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ON STRONGLY QUASISYMMETRIC HOMEOMORPHISMS

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Abstract. This short note is a continuous work of our previous paper [SW], where we were mainly concerned with the BMO theory of the universal Teichmüller space. We will endow the BMO-Teichmüller curve $\mathcal{T}_b$, the VMO-Teichmüller curve $\mathcal{T}_v$ and the BMO-Teichmüller space T_b with BMO manifold structures and prove that all these three spaces $\mathcal{T}_b$, $\mathcal{T}_v$ and T_b are contractible. We will also introduce a (new) extended BMO-Teichmüller space $\hat{T}_b$ and endow it with BMO manifold structures.

1. Introduction

We begin with some basic notations. Let $\Delta = \{z : |z| < 1\}$ denote the unit disk in the extended complex plane $\hat{\mathbf{C}}$. Then $S^1 = \partial\Delta$ is the unit circle. We also denote by $\mathbf{H} = \{z = x + iy : y > 0\}$ the upper half plane, and $\mathbf{R}$ the real line. I_0 will always denote the unit circle S^1 or the real line $\mathbf{R}$.

The universal Teichmüller space T is a universal parameter space for all Riemann surfaces and one of its models can be defined as the right coset space $T = \mathrm{QS}(I_0)/\mathrm{M\ddot{o}b}(I_0)$. Here, $\mathrm{QS}(I_0)$ denotes the group of all quasisymmetric homeomorphisms of I_0 onto itself, and $\mathrm{M\ddot{o}b}(I_0)$ the subgroup of $\mathrm{QS}(I_0)$ which consists of Möbius transformations keeping I_0 fixed. Recall that a sense preserving self-homeomorphism h of I_0 is quasisymmetric if there exists a (least) positive constant $C(h)$, called the quasisymmetric constant of h, such that

$$\frac{|h(I_1)|}{|h(I_2)|} \le C(h) \tag{1.1}$$

for all pairs of adjacent arcs I_1 and I_2 on I_0 with the same arc-length $|I_1| = |I_2|$. Beurling–Ahlfors [BA] proved that a sense preserving self-homeomorphism h of $\mathbf{R}$ is quasisymmetric if and only if there exists some quasiconformal homeomorphism of $\mathbf{H}$ onto itself which has boundary values h. Later Douady–Earle [DE] gave a quasiconformal extension of a quasisymmetric homeomorphism of S^1 which is conformally invariant. It is known that T is contractible (see [EE] and also [DE]) and is an infinite dimensional complex manifold modeled on a Banach space via the Bers embedding through the Schwarzian derivative (see [GL, Le, Na]).

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The universal Teichmüller curve $\mathcal{T}$ is a close relative of T and one of its models can be defined as the right coset space $\mathcal{T} = \mathrm{QS}(S^1)/\operatorname{Rot}(S^1)$, where $\operatorname{Rot}(S^1)$ denotes the group of all rotations about the circle S^1. It is also known that $\mathcal{T}$ is contractible, is an infinite dimensional complex manifold modeled on a Banach space via the logarithmic derivative embedding and the natural projection Φ from $\mathcal{T}$ onto T is a holomorphic split submersion (see [Be, DE, Te1-2]).

In this note, we are mainly concerned with two subclasses of quasisymmetric homeomorphisms. A sense preserving self-homeomorphism h of I_0 is said to be strongly quasisymmetric if there exist two positive constants $C_1(h)$, $C_2(h)$, called the strongly quasisymmetric constants of h, such that

$$\frac{|h(E)|}{|h(I)|} \le C_1(h) \left(\frac{|E|}{|I|} \right)^{C_2(h)} \tag{1.2}$$

whenever $I \subset I_0$ is an interval and $E \subset I$ a measurable subset. In other words, h is strongly quasisymmetric if and only if h is locally absolutely continuous so that $|h'|$ belongs to the class of weights A^∞ introduced by Muckenhoupt (see [CF, Ga]), in particular, $\log h'$ belongs to $\mathrm{BMO}(I_0)$, the space of locally integrable functions on I_0 of bounded mean oscillation (see [FS, Ga, Po, Zh] and Section 2). Here and in what follows, f' denotes the derivative of a function f defined on a set E, namely, for $z \in E$,

$$f'(z) = \lim_{E \ni \zeta \to z} \frac{f(\zeta) - f(z)}{\zeta - z},$$

provided the limit exists, while $f'(z) = 0$ otherwise. Let $\mathrm{SQS}(I_0)$ denote the set of all strongly quasisymmetric homeomorphisms of I_0 onto itself. Then $\mathrm{SQS}(I_0)$ is a sub-group of $\mathrm{QS}(I_0)$ and $T_b = \mathrm{SQS}(I_0)/\text{Möb}(I_0)$ is a model of the BMO-Teichmüller space. We say a sense preserving self-homeomorphism h of the unit circle S^1 is strongly symmetric if it is absolutely continuous such that $\log h'$ belongs to $\mathrm{VMO}(S^1)$, the space of integrable functions on S^1 of vanishing mean oscillation (see [Ga, Po, Sa, Zh] and Section 2). We denote by $\mathrm{SS}(S^1)$ the set of all strongly symmetric self-homeomorphisms of the unit circle. Then $\mathrm{SS}(S^1)$ is a sub-group of $\mathrm{SQS}(S^1)$ (see [Pa]) and $T_v = \mathrm{SS}(S^1)/\text{Möb}(S^1)$ is a model of the VMO-Teichmüller space. These two subclasses of quasisymmetric homeomorphisms and their Teichmüller spaces were much investigated (see [AZ, CZ, FH, FHS, FKP, HS, Ma, Se, SW, TWS, We, WS]) because of their great importance in the application to harmonic analysis (see [Da, FKP, Jo, Se]).

