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EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS TO THE FRACTIONAL EQUATION $\Delta^{\frac{\alpha}{2}} u = -u^\gamma$ IN BOUNDED DOMAINS

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Abstract. This paper deals with the question of the existence of a positive solution to the boundary value problem involving the fractional Laplacian

$$\begin{cases} \Delta^{\frac{\alpha}{2}} u = -u^\gamma & \text{on } D, \\ u = 0 & \text{on } D^c, \\ \lim_{x\to z\in\partial D} \delta(x)^{1-\frac{\alpha}{2}} u(x) = f(z), \end{cases}$$

for $\gamma > 1$, where $\delta(x)$ denotes the Euclidean distance from x to the boundary ∂D. We distinguish two cases of nonnegative data f: trivial and nontrivial.

1. Introduction

Let $\alpha \in]0,2[$ and $d > \alpha$. Let D be a Lipschitz domain in $\mathbf{R}^d$ and consider, for $\gamma > 1$, the problem

$$\begin{cases} \Delta^{\frac{\alpha}{2}} u = -u^\gamma & \text{in } D, \\ u = 0 & \text{on } D^c := \mathbf{R}^d \backslash D, \end{cases} \tag{1}$$

where $\Delta^{\frac{\alpha}{2}} = -(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian on $\mathbf{R}^d$. Solutions of this problem are understood in the distributional sense. The existence and nonexistence of positive solutions of problem (1) have been the main subject of some recent works [8, 14, 18, 19]. For $1 < \gamma < \frac{d+\alpha}{d-\alpha}$, it was proved in [8, 19] that problem (1) admits a positive solution in the fractional Sobolev space $H^{\frac{\alpha}{2}}(D)$. The nonexistence result in star-shaped domains has been investigated in [14, 18]. In both works, the authors proved that if $\gamma \geq \frac{d+\alpha}{d-\alpha}$ then problem (1) has no positive bounded solutions. A natural question to ask is how the solutions of (1) in [8, 19] behave near the boundary ∂D (bounded or not)? Also, what can be said about the existence of unbounded solutions of (1) for $\gamma \geq \frac{d+\alpha}{d-\alpha}$? Here, we would like to point out that this last question does not arise in the classical context $\alpha = 2$. In fact, on account of the celebrated Pohozaev identity established in [17], it is well known that, for $\gamma \geq \frac{d+2}{d-2}$, the problem

$$\begin{cases} \Delta u = -u^\gamma & \text{in } D, \\ u = 0 & \text{on } D^c, \end{cases}$$

has neither bounded nor unbounded positive solution.

We now briefly show how the boundary behavior of a solution of problem (1) is closely connected to the nonlocal character of $\Delta^{\frac{\alpha}{2}}$, and therefore some supplementary boundary condition will naturally appear in the study of problem (1). Let u be a

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positive solution of problem (1). It is obvious that $\Delta^{\frac{\alpha}{2}} u \leq 0$ in D which means that u is α-superharmonic on D. Thus, it follows from the integral representations of nonnegative α-superharmonic functions established in [4, 11] that there exists a unique nonnegative finite measure ν on ∂D such that, for every $x \in D$,

$$u(x) = \int_{\partial D} M_D^\alpha(x,z)\,\nu(dz) + \int_D G_D^\alpha(x,y) u^\gamma(y)\,dy,$$

where, M_D^α and G_D^α denotes respectively the Martin kernel and the Green kernel of $\Delta^{\frac{\alpha}{2}}$ on D. The Martin kernel M_D^α

$$\nu \to \int_{\partial D} M_D^\alpha(x,z)\,\nu(dz); \quad x \in D,$$

provides a one-to-one correspondence between positive finite measures ν on ∂D and positive singular α-harmonic functions on D. It follows from [16, Remark 3.15] that the nontangential limit

$$\lim_{x\to z} \frac{M_D^\alpha \nu(x)}{M_D^\alpha \sigma(x)} \text{ exists and is finite for } \sigma\text{-a.e. } z \in \partial D,$$

where σ is the Haussdorff surface measure on ∂D. Moreover, if ν is of density a continuous function f with respect σ then

$$\lim_{x\to z} \frac{M_D^\alpha \nu(x)}{M_D^\alpha \sigma(x)} = f(z) \text{ for every } z \in \partial D,$$

see also [6, Theorem 4.2]. In this case, $M_D^\alpha \nu$ and $M_D^\alpha \sigma$ are simply denoted by $M_D^\alpha f$ and $M_D^\alpha 1$ respectively. On the other hand, since $G_D^\alpha(u^\gamma)$ is a potential on D, it seems plausible that

$$\lim_{x\to z\in\partial D} \frac{G_D^\alpha(u^\gamma)(x)}{M_D^\alpha 1(x)} = 0,$$

but we have no proof of this. Taking into account all the aspects raised above, it is therefore reasonable to expect that every solution u of problem (1) should satisfies

$$\lim_{x\to z} \frac{u(x)}{M_D^\alpha 1(x)} \text{ exists and is finite for } \sigma\text{-a.e. } z \in \partial D.$$

The purpose of this paper is then to investigate the following appropriate reformulated problem:

$$(2) \qquad \begin{cases} \Delta^{\frac{\alpha}{2}} u = -u^\gamma & \text{on } D, \\ u = 0 & \text{on } D^c, \\ \lim_{x\to z} \frac{u(x)}{M_D^\alpha 1(x)} = f(z), & z \in \partial D. \end{cases}$$

This change in perspective allows us to gain not only finer results, but also a detailed study of problem (1). It is worth noting that several types of Neumann conditions for the fractional Laplacian, see for instance [13, 18], can also be taken into account, but we will not deal with here.

