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AVERAGING ON n-DIMENSIONAL RECTANGLES

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Abstract. In this work we investigate families of translation invariant differentiation bases $\mathscr{B}$ of rectangles in $\mathbf{R}^n$, for which $L\log^{n-1} L(\mathbf{R}^n)$ is the largest Orlicz space that $\mathscr{B}$ differentiates. In particular, we improve on techniques developed by Stokolos in [11] and [13].

1. Introduction

Recall that a *differentiation basis* in $\mathbf{R}^n$ is a collection $\mathscr{B} = \bigcup_{x\in\mathbf{R}^n} \mathscr{B}(x)$ of bounded, measurable sets with positive measure such that for each $x \in \mathbf{R}^n$ there is a subfamily $\mathscr{B}(x)$ of sets of $\mathscr{B}$ so that each $B \in \mathscr{B}(x)$ contains x and in $\mathscr{B}(x)$ there are sets arbitrarily small diameter (see e.g. de Guzmán [2, p. 104] and [3, p. 42]).

Let $\mathscr{B}$ be a differential basis in $\mathbf{R}^n$, and let $M_{\mathscr{B}}$ be the corresponding maximal functional, that is, for any summable function f in $\mathbf{R}^n$, let for $x \in \mathbf{R}^n$

$$M_{\mathscr{B}} f(x) = \sup_{\substack{B\in\mathscr{B}\\ B\ni x}} \frac{1}{|B|}\int_B |f|,$$

where$|A|$ denotes the Lebesgue measure of a measurable set $A \subseteq \mathbf{R}^n$.

In many areas of analysis a key role is played by the so-called *weak type estimates* for $M_{\mathscr{B}} f$; this is the case, in particular, for the weak $(1,1)$ estimate

$$(1) \qquad |\{x\colon M_{\mathscr{B}} f(x) > \lambda\}| \leqslant C\frac{\|f\|_1}{\lambda}, \quad \lambda > 0,$$

and the weak $L\log^d L$ estimate for $0 < d \leqslant n$

$$(2) \qquad |\{x\colon M_{\mathscr{B}} f(x) > \lambda\}| \leqslant C\int_{\mathbf{R}^n} \frac{|f|}{\lambda}\left(1 + \log_+^d \frac{|f|}{\lambda}\right), \quad \lambda > 0,$$

where we let $\log_+ t := \max(\log t, 0)$ for $t > 0$. Following e.g. Stokolos [12], if $M_{\mathscr{B}} f$ satisfies (1) or (2), we say that the basis $\mathscr{B}$ has the corresponding weak type.

As far as translation invariant bases $\mathscr{B}$ consisting of multidimensional intervals (also called rectangles or parallelepipeds, that is Cartesian products of one dimensional intervals) are concerned, an old result by Jessen, Marcinkiewicz and Zygmund [8, Theorem 4], quantified independently by Fava [5] and de Guzmán [2], ensures that $M_{\mathscr{B}}$ always enjoys weak type $L\log^{n-1} L$; this shows in particular that Zygmund's

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conjecture, stating that translation-invariant bases of n-dimensional rectangles whose sides are increasing functions of d parameters differentiate $L\log^{d-1}L(\mathbf{R}^n)$, holds for $d=n$.

In dimension 2, Stokolos [11] gave a complete characterization of all the weak estimates that translation invariant bases of rectangles can support: either the dyadic parents of elements of $\mathscr{B}$ can be, up to translation, classified into a finite number of families totally ordered by inclusion—in which case $M_{\mathscr{B}}$ satisfies a weak $(1,1)$ estimate—, or it is not the case and de Guzmán's weak $L\log L$ estimate is sharp.

In the general case of translation invariant bases of rectangles in $\mathbf{R}^n$, Fefferman and Pipher [6] gave in 2005 a covering lemma yielding $L\log^d L$ estimates for some translation-invariant bases $\mathscr{B}$ of rectangles in $\mathbf{R}^n$. Those providing a weak type $(1,1)$ estimate for $M_{\mathscr{B}}$ have been completely characterized in Stokolos [12]: a translation invariant base $\mathscr{R}$ of rectangles is of weak type $(1,1)$ if and only if the dyadic parents of elements of $\mathscr{B}$ can be, up to translation, classified into a finite number of families totally ordered by inclusion (we recall this result in Section 3 below). As far as Zygmund's conjecture is concerned, this corresponds to its validity for $d=1$.

Even though a result by Córdoba [1], stating e.g. that the translation-invariant basis $\mathscr{B}$ in $\mathbf{R}^3$ verifying

$$\mathscr{B}(0)=\{[0,s]\times[0,t]\times[0,st]\colon s,t>0\},$$

yields a weak $L\log L$ estimate for $M_{\mathscr{B}}$, hence supporting Zygmund's conjecture for $d=2$, it follows from a result by Soria [10, Proposition 5] that Zygmund's conjecture is false in general, for he there constructs continuous, increasing functions $\phi,\psi\colon(0,\infty)\to(0,\infty)$ such that the translation-invariant basis of rectangles $\mathscr{B}'$ in $\mathbf{R}^3$ satisfying

$$\mathscr{B}'(0)=\{[0,s]\times[0,t\phi(s)]\times[0,t\psi(s)]\colon s,t>0\},$$

differentiates no more than $L\log^2 L(\mathbf{R}^3)$, meaning in particular that de Guzmán's $L\log^2 L$ estimate is sharp for $M_{\mathscr{B}'}$. It is also Soria's observation (see [10, Proposition 2]) that Córdoba's result implies that the "Soria basis" $\mathscr{B}''$ verifying

$$\mathscr{B}''(0)=\{[0,s]\times[0,t]\times[0,1/t]\colon s,t>0\},$$

yields a weak $L\log L$ inequality for $M_{\mathscr{B}''}$ (note here that $\mathscr{B}''$ is not, *stricto sensu*, a differentiation basis—it lacks the "diameter" condition—, which does not prevent us to extend to such families the definitions made above).

As of today, there is no characterization of translation invariant bases of rectangles in $\mathbf{R}^n$, $n\geqslant 3$, for which de Guzmán's weak $L\log^{n-1}L$ estimate is sharp (neither is it known whether the sharpness of some weak $L\log^d L$ inequality, $d>1$, would imply that d is an integer). In 2008, Stokolos [13] gave examples of Soria bases in $\mathbf{R}^3$, i.e., bases of the form $\mathscr{B}\times\mathscr{I}$, where $\mathscr{B}$ is a basis of rectangles in $\mathbf{R}^2$ and $\mathscr{I}$ denotes the basis of all intervals in $\mathbf{R}$) for which de Guzmán's weak $L\log^2 L$ estimate is sharp.

Our intention in this work is mainly to improve on Stokolos' techniques in order to give new examples of translation invariant bases of rectangles in $\mathbf{R}^n$ for which the weak $L\log^{n-1}L$ estimate on $M_{\mathscr{B}}$ is sharp (see Section 4 below). In particular we give a way to construct, from a (strictly) decreasing sequence of rectangles in $\mathbf{R}^n$, a differentiation basis built from it and failing to differentiate spaces below $L\log^{n-1}L(\mathbf{R}^n)$ (see Theorem 7 and Remark 11 below). We finish in Section 5 by some observations showing that de Guzmán's weak $L\log^{n-1}L$ estimate can be improved once we know that, in some coordinate plane, the projections of rectangles in $\mathscr{B}$

can be classified, up to translations, into a finite set of families totally ordered by inclusion.

2. Preliminaries

Let us now precisely fix the context in which we shall work.

