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RIGIDITY THEOREMS FOR MINIMAL SUBMANIFOLDS IN A HYPERBOLIC SPACE

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Abstract. Let M be a complete immersed minimal hypersurface in a hyperbolic space. In this paper we establish conditions on the first eigenvalue of the stability and super stability operators and the L^d norm of the length of the second fundamental form of M to imply that M is totally geodesic. Similar results for minimal submanifolds in a hyperbolic space are also proven.

1. Introduction and the main results

The famous Bernstein theorem [4] states that the only complete minimal graphs in $\mathbf{R}^3$, are planes. The works of Fleming [20], De Giorgi [14], Almgren [1] and Simons [33] tell us that the Bernstein Theorem is valid for complete minimal graphs in $\mathbf{R}^{n+1}$ provided that $n \leq 7$. Moreover, the dimension restriction is necessary as indicated by the counterexamples of Bombieri, De Giorgi and Giusti [5]. Because of the stability of entire minimal graphs, one naturally hopes to know if a complete stable minimal hypersurface in $\mathbf{R}^{n+1}, n \leq 7$ is a hyperplane. It has been shown independently by do Carmo and Peng [16], Fischer-Colbrie and Schoen [19] that a complete stable minimal surface in R^3 must be a plane. Recall that a minimal submanifold in a Riemannian manifold is stable if the second variation of its volume is always nonnegative for any normal variation with compact support. For the higher dimensional case, the above question is still open. However, do Carmo and Peng [17] have shown that if M is a stable complete minimal hypersurface in $\mathbf{R}^{n+1}$ and

$$\lim_{R\to+\infty} \frac{1}{R^{2q+2}} \int_{B_p(R)} |A|^2 = 0,\ q < \sqrt{\frac{2}{n}}, \tag{1.1}$$

then M is a hyperplane. Here, $B_p(R)$ denotes the geodesic ball of radius R centered at $p \in M$ and A is the second fundamental form of M. Many interesting generalizations of the above theorems have been obtained in recent years (cf. [11, 15, 18–23, 28–33, 35–37] etc.). Another important result in this direction is due to Cao, Shen and Zhu [8] which says that a complete stable minimal hypersurface in $\mathbf{R}^{n+1}$ has only one end. On the other hand, Shen and Zhu [31] have shown that if $M^n, n \geq 3$ is a complete stable minimal hypersurface in $\mathbf{R}^{n+1}$ with finite total curvature, that is,

$$\int_M |A|^n < +\infty,$$

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then M^n is a hyperplane. We remark that this Shen–Zhu's theorem also follows from the above Cao–Shen–Zhu's theorem and a theorem of Anderson [2] stating that a complete minimal hypersurface in $\mathbf{R}^{n+1}$, $n \geq 3$, with finite total curvature and one end is a hyperplane. Wang [35] has generalized Shen–Zhu's theorem to minimal submanifolds of $\mathbf{R}^m$.

Let (M, ds^2) be a complete non-compact Riemannian manifold. Let $\mu\colon M \to \mathbf{R}$ be a continuous function and Δ the Laplacian operator acting on functions of M. We set $L_\mu = \Delta + \mu$ and we denote by $\lambda_1(L_\mu, M)$ the first eigenvalue of L_μ which can be defined as follows:

$$\lambda_1(L_\mu, M) = \inf_{f \in C_0^\infty(M), f \neq 0} \frac{\int_M (|\nabla f|^2 - \mu f^2)}{\int_M f^2}, \tag{1.2}$$

where $|\nabla f|$ denotes the magnitude of the gradient of f taken with respect to ds^2. When $\mu = 0$, we usually call $\lambda_1(L_0, M)$ the first eigenvalue of M and we denote it by $\lambda_1(M)$. It is well known that (cf. [9, 10, 26, 27])

$$\lambda_1(\mathbf{H}^n) = \frac{(n-1)^2}{4}. \tag{1.3}$$

If M is an n-dimensional complete minimal submanifold in $\mathbf{H}^m$, then we have (cf. [13])

$$\lambda_1(M) \geq \frac{(n-1)^2}{4},$$

which is equivalent to say that

$$\int_M |\nabla f|^2 \geq \frac{(n-1)^2}{4} \int_M f^2, \ \forall\, f \in C_0^\infty(M). \tag{1.4}$$

If M is a complete minimal hypersurface of $\mathbf{H}^{n+1}$, the stability operator of M is $L_{|A|^2-n}$ and M is said to be stable if $\lambda_1(L_{|A|^2-n}, M) \geq 0$, where A is the second fundamental form of M (cf. [25]). It is easy to see from (1.2) and (1.3) that the first eigenvalue of the the stability operator of a complete totally geodesic hypersurface of $\mathbf{H}^{n+1}$ is $\frac{(n-1)^2}{4} + n$.

Recently Neto, Wang and Xia [29] obtained the following result.

Theorem A. *Let M be an $n(\geq 2)$-dimensional complete immersed minimal hypersurface in $\mathbf{H}^{n+1}$ and let A be the second fundamental form of M. Suppose that there exists a number $q \in (0, \sqrt{2/n})$ such that*

$$\lim_{R \to \infty} \frac{\int_{B_p(R)} |A|^2}{R^{2q+2}} = 0. \tag{1.5}$$

i) *If $n \geq 6$ and if*

$$\lambda_1(L_{|A|^2-n}, M) > 2n - \frac{(2 - nq^2)(n-1)^2}{4n(1+q)^2}, \tag{1.6}$$

then M is totally geodesic.

ii) *If $n \leq 4$, then*

$$\lambda_1(L_{|A|^2-n}, M) \leq 2n - \frac{(2-nq^2)n}{2+2nq+n}. \tag{1.7}$$

iii) *If $n = 5$, $q \in (0, 1/5)$ and if*

$$\lambda_1(L_{|A|^2-5}, M) > 5 + \frac{25(q+1)^2}{10q+7}, \tag{1.8}$$

then M is totally geodesic.