Theorem 1. *Let f be a nontrivial nonnegative continuous function on ∂D.*

1. *If $1 < \gamma < \frac{2+\alpha}{2-\alpha}$, then there exists a positive constant $L = L(D,\alpha,\gamma)$ such that problem (2) admits a positive solution provided*

$$\|f\| := \sup_{z\in\partial D} |f(z)| \leq L.$$

2. *If $\frac{2+\alpha}{2-\alpha} \leq \gamma$, then problem (2) has no positive solutions.*

Similar results have been established in [1, Theorem 1.9], where the author investigated problem (2) replacing the data f by the constant 1 and the equation $\Delta^{\frac{\alpha}{2}}u = -u^{\gamma}$ by $\Delta^{\frac{\alpha}{2}}u = -\lambda u^{\gamma}$.

Remarks 1. 1. Let $1 < \gamma < \frac{2+\alpha}{2-\alpha}$ and $f \in \mathcal{C}(\partial D)$. If f changes sign, then the following problem

$$(3)\qquad \begin{cases} \Delta^{\frac{\alpha}{2}}u = -u\,|u|^{\gamma-1} & \text{on } D,\\ u = 0 & \text{on } D^c,\\ \lim_{x\to z} \frac{u(x)}{M_D^{\alpha}1(x)} = f(z), & z \in \partial D, \end{cases}$$

possesses a solution provided

$$\|f\| := \sup_{z\in\partial D} |f(z)| \le L.$$

The proof is a slight modification to that of the above theorem.

2. The question of whether problems (2) or (3) admits a solution when $\|f\| > L$ needs further research. We have left open this question.

Theorem 1 yields, for every $1 < \gamma < \frac{2+\alpha}{2-\alpha}$, the existence of infinitely many positive blow up boundary solutions of problem (1):

Corollary 1. *Problem (1) has infinitely positive solutions such that*

$$\lim_{x\to z} u(x) = \infty \quad \textit{for every} \quad z \in \partial D.$$

As a consequence, it turns out that the critical power characterizing the existence of a positive solution of problem (1) cannot be $\frac{d+\alpha}{d-\alpha}$ as in the classical setting ($\alpha = 2$), but it seems to be $\frac{2+\alpha}{2-\alpha}$. We note in passing that boundary blow up solutions of the fractional equation with positive semilinearity

$$\Delta^{\frac{\alpha}{2}}u = u^{\gamma}$$

have been dealt with in [15], see also [1, 2, 9]. Next, we focus on problem (2) with trivial data f.

Theorem 2. *If $f = 0$, then problem (2) has no positive solutions for $1 < \gamma < \frac{2+\alpha}{2-\alpha}$.*

This theorem immediately yields:

Corollary 2. *For $1 < \gamma < \frac{2+\alpha}{2-\alpha}$, problem (1) has no positive solution which behaves near the boundary ∂D like $\delta(x)^{-\beta}$ for some $\beta < 1 - \frac{\alpha}{2}$.*

At the first time, it seems plausible that $\frac{d+\alpha}{d-\alpha}$ is the critical exponent characterizing the existence of positive bounded solutions of problem (1) as in the classical case $\alpha = 2$, but this is not true. In fact, by taking $\beta = 0$, we deduce from the second corollary that problem (1) has no bounded positive solutions not only for $\gamma \ge \frac{d+\alpha}{d-\alpha}$ as established in [14, 18] but also for $1 < \gamma < \frac{d+\alpha}{d-\alpha}$.

Corollary 3. *For every $\gamma > 1$, problem (1) has no bounded positive solutions.*

Finally, we would like to point out that Theorem 2 fails to holds in the classical setting $\alpha = 2$. Indeed, formally taking $\alpha = 2$, we would deduce the nonexistence of a positive solution of the problem

$$\begin{cases} \Delta u = -u^{\gamma} & \text{in } D,\\ u = 0 & \text{on } D^c, \end{cases}$$

while it is well known that this classical problem admits a bounded positive solution for $1 < \gamma < \frac{d+2}{d-2}$.

This work hopefully allows us to better understand how the nonlocal property of the fractional Laplacian influences the boundary behavior of solutions. The approach that we perform in this paper is essentially based on some tools from potential theory and it is completely different from that used in [1, 8, 14, 18, 19].