This note is a continuous work of the previous paper [SW] by Wei and the third named author of the present paper, where we were mainly concerned with the BMO theory of the universal Teichmüller space. In particular, it was proved in [SW] that both T_b and T_v have complex Banach manifold structures via the Bers embedding through the Schwarzian derivative such that T_v is a complex sub-manifold of T_b. Here we shall deal with two fiber spaces $\mathcal{T}_b = \mathrm{SQS}(S^1)/\operatorname{Rot}(S^1)$ and $\mathcal{T}_v = \mathrm{SS}(S^1)/\operatorname{Rot}(S^1)$ over T_b and T_v respectively. It was also proved in [SW] that both $\mathcal{T}_b$ and $\mathcal{T}_v$ have complex Banach manifold structures via the logarithmic derivative embedding such that $\mathcal{T}_v$ is a complex sub-manifold of $\mathcal{T}_b$, and the natural projections $\Phi\colon \mathcal{T}_b \to T_b$ and $\Phi\colon \mathcal{T}_v \to T_v$ are holomorphic split submersions. In the present paper, we will endow these two fiber spaces with new BMO manifold structures (see Theorems 3.3 and 3.4). In particular, we will prove that both $\mathcal{T}_b$ and $\mathcal{T}_v$ are contractible (see Corollary 3.5).

We will also endow the BMO-Teichmüller space $T_b = \mathrm{SQS}(\mathbf{R})/\text{Möb}(\mathbf{R})$ with a new BMO manifold structure and prove that T_b is contractible (see Theorem 4.1 and

Corollary 4.2). More generally, we say a sense preserving homeomorphism h on $\mathbf{R}$ is strongly quasisymmetric if it is locally absolutely continuous so that $|h'|$ belongs to the class of weights A^∞ and it maps $\mathbf{R}$ onto a chord-arc curve passing through the point at infinity (see [Se]). Recall that a Jordan curve Γ is a chord-arc (or Lavrentiev) curve with constant $K \geq 0$ if it is locally rectifiable and

$$|s_1 - s_2| \leq (1+K)|z(s_1) - z(s_2)| \tag{1.3}$$

for all $s_1 \in \mathbf{R}$ and $s_2 \in \mathbf{R}$, where $z(s)$ is the parametrization of Γ by the arc-length $s \in \mathbf{R}$ (see [La, Po]). We denote by $\mathrm{SQS}_{\mathbf{C}}(\mathbf{R})$ the set of all strongly quasisymmetric homeomorphisms on the real line $\mathbf{R}$ and by $\text{Möb}(\mathbf{C})$ the set of all Möbius transformations in the complex plane $\mathbf{C}$ (with ∞ fixed). Then $\hat{T}_b = \mathrm{SQS}_{\mathbf{C}}(\mathbf{R})/\text{Möb}(\mathbf{C})$ is an extension of T_b, which we call it the extended BMO-Teichmüller space. In section 5, we will also endow this extended BMO-Teichmüller space $\hat{T}_b$ with a (complex) BMO manifold structure (see Theorem 5.2 and also Theorem 5.3). In the final section, we list several open problems which are suggested by our discussions in Sections 3–5.

2. Preliminaries on BMO functions

In this section, we shall recall some basic definitions and results on BMO-functions. For primary reference, see [Ga].

A locally integrable function $u \in L^1_{\mathrm{loc}}(I_0)$ is said to have bounded mean oscillation if

$$\|u\|_{I_0} = \sup \frac{1}{|I|}\int_I |u(z) - u_I||dz| < +\infty, \tag{2.1}$$

where the supremum is taken over all sub-intervals I of I_0, while u_I is the average of u on the interval I, namely,

$$u_I = \frac{1}{|I|}\int_I u(z)|dz|. \tag{2.2}$$

If u also satisfies the condition

$$\lim_{|I|\to 0} \frac{1}{|I|}\int_I |u(z) - u_I||dz| = 0, \tag{2.3}$$

we say u has vanishing mean oscillation. The classes of these functions are denoted by $\mathrm{BMO}(I_0)$ and $\mathrm{VMO}(I_0)$, respectively.