2.1. Orlicz spaces. Given an Orlicz function $\Phi\colon [0,\infty) \to [0,\infty)$ (i.e. a convex and increasing function satisfying $\Phi(0) = 0$), we define the associated Banach space $L^\Phi(\mathbf{R}^n)$ as the set of all measurable functions f on $\mathbf{R}^n$ for which one has $\Phi(|f|) \in L^1$. For the Orlicz function $\Phi_d(t) := t(1 + \log_+^d t)$, $0 < d \leqslant n$, we write $L \log^d L(\mathbf{R}^n)$ instead of $L^{\Phi_d}(\mathbf{R}^n)$. Is is clear that for $\Phi(t) = t^p$, $p \geqslant 1$, the Orlicz space $L^\Phi(\mathbf{R}^n)$ coincides with the usual Lebesgue space $L^p(\mathbf{R}^n)$.

2.2. Families of standard rectangles. In the sequel, $\mathscr{R}$ will always stand for a family of *standard rectangles*, i.e. rectangles of the form $[0,\alpha_1] \times \cdots \times [0,\alpha_n]$ in some $\mathbf{R}^n$, $n \geqslant 2$, with $0 < \alpha_i \leqslant 1$, $1 \leqslant i \leqslant n$. We will moreover say that those are *dyadic* in case one has $\alpha_i = 2^{-m_i}$ with $m_i \in \mathbf{N}$, for each $1 \leqslant i \leqslant n$ (we let as usual $\mathbf{N} = \{0, 1, 2, \dots\}$ denote the set of all natural numbers).

2.3. Weak inequalities. Given a family $\mathscr{R}$ of standard rectangles in $\mathbf{R}^n$, we associate to it a maximal operator $M_{\mathscr{R}}$ defined for a measurable function f by:

$$M_{\mathscr{R}} f(x) := \sup\left\{\frac{1}{|R|}\int_{\tau(R)} |f|\colon R \in \mathscr{R},\ \tau \text{ translation},\ x \in \tau(R)\right\}.$$

In case Φ is an Orlicz function, we shall say that $M_{\mathscr{R}}$ satisfies a weak L^Φ inequality in case there exists $C > 0$ such that, for any $\lambda > 0$ and any $f \in L^\Phi(\mathbf{R}^n)$, we have

$$|\{M_{\mathscr{R}} f > \lambda\}| \leqslant \int_{\mathbf{R}^n} \Phi\left(\frac{C|f|}{\lambda}\right).$$

2.4. Reduction to families of dyadic rectangles. Given a family $\mathscr{R}$ of standard rectangles in $\mathbf{R}^n$, we define $\mathscr{R}^* := \{R^*\colon R \in \mathscr{R}\}$, where for each standard rectangle R, we denote by R^* the standard dyadic rectangle with the smallest measure, having the property to contain R. Since it is easy to see that one has

$$\frac{1}{2^n} \cdot M_{\mathscr{R}^*} f \leqslant M_{\mathscr{R}} f \leqslant 2^n \cdot M_{\mathscr{R}^*} f,$$

on $\mathbf{R}^n$ for all measurable f, it is obvious that $M_{\mathscr{R}}$ satisfies a weak L^Φ inequality if and only if $M_{\mathscr{R}^*}$ does. For this reason, we shall always assume in the sequel that $\mathscr{R}$ is a family of standard *dyadic* rectangles.

2.5. Links to differentiation theory. Given a family of standard rectangles $\mathscr{R}$ satisfying $\inf\{\operatorname{diam} R\colon R \in \mathscr{R}\} = 0$, one can associate to it a translation invariant differentiation basis $\mathscr{B} := \{\tau(R)\colon R \in \mathscr{R},\ \tau \text{ translation}\}$ and define, for $x \in \mathbf{R}^n$, $\mathscr{B}(x) := \{R \in \mathscr{B}\colon R \ni x\}$. In many cases, it then follows from the Sawyer–Stein principle (see e.g. [7, Chapter 1]) that a weak L^Φ inequality for $M_{\mathscr{R}}$ is equivalent to having

$$f(x) = \lim_{\substack{\operatorname{diam} R \to 0\\ R \in \mathscr{B}(x)}} \frac{1}{|R|}\int_R f \quad \text{for a.e. } x \in \mathbf{R}^n,$$

for all $f \in L^\Phi(\mathbf{R}^n)$ (we shall say in this case that $\mathscr{R}$ *differentiates* $L^\Phi(\mathbf{R}^n)$).

2.6. Rademacher-type functions. Recall that one denotes by χ_A the characteristic function of a set $A \subseteq \mathbf{R}^n$. We define the sequence (r_i) of Rademacher-type functions on $\mathbf{R}$ in the following way: we let $r_1 = \chi_{\mathbf{Z}+[0,1/2)}$ and, for $i \geqslant 2$, we define r_i by asking that, for $x \in \mathbf{R}$, one has $r_i(x) = r_{i-1}(2x \mod 1)$. It is easy to see that r_i is 2^{1-i}-periodic for each $i \geqslant 1$ and that the r_i's are independent and identically distributed (IID) in the sense that we have

$$\int_I \prod_{l=1}^{k} r_{i_l} = \left(\frac{1}{2}\right)^k |I|,$$

for any finite sequence $1 \leqslant i_1 < i_2 < \cdots < i_k$ of distinct integers and any interval I whose length is a multiple of 2^{-i_1}.

3. Comparability conditions on rectangles

Following Stokolos [11] (and using the terminology introduced in Moonens and Rosenblatt [9]), we say that a family of standard dyadic rectangles in $\mathbf{R}^n$ has *finite width* in case it is a finite union of families of rectangles totally ordered by inclusion, and that it has *infinite width* otherwise. Il follows from a general result by Dilworth [4] that a family of rectangles in $\mathbf{R}^n$ has infinite width if and only if it contains families of incomparable (with respect to inclusion) rectangles having arbitrary large (finite) cardinality.

The following lemma by Stokolos [12, Lemma 1] is useful to relate the weak $(1,1)$ behaviour of the maximal operator $M_{\mathscr{R}}$ associated to a family of rectangles, and to their comparability properties.

Lemma 1. *A family of rectangles in $\mathbf{R}^n$ is a chain (with respect to inclusion) if and only if the projections of its elements on the x_1x_j plane form a chain of rectangles in $\mathbf{R}^2$, for any $j = 2, \ldots, n$. In particular, a family $\mathscr{R}$ of rectangles in $\mathbf{R}^n$ has finite width if and only if, for each $j = 2, \ldots, n$, the family*

$$\{p_j(R) \colon R \in \mathscr{R}\}$$

has finite width, where $p_j \colon \mathbf{R}^n \to \mathbf{R}^2$ denotes the projection on the x_1x_j plane.

Using this lemma and [11, Lemma 1], Stokolos [12, Theorem 2] obtains the following geometrical characterization of families of rectangles providing a weak $(1,1)$ inequality.

Theorem 2. (Stokolos) *Assume $\mathscr{R}$ is a differentiating family of standard, dyadic rectangles in $\mathbf{R}^n$. The maximal operator $M_{\mathscr{R}}$ satisfies a weak $(1,1)$ inequality if and only if $\mathscr{R}$ has finite width.*

We intend now to discuss some examples of families of rectangles having infinite width, but providing different optimal weak type inequalities. Our first example will deal with families of rectangles in $\mathbf{R}^n$ for which the de Guzmán's $L \log^{n-1} L$ weak inequality is optimal.

4. Families of rectangles for which $L \log^{n-1} L$ is sharp

In this section we provide some examples of families of rectangles $\mathscr{R}$ in $\mathbf{R}^n$ for which de Guzmán's $L \log^{n-1} L$ estimate is sharp. To this purpose, we now state a sufficient condition (in the spirit of Stokolos' [11, Lemma 1]) on a family of rectangles providing this sharpness.