2. Preliminaries

For an open set U of $\mathbf{R}^d$, let $\mathcal{B}(U)$ be the set of all Borel measurable functions on U, $\mathcal{B}_b(U)$ be the set of all bounded Borel measurable functions on U, $\mathcal{C}(U)$ be the set of all continuous functions on U and $\mathcal{C}_0(U)$ be the set of all continuous functions on U such that $u = 0$ on ∂U. We denote by $\mathcal{C}_c^\infty(U)$ the set of all infinitely differentiable functions on U with compact support. For two nonnegative functions f and g, the notation $f \approx g$ on U means that there exist two positive constants c_1 and c_2 such that $c_1 f(x) \leq g(x) \leq c_2 f(x)$ for all $x \in U$.

Let $\alpha \in]0, 2[$ and $d > \alpha$. The fractional Laplacian $\Delta^{\frac{\alpha}{2}}$ is a prototype of non-local operators, it is defined, for $\varphi \in \mathcal{C}_c^\infty(\mathbf{R}^d)$, by

$$\begin{aligned}\Delta^{\frac{\alpha}{2}} u(x) &= C(d,\alpha) P.V \int_{\mathbf{R}^d} \frac{u(x+y) - u(x)}{|y|^{d+\alpha}}\, dy \\ &= C(d,\alpha) \lim_{\varepsilon \to 0} \int_{\{|y| \geq \varepsilon\}} \frac{u(x+y) - u(x)}{|y|^{d+\alpha}}\, dy,\end{aligned}$$

where the constant $C(d,\alpha)$ depending only on d and α. Throughout this paper, we fix a bounded Lipschitz open subset D of $\mathbf{R}^d$. For every locally bounded function v on D and every $u \in \mathcal{B}(\mathbf{R}^d)$ such that

$$\int_{\mathbf{R}^d} \frac{|u(x)|}{(1+|x|)^{d+\alpha}}\, dy < \infty, \tag{4}$$

we say that $\Delta^{\frac{\alpha}{2}} u = v$ in D *in the distributional sense* if for every nonnegative $\varphi \in \mathcal{C}_c^\infty(D)$,

$$\int_{\mathbf{R}^d} u(x) \Delta^{\frac{\alpha}{2}} \varphi(x)\, dx = \int_{\mathbf{R}^d} v(x) \varphi(x)\, dx.$$

Let us make the observation that if u is identically zero on D^c, then the integrability condition (4) simply means that $u \in L^1(D)$ the set of all Lebesgue integrable functions on D.

Definition 1. A function h satisfying (4) is said to be:

(a) α-harmonic in D if $h \in \mathcal{C}(D)$ and $\Delta^{\frac{\alpha}{2}} h = 0$ in D. If, in addition, $h = 0$ on D^c, we say that h is singular α-harmonic in D.

(b) α-superharmonic in D if h is lower semi-continuous on D and $\Delta^{\frac{\alpha}{2}} h \leq 0$.

As in the classical case ($\alpha = 2$), there exist probabilistic equivalent definitions for α-superharmonic and α-harmonic functions using the symmetric α-stable process (see [5, Theorem 3.9]).

By a solution of problem (1), we shall mean every function $u \in C(D) \cap L^1(D)$ such that $u = 0$ on D^c and

$$\int_{\mathbf{R}^d} u(x) \Delta^{\frac{\alpha}{2}} \varphi(x)\, dx = -\int_{\mathbf{R}^d} u^\gamma(x) \varphi(x)\, dx$$

for every $\varphi \in \mathcal{C}_c^\infty(D)$.

Sometimes it is more expedient to define the fractional Laplacian in probabilistic terms. We denote by (Ω, X_t, P^x) the standard rotation invariant (symmetric) stable process in $\mathbf{R}^d$, with index of stability α, and characteristic function

$$E^0[e^{i<\xi, X_t>}] = e^{-t|\xi|^\alpha}; \quad \xi \in \mathbf{R}^d,\ t \geq 0,$$

where E^x is the expectation with respect to the distribution P^x of the process starting from $x \in \mathbf{R}^d$. The limiting classical case $\alpha = 2$ corresponds to the Brownian motion generated by the usual Laplacian $\Delta = \sum_{i=1}^N \partial_i^2$. Nevertheless, when $0 < \alpha < 2$, the process is generated by the fractional Laplacian $\Delta^{\frac{\alpha}{2}}$. Let (X_t^D) be the killed stable process in D defined by

$$X_t^D = \begin{cases} X_t & \text{if } t < \tau_D, \\ \partial & \text{if } t \geq \tau_D, \end{cases}$$

where τ_D is the first exit time from D and ∂ is a cemetery point.