We need some basic results on BMO functions. By the well-known theorem of John–Nirenberg for BMO functions (see [Ga]), there exist two universal positive constants C_1 and C_2 such that for any BMO function u, any subinterval I of I_0 and any $\lambda > 0$, it holds that

$$\frac{|\{z \in I \colon |u(z) - u_I| \geq \lambda\}|}{|I|} \leq C_1 \exp\left(\frac{-C_2\lambda}{\|u\|_{I_0}}\right). \tag{2.4}$$

By means of (2.4), it is easy to obtain the following basic results (see [Sh] for a proof).

Proposition 2.1. *Let $u \in \mathrm{BMO}(S^1)$ and $p \geq 1$. Then $e^u \in L^p(S^1)$ when $p\|u\|_{S^1}$ is small. In particular, if $u \in \mathrm{VMO}(S^1)$, then $e^u \in L^p(S^1)$ for any real number $p \geq 1$.*

Proposition 2.2. *Let $u \in \mathrm{VMO}(S^1)$ and $u_n \in \mathrm{BMO}(S^1)$. If $\|u_n - u\|_{S^1} \to 0$ and $(u_n - u)_{S^1} \to 0$ when $n \to \infty$, then for any $p \geq 1$, $\|e^{u_n} - e^u\|_{L^p} \to 0$ as $n \to \infty$.*

(2.4) also plays an important role in the proof of the following well-known result.

Proposition 2.3. [Jo] *Let $h \in \mathrm{SQS}(I_0)$. Then the pull-back operator P_h defined by $P_h(u) = u \circ h$ is a bounded isomorphism from $\mathrm{BMO}(I_0)$ onto itself.*

We also need the following basic result. A proof can be found in [Pa].

Proposition 2.4. [Pa] *Let $f \in L^1(S^1)$ and $\hat{f} \in L^1_{\mathrm{loc}}(\mathbf{R})$ be related by $\hat{f}(x) = f(e^{ix})$ for $x \in \mathbf{R}$. Then $f \in \mathrm{BMO}(S^1)$ if and only if $\hat{f} \in \mathrm{BMO}(\mathbf{R})$, and $\|f\|_{S^1} \le \|\hat{f}\|_{\mathbf{R}} \le 3\|f\|_{S^1}$.*

3. On the structures of $\mathcal{T}_b$ and $\mathcal{T}_v$

We first recall the following result from our paper [SW].

Proposition 3.1. [SW] *The mapping $\Psi\colon \mathcal{T}_b \to \mathrm{BMO}(S^1)/\mathbf{C}$ defined by $\Psi(h) = \log h'$ is a homeomorphism from $\mathcal{T}_b$ onto its image.*[1]

By means of Proposition 3.1, it is natural to assign the following metric to $\mathcal{T}_b$ by

$$d(h_1, h_2) = \| \log h_1' - \log h_2' \|_{S^1}. \tag{3.1}$$

As will be seen below, a more natural metric on $\mathcal{T}_b$ is

$$d_{\mathbf{R}}(h_1, h_2) = \| \log |h_1'| - \log |h_2'| \|_{S^1}. \tag{3.2}$$

Lemma 3.2. *$d_{\mathbf{R}}$ and d induce the same topology on $\mathcal{T}_b$.*

Proof. We use some discussion from [Sh] by the third author of the present paper. Each point h in $\mathcal{T}_b$ can be considered as a strongly quasisymmetric homeomorphism of S^1 onto itself which keeps 1 fixed. Writing $h(e^{i\theta}) = e^{i\phi(\theta)}$, ϕ is a strictly increasing and absolutely continuous function on the real line $\mathbf{R}$ such that $\phi(0) = 0$, $\phi(\theta+2\pi) = \phi(\theta) + 2\pi$. Then it holds that

$$h'(e^{i\theta}) = e^{i(\phi(\theta)-\theta)}\phi'(\theta). \tag{3.3}$$

Recall that $\phi'(\theta) = |h'(e^{i\theta})| \in A^\infty$.

We first show that, as $\| \log |h'| \|_{S^1} \to 0$, $\|\Im(\log h')\|_{S^1} \to 0$, which implies that $\| \log h' \|_{S^1} \to 0$. For simplicity, we set $f = \Im(\log h')$ so that $\hat{f}(\theta) = \phi(\theta) - \theta$. Recall that for any $s > 0$, the Sobolev space H^s consists of all integrable functions $u \in L^1(S^1)$ on the unit circle with semi-norm

$$\|u\|_{H^s} = \left(\sum_{n=-\infty}^{+\infty} |n|^{2s} |a_n(u)|^2 \right)^{\frac{1}{2}}, \tag{3.4}$$

where, as usual, $a_n(u)$ is the n-th Fourier coefficient of u, namely,

$$a_n(u) = \frac{1}{2\pi} \int_0^{2\pi} \hat{u}(\theta) e^{-in\theta} \, d\theta. \tag{3.5}$$

In particular, $a_0(u) = u_{S^1}$. Then it is well known that $H^{\frac{1}{2}} \subset \mathrm{VMO}(S^1)$, and the inclusion map is continuous (see [Zh]). Therefore, it is sufficient to show that $\|f\|_{H^{\frac{1}{2}}} \to 0$ as $\| \log |h'| \|_{S^1} \to 0$.