iv) *If $n = 5$ and if $q \in [1/5, \sqrt{2/5})$, then*

$$\lambda_1(L_{|A|^2-5}, M) \le 5 + \frac{25(q+1)^2}{10q+7}. \tag{1.9}$$

The first result in the present paper is the following

Theorem 1.1. *Let M be an $n(\ge 2, \ne 3)$-dimensional complete immersed minimal hypersurface in $\boldsymbol{H}^{n+1}$ and A the second fundamental form of M.*

i) *If*

$$\lambda_1(L_{|A|^2-n}, M) > 2n - \frac{(8-n(d-2)^2)(n-1)^2}{4nd^2} \tag{1.10}$$

and

$$\lim_{R\to+\infty} \frac{1}{R^2} \int_{B_p(R)} |A|^d = 0, \tag{1.11}$$

for some constant d satisfying

$$d \in \begin{cases} \left(0, \frac{1}{2}\right), & \text{for } n = 2, \\ \left(\frac{n-1}{n}, \frac{(n-1)(n-2)}{n}\right), & \text{for } n = 4 \text{ or } 5, \\ \left(2 - 2\sqrt{\frac{2}{n}}, 2 + 2\sqrt{\frac{2}{n}}\right), & \text{for } n \ge 6, \end{cases} \tag{1.12}$$

then M is totally geodesic.

ii) *When $n = 2$, if M is stable and if there exists a constant $d \in \left(0, \frac{4}{17}\right)$ such that (1.11) is satisfied, then M is totally geodesic.*

Remark 1.1. One can check that d satisfies (1.12) if and only if the constant on the right hand side of (1.10) is less than $\frac{(n-1)^2}{4} + n$. Thus Theorem 1.1 is a gap phenomenon for minimal hypersurfaces in a hyperbolic space. Both Theorem A and Theorem 1.1 provide conditions on the growth of the norm of the second fundamental form and on the first eigenvalue of the stability operator of a complete n-dimensional minimal hypersurface in a hyperbolic space to imply that it is totally geodesic, where, the dimension n is no less than 6 in the first result and in the second one, it is no less than 2 and not equal to 3.

Remark 1.2. The method in the proof of Theorem 1.1 does not work for the three dimensional case. We believe that a similar result also holds for 3-dimensional minimal hypersurfaces in $\mathbf{H}^4$.

Remark 1.3. With do Carmo–Peng and Fischer-Colbrie–Schoen's theorem in mind, it is interesting to know if the condition (1.11) in item ii) of Theorem 1.1 is necessary.

Let M^n be an n-dimensional stable minimal submanifold in $\mathbf{R}^{n+p}$. When $p = 1$, the stability of M is equivalent to the condition that

$$\int_M (|\nabla f|^2 - |A|^2 f^2) \ge 0, \quad \forall\, f \in C_0^\infty(M).$$

In higher codimensional case, Spruck (cf. [34]) proved that for a variation vector field $E = \phi\nu$, the second variation of $\mathrm{Vol}(M_t)$ satisfies

$$\left.\frac{d^2\mathrm{Vol}(M_t)}{dt^2}\right|_{t=0} \ge \int_M (|\nabla\phi|^2 - |A|^2\phi^2),$$

where ν is the unit normal vector field and $\phi \in W_0^{1,2}(M)$. Motivated by this, Wang [13] introduced the concept of super stability:

Definition 1.1. Let M be an n-dimensional complete minimal immersed submanifold in $\mathbf{R}^{n+p}$. M is super stable if

$$\int_M (|\nabla f|^2 - |A|^2 f^2) \geq 0, \quad \forall\, f \in C_0^\infty(M). \tag{1.13}$$

In the same direction, some authors developed the concept of super stability for submanifolds of $\mathbf{H}^{n+p}$. In [30], Seo introduced the following

Definition 1.2. Let M be an n-dimensional complete minimal immersed submanifold in $\mathbf{H}^{n+p}$. The super index of M is defined to be the limit of the indices of an increasing sequence of exhausting compact domains in M. The index of a compact domain D is the number of negative eigenvalues of the eigenvalue problem

$$\begin{cases} (\triangle + |A|^2 - n)f + \lambda f = 0 \quad \text{on } D, \\ f\ |_{\partial D} = 0. \end{cases} \tag{1.14}$$

We say that M is super stable if it has super index zero, which means that

$$\int_M (|\nabla f|^2 - (|A|^2 - n) f^2) \geq 0, \quad \forall\, f \in C_0^\infty(M). \tag{1.15}$$

When M has codimension one, the concept "super stability" is the same as the usual definition of "stability" and the "super index" of M equals to the index of it.

For minimal submanifolds of lower dimensions of a hyperbolic space, we have

Theorem 1.2. *Let M be an n-dimensional complete immersed minimal submanifold in $\mathbf{H}^{n+p}$, $2 \leq n \leq 5$.*

i) *When $n \neq 3$, if*

$$\lambda_1(L_{|A|^2-n}, M) > \frac{5}{3}n + \frac{(3nd^2 - 8nd + 8(n-2))(n-1)^2}{12nd^2}, \tag{1.16}$$

and (1.11) is satisfied, where

$$d \in \begin{cases} \left(0, \frac{1}{2}\right), & \text{when } n = 2, \\ \left(\frac{3}{4}, \frac{3}{2}\right), & \text{when } n = 4, \\ \left(\frac{4}{3} - \frac{2}{3}\sqrt{\frac{2}{5}}, \frac{4}{3} + \frac{2}{3}\sqrt{\frac{2}{5}}\right), & \text{when } n = 5, \end{cases} \tag{1.17}$$

then M is totally geodesic.