For $x \in D$, let $H_D^\alpha(x, \cdot)$ be the α-harmonic measure relative to x and D defined, for $f \in \mathcal{B}_b(\mathbf{R}^d)$, by

$$H_D^\alpha f(x) = E^x[f(X_{\tau_D})].$$

$H_D^\alpha(x, \cdot)$ is concentrated on $\overline{D}^c$ and is absolutely continuous with respect to the Lebesgue measure. Its corresponding density function $K_D^\alpha(x, y)$, called the Poisson kernel of D, is continuous in $(x, y) \in D \times \overline{D}^c$. The function $H_D^\alpha f$ is a fundamental example of α-harmonic function in D. Furthermore, if $f \in \mathcal{C}_b(D^c)$ then $H_D^\alpha f$ is the unique solution of the fractional Dirichlet problem

$$\begin{cases} \Delta^{\frac{\alpha}{2}} u = 0 & \text{in } D, \\ u = f & \text{on } D^c. \end{cases}$$

The potential operator G_D^α of the killed process (X_t^D) is defined, for every $f \in \mathcal{B}(D)$ for which the following identity exists, by

$$G_D^\alpha f(x) = E^x\left[\int_0^\infty f(X_t^D)\, dt\right]; \quad x \in D.$$

If f is bounded then $G_D^\alpha f \in \mathcal{C}_0(D)$. Moreover, if $G_D^\alpha f \not\equiv \infty$ then

$$\Delta^{\frac{\alpha}{2}} G_D^\alpha f = -f \quad \text{in} \quad D \tag{5}$$

in the distributional sense. The kernel $G_D^\alpha(\cdot, \cdot)$ of the operator G_D^α is called the Green function of (X_t^D), i.e.,

$$G_D^\alpha f(x) = \int_D G_D^\alpha(x, y) f(y)\, dy; \quad x \in D.$$

The Green function $G_D^\alpha(\cdot, \cdot)$ defined on $D \times D$ is positive, symmetric and continuous except along the diagonal. Moreover,

$$G_D^\alpha(x, y) \leq c \min\left\{\frac{1}{|x - y|^{d-\alpha}}, \frac{\delta(x)^{\frac{\alpha}{2}} \delta(y)^{\frac{\alpha}{2}}}{|x - y|^d}\right\}, \tag{6}$$

$$G_D^\alpha(x, y) \geq c\, \frac{\delta(x)^{\frac{\alpha}{2}} \delta(y)^{\frac{\alpha}{2}}}{|x - y|^d} \quad \text{if} \quad |x - y| > \max\left\{\frac{\delta(x)}{2}, \frac{\delta(y)}{2}\right\}. \tag{7}$$

Let $x_0 \in D$ be a reference point. The Martin kernel of the killed symmetric stable process is defined by

$$M_D^\alpha(x, z) = \lim_{y \to z} \frac{G_D^\alpha(x, y)}{G_D^\alpha(x_0, y)}, \quad \text{for } x \in D \text{ and } z \in \partial D.$$

The mapping $(x,z) \mapsto M_D^\alpha(x,z)$ is continuous on $D \times \partial D$. Moreover, for every $z \in \partial D$, $M_D^\alpha(\cdot, z)$ is a positive singular α-harmonic in D and for every $z, w \in \partial D$ such that $z \neq w$ we have $\lim_{x \to w} M_D^\alpha(x,z) = 0$. For every $f \in \mathcal{C}(\partial D)$, the function $M_D^\alpha f$ defined on $\mathbf{R}^d$ by

$$M_D^\alpha f(x) = \begin{cases} \displaystyle\int_{\partial D} M_D^\alpha(x,z)\, f(z)\, \sigma(dz) & \text{if } x \in D, \\ 0 & \text{if } x \notin D, \end{cases}$$

is the unique positive solution h of the following boundary value problem:

$$\begin{cases} \Delta^{\frac{\alpha}{2}} h = 0 & \text{in } D, \\ h = 0 & \text{on } D^c, \\ \lim_{x \to z \in \partial D} \frac{h(x)}{M_D^\alpha 1(x)} = f(z). \end{cases}$$

Furthermore,

$$M_D^\alpha 1 \approx \delta^{\frac{\alpha}{2}-1} \text{ on } D. \tag{8}$$

The explicit formula of $M_D^\alpha 1$ is known only for few choices of D, namely, for the unit ball B of $\mathbf{R}^d$:

$$M_B^\alpha 1(x) = \frac{1}{(1 - |x|^2)^{1-\frac{\alpha}{2}}}; \quad x \in B.$$

Concluding this section, we refer to [3, 4, 5, 6, 7, 10, 11, 16] for more details on the properties of the Poisson kernel K_D^α, the Green function G_D^α, the Martin kernel M_D^α and also on further background material for the potential theory of the fractional Laplacian.

3. Proofs of Theorem 1 and 2

We shall apply Schauder's fixed point theorem to prove the existence of positive solution of problem (2). Before going to the proof, we first give some interesting properties of the potential $G_D^\alpha(\delta^{-\lambda})$ which are technically needed. The upcoming lemma has been proved in [2], but for the reader's convenience we include the complete proof.