Set $u = \log |h'| - a_0(\log |h'|)$ so that $a_0(u) = 0$. Then, by Proposition 2.2, we obtain $\|e^u - 1\|_{L^1} \to 0$ as $\| \log |h'| \|_{S^1} \to 0$. Noting that

$$|a_0(e^u) - 1| = \frac{1}{2\pi} \left| \int_0^{2\pi} (e^{\hat{u}(\theta)} - 1) \, d\theta \right| \le \|e^u - 1\|_{L^1},$$

[1]The correspondence $h \mapsto \log h'$ induces a map from $\mathrm{QS}(S^1)$ into $\mathrm{BMO}(S^1)$. It projects to a well-defined map Ψ sending an equivalence $[h]$ in $\mathcal{T}_b$ to the equivalence class $[\log h']$ in $\mathrm{BMO}(S^1)/\mathbf{C}$. For simplicity, we still use $h \mapsto \log h'$ to denote the correspondence by Ψ, namely, $\Psi(h) = \log h'$. The correspondences in the following Theorems 3.3, 4.1 and 5.2 will also be understood in this sense.

we conclude that $a_0(e^u) \to 1$ as $\|\log|h'|\|_{S^1} \to 0$. Since $e^u = |h'|/e^{a_0(\log|h'|)}$, and $a_0(|h'|) = 1$, we have $a_0(\log|h'|) \to 0$ as $\|\log|h'|\|_{S^1} \to 0$. By Proposition 2.2 again, we obtain $\||h'| - 1\|_{L^2} \to 0$ as $\|\log|h'|\|_{S^1} \to 0$.

On the other hand, the n-th ($n \neq 0$) Fourier coefficient of f is

$$a_n(f) = \frac{1}{2\pi}\int_0^{2\pi}(\phi(\theta) - \theta)e^{-in\theta}\,d\theta = \frac{1}{2n\pi i}\int_0^{2\pi}(\phi'(\theta) - 1)e^{-in\theta}\,d\theta.$$

Thus, by Parseval's equality, we have

$$\|f\|_{H^1}^2 = \sum_{n\neq 0} n^2|a_n(f)|^2 = \frac{1}{4\pi^2}\sum_{n\neq 0}\left|\int_0^{2\pi}(\phi'(\theta) - 1)e^{-in\theta}\,d\theta\right|^2 = \||h'| - 1\|_{L^2}^2.$$

Thus, $\|f\|_{H^{\frac{1}{2}}} \leq \|f\|_{H^1} \to 0$ as $\|\log|h'|\|_{S^1} \to 0$.

Now let h_0 and h be given in $\mathcal{T}_b$ such that $d_{\mathbf{R}}(h, h_0) \to 0$. We consider $g = h \circ h_0^{-1}$. Then

$$\log g' = (\log h' - \log h_0') \circ h_0^{-1}. \tag{3.6}$$

By Proposition 2.3, we obtain

$$\|\log g'\|_{S^1} = \|(\log h' - \log h_0') \circ h_0^{-1}\|_{S^1} \asymp \|\log h' - \log h_0'\|_{S^1} = d(h, h_0). \tag{3.7}$$

Here and in what follows, the notation $A \asymp B$ means that there is a constant C independent of A and B such that $A/C \leq B \leq CA$. Writing $h_0(e^{i\theta}) = e^{i\phi_0(\theta)}$ and $h(e^{i\theta}) = e^{i\phi(\theta)}$ as above, then $g(e^{i\tau}) = e^{i\psi(\tau)}$ with $\psi = \phi \circ \phi_0^{-1}$. Thus,

$$\log \psi' = (\log \phi' - \log \phi_0') \circ \phi_0^{-1}. \tag{3.8}$$

Noting that $\phi_0'(\theta) = |h_0'(e^{i\theta})|$, $\phi'(\theta) = |h'(e^{i\theta})|$ and $\psi'(\tau) = |g'(e^{i\tau})|$, we conclude by Propositions 2.3 and 2.4 that

$$\begin{aligned}\|\log|g'|\|_{S^1} &\asymp \|\log\psi'\|_{\mathbf{R}} = \|(\log\phi' - \log\phi_0') \circ \phi_0^{-1}\|_{\mathbf{R}}\\ &\asymp \|\log\phi' - \log\phi_0'\|_{\mathbf{R}} \asymp \|\log|h'| - \log|h_0'|\|_{S^1}\\ &= d_{\mathbf{R}}(h, h_0).\end{aligned} \tag{3.9}$$

As a matter of fact the properties (3.7) and (3.9) imply that $d(h, h_0) \to 0$ if and only if $d_{\mathbf{R}}(h, h_0) \to 0$. □

Let $\mathrm{BMO}_{\mathbf{R}}(I_0)$ ($\mathrm{VMO}_{\mathbf{R}}(I_0)$) denote the real valued BMO (VMO) functions on I_0. Then we have

