Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 42, 2017, 755–770

SOME METRIC PROPERTIES OF THE TEICHMÜLLER SPACE OF A CLOSED SET IN THE RIEMANN SPHERE

Nishan Chatterjee

Graduate Center of the City University of New York, Department of Mathematics
365 Fifth Avenue, New York, NY 10016, U.S.A.; nchatterjee@gradcenter.cuny.edu

Abstract. Let E be an infinite closed set in the Riemann sphere, and let $T(E)$ denote its Teichmüller space. In this paper, we study some metric properties of $T(E)$. We prove Earle's form of Teichmüller contraction for $T(E)$, holomorphic isometries from the open unit disk into $T(E)$, extend Earle's form of Schwarz's lemma for classical Teichmüller spaces to $T(E)$, and finally study complex geodesics and unique extremality for $T(E)$.

Introduction

Let $\mathbf{C}$ denote the complex plane, $\Delta := \{z \in \mathbf{C} \colon |z| < 1\}$ denote the open unit disk and $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ denote the Riemann sphere. Throughout this paper, we will assume that E is a closed set in $\widehat{\mathbf{C}}$ and that 0, 1, and ∞ belong to E. The Teichmüller space of E, denoted by $T(E)$, was first studied by Lieb in his 1990 Cornell University dissertation [14], written under the direction of Earle. It has several applications in holomorphic motions, geometric function theory, and holomorphic families of Möbius groups; see the papers [7, 12, 15, 18]. In this paper, we study some metric properties of $T(E)$. Our paper is arranged as follows. In §1, we give the relevant definitions and also state various properties of $T(E)$ that will be necessary in our paper. In §2, we state the main theorems of our paper and also the motivations for these results. In §§3–6, we give the proofs of our main theorems.

1. Teichmüller space of a closed set in $\widehat{\mathbf{C}}$

We call a homeomorphism of $\widehat{\mathbf{C}}$ *normalized* if it fixes the points 0, 1, and ∞. Let $M(\mathbf{C})$ denote the open unit ball of the complex Banach space $L^\infty(\mathbf{C})$. For each μ in $M(\mathbf{C})$, there exists a unique normalized quasiconformal homeomorphism of $\widehat{\mathbf{C}}$ onto itself that has Beltrami coefficient μ, denoted by w^μ.

Definition 1.1. The normalized quasiconformal self-mappings f and g of $\widehat{\mathbf{C}}$ are said to be E-equivalent if and only if $f^{-1} \circ g$ is isotopic to the identity rel E. The *Teichmüller space* $T(E)$ is the set of all E-equivalence classes of normalized quasiconformal self-mappings of $\widehat{\mathbf{C}}$. The *basepoint* of $T(E)$ is the E-equivalence class of the identity map.

We define the projection

$$P_E \colon M(\mathbf{C}) \to T(E)$$

https://doi.org/10.5186/aasfm.2017.4247
2010 Mathematics Subject Classification: Primary 32G15, 30F60; Secondary 32H02.
Key words: Teichmüller space of a closed set, Teichmüller contraction, holomorphic isometries, Schwarz's lemma, complex geodesics.

by setting $P_E(\mu)$ equal to the E-equivalence class of w^μ, written as $[w^\mu]_E$. Clearly, P_E maps the basepoint of $M(\mathbf{C})$ to the basepoint of $T(E)$. (We will use the same notation 0 for the basepoints in $M(\mathbf{C})$ and $T(E)$.)

In his doctoral dissertation [14], Lieb proved that $T(E)$ is a complex Banach manifold such that the projection map P_E is a holomorphic split submersion. For more details, see [7].

The tangent space at the basepoint. Let $A(E)$ be the closed subspace of $L^1(\mathbf{C})$ consisting of the functions f in $L^1(\mathbf{C})$ whose restriction to E^c is holomorphic. We identify $L^\infty(\mathbf{C})$ with the dual space of $L^1(\mathbf{C})$ in the usual way. Set

$$A(E)^\perp = \{\mu \in L^\infty(\mathbf{C})\colon \ell_\mu(f) = \iint_{\mathbf{C}} \mu(z) f(z)\, dx\, dy = 0 \text{ for all } f \text{ in } A(E)\}.$$

Proposition 1.2. (Teichmüller's lemma for $T(E)$) $\ker(P_E'(0)) = A(E)^\perp$.

See Proposition 7.18 in [7].

Corollary 1.3. *The tangent space to $T(E)$ at its basepoint is naturally isomorphic to $A(E)^*$, the dual space of $A(E)$.*

The natural isomorphism sends the tangent vector $P_E'(0)\mu$ to the linear functional $f \mapsto \ell_\mu(f)$ on $A(E)$.

Changing the basepoint. Let h be a normalized quasiconformal self-mapping of $\widehat{\mathbf{C}}$, and let $\widetilde{E} = h(E)$. By definition, the *allowable map* h^* from $T(\widetilde{E})$ to $T(E)$ maps the $\widetilde{E}$-equivalence class of g to the E-equivalence class of $g \circ h$ for every normalized quasiconformal self-mapping g of $\widehat{\mathbf{C}}$.

Proposition 1.4. *The allowable map $h^*\colon T(\widetilde{E}) \to T(E)$ is biholomorphic. If μ is the Beltrami coefficient of h, then h^* maps the basepoint of $T(\widetilde{E})$ to the point $P_E(\mu)$ in $T(E)$.*

Forgetful maps. If E is a subset of the closed set $\widetilde{E}$ and μ is in $M(\mathbf{C})$, then the $\widetilde{E}$-equivalence class of w^μ is contained in the E-equivalence class of w^μ. Therefore, there is a well-defined *forgetful map* $p_{\widetilde{E},E}$ from $T(\widetilde{E})$ to $T(E)$ such that $P_E = p_{\widetilde{E},E} \circ P_{\widetilde{E}}$.