Lemma 1. *For every $\lambda < 1 + \frac{\alpha}{2}$, the potential $G_D^\alpha\left(\delta^{-\lambda}\right)$ is continuous on D.*

Proof. Let $\lambda < 1 + \frac{\alpha}{2}$. Let $x_0 \in D$ and let $r > 0$ such that $B(x_0, 2r) \subset D$. We write $G_D^\alpha\left(\delta^{-\lambda}\right) = h_1 + h_2$, where

$$h_1(x) := G_D^\alpha\left(1_{B(x_0,2r)}\delta^{-\lambda}\right)(x) \quad \text{and} \quad h_2(x) := G_D^\alpha\left(1_{D \setminus B(x_0,2r)}\delta^{-\lambda}\right)(x).$$

The approach is as follows. We check that $\sup_{|x-x_0| \le r} |h_1(x) - h_1(x_0)|$ can be made arbitrarily small with a suitably chosen $r > 0$ and that $\lim_{x \to x_0} |h_2(x) - h_2(x_0)| = 0$ for this choose of r. In this proof, the letter c signifies a positive constant which may change from one location to another. Seeing that $\delta(y) \ge \delta(x_0) - 2r$ for every $y \in B(x_0, 2r)$, it follows from (6) that, for every $x \in B(x_0, r)$,

$$G_D^\alpha(x,y)\delta^{-\lambda}(y) \le c\left(\delta(x_0) - 2r\right)^{-\lambda} \frac{1}{|x-y|^{d-\alpha}}.$$

Then, recalling the Lebesgue measure of balls $|B(x_0, 2r)| = c\, r^d$, we obtain

$$
\begin{aligned}
|h_1(x) - h_1(x_0)| &\le \int_{B(x_0,2r)} |G_D^\alpha(x,y) - G_D^\alpha(x_0,y)| \delta(y)^{-\lambda}\, dy \\
&\le c \int_{B(x_0,2r)} \left(G_D^\alpha(x,y) + G_D^\alpha(x_0,y)\right) \delta^{-\lambda}(y)\, dy \\
&\le c\,(\delta(x_0) - 2r)^{-\lambda} \left(\int_{B(x_0,2r)} \frac{dy}{|x-y|^{d-\alpha}} + \int_{B(x_0,2r)} \frac{dy}{|x_0-y|^{d-\alpha}} \right) \\
&\le c\,(\delta(x_0) - 2r)^{-\lambda} \left(\int_{B(x,r)} \frac{dy}{|x-y|^{d-\alpha}} + \int_{B(x_0,2r)\setminus B(x,r)} \frac{dy}{|x-y|^{d-\alpha}} + (2r)^\alpha \right) \\
&\le c\,(\delta(x_0) - 2r)^{-\lambda} \left(r^\alpha + \frac{|B(x_0,2r)|}{r^{d-\alpha}} + r^\alpha \right) \\
&\le c\,(\delta(x_0) - 2r)^{-\lambda}\, r^\alpha.
\end{aligned}
$$

It is clear that the last term is arbitrarily small for a careful choose of $r > 0$. Now, having chosen r, let us turn to show that $\lim_{x\to x_0} |h_2(x) - h_2(x_0)| = 0$. For every $y \in B\setminus B(x_0, 2r)$ and every $x \in B(x_0, r)$, we have $|x-y| > r$ and $|x_0 - y| > r$. Then it follows from (6) that

$$G_D^\alpha(x,y) + G_D^\alpha(x_0,y) \le \frac{c}{r^d} \delta(y)^{\frac{\alpha}{2}}.$$

Thus, for every $x \in B(x_0, r)$ and every $y \in D\setminus B(x_0, 2r)$, we obtain

$$|G_D^\alpha(x,y) - G_D^\alpha(x_0,y)|\, \delta(y)^{-\lambda} \le \left(G_D^\alpha(x,y) + G_D^\alpha(x_0,y)\right) \delta(y)^{-\lambda} \le \frac{c}{r^d}\, \delta(y)^{\frac{\alpha}{2}-\lambda}.$$

Noting that $-1 < \frac{\alpha}{2} - \lambda$, we get

$$\int_{D\setminus B(x_0,2r)} \delta(y)^{\frac{\alpha}{2}-\lambda} dy \le \int_D \delta(y)^{\frac{\alpha}{2}-\lambda} dy < \infty.$$

Hence, by dominated convergence theorem,

$$\lim_{x\to x_0} |h_2(x) - h_2(x_0)| = \lim_{x\to x_0} \int_{D\setminus B(x_0,2r)} |G_D^\alpha(x,y) - G_D^\alpha(x_0,y)|\, \delta(y)^{-\lambda}\, dy = 0.$$

This finish the proof of the lemma. □

The following important result is due to [12, Proposition 7] and provides some useful estimates of the potential $G_D^\alpha(\delta^{-\lambda})$ on D.