ii) *When $n = 3$, if*

$$\lim_{R \to +\infty} \frac{1}{R^2} \int_{B_p(R)} |A|^{2/3} = 0, \tag{1.18}$$

then

$$\lambda_1(L_{|A|^2-n}, M) \leq 4. \tag{1.19}$$

Theorem 1.3. *Let M be an n-dimensional complete immersed minimal submanifold in $\mathbf{H}^{n+p}$, $2 \leq n \leq 5$.*

i) *When $n \neq 3$, if*

$$\lambda_1(L_{|A|^2-n}, M) > \frac{5}{3}n + \frac{(3n(k-1)^2 - 8n(k-1) + 8(n-2))(n-1)^2}{12n(k-1)^2} \tag{1.20}$$

and

$$\lim_{R\to+\infty}\frac{1}{R^k}\int_{B_p(R)}|A|=0, \tag{1.21}$$

where

$$k\in\begin{cases}\left(1,\frac{3}{2}\right), & \text{when } n=2,\\ \left(\frac{7}{4},\frac{5}{2}\right), & \text{when } n=4,\\ \left(\frac{7}{3}-\frac{2}{3}\sqrt{\frac{2}{5}},\frac{7}{3}+\frac{2}{3}\sqrt{\frac{2}{5}}\right), & \text{when } n=5,\end{cases} \tag{1.22}$$

then M is totally geodesic.

ii) *When $n=3$, if*

$$\lim_{R\to+\infty}\frac{1}{R^{5/3}}\int_{B_p(R)}|A|=0, \tag{1.23}$$

then

$$\lambda_1(L_{|A|^2-n},M)\le 4. \tag{1.24}$$

The next results give estimates for the first eigenvalue of the super stability operator of minimal submanifolds of dimension no larger than 3 in a hyperbolic space.

Theorem 1.4. *Let M be an n-dimensional complete immersed minimal submanifold in $\boldsymbol{H}^{n+p}$, $n=2$ or 3. Suppose that there exists a constant d in the interval*

$$\left[2,\ 2+\frac{2}{3}\left(\sqrt{\frac{12}{n}-2}-1\right)\right), \tag{1.25}$$

such that (1.11) *is satisfied. Then*

$$\lambda_1(L_{|A|^2-n},M)\le\frac{5}{3}n+\frac{(3nd^2-8nd+8(n-2))(n-1)^2}{12nd^2}. \tag{1.26}$$

Theorem 1.5. *Let M be an n-dimensional complete immersed minimal submanifold in $\boldsymbol{H}^{n+p}$, $n=2$ or 3. Suppose that there exists a constant*

$$k\in\left[3,\ 3+\frac{2}{3}\left(\sqrt{\frac{12}{n}-2}-1\right)\right), \tag{1.27}$$

such that (1.21) *is satisfied. Then*

$$\lambda_1(L_{|A|^2-n},M)\le\frac{5}{3}n+\frac{(3n(k-1)^2-8n(k-1)+8(n-2))(n-1)^2}{12n(k-1)^2}. \tag{1.28}$$

Remark 1.4. The method of the proofs is classic and has been used in many of the articles in the bibliography. The article by Bérard [3] is a pioneer reference for the generalized Simons' equation satisfied by the second fundamental form of an immersion in a Riemannian manifold. Simons' inequalities here can be deduced from that article of Bérard.

Remark 1.5. An interesting problem suggested by the referee is to understand the relation between the assumption on the total curvature and the volume entropy of the hypersurface M. For some ideas, one can see the pioneer articles [6, 7] and the more recent [24] and references therein. In particular [24, Theorem 6.1].

2. Proofs of the results

Proof of Theorem 1.1. Since M is a minimal hypersurface of $\mathbf{H}^{n+1}$, we have the following Simons' formula (cf. [12, 36, 37]):

$$\frac{1}{2}\triangle|A|^2 = |\nabla A|^2 - |A|^4 - n|A|^2. \tag{2.1}$$

Using equation (2.1) and the Lemma 2.1 in [36] we obtain that

$$|A|\triangle|A| + |A|^4 + n|A|^2 \geq \frac{2}{n}|\nabla|A||^2. \tag{2.2}$$

Let α be a fixed positive constant. It follows from (2.2) that

$$\begin{aligned}
|A|^\alpha\triangle|A|^\alpha &= |A|^\alpha(\alpha\nabla|A|^{\alpha-1}\nabla|A| + \alpha|A|^{\alpha-1}\triangle|A|) \\
&= \frac{(\alpha-1)}{\alpha}\alpha^2|A|^{2(\alpha-1)}|\nabla|A||^2 + \alpha|A|^{2\alpha-2}|A|\triangle|A| \\
&= \frac{(\alpha-1)}{\alpha}|\nabla|A|^\alpha|^2 + \alpha|A|^{2(\alpha-1)}|A|\triangle|A| \\
&\geq \frac{(\alpha-1)}{\alpha}|\nabla|A|^\alpha|^2 + \frac{2\alpha}{n}|A|^{2\alpha-2}|\nabla|A||^2 - \alpha|A|^{2\alpha+2} - \alpha n|A|^{2\alpha} \\
&= \frac{(\alpha-1)}{\alpha}|\nabla|A|^\alpha|^2 + \frac{2}{n\alpha}|\nabla|A|^\alpha|^2 - \alpha|A|^{2\alpha+2} - \alpha n|A|^{2\alpha},
\end{aligned}$$

which gives

$$|A|^\alpha\triangle|A|^\alpha \geq \left(1 - \frac{n-2}{\alpha n}\right)|\nabla|A|^\alpha|^2 - \alpha|A|^{2\alpha+2} - \alpha n|A|^{2\alpha}. \tag{2.3}$$

Let q be a non-negative constant and $f \in C_0^\infty(M)$. Multiplying (2.3) by $|A|^{2\alpha q}f^2$ and then integrating on M we obtain

$$\begin{aligned}
&\left(1 - \frac{n-2}{\alpha n}\right)\int_M |\nabla|A|^\alpha|^2|A|^{2\alpha q}f^2 \\
&\leq \alpha\int_M |A|^{2(q+1)\alpha}f^2|A|^2 + \alpha n\int_M |A|^{2(q+1)\alpha}f^2 + \int_M |A|^{(2q+1)\alpha}f^2\triangle|A|^\alpha.
\end{aligned} \tag{2.4}$$