Theorem 3.3. *The mapping $\Psi_{\mathbf{R}}\colon \mathcal{T}_b \to \mathrm{BMO}_{\mathbf{R}}(S^1)/\mathbf{R}$ defined by $\Psi_{\mathbf{R}}(h) = \log|h'|$ is a homeomorphism from $\mathcal{T}_b$ onto its image. The image $\Psi_{\mathbf{R}}(\mathcal{T}_b)$ is a starlike open subset of $\mathrm{BMO}_{\mathbf{R}}(S^1)/\mathbf{R}$.*

Proof. The first statement follows directly from Proposition 3.1 and Lemma 3.2. To prove the openness of the image, we need a basic property of A^∞ weight (see [Ga]): There exists some universal constant $C > 0$ such that $e^u \in A^\infty$ whenever $u \in \mathrm{BMO}_{\mathbf{R}}(I_0)$ has norm $\|u\|_{I_0} < C$. Now let $h_0(e^{i\theta}) = e^{i\phi_0(\theta)}$ be given in $\mathcal{T}_b$. For any $u \in \mathrm{BMO}_{\mathbf{R}}(S^1)/\mathbf{R}$, by adding to a constant to u if necessary, we may assume that $a_0(e^u) = 1$. Set $h(e^{i\theta}) = e^{i\phi(\theta)}$ by

$$\phi(\theta) = \int_0^\theta e^{\hat{u}(t)}\,dt, \quad \theta \in \mathbf{R}. \tag{3.10}$$

Then h is an absolutely continuous sense-preserving homeomorphism of the unit circle with $\log|h'(e^{i\theta})| = \log\phi'(\theta) = u(e^{i\theta})$. As in the proof of Lemma 3.2, set

$g = h \circ h_0^{-1}$ and $\psi = \phi \circ \phi_0^{-1}$ so that $g(e^{i\tau}) = e^{\psi(\tau)}$. We conclude by (3.9) that, when $\|u - \log|h_0'|\|_{S^1}$ is small, $\|\log|g'|\|_{S^1}$ is also small, which implies by the above mentioned A^∞ property that $|g'| \in A^\infty$, that is, $g \in \mathcal{T}_b$. Since $\mathcal{T}_b$ is a group, we obtain $h = g \circ h_0 \in \mathcal{T}_b$, or equivalently, $u \in \Psi_{\mathbf{R}}(\mathcal{T}_b)$ when $\|u - \log|h_0'|\|_{S^1}$ is small. Consequently, $\Psi_{\mathbf{R}}(\mathcal{T}_b)$ is an open subset of $\mathrm{BMO}_{\mathbf{R}}(S^1)/\mathbf{R}$.

To prove the starlikeness of the image, we need another basic property of A^∞ weight (see [Ga]): Let $\omega \in A^\infty$ be given. Then for each $0 < t < 1$, $\omega^t \in A^\infty$. Let $h(e^{i\theta}) = e^{i\phi(\theta)}$ be given in $\mathcal{T}_b$. Then $\phi' \in A^\infty$. For $0 < t < 1$, define $h_t(e^{i\theta}) = e^{i\phi_t(\theta)}$ by

$$\phi_t(\theta) = \frac{2\pi}{\int_0^{2\pi} (\phi')^t(\tau)\, d\tau} \int_0^\theta (\phi')^t(\tau)\, d\tau, \quad \theta \in \mathbf{R}. \tag{3.11}$$

Since $\phi_t' = (\phi')^t / a_0((\phi')^t) \in A^\infty$, $h_t \in \mathcal{T}_b$. Noting that $\log|h_t'| = t \log|h'| - \log a_0((\phi')^t)$, we conclude that $\Psi_{\mathbf{R}}(\mathcal{T}_b)$ is a starlike subset (with respect to the zero element) of $\mathrm{BMO}_{\mathbf{R}}(S^1)/\mathbf{R}$. $\square$

Theorem 3.4. *The mapping $\Psi_{\mathbf{R}}$ is a homeomorphism from $\mathcal{T}_v$ onto* $\mathrm{VMO}_{\mathbf{R}}(S^1)/\mathbf{R}$.

Proof. It is sufficient to show that $\Psi_{\mathbf{R}}$ maps $\mathcal{T}_v$ onto $\mathrm{VMO}_{\mathbf{R}}(S^1)/\mathbf{R}$. This is easy. For any $u \in \mathrm{VMO}_{\mathbf{R}}(S^1)/\mathbf{R}$, by adding to a constant to u if necessary, we may assume that $a_0(e^u) = 1$. Define $h(e^{i\theta}) = e^{i\phi(\theta)}$ by (3.10). Then h is an absolutely continuous sense-preserving homeomorphism of the unit circle with $\log|h'(e^{i\theta})| = \log \phi'(\theta) = u(e^{i\theta})$. As in the proof of Lemma 3.2, consider $f = \Im(\log h')$ so that $\hat{f}(\theta) = \phi(\theta) - \theta$. Thus, f is continuous on the unit circle S^1, in particular, $f \in \mathrm{VMO}(S^1)$. Thus, $\log h' = \log|h'| + if \in \mathrm{VMO}(S^1)$, that is, $h \in \mathcal{T}_v$, and $u \in \Psi_{\mathbf{R}}(\mathcal{T}_v)$. $\square$