Proposition 3. *Assume that $\mathscr{R}$ is a family of standard dyadic rectangles in $\mathbf{R}^n$ and that there exists and integer $1 \leqslant d \leqslant n-1$, together with positive constants c and*

$c' \geqslant 1$ depending only on n and d, having the following property: for each sufficiently large $k \in \mathbf{N}$, there exist sets Θ_k and Y_k in $\mathbf{R}^n$ with the following properties:

(i) $\Theta_k \subseteq Y_k$;

(ii) $|Y_k| \geqslant c \cdot 2^{dk} k^d |\Theta_k|$;

(iii) *for each $x \in Y_k$, one has $M_{\mathscr{R}} \chi_{\Theta_k}(x) \geqslant c' 2^{-dk}$.*

Under these assumptions, if Φ is an Orlicz function satisfying $\Phi = o(\Phi_d)$ at ∞, then $M_{\mathscr{R}}$ does not *satisfy a weak L^Φ estimate. In particular, $M_{\mathscr{R}}$ does not satisfy a weak $(1,1)$ estimate.*

Proof. Define, for k sufficiently large, $f_k := (1/c') \cdot 2^{dk} \chi_{\Theta_k}$, where Θ_k and Y_k are associated to k and $\mathscr{R}$ according to (i–iii).

Claim 1. For each sufficiently large k, we have

$$|\{M_{\mathscr{R}} f_k \geqslant 1\}| \geqslant \kappa(n,d) \int_{\mathbf{R}^n} \Phi_d(f_k),$$

where $\kappa(n,d) := \frac{d^d}{cc'}$ is a constant depending only on n and d.

Proof of the claim. To prove this claim, one observes that for $x \in Y_k$ we have $M_{\mathscr{R}} f_k(x) \geqslant 1$ according to assumption (iii). Yet, on the other hand, one computes, for k sufficiently large

$$\int_{\mathbf{R}^n} \Phi_d(f_k) \leqslant \frac{1}{c'} \cdot 2^{dk} |\Theta_k| [1 + (dk \log 2)^d] \leqslant \frac{d^d}{c'} \cdot 2^{dk} k^d |\Theta_k| \leqslant \kappa(n,d) \cdot |Y_k|,$$

and the claim follows. □

Claim 2. For any Φ satisfying $\Phi = o(\Phi_d)$ at ∞ and for each $C > 0$, we have

$$\lim_{k \to \infty} \frac{\int_{\mathbf{R}^n} \Phi_d(|f_k|)}{\int_{\mathbf{R}^n} \Phi(C|f_k|)} = \infty.$$

Proof of the claim. Compute for any k

$$\frac{\int_{\mathbf{R}^n} \Phi(C|f_k|)}{\int_{\mathbf{R}^n} \Phi_d(|f_k|)} = \frac{\Phi(2^{dk} C/c')}{\Phi_d(2^{dk}/c')} = \frac{\Phi(2^{dk} C/c')}{\Phi_d(2^{dk} C/c')} \frac{\Phi_d(2^{dk} C/c')}{\Phi_d(2^{dk}/c')},$$

observe that the quotient $\frac{\Phi_d(2^{dk} C/c')}{\Phi_d(2^{dk}/c')}$ is bounded as $k \to \infty$ by a constant independent of k, while by assumption the quotient $\frac{\Phi(2^{dk} C/c')}{\Phi_d(2^{dk} C/c')}$ tends to zero as $k \to \infty$. The claim is proved. □

We now finish the proof of Proposition 3. To this purpose, fix Φ an Orlicz function satisfying $\Phi = o(\Phi_d)$ at ∞ and assume that there exists a constant $C > 0$ such that, for any $\lambda > 0$, one has

$$|\{M_{\mathscr{R}} f > \lambda\}| \leqslant \int_{\mathbf{R}^n} \Phi\left(\frac{C|f|}{\lambda}\right).$$

Using Claim 1, we would then get, for each k sufficiently large

$$0 < \kappa(n,d) \int_{\mathbf{R}^n} \Phi_d(f_k) \leqslant \left|\left\{M_{\mathscr{R}} f_k > \frac{1}{2}\right\}\right| \leqslant \int_{\mathbf{R}^n} \Phi(2C f_k),$$

contradicting the previous claim and proving the proposition. □

A first situation to which the previous proposition can be applied concerns some families of rectangles containing arbitrary large subsets of rectangles having the same n-dimensional measure.

4.1. A first example. We shall need the following Lemma.

Lemma 4. *Fix $\alpha > 0$. For each $k \in \mathbf{N} \setminus \{0\}$, let $\mathbf{D}_k := \{2^{-j} \colon 0 \leqslant j \leqslant k\}$ and define*

$$\mathscr{R}_k := \left\{[0, s_1] \times \ldots \times [0, s_{n-1}] \times \left[0, \frac{\alpha}{s_1 \cdot \ldots \cdot s_{n-1}}\right] : s_1, \ldots, s_{n-1} \in \mathbf{D}_k\right\}.$$

Then

$$|\cup \mathscr{R}_k| \geqslant \frac{1}{3 \cdot 2^{n-2}} k^{n-1} \alpha.$$

Proof. Fix $k \in \mathbf{N}$. We prove this claim by induction on the dimension of the space n, and we observe that it follows from Moonens and Rosenblatt [9, Claim 15] that the inequality holds true in dimension $n = 2$.

Assume now that it holds in dimension $n-1$ and let us prove it in dimension n. To that purpose, define $E_j := [2^{-j-1}, 2^{-j}] \times [0,1]^{n-1}$ for each $0 \leqslant j \leqslant k$ and observe that one has

$$|\cup \mathscr{R}_k| \geqslant \sum_{j=0}^{k} |(\cup \mathscr{R}_k) \cap E_j|\,.$$

Denoting by $s_1(R)$ the length of the first side of $R \in \mathscr{R}_k$, and by $p_1' \colon \mathbf{R}^n \to \mathbf{R}^{n-1}$ the projection on the $n-1$ last coordinates defined by $p_1'(x_1, \ldots, x_n) = (x_2, \ldots, x_n)$, we obviously have, for $0 \leqslant j \leqslant k$

$$(\cup \mathscr{R}_k) \cap E_j \supseteq (\cup\{R \in \mathscr{R}_k \colon s_1(R) = 2^{-j}\}) \cap E_j.$$

Since we compute, for the same j

$$|(\cup\{R \in \mathscr{R}_k \colon s_1(R) = 2^{-j}\}) \cap E_j| = 2^{-j-1} |\cup \{p_1'(R) \colon R \in \mathscr{R}_k, s_1(R) = 2^{-j}\}|,$$

and since we have

$$\begin{aligned}\mathscr{R}_k' &:= \cup\{p_1'(R) \colon R \in \mathscr{R}_k, s_1(R) = 2^{-j}\} \\ &= \left\{[0, s_2] \times \cdots \times [0, s_{n-1}] \times \left[0, \frac{2^j \alpha}{s_2 \cdot \cdots \cdot s_{n-1}}\right] : s_2, \ldots, s_{n-1} \in \mathscr{D}_k\right\},\end{aligned}$$

the induction hypothesis, applied to $\mathscr{R}_k'$ and 2^j, yields

$$|\cup \{p_1'(R) \colon R \in \mathscr{R}_k, s_1(R) = 2^{-j}\}| \geqslant \frac{1}{3 \cdot 2^{n-3}} 2^j k^{n-2} \alpha.$$

Now we compute

$$|\cup \mathscr{R}_k| \geqslant \frac{1}{3 \cdot 2^{n-3}} k^{n-2} \alpha \sum_{j=0}^{k} 2^{-j-1} \cdot 2^j \geqslant \frac{1}{3 \cdot 2^{n-2}} k^{n-1} \alpha;$$

the proof is complete. □

Using the previous estimate, we can show the following theorem.