For any $\epsilon > 0$, it follows from integration by parts and Young's inequality that

$$\begin{aligned}
&\left(2(q+1) - \frac{n-2}{\alpha n} - \epsilon\right)\int_M |\nabla|A|^\alpha|^2|A|^{2\alpha q}f^2 \\
&\leq \alpha\int_M |A|^{2(q+1)\alpha}f^2|A|^2 + \alpha n\int_M |A|^{2(q+1)\alpha}f^2 + \frac{1}{\epsilon}\int_M |A|^{2(q+1)\alpha}|\nabla f|^2.
\end{aligned} \tag{2.5}$$

One gets from the definition of $\lambda_1(L_{|A|^2-n}, M)$ that

$$\int_M |\nabla f|^2 \geq \int_M |A|^2f^2 - n\int_M f^2 + \lambda_1\int_M f^2, \ \forall\ f \in C_0^\infty(M).$$

Setting $\theta = \lambda_1 - n$, we get

$$\int_M |\nabla f|^2 \geq \int_M |A|^2f^2 + \theta\int_M f^2, \ \forall\ f \in C_0^\infty(M). \tag{2.6}$$

Letting $\gamma = \frac{(n-1)^2}{4}$, we have from (1.4) that

$$\int_M |\nabla f|^2 \geq \gamma\int_M f^2, \ \forall\ f \in C_0^\infty(M). \tag{2.7}$$

Fixing an $x \in [0,1]$, one deduces from (2.6) and (2.7) that

$$x\int_M |A|^2 f^2 + (\theta x + (1-x)\gamma)\int_M f^2 \le \int_M |\nabla f|^2. \tag{2.8}$$

Plugging $f|A|^{(q+1)\alpha}$ in (2.8) and using Young's inequality we obtain

$$\begin{aligned} & x\int_M |A|^{2(q+1)\alpha+2} f^2 + (\theta x + (1-x)\gamma)\int_M |A|^{2(q+1)\alpha} f^2 \\ & \le \left(1 + \frac{(q+1)\alpha}{\epsilon}\right)\int_M |A|^{2(q+1)\alpha}|\nabla f|^2 \\ & \quad + \frac{(q+1)}{\alpha}(\epsilon + (q+1)\alpha)\int_M |\nabla |A|^\alpha|^2 |A|^{2q\alpha} f^2. \end{aligned} \tag{2.9}$$

Supposing $\left(2(q+1) - \frac{n-2}{n\alpha} - \epsilon\right) > 0$, we can multiply (2.5) by $\frac{(q+1)}{\alpha}(\epsilon + (q+1)\alpha)$ and (2.9) by $\left(2(q+1) - \frac{n-2}{n\alpha} - \epsilon\right)$, respectively, and combine the two resulted expressions to get

$$\begin{aligned} & \left(2(q+1) - \frac{n-2}{n\alpha} - \epsilon\right) \\ & \quad \cdot \left(x\int_M |A|^{2(q+1)\alpha+2} f^2 + (\theta x + (1-x)\gamma)\int_M |A|^{2(q+1)\alpha} f^2\right) \\ & \le \left(2(q+1) - \frac{n-2}{n\alpha} - \epsilon\right)\left(1 + \frac{(q+1)\alpha}{\epsilon}\right)\int_M |A|^{2(q+1)\alpha}|\nabla f|^2 \\ & \quad + (q+1)(\epsilon + (q+1)\alpha)\int_M |A|^{2(q+1)\alpha+2} f^2 \\ & \quad + (q+1)(\epsilon + (q+1)\alpha)n\int_M |A|^{2(q+1)\alpha} f^2 \\ & \quad + \frac{(q+1)}{\alpha\epsilon}(\epsilon + (q+1)\alpha)\int_M |A|^{2(q+1)\alpha}|\nabla f|^2. \end{aligned} \tag{2.10}$$

Now we prove item i) of Theorem 1.1. Let

$$d := 2(q+1)\alpha. \tag{2.11}$$

Setting

$$\beta = n - \frac{(8 - n(d-2)^2)(n-1)^2}{4nd^2}, \tag{2.12}$$

we know from (1.5) and $\theta = \lambda_1 - n$ that there exists a constant $\rho > 0$ such that

$$\theta \ge \beta + \rho. \tag{2.13}$$

Since $d \in \left(2 - 2\sqrt{\frac{2}{n}},\ 2 + 2\sqrt{\frac{2}{n}}\right)$, we can find an $\epsilon > 0$ satisfying

$$\frac{(q+1)(\epsilon + (q+1)\alpha)}{2(q+1) - \frac{n-2}{n\alpha} - \epsilon} + \epsilon < 1 \tag{2.14}$$

and

$$\rho + \left(\frac{1}{\frac{(q+1)(\epsilon+(q+1)\alpha)}{2(q+1)-\frac{n-2}{n\alpha}-\epsilon} + \epsilon} - 1 - \frac{\frac{2}{n} - ((q+1)\alpha - 1)^2}{((q+1)\alpha)^2}\right)\gamma > 0. \tag{2.15}$$