Corollary 3.5. *Both $\mathcal{T}_b$ and $\mathcal{T}_v$ are contractible.*

Proof. Since $\Psi_{\mathbf{R}}(\mathcal{T}_b)$ and $\Psi_{\mathbf{R}}(\mathcal{T}_v)$ are starlike, and the latter one is even the whole real Banach space $\mathrm{VMO}_{\mathbf{R}}(S^1)/\mathbf{R}$, they are definitely contractible. As the images of contractible spaces under homeomorphic maps, $\mathcal{T}_b$ and $\mathcal{T}_v$ are also contractible. $\square$

4. On the structure of T_b

Theorem 4.1. *The mapping $\Psi \colon T_b \to \mathrm{BMO}_{\mathbf{R}}(\mathbf{R})/\mathbf{R}$ defined by $\Psi(h) = \log h'$ is a homeomorphism from $T_b = \mathrm{SQS}(\mathbf{R})/\mathrm{Möb}(\mathbf{R})$ onto its image. The image $\Psi(T_b)$ is a starlike open subset of* $\mathrm{BMO}_{\mathbf{R}}(\mathbf{R})/\mathbf{R}$.

Proof. A proof of the first statement was given in our previous paper [SW], which uses a series of results from the papers [AZ, Da2, Se]. For details, see Theorem 7.1 and its proof in [SW].

The proof of the second statement is similar to that of Theorem 3.3. Each point in $T_b = \mathrm{SQS}(\mathbf{R})/\mathrm{Möb}(\mathbf{R})$ can be considered as a strongly quasisymmetric homeomorphism h which maps the real line $\mathbf{R}$ strictly increasingly onto itself and keeps the points 0 and 1 fixed. Now let $h_0 \in T_b$ be given. For any $u \in \mathrm{BMO}_{\mathbf{R}}(\mathbf{R})/\mathbf{R}$, by adding a constant to u if necessary, we may assume that $\int_0^1 e^{u(t)}\, dt = 1$. Set

$$h(x) = \int_0^x e^{u(t)}\, dt, \quad x \in \mathbf{R}. \tag{4.1}$$

Then h is a locally absolutely continuous and strictly increasing homeomorphism of the real line $\mathbf{R}$ onto itself with $\log h' = u$. Setting $g = h \circ h_0^{-1}$, we conclude by Proposition 2.3 (see (3.7)) that, when $\|u - \log h_0'\|_{\mathbf{R}}$ is small, $\|\log g'\|_{\mathbf{R}}$ is also small, which implies by the A^∞ property that $g' \in A^\infty$, that is, $g \in T_b$. Since T_b is a group, we obtain $h = g \circ h_0 \in T_b$, or equivalently, $u \in \Psi(T_b)$ when $\|u - \log h_0'\|_{\mathbf{R}}$ is small. Consequently, $\Psi(T_b)$ is an open subset of $\mathrm{BMO}_{\mathbf{R}}(\mathbf{R})/\mathbf{R}$.

To prove the starlikeness of the image, we fix $h \in T_b$ so that $h' \in A^\infty$. For $0 < t < 1$, define

$$h_t(x) = \frac{\int_0^x (h')^t(\tau)\, d\tau}{\int_0^1 (h')^t(\tau)\, d\tau}, \quad x \in \mathbf{R}. \tag{4.2}$$

Clearly, $h_t' \in A^\infty$ and $h_t \in T_b$. Noting that $\log h_t' = t \log h'$ up to a constant, we conclude that $\Psi(T_b)$ is a starlike subset (with respect to the zero element) of $\mathrm{BMO}_{\mathbf{R}}(\mathbf{R})/\mathbf{R}$. □

Corollary 4.2. *T_b is contractible.*

Proof. As the image of a starlike set under a homeomorphism, T_b contractible. □

Remark 4.3. The contractibility of the VMO-Teichmüller space T_v was proved recently in our paper [TWS] by a very different approach. In the recent paper [FH], Fan–Hu even proved that the VMO-Teichmüller space T_v is holomorphically contractible.

Remark 4.4. As stated in section 1, T_b is a complex manifold modeled on a Banach space (see [SW]). Theorem 4.2 implies that T_b can be endowed with the BMO manifold structure by the homeomorphism $\Psi\colon T_b \to \mathrm{BMO}_{\mathbf{R}}(\mathbf{R})/\mathbf{R}$ defined by $\Psi(h) = \log h'$. It can be proved that the original complex manifold structure on T_b is compatible with this new BMO manifold structure. Details will appear elsewhere.

5. On the structure of $\hat{T}_b$

We first recall some basic results on chord-arc curves. Let Γ be a chord-arc curve passing through the point at infinity with parametrization $z(s)$ by the arc-length $s \in \mathbf{R}$. David [Da1] proved that there exists some $b \in \mathrm{BMO}_{\mathbf{R}}(\mathbf{R})$ such that $z'(s) = e^{ib(s)}$. Furthermore, these BMO functions $b's$ form an open subset Ω of $\mathrm{BMO}_{\mathbf{R}}(\mathbf{R})/\mathbf{R}$. Actually, Coifman–Meyer [CM] proved the following stronger result.