Theorem 5. *Let $\mathscr{R} := \bigcup_{k \in \mathbf{N}} \mathscr{R}_k$, where for each $k \in \mathbf{N}$, $\mathscr{R}_k$ is defined as in the statement of Lemma 4 with α replaced by $\alpha_k := 2^{-nk}$. Under those conditions, de Guzmán's $L \log^{n-1} L$ weak estimate for the maximal operator $M_{\mathscr{R}}$ is optimal in the following sense: if Φ is an Orlicz function satisfying $\Phi = o(\Phi_{n-1})$ at ∞, then $M_{\mathscr{R}}$ does* not *satisfy a weak L^Φ estimate.*

It is clear, according to Proposition 3, that in order to prove Theorem 5 we simply need to show the following lemma.

Lemma 6. *Let $\mathscr{R} := \bigcup_{k\in\mathbf{N}} \mathscr{R}_k$, where for each $k \in \mathbf{N}$, $\mathscr{R}_k$ is defined as in the statement of Lemma 4 with α replaced by $\alpha_k := 2^{-nk}$. Under those assumptions, for each $k \in \mathbf{N}$, there exist sets $\Theta_k \subseteq [0,1]^n$ and $Y_k \subseteq [0,1]^n$ satisfying the following properties:*

(i) $\Theta_k \subseteq Y_k$;
(ii) $|Y_k| \geqslant \frac{1}{3\cdot 2^{n-2}} \cdot 2^{(n-1)k} k^{n-1} |\Theta_k|$;
(iii) *for each $x \in Y_k$, one has $M_{\mathscr{R}} \chi_{\Theta_k}(x) \geqslant 2^{(1-n)k}$.*

Proof. To prove this lemma, fix $k \in \mathbf{N}$ and let $\Theta_k := \cap \mathscr{R}_k$ and $Y_k := \cup \mathscr{R}_k$, so that (i) is obvious. One also observes that Θ_k is a rectangle whose measure satisfies $|\Theta_k| = 2^{(1-n)k}\alpha_k = 2^{(1-2n)k}$. According to Lemma 4, we get

$$|Y_k| \geqslant \frac{1}{3\cdot 2^{n-2}} k^{n-1} 2^{-nk} = \frac{1}{3\cdot 2^{n-2}} 2^{(n-1)k} k^{n-1} \cdot 2^{(1-2n)k} = \frac{1}{3\cdot 2^{n-2}} \cdot 2^{(n-1)k} k^{n-1} |\Theta_k|,$$

which completes the proof of (ii).

To show (iii), observe that, for $x \in Y_k$, there exists $R \in \mathscr{R}_k$ such that one has $x \in R$; since we have $\Theta_k \subseteq R$, this yields

$$M_{\mathscr{R}} \chi_{\Theta_k}(x) \geqslant \frac{|\Theta_k \cap R|}{|R|} = \frac{|\Theta_k|}{|R|} = 2^{(1-n)k},$$

and so (iii) is proved. □

Another interesting situation can be dealt with according to Proposition 3.

4.2. Another series of examples after Stokolos' study of Soria bases in $\mathbf{R}^3$. [13] In this section, we write $R \prec R'$ for two standard dyadic rectangles $R = \prod_{i=1}^n [0, a_i]$ and $R' = \prod_{i=1}^n [0, b_i]$ in case one has $a_i < b_i$ for all $1 \leqslant i \leqslant n$. A *strict chain* of dyadic rectangles is then a finite family of standard dyadic rectangles, for which one has $R_0 \prec R_1 \prec \cdots \prec R_k$.

Given $\mathscr{C} = \{R_0, \ldots, R_k\}$ ($k \in \mathbf{N}$) a strict chain of standard dyadic rectangles (say that one has $R_0 \prec R_1 \prec \cdots \prec R_k$), we denote by $\widehat{\mathscr{C}}$ the family of standard dyadic rectangles obtained as intersections $\bigcap_{i=1}^n R_{j_i}^i$, where $k \geqslant j_1 \geqslant \cdots \geqslant j_n \geqslant 0$ is a nonincreasing sequence of integers, and where we let, for $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant k$

$$R_j^i := \begin{cases} R_j & \text{if } i = 1, \\ p_{1,\ldots,i-1}(R_k) \times p_{i,i+1,\ldots,n}(R_j) & \text{if } i \geqslant 2, \end{cases}$$

with $p_{1,\ldots,i-1}$ (resp. $p_{i,i+1,\ldots,n}$) denoting the projection on the $i-1$ first (resp. $n-i+1$ last) coordinates.

On Figure 1, we represent a strict chain of rectangles $\mathscr{C}$ and the 6 rectangles belonging to the associated family $\widehat{\mathscr{C}}$; for the sake of clarity, we give a picture of each element of $\widehat{\mathscr{C}}$ in Figure 2.

Theorem 7. *Assume that $\mathscr{R}$ is a family of standard, dyadic rectangles in $\mathbf{R}^n$ having the following property: for all $k \in \mathbf{N}$, there exist a dyadic number $\alpha \leqslant 2^{-k-1}$ and a strict chain $\mathscr{C}$ of standard dyadic rectangles in $\mathbf{R}^{n-1}$ satisfying $\#\mathscr{C} = k+1$ and such that one has:*

$$\mathscr{R} \supseteq \left\{ R \times [0, \alpha] \colon R \in \widehat{\mathscr{C}} \right\}.$$

Then de Guzmán's $L \log^{n-1} L$ weak estimate for the maximal operator $M_{\mathscr{R}}$ is optimal in the following sense: if Φ is an Orlicz function satisfying $\Phi = o(\Phi_{n-1})$ at ∞, then $M_{\mathscr{R}}$ does not *satisfy a weak L^Φ estimate.*

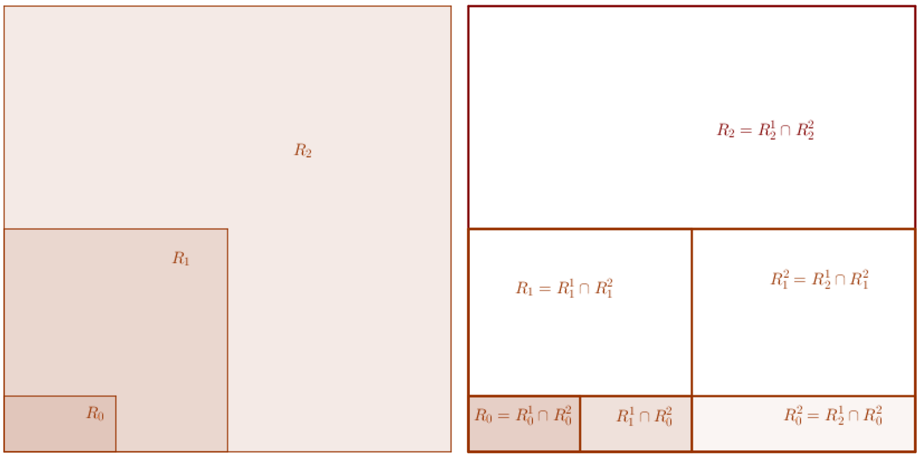

Figure 1. A strict chain of dyadic rectangles $\mathscr{C} = \{R_0, R_1, R_2\}$ (left) and the associated family $\widehat{\mathscr{C}}$ (right).

Remark 8. The preceeding theorem includes as a particular case Stokolos' result [13] about Soria bases in $\mathbf{R}^3$. More precisely, Stokolos there shows an estimate similar to the one we get in Claim 1 above with $l = n - 1$, for families of standard dyadic rectangles in $\mathbf{R}^3$ of the form

$$\mathscr{R} = \{R \times I \colon R \in \mathscr{R}', I \in \mathscr{I}_d\},$$

where $\mathscr{I}_d$ is the family of all dyadic intervals, and where $\mathscr{R}'$ is a family of standard dyadic rectangles in $\mathbf{R}^2$ containing arbitrary large (in the sense of cardinality) finite subsets $\mathscr{R}''$ having the following property

(is) for all $R, R' \in \mathscr{R}''$ with $R \neq R'$, we have $R \nsim R'$ and $R \cap R' \in \mathscr{R}'$;

here we mean by $R \nsim R'$ that neither $R \subseteq R'$ nor $R' \subseteq R$ holds.