Taking

$$x = \frac{(q+1)(\epsilon + (q+1)\alpha)}{2(q+1) - \frac{n-2}{n\alpha} - \epsilon} + \epsilon \tag{2.16}$$

in (2.10) and dividing by $\left(2(q+1) - \dfrac{n-2}{n\alpha} - \epsilon\right)$, one obtains that

$$\begin{aligned}&\epsilon \int_M |A|^{2(q+1)\alpha+2} f^2 + (\theta x + (1-x)\gamma - nx + n\epsilon) \int_M |A|^{2(q+1)\alpha} f^2 \\ &\leq C \int_M |A|^{2(q+1)\alpha} |\nabla f|^2,\end{aligned} \tag{2.17}$$

for some positive constant C depending only on n, q, α and ϵ. It follows from (2.13), (2.15) and (2.16) that

$$\begin{aligned}&\gamma + x(\theta - \gamma) - nx = x\left(\left(\frac{1}{x} - 1\right)\gamma + \theta - n\right) \\ &\geq x\left(\left(\frac{1}{\frac{(q+1)(\epsilon+(q+1)\alpha)}{2(q+1)-\frac{n-2}{n\alpha}-\epsilon} + \epsilon} - 1\right)\gamma - \frac{\left(\frac{2}{n} - ((q+1)\alpha - 1)^2\right)(n-1)^2}{4((q+1)\alpha)^2} + \rho\right) \\ &= x\left(\rho + \left(\frac{1}{\frac{(q+1)(\epsilon+(q+1)\alpha)}{2(q+1)-\frac{n-2}{n\alpha}-\epsilon} + \epsilon} - 1 - \frac{\frac{2}{n} - ((q+1)\alpha-1)^2}{((q+1)\alpha)^2}\right)\gamma\right) \geq 0.\end{aligned}$$

Thus, we can find an $\epsilon > 0$ and a positive constant C_2 depending only on n, q, α and ϵ such that

$$\int_M |A|^{d+2} f^2 \leq C_2 \int_M |A|^d |\nabla f|^2, \ \forall\ f \in C_0^\infty(M). \tag{2.18}$$

Since the first eigenvalue of the stability operator of a complete totally geodesic hypersurface is equal to

$$\frac{(n-1)^2}{4} + n,$$

Item i) in Theorem 1.1 makes sense if

$$\frac{(n-1)^2}{4} + n > 2n - \frac{(8 - n(d-2)^2)(n-1)^2}{4nd^2}. \tag{2.19}$$

So we know that (2.19) is valid if and only if

$$d \in \begin{cases} \left(0, \frac{1}{2}\right), & \text{for } n = 2, \\ \left(\frac{n-1}{n}, \frac{(n-1)(n-2)}{n}\right), & \text{for } n > 2 \end{cases} \tag{2.20}$$

If $n < 6$ and if d satisfies (2.20), it is easy to see that

$$d \in \left(2 - 2\sqrt{\frac{2}{n}}, 2 + 2\sqrt{\frac{2}{n}}\right).$$

If $n \geq 6$, we have

$$\left(2 - 2\sqrt{\frac{2}{n}}, 2 + 2\sqrt{\frac{2}{n}}\right) \subset \left(\frac{n-1}{n}, \frac{(n-1)(n-2)}{n}\right).$$

So the condition (1.12) implies that (2.14) and (2.19) are justified.

Let f be a smooth function on $[0,\infty)$ such that $f\geq 0$, $f=1$ on $[0,R]$ and $f=0$ in $[2R,+\infty)$ with $|f'|\leq \frac{2}{R}$. Considering $f\circ r$, where r is the distance function from the point p, we have from (2.18) that

$$\int_{B_p(R)}|A|^{d+2}\leq \frac{4C_2}{R^2}\int_{B_p(2R)}|A|^d.$$

Letting $R\longrightarrow\infty$ and using (1.11), we conclude $|A|=0$ in M, that is, M is totally geodesic.

As for item ii) of Theorem 1.1, we also set $d=2(q+1)\alpha$. Since $d\in\left(0,\frac{4}{17}\right)$, one can find an $\epsilon>0$ such that

$$0<\frac{(q+1)(\epsilon+(q+1)\alpha)}{2(q+1)-\epsilon}+\epsilon<\frac{1}{17}. \tag{2.21}$$

Taking $n=2$ and substituting

$$x=\frac{(q+1)(\epsilon+(q+1)\alpha)}{2(q+1)-\epsilon}+\epsilon$$

in (2.10) and dividing by $(2(q+1)-\epsilon)$ we get

$$\begin{aligned}&\epsilon\int_M|A|^{2(q+1)\alpha+2}f^2+(\theta x+(1-x)\gamma-2x+2\epsilon)\int_M|A|^{2(q+1)\alpha}f^2\\ &\leq C_1\int_M|A|^{2(q+1)\alpha}|\nabla f|^2,\end{aligned}$$

for some positive constant C_1 depending only on n, q, α and ϵ. Since M is stable, we have $\lambda_1\geq 0$. Using (2.21) and remembering that $\theta:=\lambda_1-2$ we have

$$\begin{aligned}\theta x+(1-x)\gamma-2x&=x\left(\left(\frac{1}{x}-1\right)\gamma+\theta-2\right)\\ &=x\left(\left(\frac{1}{x}-1\right)\gamma+\lambda_1-4\right)\\ &\geq x\left(\left(\frac{1}{x}-1\right)\gamma-4\right)\geq 0,\end{aligned}$$

Thus, we can find an $\epsilon>0$ and a positive constant C_2 depending only on q, α and ϵ such that

$$\int_M|A|^{d+2}f^2\leq C_2\int_M|A|^d|\nabla f|^2,\ \forall\ f\in C_0^\infty(M).$$

The remaining proof now follows exactly as in the final part of the proof of item i). $\square$

Proof of Theorem 1.2. Since M is a minimal submanifold of $\mathbf{H}^{n+p}$, we have the following Simons type inequality (cf. [36, 37]):

$$|A|\triangle|A|+\frac{3}{2}|A|^4+n|A|^2\geq\frac{2}{n}|\nabla|A||^2 \tag{2.22}$$

Let $\alpha>0$. Using the same calculations as in deriving (2.2) we obtain

$$|A|^\alpha|\triangle|A|^\alpha\geq\left(1-\frac{n-2}{\alpha n}\right)|\nabla|A|^\alpha|^2-\frac{3}{2}\alpha|A|^{2\alpha+2}-\alpha n|A|^{2\alpha}. \tag{2.23}$$