Proposition 5.1. [CM] *For any $b \in \Omega$, there exists some $\delta > 0$ such that the following equation*

$$\tilde{z}(x) = z_0 + \int_0^x e^{i(b(t)+w(t))}\, dt \tag{5.1}$$

represents a chord-arc curve whenever $w \in \mathrm{BMO}(\mathbf{R})$ with $\|w\|_{\mathbf{R}} < \delta$.

Theorem 5.2. *The mapping Ψ defined by $\Psi(h) = \log h'$ is a one-to-one map from $\hat{T}_b = \mathrm{SQS}_{\mathbf{C}}(\mathbf{R})/\mathrm{Möb}(\mathbf{C})$ into $\mathrm{BMO}(\mathbf{R})/\mathbf{C}$. The image $\Psi(\hat{T}_b)$ is an open subset of $\mathrm{BMO}(\mathbf{R})/\mathbf{C}$.*

Proof. Let $h_0 \in \mathrm{SQS}_{\mathbf{C}}(\mathbf{R})$ so that h_0 is a locally absolutely continuous sense-preserving homeomorphism from $\mathbf{R}$ onto a chord-arc curve Γ_0 so that $|h_0'| \in A^\infty$. We first show that $\log h_0' \in \mathrm{BMO}(\mathbf{R})$. Let $z_0(s)$ be the parametrization of Γ_0 by the arc-length $s \in \mathbf{R}$. Then there exists $b_0 \in \Omega$ such that $z_0'(s) = e^{ib_0(s)}$. Set $g_0 = z_0^{-1} \circ h_0$ so that $z_0 \circ g_0 = h_0$. Then g_0 is a locally absolutely continuous self-homeomorphism of $\mathbf{R}$, and $(z_0' \circ g_0)g_0' = h_0'$ so that $g_0' = |h_0'| \in A^\infty$, that is, $g_0 \in \mathrm{SQS}(\mathbf{R})$. Consequently, $\log h_0' = ib_0 \circ g_0 + \log g_0' \in \mathrm{BMO}(\mathbf{R})$. Clearly, $\Psi(h) = \log h'$ determines a one-to-one map Ψ from $\hat{T}_b = \mathrm{SQS}_{\mathbf{C}}(\mathbf{R})/\mathrm{Möb}(\mathbf{C})$ into $\mathrm{BMO}(\mathbf{R})/\mathbf{C}$.

We proceed to show that $\log h_0'$ is an interior point of $\Psi(\hat{T}_b)$. Let $w = u + iv \in \mathrm{BMO}(\mathbf{R})$ be given with small norm $\|w\|_{\mathbf{R}}$. We need to find $h \in \hat{T}_b$ with $\log h' = \log h_0' + w$.

To do so, let $z_0'(s) = e^{ib_0(s)}$ and $z_0 \circ g_0 = h_0$ as above. Consider

$$\tilde{z}(x) = \int_0^x e^{i(b_0(t) - i(w \circ g_0^{-1})(t))}\, dt. \tag{5.2}$$

Since $\|w\|_{\mathbf{R}}$ is small, and $g_0 \in \mathrm{SQS}(\mathbf{R})$, $\|w \circ g_0^{-1}\|_{\mathbf{R}}$ is also small. We conclude by Proposition 5.1 that the equation (5.2) represents a chord-arc curve Γ. Set $h = \tilde{z} \circ g_0$ so that h maps $\mathbf{R}$ onto Γ. Then

$$h' = (\tilde{z}' \circ g_0) g_0' = (z_0' \circ g_0) g_0' e^w = h_0' e^w,$$

which implies that $\log h' = \log h_0' + w$. We also have $|h'| = g_0' e^u$, or equivalently, $\log |(h \circ g_0^{-1})'| = u \circ g_0^{-1}$. Since $\|u\|_{\mathbf{R}} \leq \|w\|_{\mathbf{R}}$ is small, and $g_0 \in \mathrm{SQS}(\mathbf{R})$, $\|u \circ g_0^{-1}\|_{\mathbf{R}}$ is also small. By A^∞ property we obtain $|(h \circ g_0^{-1})'| \in A^\infty$ and so $|h'| \in A^\infty$ due to the fact that $\mathrm{SQS}(\mathbf{R})$ is a group. Consequently, $h \in \hat{T}_b$ is the required mapping. This completes the proof of Theorem 5.2. □

During the proof of Theorem 5.2, we use the chord-arc curve Γ by (5.2). In general, (5.2) is not the parametrization of Γ by the arc-length, and it is really so only when $u = 0$. For completeness, we now find the parametrization of Γ by the arc-length $s \in \mathbf{R}$ even if $u \neq 0$. To approach this, we set

$$f_0(x) = \int_0^x e^{u \circ g_0^{-1}}\, dt. \tag{5.3}$$

Since $\|u \circ g_0^{-1}\|_{\mathbf{R}}$ is small, $f_0' \in A^\infty$ and $f_0 \in \mathrm{SQS}(\mathbf{R})$. Set $g = f_0 \circ g_0$. Then $g \in \mathrm{SQS}(\mathbf{R})$, and $g' = (f_0' \circ g_0) g_0' = e^u g_0'$.