Lemma 2.

$$G_D^\alpha(\delta^{-\lambda}) \approx \begin{cases} \delta^{\alpha-\lambda}, & \text{if } \frac{\alpha}{2} < \lambda < 1 + \frac{\alpha}{2}, \\ \delta^{\frac{\alpha}{2}} \ln\left(\frac{2d}{\delta}\right), & \text{if } \lambda = \frac{\alpha}{2}, \\ \delta^{\frac{\alpha}{2}}, & \text{if } \lambda < \frac{\alpha}{2}. \end{cases}$$

Lemma 1 and Lemma 2 bear as a consequence the following:

Corollary 4. *Let $\lambda < 1 + \frac{\alpha}{2}$. Then, for every $f \in \mathcal{B}_b(D)$, $G_D^\alpha\left(\delta^{-\lambda} f\right) \in \mathcal{C}(D)$ and*

$$\delta^{1-\frac{\alpha}{2}} G_D^\alpha\left(\delta^{-\lambda} f\right) \in \mathcal{C}_0(D). \tag{9}$$

Proof. Let $f \in \mathcal{B}_b(D)$ and denote $\sup_{x\in D} |f(x)|$ by M. We can clearly assume that $f \ge 0$, else we break up f as $f^+ - f^+$, where $f^+ = \max(f, 0)$ and $f^- =$

$\max(-f,0)$. Put $g := M - f$. Multiplying by $\delta^{-\lambda}$ and then applying the operator G_D^α, we obtain

$$G_D^\alpha\left(\delta^{-\lambda} f\right)(x) + G_D^\alpha\left(\delta^{-\lambda} g\right)(x) = M\, G_D^\alpha\left(\delta^{-\lambda}\right)(x).$$

Now, seeing that $G_D^\alpha\left(\delta^{-\lambda} f\right)$ and $G_D^\alpha\left(\delta^{-\lambda} g\right)$ are lower semi-continuous on D and that $G_D^\alpha(\delta^{-\lambda}) \in \mathcal{C}(D)$, we immediately deduce that $G_D^\alpha\left(\delta^{-\lambda} f\right) \in \mathcal{C}(D)$. The statement (9) follows immediately from Lemma 2. □

Lemma 3. *Let $\lambda < 1 + \frac{\alpha}{2}$. Then, for every $M > 0$, the family*

$$\left\{\delta^{1-\frac{\alpha}{2}} G_D^\alpha\left(\delta^{-\lambda} f\right);\ f \in \mathcal{B}_b(D) \ \text{ and } \ \|f\| \le M\right\}$$

is relatively compact in $\mathcal{C}_0(D)$ endowed with the uniform norm $\|\cdot\|$.

Proof. Let $M > 0$. For every $f \in \mathcal{B}_b(D)$ such that $\|f\| \le M$ and for every $x \in D$, we have

$$\left|\delta(x)^{1-\frac{\alpha}{2}} G_D^\alpha\left(\delta^{-\lambda} f\right)(x)\right| \le M \sup_{x\in D}\left(\delta(x)^{1-\frac{\alpha}{2}} G_D^\alpha\left(\delta^{-\lambda}\right)(x)\right) < \infty.$$

Thus the family $\{\delta^{1-\frac{\alpha}{2}} G_D^\alpha\left(\delta^{-\lambda} f\right),\ \|f\| \le M\}$ is uniformly bounded in $\mathcal{C}_0(D)$. Then, in virtue of the Arzelà–Ascoli theorem, we need only to show that this family is equicontinuous on D. Let $x_0 \in D$. Since

$$\begin{aligned}&\sup_{\|f\|\le M}\left|\delta(x)^{1-\frac{\alpha}{2}} G_D^\alpha\left(\delta^{-\lambda} f\right)(x) - \delta(x_0)^{1-\frac{\alpha}{2}} G_D^\alpha\left(\delta^{-\lambda} f\right)(x_0)\right|\\ &\le M \int_D \left|\delta(x)^{1-\frac{\alpha}{2}} G_D^\alpha(x,y) - \delta(x_0)^{1-\frac{\alpha}{2}} G_D^\alpha(x_0,y)\right| \delta(y)^{-\lambda}\, dy =: k(x),\end{aligned}$$

it will be sufficient to show that $\lim_{x\to x_0} k(x) = 0$. Let $r > 0$ such that $B(x_0, 2r) \subset D$. We write $k = k_1 + k_2$, where

$$k_1(x) = \int_{B(x_0,2r)} \left|\delta(x)^{1-\frac{\alpha}{2}} G_D^\alpha(x,y) - \delta(x_0)^{1-\frac{\alpha}{2}} G_D^\alpha(x_0,y)\right| \delta(y)^{-\lambda}\, dy$$

and

$$k_2(x) = \int_{D\setminus B(x_0,2r)} \left|\delta(x)^{1-\frac{\alpha}{2}} G_D^\alpha(x,y) - \delta(x_0)^{1-\frac{\alpha}{2}} G_D^\alpha(x_0,y)\right| \delta(y)^{-\lambda}\, dy.$$

Now, following steps analogous to those in the proof the continuity of $G_D^\alpha\left(\delta^{-\lambda}\right)$ at x_0 in Lemma 1, we show that k_1 can be made arbitrarily small on $B(x_0, r)$ with a suitably chosen $r > 0$ and that $\lim_{x\to x_0} k_2(x) = 0$ for this choose of r. To avoid repetition, we omit the computation. □

Now we are in position to prove our theorems.