Let q be a non-negative constant and $f \in C_0^\infty(M)$. Multiplying (2.23) by $|A|^{2\alpha q} f^2$ and integrating on M, using integration by parts and Young's inequality we obtain

$$\begin{aligned}&\left(2(q+1) - \frac{n-2}{\alpha n} - \epsilon\right) \int_M |\nabla |A|^\alpha|^2 |A|^{2\alpha q} f^2 \\ &\le \frac{3}{2}\alpha \int_M |A|^{2(q+1)\alpha+2} f^2 + \alpha n \int_M |A|^{2(q+1)\alpha} f^2 + \frac{1}{\epsilon}\int_M |A|^{2(q+1)\alpha} |\nabla f|^2.\end{aligned} \tag{2.24}$$

Since (1.4) is true for minimal submanifolds in $\mathbf{H}^{n+p}$, we can use the inequality (2.9). Suppose that

$$2(q+1) - \frac{n-2}{\alpha n} - \epsilon > 0.$$

Multiplying (2.24) by $\frac{(q+1)}{\alpha}(\epsilon + (q+1)\alpha)$ and (2.9) by $\left(2(q+1) - \frac{n-2}{\alpha n} - \epsilon\right)$ and joining the two inequalities we get

$$\begin{aligned}&\left(2(q+1) - \frac{n-2}{n\alpha} - \epsilon\right)\left(x\int_M |A|^{2(q+1)\alpha+2} f^2 + (\theta x + (1-x)\gamma)\int_M |A|^{2(q+1)\alpha} f^2\right) \\ &\le \left(2(q+1) - \frac{n-2}{n\alpha} - \epsilon\right)\left(1 + \frac{(q+1)\alpha}{\epsilon}\right)\int_M |A|^{2(q+1)\alpha}|\nabla f|^2 \\ &\quad + \frac{3}{2}(q+1)(\epsilon + (q+1)\alpha)\int_M |A|^{2(q+1)\alpha+2} f^2 \\ &\quad + (q+1)(\epsilon + (q+1)\alpha) n \int_M |A|^{2(q+1)\alpha} f^2 \\ &\quad + \frac{(q+1)}{\alpha\epsilon}(\epsilon + (q+1)\alpha)\int_M |A|^{2(q+1)\alpha}|\nabla f|^2.\end{aligned} \tag{2.25}$$

We define again

$$d := 2(q+1)\alpha.$$

Now we consider two cases.

Case i): $n \neq 3$. Setting

$$\beta = \frac{2}{3}n - \frac{(3nd^2 - 8nd + 8(n-2))(n-1)^2}{12nd^2}, \quad \theta = \lambda_1 - n, \tag{2.26}$$

we know from (1.16) that there exists a constant $\rho > 0$ such that $\theta \ge \beta + \rho$. Since

$$d \in \left(\frac{4}{3} - \frac{2}{3}\sqrt{\frac{12}{n} - 2}, \frac{4}{3} + \frac{2}{3}\sqrt{\frac{12}{n} - 2}\right),$$

we can find an $\epsilon > 0$ satisfying

$$\frac{\frac{3}{2}(q+1)(\epsilon + (q+1)\alpha)}{2(q+1) - \frac{n-2}{n\alpha} - \epsilon} + \epsilon < 1, \tag{2.27}$$

and

$$\begin{aligned}&\left(\frac{1}{\frac{\frac{3}{2}(q+1)(\epsilon+(q+1)\alpha)}{2(q+1)-\frac{n-2}{n\alpha}-\epsilon} + \epsilon} - 1 + \frac{12n((q+1)\alpha)^2 - 16n(q+1)\alpha + 8(n-2)}{12n((q+1)\alpha)^2}\right)\gamma + \rho \\ &> 0.\end{aligned} \tag{2.28}$$

Supposing

$$2(q+1) - \frac{n-2}{n\alpha} - \epsilon > 0$$

and substituting $x = \dfrac{\frac{3}{2}(q+1)(\epsilon+(q+1)\alpha)}{2(q+1)-\frac{n-2}{n\alpha}-\epsilon}+\epsilon$ in (2.25) we have

$$\begin{aligned}&\epsilon\int_M |A|^{2(q+1)\alpha+2}f^2+\left(\theta x+(1-x)\gamma-\frac{2}{3}nx+\frac{2}{3}n\epsilon\right)\int_M |A|^{2(q+1)\alpha}f^2\\&\le C\int_M |A|^{2(q+1)\alpha}|\nabla f|^2,\end{aligned}\tag{2.29}$$

for some positive constant C depending only on n, q, α and ϵ. It follows from (2.26)–(2.28) that $\theta x+(1-x)\gamma-\frac{2}{3}nx\ge 0$. Thus, we can find an $\epsilon>0$ and a a positive constant C_2 depending only on n, q, α and ϵ such that (2.30) becomes

$$\int_M |A|^{d+2}f^2\le C_2\int_M |A|^d|\nabla f|^2,\ \forall\ f\in C_0^\infty(M).$$

Observe that the first eigenvalue of the super stability operator of a complete totally geodesic submanifold $\mathbf{H}^n$ in $\mathbf{H}^{n+p}$ is

$$\frac{(n-1)^2}{4}+n.$$

Therefore Theorem 1.2 makes sense if

$$\frac{(n-1)^2}{4}+n>\frac{5}{3}n+\frac{(3nd^2-8nd+8(n-2))(n-1)^2}{12nd^2}.\tag{2.30}$$

We can see that this is true if and only if

$$d\in\begin{cases}\left(0,\frac{1}{2}\right), & \text{for } n=2,\\ \left(\frac{n-1}{n},\frac{(n-1)(n-2)}{n}\right), & \text{for } n>2.\end{cases}\tag{2.31}$$