We now show that our Theorem 7 applies to this situation. For a given $k \in \mathbf{N}$, we denote by $\mathscr{R}'_k$ a subfamily of $\mathscr{R}'$ satisfying $\#\mathscr{R}'_k = 2k+1$ as well as property (is), and we write $\mathscr{R}'_k = \{[0, \alpha_j] \times [0, \beta_j] \colon 0 \leqslant j \leqslant 2k\}$ with $\alpha_0 < \alpha_1 < \cdots < \alpha_{2k}$. Then, we have $\beta_0 > \beta_1 > \cdots > \beta_{2k}$ (using the pairwise incomparability of elements in $\mathscr{R}'_k$). Now let $\alpha_j^1 := \alpha_j$ and $\alpha_j^2 := \beta_{2k-j}$ for $0 \leqslant j \leqslant k$, define for $0 \leqslant j \leqslant k$

$$R_j := [0, \alpha_j^1] \times [0, \alpha_j^2],$$

and observe that one has $R_0 \prec R_1 \prec \cdots \prec R_k$. Letting $\mathscr{C} := \{R_0, \dots, R_k\}$ we see that for $k \geqslant j_1 \geqslant j_2 \geqslant 0$ we have, using property (is)

$$\begin{aligned} R_{j_1}^1 \cap R_{j_2}^2 &= R_{j_1} \cap [p_1(R_k) \times p_2(R_{j_2})] = ([0, \alpha_{j_1}^1] \times [0, \alpha_{j_1}^2]) \cap ([0, \alpha_k^1] \times [0, \alpha_{j_2}^2]) \\ &= [0, \alpha_{j_1}^1] \times [0, \alpha_{j_2}^2] = ([0, \alpha_{j_1}] \times [0, \beta_{j_1}]) \cap ([0, \alpha_{2k-j_2}] \times [0, \beta_{2k-j_2}]) \in \mathscr{R}'. \end{aligned}$$

Hence we get $\mathscr{R} \supseteq \{R \times [0, 2^{-k-1}] \colon R \in \widehat{\mathscr{C}}\}$, and it is now clear that the hypotheses of Theorem 7 are satisfied.

Remark 9. Assume that $k \in \mathbf{N}$ is an integer and that $\mathscr{C} = \{R_0, \dots, R_k\}$ is a strict chain of standard dyadic rectangles in $\mathbf{R}^n$ with $R_0 \prec R_1 \prec \cdots \prec R_k$. Write, for each $0 \leqslant j \leqslant k$, $R_j := \prod_{i=1}^n [0, \alpha_j^i]$. Observe now that, for $k \geqslant j_1 \geqslant \cdots \geqslant j_n \geqslant 0$, one computes

$$\prod_{i=1}^n [0, \alpha_{j_i}^i] = \bigcap_{i=1}^n R_{j_i}^i \in \widehat{\mathscr{C}}.$$

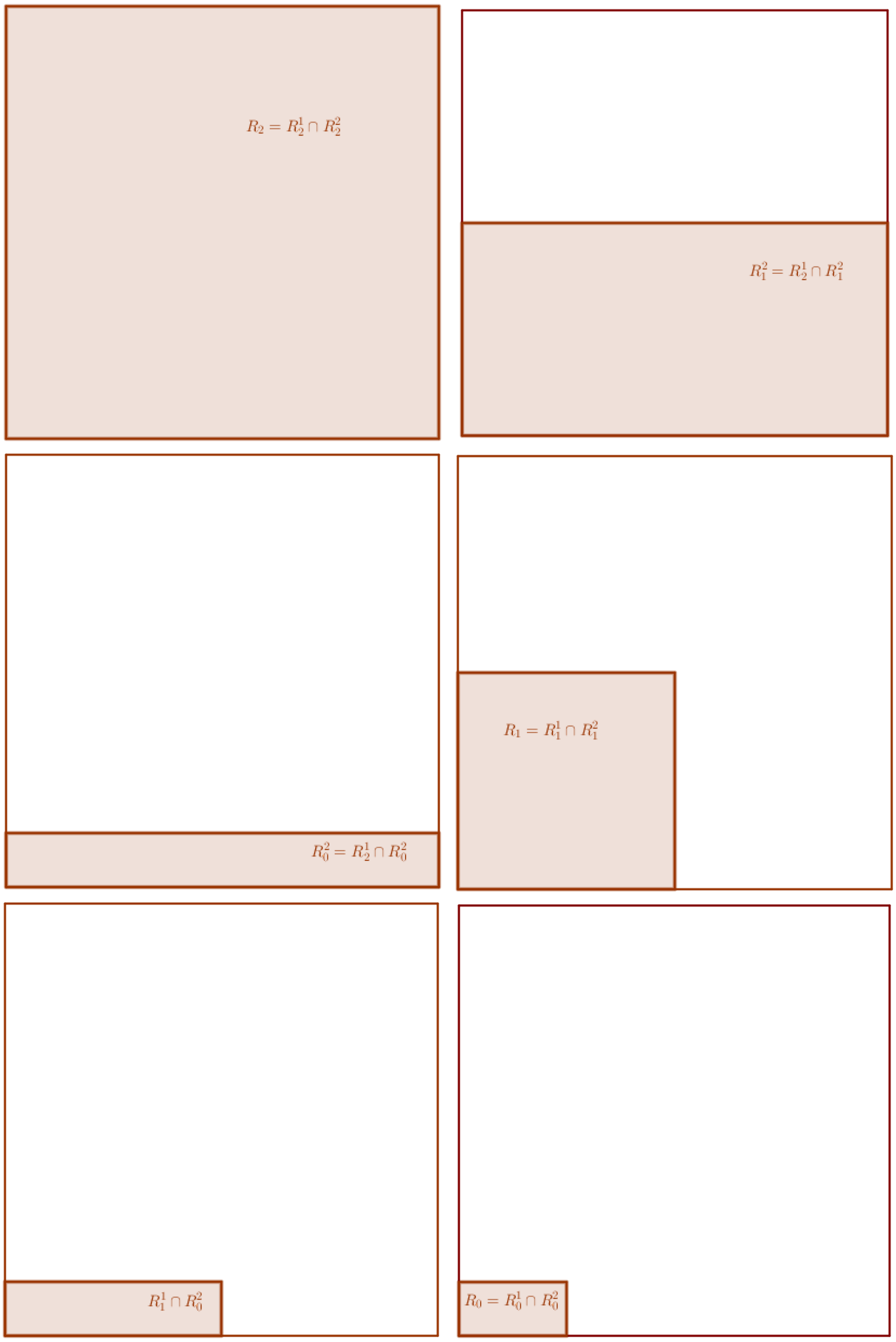

Figure 2. The rectangles belonging to $\widehat{\mathscr{C}}$ for $\mathscr{C} = \{R_0, R_1, R_2\}$ as in Figure 1.

In order to prove Theorem 7, it is sufficient, according to Proposition 3, to prove the following lemma, improving on Stokolos techniques in [11] and [13], and using Rademacher functions as in [9].

Lemma 10. *Assume the family $\mathscr{R}$ satisfies the hypotheses of Theorem 7. Then, for each $k \geqslant 2n - 6$, there exist sets Θ_k and Y_k satisfying the following conditions:*

(i) $\Theta_k \subseteq Y_k$;
(ii) $|Y_k| \geqslant \frac{2^{5-3n}}{(n-1)!} \cdot 2^{(n-1)k} k^{n-1} |\Theta_k|$;
(iii) *for each $x \in Y_k$, one has* $M_{\mathscr{R}} \chi_\Theta(x) \geqslant 2^{n-1} \cdot 2^{(1-n)k}$.