Proof of Theorem 1. For the sake of simplicity, we denote $M_D^\alpha 1$ by h.

1. Let f be a nontrivial nonnegative continuous function on ∂D and assume that $1 < \gamma < \frac{2+\alpha}{2-\alpha}$. Seeing that $\gamma(1 - \frac{\alpha}{2}) < 1 + \frac{\alpha}{2}$ and using the fact that $h \approx \delta^{\frac{\alpha}{2}-1}$, it follows from (9) that

$$c := \sup_{x\in D}\left[\frac{G_D^\alpha\left(h^\gamma\right)(x)}{h(x)}\right] < \infty.$$

Let λ_0 and L be two positive constants chosen so that

$$\frac{\lambda_0 - 1}{\lambda_0^\gamma} = \max_{\lambda>1} \frac{\lambda - 1}{\lambda^\gamma} \quad \text{and} \quad L = \left(\frac{\lambda_0 - 1}{c\lambda_0^\gamma}\right)^{\frac{1}{\gamma-1}}.$$

Let

$$\Lambda := \{v \in \mathcal{C}(\overline{D});\ 0 < v \le \lambda_0\|f\|\}$$

and consider the operator $T\colon \Lambda \to \mathcal{C}(\overline{D})$ defined by

$$Tv(x) := \frac{M_D^{\alpha} f(x)}{h(x)} + \frac{G_D^{\alpha}\left(h^{\gamma}v^{\gamma}\right)(x)}{h(x)}.$$

Then, for every $v \in \Lambda$ and every $x \in D$, we have

$$0 < Tv(x) \le \|f\| + c\,(\lambda_0\|f\|)^{\gamma}.$$

Now, the assumption $\|f\| \le L$ yields $\|f\| + c\,(\lambda_0\|f\|)^{\gamma} \le \lambda_0\|f\|$ and hence

$$T(\Lambda) \subset \Lambda.$$

Let $(v_n)_n$ be a sequence in Λ converging uniformly to $v \in \Lambda$ and let $\varepsilon > 0$. Since the function $t \mapsto t^{\gamma}$ is uniformly continuous on the interval $[0, \lambda_0\|f\|]$, there exists $n_0 \in \mathbf{N}$ such that, for every $n \ge n_0$,

$$\sup_{x\in D} |v_n^{\gamma}(x) - v^{\gamma}(x)| \le \varepsilon$$

and hence, for every $n \ge n_0$ and every $x \in D$,

$$|Tv_n(x) - Tv(x)| \le \varepsilon\,\frac{1}{h(x)} G_D^{\alpha}\left(h^{\gamma}\right)(x) \le c\varepsilon.$$

This shows that $(Tv_n)_n$ converges uniformly to Tv and therefore the operator T is continuous. On the other hand, thanks to Lemma 3, $T(\Lambda)$ is relatively compact. Since Λ is a closed bounded convex subset of $\mathcal{C}(\overline{D})$, the Schauder fixed point theorem ensures the existence of a function $v \in \Lambda$ such that $v = Tv$ on D, that is, for every $x \in D$,

$$h(x)v(x) = M_D^{\alpha} f(x) + G_D^{\alpha}\left((hv)^{\gamma}\right)(x).$$

Now, put $u = hv$. Then, we readily deduce that

$$u(x) = M_D^{\alpha} f(x) + G_D^{\alpha}(u^{\gamma})(x); \quad x \in D,$$

and hence, by (5), $\Delta^{\frac{\alpha}{2}}u = -u^{\gamma}$ in D. Moreover, for every $z \in \partial D$,

$$\lim_{x\to z} u(x)/h(x) = \lim_{x\to z} M_D^{\alpha} f(x)/h(x) = f(z)$$

since

$$\lim_{x\to z} \frac{G_D^{\alpha}(u^{\gamma})(x)}{h(x)} \le (\lambda_0\|f\|)^{\gamma} \lim_{x\to z} \frac{G_D^{\alpha}(h^{\gamma})(x)}{h(x)} = 0.$$

This completes the proof of the first statement of the theorem.