On the other hand, if $n=5$, we have

$$\left(\frac{4}{3}-\frac{2}{3}\sqrt{\frac{12}{n}-2},\frac{4}{3}+\frac{2}{3}\sqrt{\frac{12}{n}-2}\right)=\left(\frac{4}{3}-\frac{2}{3}\sqrt{\frac{2}{5}},\frac{4}{3}+\frac{2}{3}\sqrt{\frac{2}{5}}\right)$$

and

$$\left(\frac{4}{3}-\frac{2}{3}\sqrt{\frac{2}{5}},\frac{4}{3}+\frac{2}{3}\sqrt{\frac{2}{5}}\right)\subset\left(\frac{4}{5},\frac{12}{5}\right)=\left(\frac{n-1}{n},\frac{(n-1)(n-2)}{n}\right).$$

When $n=2$ or 4, if d satisfies (2.32), then

$$d\in\left(\frac{4}{3}-\frac{2}{3}\sqrt{\frac{12}{n}-2},\frac{4}{3}+\frac{2}{3}\sqrt{\frac{12}{n}-2}\right).$$

Therefore (2.27) and (2.30) hold. As in the proof of Theorem 1.1 we get

$$\int_M |A|^{d+2}f^2\le C_2\int_M |A|^d|\nabla f|^2,\ \forall\ f\in C_0^\infty(M),$$

and we can conclude that M is totally geodesic.

Case ii): $n=3$. Suppose that $\lambda_1(L_{|A|^2-3},M)>4$. Replacing $n=3$ in (1.16) and observing

$$4=\frac{5}{3}\cdot 3+\frac{\left(3\cdot 3\left(\frac{2}{3}\right)^2-8\cdot 3\cdot\frac{2}{3}+8(3-2)\right)(3-1)^2}{12\cdot 3\cdot\left(\frac{2}{3}\right)^2}$$

and

$$\frac{2}{3} \in \left(\frac{4}{3} - \frac{2}{3}\sqrt{2}, \frac{4}{3} + \frac{2}{3}\sqrt{2}\right),$$

we can repeat the above proof by taking $n = 3$ and $d = \frac{2}{3}$ to conclude that M is totally geodesic, which, implies that $\lambda_1(L_{|A|^2-3,}M) = 4$. This is a contradiction. Thus, we have $\lambda_1(L_{|A|^2-3,}M) \leq 4$. □

Proof of Theorem 1.3. Setting $k = d+1$, by hypothesis we have

$$\begin{aligned}\lambda_1(L_{|A|^2-n}, M) &> \frac{5}{3}n + \frac{(3n(k-1)^2 - 8n(k-1) + 8(n-2))(n-1)^2}{12n(k-1)^2}\\ &= \frac{5}{3}n + \frac{(3nd^2 - 8nd + 8(n-2))(n-1)^2}{12nd^2}.\end{aligned}$$

Case i): $n \neq 3$. As in the proof of the Theorem 1.2 we can obtain the following inequality

$$\int_M |A|^{d+2} f^2 \leq C \int_M |A|^d |\nabla f|^2,$$

where C is a positive constant. Plugging $f^{\frac{d+1}{2}}$ in the above expression and using Hölder's inequality we get

$$\begin{aligned}\int_M |A|^{d+2} f^{d+1} &\leq c_3 \int_M |A|^d f^{d-1} |\nabla f|^2\\ &\leq c_3 \left(\int_M |A|^{d+2} f^{d+1}\right)^{\frac{d-1}{d+1}} \left(\int_M |A| |\nabla f|^{d+1}\right)^{\frac{2}{d+1}}.\end{aligned} \tag{2.32}$$

Let f be a smooth function on $[0, \infty)$ such that $f \geq 0$, $f = 1$ on $[0, R]$ and $f = 0$ in $[2R, +\infty)$ with $|f'| \leq \frac{2}{R}$. Then considering $f \circ r$, where r is the distance from p (2.32) becomes

$$\left(\int_{B_p(R)} |A|^{d+2}\right)^{\frac{2}{d+1}} \leq c_4 \left(\frac{1}{R^{d+1}} \int_{B_p(R)} |A|\right)^{\frac{2}{d+1}}. \tag{2.33}$$

As $k = d+1$, we have by taking $R \longrightarrow \infty$ that $|A| = 0$ on M, that is, M is totally geodesic.

Case ii): $n = 3$. Suppose that $\lambda_1(L_{|A|^2-3,}M) > 4$. We can repeat the above arguments by taking $n = 3$ and $k = \frac{5}{3}$ to conclude that M is totally geodesic, which implies $\lambda_1(L_{|A|^2-3,}M) = 4$. This is a contradiction. Thus we have $\lambda_1(L_{|A|^2-3,}M) \leq 4$. □

Proof of Theorem 1.4. Suppose that

$$\lambda_1(L_{|A|^2-n}) > \frac{5}{3}n + \frac{(3nd^2 - 8nd + 8(n-2))(n-1)^2}{12nd^2}. \tag{2.34}$$

Let q be a non-negative constant and $f \in C_0^\infty(M)$. Multiplying (2.22) by $(1+q)f^2|A|^{2q}$, integrating the resulted equation on M and using the divergence theorem we obtain

$$\begin{aligned}\left(\frac{2}{n} + 2q + 1\right) \int_M |\nabla|A||^2 |A|^{2q} f^2 &\leq \frac{3}{2} \int_M |A|^{2q+4} f^2 + n \int_M |A|^{2q+2} f^2\\ &\quad - 2 \int_M |A|^{2q+1} f \langle \nabla f, \nabla |A| \rangle.\end{aligned} \tag{2.35}$$