Proof. According to Remark 9, the hypothesis made on $\mathscr{R}$ implies that, for each $k \in \mathbf{N}$, one can find increasing sequences of dyadic numbers $\alpha_0^i < \alpha_1^i < \cdots < \alpha_k^i$, $1 \leqslant i \leqslant n-1$ and an integer $p \geqslant k+1$ such that, for any nonincreasing sequence $k \geqslant j_1 \geqslant j_2 \geqslant \cdots \geqslant j_{n-1} \geqslant 0$, one has

$$\left(\prod_{i=1}^{n-1}[0,\alpha_{j_i}^i]\right) \times [0,2^{-p}] \in \mathscr{R}. \tag{3}$$

Given $0 \leqslant j \leqslant k$, let $\alpha_j^i = 2^{-m_j^i}$ for $1 \leqslant i \leqslant n$ and define $m_j^n := p - j \geqslant 1$. Define also $R_0 := \prod_{i=1}^n [0, 2^{-m_k^i}]$. Denote by $C(k+1, n-1)$ the set of all nonincreasing $(n-1)$-tuples $J = (j_1, \ldots, j_{n-1})$ of integers in $\{0, 1, \ldots, k\}$ and note that one has

$$\#C(k+1,n-1) \geqslant C_{k+1}^{n-1} = \frac{(k+1)k\cdots(k-n+3)}{(n-1)!} \geqslant \frac{1}{2^{n-3}(n-1)!}k^{n-1},$$

given that $k \geqslant 2n-6$.

Define a set $\Theta \subseteq \mathbf{R}^n$ (to avoid unnecessary indices here, we write Θ and Y instead of Θ_k and Y_k, since k remains unchanged in this whole proof) by asking that, for any $x \in \mathbf{R}^n$, one has

$$\chi_\Theta(x) = \prod_{i=1}^n \prod_{j_i=1}^k r_{m_{j_i}^i}(x_i).$$

For $J = (j_1, \ldots, j_{n-1}) \in C(k+1, n-1)$, let $j_0 := k$, $j_n := 0$ and define a set $Y_J \subseteq \mathbf{R}^n$ by asking that, for any $x \in \mathbf{R}^n$, one has

$$\chi_{Y_J}(x) = \prod_{i=1}^n \prod_{\mu_i=j_i}^{j_{i-1}} r_{m_{\mu_i}^i}(x_i),$$

and let $Y := \bigcup_{J \in C(k+1,n-1)} Y_J$. Clearly, (i) holds.

It is clear, since the Rademacher functions (r_i) form an IID sequence, that one has $|\Theta| = 2^{-nk}|R_0|$ and, for $J \in C(k+1, n-1)$, $|Y_J| = 2^{1-k-n}|R_0|$.

We now write, for $1 \leqslant i \leqslant n-1$

$$Y = \bigcup_{j_1=0}^{k} \bigcup_{j_2=0}^{j_1} \cdots \bigcup_{j_{i-1}=0}^{j_{i-2}} \bigcup_{J' \in C(j_{i-1}+1, n-i)} Y_{j_1, \ldots j_{i-1}, J'}. \tag{4}$$

Hence, we define, for $1 \leqslant i \leqslant n$ and $k \geqslant j_1 \geqslant j_2 \geqslant \cdots \geqslant j_{i-1} \geqslant 0$

$$E_{j_1, \ldots, j_{i-1}} := \bigcup_{J' \in C(j_{i-1}+1, n-i)} Y_{j_1, \ldots j_{i-1}, J'};$$

with this definition, we get in particular $E_{j_1, j_2, \ldots, j_{n-1}} = Y_{j_1, \ldots, j_{n-1}}$ for $k \geqslant j_1 \geqslant j_2 \geqslant \cdots \geqslant j_{n-1} \geqslant 0$.

Claim 1. For all $1 \leqslant r \leqslant n$ and $k \geqslant j_1 \geqslant j_2 \geqslant \cdots \geqslant j_{r-2} \geqslant 0$, we have

$$\left|\bigcup_{j=0}^{j_{r-2}} E_{j_1, \ldots, j_{r-2}, j}\right| \geqslant \frac{1}{2} \sum_{j=0}^{j_{r-2}} |E_{j_1, \ldots, j_{r-2}, j}|.$$

Proof of the claim. To prove this claim, we write

$$\left|\bigcup_{j=0}^{j_{r-2}} E_{j_1, \ldots, j_{r-2}, j}\right| = \sum_{j=0}^{j_{r-2}} \left[|E_{j_1, \ldots, j_{r-2}, j}| - \left| E_{j_1, \ldots, j_{r-2}, j} \cap \bigcup_{l=0}^{j-1} E_{j_1, \ldots, j_{r-2}, l} \right| \right].$$

Assuming that $0 \leqslant j \leqslant j_{r-2}$ and $x \in E_{j_1,\dots,j_{r-2},j} \cap \bigcup_{l=0}^{j-1} E_{j_1,\dots,j_{r-2},l}$ are given, we find both

$$\prod_{i=1}^{r-2}\left[\prod_{\mu_i=j_i}^{j_{i-1}} r_{m^i_{\mu_i}}(x_i)\right]\cdot\prod_{\nu=j}^{j_{r-2}} r_{m^{r-1}_\nu}(x_{r-1}) \cdot \max_{J'\in C(j+1,n-r)}\left[\prod_{\xi=j_{r+1}}^{j} r_{m^r_\xi}(x_r)\prod_{i=r+1}^{n}\prod_{\mu_i=j_i}^{j_{i-1}} r_{m^i_{\mu_i}}(x_i)\right]=1,$$

and, for some $1 \leqslant l \leqslant j-1$

$$\prod_{i=1}^{r-2}\left[\prod_{\mu_i=j_i}^{j_{i-1}} r_{m^i_{\mu_i}}(x_i)\right]\cdot\prod_{\nu=l}^{j_{r-2}} r_{m^{r-1}_\nu}(x_{r-1}) \cdot \max_{J'\in C(l+1,n-r)}\left[\prod_{\xi=j_{r+1}}^{l} r_{m^r_\xi}(x_r)\prod_{i=r+1}^{n}\prod_{\mu_i=j_i}^{j_{i-1}} r_{m^i_{\mu_i}}(x_i)\right]=1,$$

so that we have

$$\begin{aligned} & r_{m^{r-1}_{j-1}}(x_{r-1})\chi_{E_{j_1,\dots,j_{r-2},j}}(x) \\ (5)\qquad &= \prod_{i=1}^{r-2}\left[\prod_{\mu_i=j_i}^{j_{i-1}} r_{m^i_{\mu_i}}(x_i)\right]\cdot\prod_{\nu=j-1}^{j_{r-2}} r_{m^{r-1}_\nu}(x_{r-1}) \\ &\quad \cdot \max_{J'\in C(j+1,n-r)}\left[\prod_{\xi=j_{r+1}}^{j} r_{m^r_\xi}(x_r)\prod_{i=r+1}^{n}\prod_{\mu_i=j_i}^{j_{i-1}} r_{m^i_{\mu_i}}(x_i)\right] \\ &= 1. \end{aligned}$$