2. Let $\gamma \ge \frac{2+\alpha}{2-\alpha}$. Arguing by contradiction, we assume that problem (2) admits a positive solution u. Since $\Delta^{\frac{\alpha}{2}}u = -u^{\gamma} \le 0$ in D, it follows from [11, Theorem 3.8] that u can be written uniquely as

$$u(x) = M_D^{\alpha}\nu(x) + G_D^{\alpha}(u^{\gamma})(x); \quad x \in D,$$

where ν is finite measure on ∂D. This implies, in particular, that

$$G_D^{\alpha}(u^{\gamma})(x) < \infty; \quad x \in D.$$

On the other hand, let $z_0 \in \partial D$ such that $f(z_0) > 0$. By the hypothesis

$$\lim_{x\to z_0} \frac{u(x)}{h(x)} = f(z_0),$$

there exist two constants $r, \eta > 0$ small enough such that $u(x) \geq \eta\, h(x)$ for every $x \in D \cap B(z_0, r)$. Let $x_0 \in D$ be fixed and consider

$$K = \{y \in D;\ 2\delta(y) < \delta(x_0)\}.$$

It is easy to verify that, for every $y \in K$,

$$|x_0 - y| > \max\left(\frac{\delta(x_0)}{2}, \frac{\delta(y)}{2}\right).$$

Then, invoking (8) and (7), we obtain

$$\begin{aligned} G_D^\alpha(u^\gamma)(x_0) &\geq \eta^\gamma \int_{D\cap B(z_0,r)} G_D^\alpha(x_0, y) h^\gamma(y)\, dy \\ &\geq c_1 \int_{K\cap D\cap B(z_0,r)} \frac{\delta(x_0)^{\frac{\alpha}{2}} \delta(y)^{\frac{\alpha}{2}}}{|x_0 - y|^d}\, \delta(y)^{\gamma(\frac{\alpha}{2}-1)}\, dy \\ &\geq c_2 \int_{K\cap D\cap B(z_0,r)} \delta(y)^{\frac{\alpha}{2}+\gamma(\frac{\alpha}{2}-1)}\, dy = \infty \end{aligned}$$

since $\frac{\alpha}{2} + \gamma(\frac{\alpha}{2} - 1) \leq -1$. But this is impossible since $G_D^\alpha(u^\gamma)(x_0) < \infty$. Hence problem (2) has no positive solutions as desired. □

Proof of Theorem 2. Summarizing the first statement of Theorem 1, whenever $0 < \|f\| \leq L$ and $1 < \gamma < \frac{2+\alpha}{2-\alpha}$, we have constructed a solution u of problem (2) such that, for every $x \in D$,

$$0 < u(x) \leq \lambda_0\, \|f\|\, M_D^\alpha 1(x). \tag{10}$$

Let $n_0 \in \mathbf{N}$ such that $1 \leq n_0\, L$. For every $n \geq n_0$, let u_n be the solution of problem (2) with boundary data $f \equiv \frac{1}{n}$. It follows from (10) that the sequence $(u_n)_n$ converges to zero pointwise. Let u be a nonnegative solution of problem (2) with $f \equiv 0$. We now prove that $u \leq u_n$ for every $n \geq n_0$ and hence u will be identically zero as desired. Indeed, suppose otherwise. Then, there exists $m \geq n_0$ such that the open set

$$V := \{x \in D;\ u > u_m\}$$

is not empty. It follows from [11, Theorem 3.8] that, for every $x \in D$,

$$u(x) = G_D^\alpha(u^\gamma)(x) \quad \text{and} \quad u_m(x) = M_D^\alpha \nu_m(x) + G_D^\alpha(u_m^\gamma)(x),$$

where ν_m is (the unique) finite measure on ∂D. Let $w := u_m - u$ and $\rho := u_m^\gamma - u^\gamma$. Since u and u_m are identically zero on D^c, we thus obtain

$$w(x) = M_D^\alpha \nu_m(x) + G_D^\alpha(\rho)(x); \quad x \in \mathbf{R}^d \backslash \partial D.$$

Integrating with respect to the harmonic measure $H_V^\alpha(x, \cdot), x \in V$, shows that

$$\begin{aligned} H_V^\alpha w(x) &= M_D^\alpha \nu_m(x) + H_V^\alpha\left(G_D^\alpha(\rho)\right)(x) = M_D^\alpha \nu_m(x) + G_D^\alpha(\rho)(x) - G_V^\alpha(\rho)(x) \\ &= w(x) - G_V^\alpha(\rho)(x) \leq -G_V^\alpha(\rho)(x). \end{aligned}$$

This implies that $H_V^\alpha w \equiv 0$ on V since $G_V^\alpha(-\rho)$ is a potential. But this is impossible because, for $x \in V$, the harmonic measure $H_V^\alpha(x, \cdot)$ is absolutely continuous on V^c with respect to the Lebesgue measure and therefore must charges the nonempty open set $\{w > 0\} = D\backslash\overline{V}$ which implies that $H_V^\alpha w(x) \neq 0$. Here, the fact that the set $\{w > 0\}$ is not empty follows immediately from the boundary conditions

$$\lim_{x\to z} \frac{w(x)}{M_D^\alpha 1(x)} = \lim_{x\to z} \frac{u_m(x)}{M_D^\alpha 1(x)} - \lim_{x\to z} \frac{u(x)}{M_D^\alpha 1(x)} = \frac{1}{m} > 0 \quad \text{for all } z \in \partial D.$$

This completes the proof of theorem 2. □

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