{F}_1^*\mathcal{F}_2)\,(\mathcal{F}_1^*\mathcal{F}_2)^* = \mathcal{F}_1^*\mathcal{F}_2\,\mathcal{F}_2^*\,\mathcal{F}_1 = d_2\mathcal{F}_1^*\,\mathcal{F}_1.$$

Thus we have

$$d_1 f_2^* \circ f_{2*} = d_2 f_1^* \circ f_{1*}.$$

This means that there is an automorphism h of Y with $f_1 = h \circ f_2$ by Kani's rigidity theorem [6, p. 186, Theorem 2]. □

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