We then get

$$\begin{aligned} \left|E_{j_1,\dots,j_{r-2},j} \cap \bigcup_{l=0}^{j-1} E_{j_1,\dots,j_{r-2},l}\right| &\leqslant \int_{R_0} r_{m^{r-1}_{j-1}}(x_{r-1})\chi_{E_{j_1,\dots,j_{r-2},j}}(x)\,dx \\ &= \frac{1}{2}\int_{R_0}\chi_{E_{j_1,\dots,j_{r-2},j}} = \frac{1}{2}|E_{j_1,\dots,j_{r-2},j}|, \end{aligned}$$

using the IID property of the Rademacher functions, Fubini's theorem and the fact that the first series of factors in (5) only depend on $x_1,\dots,x_{r-1}$. The proof of the claim is hence complete. □

We now turn to the proof of (ii). Observe that it now follows from (4) and from Claim 1 that we have

$$|Y| \geqslant \frac{1}{2^{n-1}}\sum_{J\in C(k+1,n-1)}|Y_J|,$$

for we recall that given $J=(j_1,\dots,j_{n-1})\in C(k+1,n-1)$, one has $E_{j_1,\dots,j_{n-1}} = Y_J$. This yields

$$|Y| \geqslant 2^{2-2n-k}|R_0|\#C(k+1,n-1) \geqslant \frac{2^{5-3n-k}}{(n-1)!}k^{n-1}|R_0| = \frac{2^{5-3n}}{(n-1)!}\cdot 2^{(n-1)k}k^{n-1}|\Theta|,$$

which completes the proof of (ii).

To prove (iii), fix $x \in Y$ and choose $J \in C(k+1, n-1)$ such that one has $x \in Y_J$. Now observe that Y_J is the disjoint union of rectangles, the lenghts of whose n sides are $2^{-m^1_{j_1}}$, $2^{-m^2_{j_2}}$, $\cdots$, $2^{-m^{n-1}_{j_{n-1}}}$ and 2^{-p}; letting $R_J := [0, 2^{-m^1_{j_1}}] \times [0, 2^{-m^2_{j_2}}] \times \cdots \times [0, 2^{-m^{n-1}_{j_{n-1}}}] \times [0, 2^{-p}]$, we see that $R_J \in \mathscr{R}$ and that there exists a translation τ of $\mathbf{R}^n$ for which we have $x \in \tau(R_j)$. Periodicity conditions moreover show that we have

$$\frac{|\tau(R_J) \cap \Theta|}{|R_J|} = \frac{|R_J \cap \Theta|}{|R_J|} = \frac{|Y_J \cap \Theta|}{|Y_J|} = \frac{|\Theta|}{|Y_J|} = 2^{n-1} \cdot 2^{(1-n)k},$$

so that we can write

$$M_{\mathscr{R}} \chi_\Theta(x) \geqslant \frac{1}{|R_J|} \int_{\tau(R_J)} \chi_\Theta = \frac{|\tau(R_J) \cap \Theta|}{|R_J|} \geqslant 2^{n-1} \cdot 2^{(1-n)k},$$

which finishes to prove (iii), and completes the proof of the lemma. □

Remark 11. A careful look as Stokolos' proof of [13, Theorem, p. 491] on Soria bases (see Remark 8 above) shows that, in case $n = 3$, it is a condition of type (3) that Stokolos actually uses in his proof to get an inequality similar to the one obtained in Claim 1 for the maximal operator $M_{\mathscr{R}}$ when $\mathscr{R}$ is a Soria basis.

It is clear also from the proof of Lemma 10 that condition (3) is the one we really use to prove Theorem 7. Formulated as in Theorem 7 though, this condition reads as a possibility of finding a sequence of dyadic numbers (α_k) tending to zero, such that the family obtained by projecting, on the $n-1$ first coordinates, the rectangles in $\mathscr{R}$ having their last side equal to α_k, contains a large number of rectangles, many of them being incomparable, and which enjoys some intersection property.

Theorem 7 also gives a way of constructing, from a strict chain of standard, dyadic rectangles, a translation-invariant basis $\mathscr{B}$ in $\mathbf{R}^n$ for which $L \log^{n-1} L(\mathbf{R}^n)$ is the optimal Orlicz space to be differentiated by $\mathscr{B}$.

We now turn to examine how the $L \log^{n-1} L$ weak estimate can be improved with some additional information on the *projections on a coordinate plane* of the rectangles belonging to the family under study.

5. Some conditions on projections

When a family of rectangles projects on a coordinate plane onto a family of 2-dimensional rectangles having finite width, a weak $L \log^{n-2} L$ inequality holds.

Proposition 12. *Fix $\mathscr{R}$ a family of standard dyadic rectangles in $\mathbf{R}^n$, $n \geqslant 2$, and assume that there exists a projection $p \colon \mathbf{R}^n \to \mathbf{R}^2$ onto one of the coordinate planes, such that the family $\mathscr{R}' := \{p(R) \colon R \in \mathscr{R}\}$ has finite width. Under these assumptions, there exists a constant $c > 0$ such that, for any measurable f and any $\lambda > 0$, one has*

$$|\{M_{\mathscr{R}} f > \lambda\}| \leqslant c \int_{\mathbf{R}^n} \frac{|f|}{\lambda} \left(1 + \log_+^{n-2} \frac{|f|}{\lambda}\right).$$

Remark 13. When $n = 3$, this result is due to Stokolos (see [12]). We mention it here in dimension n since the proof is straightforward and yields an interesting comparison with the examples in Section 4 above.

Proof. We can assume without loss of generality that $p \colon \mathbf{R}^n \to \mathbf{R}^2$, $(x_1, \ldots, x_n) \mapsto (x_{n-1}, x_n)$ is the projection onto the $x_{n-1}x_n$-plane. Let us prove the result by recurrence on the dimension n. It is clear that the result holds if $n = 2$, for then $\mathscr{R} = \mathscr{R}'$

and hence, it is standard (see e.g. [9, Theorem 10]) to see that the maximal operator $M_{\mathscr{R}}$ satisfies a weak $(1,1)$ inequality.

So fix now $n \geqslant 3$ and assume that the results holds in dimension $n-1$. Denote by $p_1 \colon \mathbf{R}^n \to \mathbf{R}$, $(x_1 \dots, x_n) \mapsto x_1$ the projection on the first axis and by $p_1' \colon \mathbf{R}^n \to \mathbf{R}^{n-1}$, $(x_1, \dots, x_n) \mapsto (x_2, \dots, x_n)$ the projection on the last $n-1$ coordinates, and observe that, by hypothesis, the families

$$\mathscr{R}_1 := \{p_1(R) \colon R \in \mathscr{R}\} \quad \text{and} \quad \mathscr{R}_2 := \{p_1'(R) \colon R \in \mathscr{R}\}$$

satisfy weak inequalities as in de Guzmán [2, p. 186] with $\varphi_1(t) := t$ and $\varphi_2(t) := t(1+\log_+^{n-3} t)$. According to [3, Theorem, p. 50] and to the obvious inclusion $\mathscr{R}_1 \times \mathscr{R}_2 \supseteq \mathscr{R}$, we then have, for each measurable f and each $\lambda > 0$

$$|\{M_{\mathscr{R}} f > \lambda\}| \leqslant \varphi_2(1) \int_{\mathbf{R}^n} \varphi_1\left(\frac{2|f|}{\lambda}\right) + \int_{\mathbf{R}^n} \left[\int_1^{\frac{4|f(x)|}{\lambda}} \varphi_1\left(\frac{4|f(x)|}{\lambda\sigma}\right) d\varphi_2(\sigma)\right] dx.$$