Again from the equations (1.2) and (1.4), we can find an $x \in [0,1]$ such that

$$x \int_M |A|^2 f^2 + (\theta x + (1-x)\gamma) \int_M f^2 \leq \int_M |\nabla f|^2,$$

where $\theta = \lambda_1 - n$. Plugging $f|A|^{q+1}$ in the above equation we have

$$\begin{aligned} & x \int_M |A|^{2q+4} f^2 + (\theta x + (1-x)\gamma) \int_M |A|^{2q+2} f^2 \\ & \leq \int_M |A|^{2q+2} |\nabla f|^2 + (1+q)^2 \int_M |A|^{2q} f^2 |\nabla |A||^2 \\ & \quad + 2(q+1) \int_M |A|^{2q+1} f \langle \nabla f, \nabla |A| \rangle. \end{aligned} \tag{2.36}$$

Multiplying (2.35) by $(q+1)$ and summing with (2.36) we get

$$\begin{aligned} & (1+q)\left(\frac{2}{n}+q\right) \int_M |\nabla |A||^2 |A|^{2q} f^2 + x \int_M |A|^{2q+4} f^2 \\ & + (\theta x + (1-x)\gamma) \int_M |A|^{2q+2} f^2 \\ & \leq \int_M |A|^{2q+2} |\nabla f|^2 + \frac{3}{2}(1+q) \int_M |A|^{2q+4} f^2 + n(q+1) \int_M |A|^{2q+2} f^2. \end{aligned} \tag{2.37}$$

Multiplying (2.36) by $\frac{\frac{2}{n}+q}{1+q+\epsilon}$ and summing with (2.37) we obtain by using Young's inequality that

$$\begin{aligned} & \left(1 + \frac{\frac{2}{n}+q}{1+q+\epsilon}\right)\left(x \int_M |A|^{2q+4} f^2 + (\theta x + (1-x)\gamma) \int_M |A|^{2q+2} f^2\right) \\ & \leq \left(1 + \frac{\frac{2}{n}+q}{\epsilon}\right) \int_M |A|^{2q+2} |\nabla f|^2 + \frac{3}{2}(1+q) \int_M |A|^{2q+4} f^2 \\ & \quad + n(q+1) \int_M |A|^{2q+2} f^2. \end{aligned} \tag{2.38}$$

Setting

$$\beta = \frac{2}{3} n + \frac{(3nq^2 + 2nq + n - 4)(n-1)^2}{12n(q+1)^2}, \tag{2.39}$$

by (2.34) and $\theta = \lambda_1 - n$, we know that exists a constant $\rho > 0$ such that

$$\theta \geq \beta + \rho.$$

Setting $d := 2q+2$, we have by hypothesis that $q < \frac{1}{3}\left(\sqrt{\frac{12}{n} - 2} - 1\right)$. Since $n = 2$ or 3, we can then find an $\epsilon > 0$ satisfying

$$\frac{\frac{3}{2}(q+1)(q+1+\epsilon)}{\frac{2}{n} + 2q + 1 + \epsilon} + \epsilon < 1 \tag{2.40}$$

and

$$\rho + \left(\frac{1}{\frac{\frac{3}{2}(q+1)(q+1+\epsilon)}{\frac{2}{n}+2q+1+\epsilon} + \epsilon} - 1 + \frac{\frac{3}{2}q^2 + \frac{1}{2} + q - \frac{2}{n}}{\frac{3}{2}(q+1)^2}\right)\gamma > 0. \tag{2.41}$$

Thus, dividing by $\left(1+\frac{\frac{2}{n}+q}{1+q+\epsilon}\right)$ and substituting

$$x=\frac{\frac{3}{2}(q+1)(q+1+\epsilon)}{\frac{2}{n}+2q+1+\epsilon}+\epsilon \tag{2.42}$$

in (2.38), we get

$$\begin{aligned}&\epsilon\int_M |A|^{2q+4}f^2+\left(\theta x+(1-x)\gamma-\frac{2}{3}nx+\frac{2}{3}n\epsilon\right)\int_M |A|^{2q+2}f^2\\ &\le C\int_M |A|^{2(q+1)\alpha}|\nabla f|^2,\end{aligned} \tag{2.43}$$

for some positive constant C depending only on n, q and ϵ. It follows from the definition of θ and (2.40)–(2.42) that $\theta x+(1-x)\gamma-\frac{2}{3}nx>0$. Consequently, we can find an $\epsilon>0$ and a positive constant C_2 depending only on n, q and ϵ such that

$$\int_M |A|^{d+2}f^2\le C_2\int_M |A|^d|\nabla f|^2,\ \forall\ f\in C_0^\infty(M),$$

where $d:=2q+2$. Proceeding as in the proof of Theorem 1.1, we can conclude that M is totally geodesic. Therefore if, $n=2$ then $\lambda_1(L_{|A|^2-2},M)=\frac{9}{4}$, and if $n=3$ then $\lambda_1(L_{|A|^2-3},M)=4$, which together with (2.34), gives us a contradiction. □

Proof of Theorem 1.5. Suppose that

$$\lambda_1(L_{|A|^2-n})>\frac{5}{3}n+\frac{(3n(k-1)^2-8n(k-1)+8(n-2)(n-1)^2}{12n(k-1)^2}. \tag{2.44}$$

By using the same arguments as in the proof of Theorem 1.4 we obtain again

$$\int_M |A|^{d+2}f^2\le c\int_M |A|^d|\nabla f|^2.$$

Plugging $f^{\frac{d+1}{2}}$ in the above inequality one gets

$$\left(\int_{B_p(R)}|A|^{d+2}\right)^{\frac{2}{d+1}}\le c_5\left(\frac{1}{R^{d+1}}\int_{B_p(R)}|A|\right)^{\frac{2}{d+1}}. \tag{2.45}$$

Since $k=d+1$, making $R\to+\infty$ we conclude that $|A|=0$ on M, that is, M totally geodesic. Therefore, we have that if $n=2$ then $\lambda_1(L_{|A|^2-2},M)=\frac{9}{4}$, and if $n=3$ then $\lambda_1(L_{|A|^2-3},M)=4$, which together with (2.44), gives us a contradiction. □

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