Let

$$b = b_0 \circ f_0^{-1} + v \circ g^{-1}, \tag{5.4}$$

and define

$$z(s) = \int_0^s e^{ib(t)}\, dt. \tag{5.5}$$

Then $z(s)$ is the the parametrization of Γ by the arc-length $s \in \mathbf{R}$. This follows from the following computation:

$$\begin{aligned}
\tilde{z}(x) &\overset{\text{by (5.2)}}{=} \int_0^x e^{i(b_0 - i(w \circ g_0^{-1}))}\, dt \overset{\text{by } w = u + iv}{=} \int_0^x e^{i(b_0 + v \circ g_0^{-1} - iu \circ g_0^{-1})}\, dt \\
&\overset{\text{by (5.3)}}{=} \int_0^x e^{i(b_0 + v \circ g_0^{-1} - i \log f_0')}\, dt = \int_0^x e^{i(b_0 + v \circ g_0^{-1})} f_0'\, dt = \int_0^{f_0(x)} e^{i(b_0 + v \circ g_0^{-1}) \circ f_0^{-1}}\, dt \\
&\overset{\text{by (5.4)}}{=} \int_0^{f_0(x)} e^{ib(t)}\, dt \overset{\text{by (5.5)}}{=} z(f_0(x)).
\end{aligned}$$

Note that the computation also implies that $h = \tilde{z} \circ g_0 = z \circ g$ by $g = f_0 \circ g_0$. Since $z(s)$ is the the parametrization of Γ, we also have $b \in \Omega$.

By the factorization in the proof of Theorem 5.1, the extended Teichmüller space $\hat{T}_b$ has another model.

Theorem 5.3. *There is a one-to-one map from $\hat{T}_b$ onto $T_b \times \Omega$.*

Proof. From the proof of Theorem 5.2, each $h \in \mathrm{SQS}_{\mathbf{C}}(\mathbf{R})$ induces a $g \in \mathrm{SQS}(\mathbf{R})$ and a $b \in \Omega$ such that $h = z \circ g$ maps $\mathbf{R}$ onto a chord-arc curve Γ whose parametrization $z(s)$ by the arc-length $s \in \mathbf{R}$ satisfies $z'(s) = e^{ib(s)}$. This induces a one-to-one map $\hat{\Psi}$ from $\hat{T}_b$ onto $T_b \times \Omega$ by letting $\hat{\Psi}(h) = (g, b)$. Replacing h by $\tilde{h}$ defined as

$$\tilde{h}(x) = \frac{|h(1) - h(0)|}{h(1) - h(0)} \frac{h(x) - h(0)}{\int_0^1 |h'(t)|\,dt}$$

if necessary, we may assume that each $h \in \hat{T}_b$ satisfies the normalized condition $h(0) = 0$, $h(1) > 0$ and $\int_0^1 |h'(t)|\,dt = 1$. Then the corresponding function $g \in T_b$ satisfies the normalized condition $g(0) = 0$, $g(1) = 1$. □

We end the paper with the following problems, which are suggested by our discussions in the proof of Theorems 5.2 and 5.3.

Problem 5.4. Theorem 5.2 implies that the extended BMO-Teichmüller space $\hat{T}_b$ has a (complex) BMO manifold structure by the embedding $\Psi : \hat{T}_b \to \mathrm{BMO}(\mathbf{R})/\mathbf{C}$ defined by $\Psi(h) = \log h'$. On the other hand, by Theorem 4.1 and David's result, each of T_b and Ω is an open subset of $\mathrm{BMO}_{\mathbf{R}}(\mathbf{R})/\mathbf{R}$, which implies that $T_b \times \Omega$ is an open subset of $\mathrm{BMO}_{\mathbf{R}}(\mathbf{R})/\mathbf{R} \times \mathrm{BMO}_{\mathbf{R}}(\mathbf{R})/\mathbf{R}$. Then Theorem 5.3 implies that $\hat{T}_b$ has another (real) BMO manifold structure by the bijection $\hat{\Psi} \colon \hat{T}_b \to T_b \times \Omega$. Determine whether or not these two manifold structures on $\hat{T}_b$ are compatible with each other. It should be pointed out that in this case it is even not clear whether these two manifold structures induce the same topology on $\hat{T}_b$, although it seems to be so.

Problem 5.5. By Corollaries 3.5, 4.2 and Remark 4.3, we know that each of $\mathcal{T}_b$, $\mathcal{T}_v$, T_b and T_v is contractible. Determine whether or not the extended BMO-Teichmüller space $\hat{T}_b$ is connected under the embedding $\Psi \colon \hat{T}_b \to \mathrm{BMO}(\mathbf{R})/\mathbf{C}$ and/or the bijection $\hat{\Psi} \colon \hat{T}_b \to T_b \times \Omega$. In the latter case, the problem is closely related to the one about the connectedness of the chord-arc curve space Ω, which is known to be a difficult open problem (see [AZ, CM]).

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