Now compute, for $t > 1$ and $n \geqslant 4$

$$\varphi_2'(t) = 1 + \log^{n-3} t + (n-3)\log^{n-4} t \leqslant 1 + (n-2)\log^{n-3} t,$$

inequality which also holds for $n = 3$. We hence have, for $x \in \mathbf{R}^n$ and $\lambda > 0$

$$\int_1^{\frac{4|f(x)|}{\lambda}} \varphi_1\left(\frac{4|f(x)|}{\lambda\sigma}\right) d\varphi_2(\sigma) \leqslant \int_{\mathbf{R}^n} \frac{4|f(x)|}{\lambda} \left[\int_1^{\frac{4|f(x)|}{\sigma}} \frac{1 + (n-2)\log^{n-3}\sigma}{\sigma}\, d\sigma\right] dx,$$

yet we compute

$$\int_1^{\frac{4|f(x)|}{\sigma}} \frac{1 + (n-2)\log^{n-3}\sigma}{\sigma}\, d\sigma \leqslant \log_+ \frac{4|f(x)|}{\lambda} + \log_+^{n-2} \frac{4|f(x)|}{\lambda},$$

so that we can write

$$|\{M_{\mathscr{R}} f > \lambda\}| \leqslant 2 \int_{\mathbf{R}^n} \frac{|f|}{\lambda}\left[1 + 2\log_+ \frac{4|f|}{\lambda} + 2\log_+^{n-2} \frac{4|f|}{\lambda}\right].$$

Yet we have on one hand

$$\int_{\{4|f| \leqslant \lambda e\}} \frac{|f|}{\lambda}\left[1 + 2\log_+ \frac{4|f|}{\lambda} + 2\log_+^{n-2} \frac{4|f|}{\lambda}\right] \leqslant 5 \int_{\{4|f| \leqslant \lambda e\}} \frac{|f|}{\lambda},$$

and on the other hand

$$\int_{\{4|f| > \lambda e\}} \frac{|f|}{\lambda}\left[1 + 2\log_+ \frac{4|f|}{\lambda} + 2\log_+^{n-2} \frac{4|f|}{\lambda}\right] \leqslant \int_{\{4|f| > \lambda e\}} \frac{|f|}{\lambda}\left[1 + 4\log_+^{n-2} \frac{4|f|}{\lambda}\right].$$

Summing up these inequalities we obtain

$$|\{M_{\mathscr{R}} f > \lambda\}| \leqslant 10 \int_{\mathbf{R}^n} \frac{|f|}{\lambda} + 8\log^{n-2} 4 \int_{\mathbf{R}^n} \frac{|f|}{\lambda}\left(1 + \log_+ \frac{|f|}{\lambda}\right)^{n-2}.$$

Computing again

$$\int_{\{|f| \leqslant \lambda e\}} \frac{|f|}{\lambda}\left(1 + \log_+ \frac{|f|}{\lambda}\right)^{n-2} \leqslant 2^{n-2} \int_{\mathbf{R}^n} \frac{|f|}{\lambda},$$

as well as

$$\int_{\{|f| > \lambda e\}} \frac{|f|}{\lambda}\left(1 + \log_+ \frac{|f|}{\lambda}\right)^{n-2} \leqslant 2^{n-2} \int_{\mathbf{R}^n} \frac{|f|}{\lambda} \log_+^{n-2} \frac{|f|}{\lambda}.$$

We finally get

$$|\{M_{\mathscr{R}} f > \lambda\}| \leqslant (10 + 2^{n+1} \log^{n-2} 4) \int_{\mathbf{R}^n} \frac{|f|}{\lambda} \left(1 + \log_+^{n-2} \frac{|f|}{\lambda}\right),$$

and the proof is complete. □

Using the results in Section 4, it is easy to provide an example in $\mathbf{R}^n$, $n \geqslant 3$ satisfying the hypotheses of the previous proposition, for which the $L \log^{n-2} L$ estimate is sharp.

Example 14. Assume $n \geqslant 3$ and denote by $\mathscr{R}$ the family of rectangles in $\mathbf{R}^{n-1}$ defined in Lemma 6, with n replaced by $n-1$. For each $k \in \mathbf{N}$, denote by Θ_k and Y_k the subsets of $\mathbf{R}^{n-1}$ associated to $\mathscr{R}$ as in Lemma 6.

Now define

$$\tilde{\mathscr{R}} := \{R \times [0,1] \colon R \in \mathscr{R}\},$$

let $\tilde{Y}_k := Y_k \times [0,1]$ and $\tilde{\Theta}_k := \Theta_k \times [0,1] \subseteq \tilde{Y}_k$. It is clear that one has $|\tilde{Y}_k| \geqslant c(n) 2^{(n-2)k} k^{n-2} |\tilde{\Theta}_k|$ with $c(n) := \frac{1}{3 \cdot 2^{n-3}}$.

Observe, finally, that if $x = (x_1, \dots, x_{n-1}, x_n) \in \tilde{Y}_k$ is given, one can find, according to the proof of Lemma 6, a rectangle $R \in \mathscr{R}$ in $\mathbf{R}^{n-1}$ such that we have $(x_1, \dots, x_{n-1}) \in R$ and

$$\frac{|R \cap \Theta_k|}{|R|} \geqslant 2^{(2-n)k}.$$

Letting $\tilde{R} := R \times [0,1]$, we get $x \in \tilde{R}$ and

$$M_{\tilde{\mathscr{R}}} \chi_{\tilde{\Theta}_k}(x) \geqslant \frac{|(R \times [0,1]) \cap (\Theta \times [0,1])|}{|R \times [0,1]|} \geqslant 2^{(2-n)k}.$$

It hence follows that $\tilde{R}$ satisfies the hypotheses of Proposition 3 with $d = n-2$. According to this proposition, we see that the $L \log^{n-2} L$ weak estimate for $M_{\tilde{\mathscr{R}}}$ is sharp. On the other hand, it is clear that the projections of rectangles in $\tilde{\mathscr{R}}$ onto the $x_{n-1}x_n$ coordinate plane form a family of rectangles in $\mathbf{R}^2$ having finite width.

The following property, introduced by Stokolos [12], also generalizes in a straightforward way to the n-dimensional case.

Definition 15. A family of standard, dyadic rectangles in $\mathbf{R}^n$ satisfies property (C) if there exists a projection p onto one coordinate plane enjoying the following property:

(C) there exists an integer $k \in \mathbf{N}^*$ such that for any finite family of rectangles $R_1, \dots, R_k \in \mathscr{R}$ whose projections $p(R_i)$ are *pairwise comparable*, one can find integers $1 \leqslant i < j \leqslant k$ with $R_i \sim R_j$.

Proposition 16. *If a family of rectangles in $\mathbf{R}^n$, $n \geqslant 3$, satisfies property (C), then it has weak type $L \log^{n-2} L$.*

Proof. Without loss of generality we can assume that $p \colon \mathbf{R}^n \to \mathbf{R}^2$, $(x_1, \dots, x_n) \mapsto (x_{n-1}, x_n)$ is the projection onto the $x_{n-1}x_n$-plane. Let us prove the result by recurrence on the dimension n. For $n = 3$, the conclusion follows from Theorem 3 in [12]. So fix now $n \geqslant 4$ and assume that the result holds in dimension $n-1$. Denote by $p_1 \colon \mathbf{R}^n \to \mathbf{R}$, $(x_1 \dots, x_n) \mapsto x_1$ the projection on the first axis and by $p_1' \colon \mathbf{R}^n \to \mathbf{R}^{n-1}$, $(x_1, \dots, x_n) \mapsto (x_2, \dots, x_n)$ the projection on the last $n-1$ coordinates, and observe that, by hypothesis, the families:

$$\mathscr{R}_1 := \{p_1(R) \colon R \in \mathscr{R}\} \quad \text{and} \quad \mathscr{R}_2 := \{p_1'(R) \colon R \in \mathscr{R}\}$$

satisfy weak inequalities as in de Guzmán [2, p. 185] with $\varphi_1(t) := t$ and $\varphi(t) := t(1+\log_+^{n-3} t)$. We proceed as in Proposition 12. $\square$

Remark 17. When $n = 3$, the above proposition is shown in [12] to be sharp. Example 14 shows again that this estimate cannot be improved in general.

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