Proposition 1.5. *The forgetful map $p_{\widetilde{E},E}$ is a basepoint preserving holomorphic split submersion.*

Proof. Since $P_E = p_{\widetilde{E},E} \circ P_{\widetilde{E}}$ and P_E and $P_{\widetilde{E}}$ are holomorphic split submersions, so is $p_{\widetilde{E},E}$. □

The following proposition will be very useful in our paper.

Proposition 1.6. *Let f be any holomorphic map of Δ into $T(E)$ and let μ be any point in $M(\mathbf{C})$ such that $P_E(\mu) = f(0)$. There is a holomorphic map $\widehat{f}$ from Δ to $M(\mathbf{C})$ such that $\widehat{f}(0) = \mu$ and $P_E \circ \widehat{f} = f$.*

For proofs see Proposition 7.27 in [7] or Proposition 5.1 in [17]. This is an easy consequence of the "universal" property of $T(E)$ (see [15]) and Slodkowski's theorem on extensions of holomorphic motions (see [21]).

The Kobayashi and Teichmüller metrics on $T(E)$.

Proposition 1.7. *The Kobayashi metric on $M(\mathbf{C})$ is given by*

$$\rho_M(\mu, \nu) = \tanh^{-1} \left\| \frac{(\mu - \nu)}{(1 - \bar{\mu}\nu)} \right\|_\infty$$

for all μ, ν in $M(\mathbf{C})$. The infinitesimal Kobayashi metric on $M(\mathbf{C})$ is given by

$$K_M(\mu,\lambda) = \left\| \frac{\lambda}{(1-|\mu|^2)} \right\|_\infty$$

for μ in $M(\mathbf{C})$ and λ in $L^\infty(\mathbf{C})$.

See Proposition 7.25 in [7].

By definition, the Teichmüller metric $d_{T(E)}$ on $T(E)$ is given by

$$d_{T(E)}(P_E(\mu), t) = \inf\{\rho_M(\mu,\nu)\colon \nu \in M(\mathbf{C}) \text{ and } P_E(\nu) = t\}$$

for all μ in $M(\mathbf{C})$ and t in $T(E)$.

The infinitesimal Teichmüller metric $F_{T(E)}$ is defined on the tangent bundle of $T(E)$ by the formula

$$F_{T(E)}(P_E(\mu), v) = \inf\{K_M(\mu,\lambda)\colon \lambda \in L^\infty(\mathbf{C}) \text{ and } P'_E(\mu)\lambda = v\},$$

for any μ in $M(\mathbf{C})$ and tangent vector v to $T(E)$ at the point $P_E(\mu)$.

Proposition 1.8. *The Teichmüller and Kobayashi metrics on $T(E)$ are equal, and the infinitesimal Teichmüller and Kobayashi metrics are also equal.*

See Proposition 7.30 in [7].

Definition 1.9. A map $f\colon \Delta \to T(E)$ is called a *holomorphic isometry* if f is holomorphic and for any pair t, t' in Δ, $d_{T(E)}(f(t), f(t')) = \rho_\Delta(t,t')$.

Recall that the Poincaré metric on Δ is given by

$$\rho_\Delta(z,w) = \tanh^{-1}\left|\frac{z-w}{1-\bar{z}w}\right|$$

for all z and w in Δ.

Definition 1.10. A Beltrami coefficient μ in $M(\mathbf{C})$ is called *extremal* in its E-equivalence class, if $P_E(\mu) = P_E(\nu)$ and $\|\mu\|_\infty \le \|\nu\|_\infty$. Equivalently, μ in $M(\mathbf{C})$ is extremal in its E-equivalence class if $d_{T(E)}(0_T, P_E(\mu)) = \rho_M(0,\mu)$.

We defined a natural isomorphism mapping the tangent space to $T(E)$ at its basepoint onto a Banach space $A(E)^*$. That isomorphism is an isometry with respect to the infinitesimal Teichmüller metric on the tangent space and the usual norm on $A(E)^*$. Throughout this paper we will denote this infinitesimal Teichmüller norm by ℓ_μ; so ℓ_μ is the norm of the linear functional

$$\ell_\mu(\phi) = \iint_{\mathbf{C}} \mu\phi \, dx\, dy \quad \text{on } A(E).$$

Henceforth, we will denote this by

$$\|\ell_\mu\|_{T(E)} = \sup_{\|\phi\|=1} \left| \iint_{\mathbf{C}} \mu\phi\, dx\, dy \right|, \quad \phi \in A(E).$$

It is clear that $\|\ell_\mu\|_{T(E)} \le \|\mu\|_\infty$ for μ in $L^\infty(\mathbf{C})$.

Definition 1.11. A Beltrami coefficient μ is *infinitesimally extremal* in its E-equivalence class, if $\|\ell_\mu\|_{T(E)} = \|\mu\|_\infty$.

The following proposition is obvious.

Proposition 1.12. *If E is a subset of $\widetilde{E}$ and $p_{\widetilde{E},E}\colon T(\widetilde{E}) \to T(E)$ is the forgetful map, then*

$$d_{T(E)}(p_{\widetilde{E},E}(s), p_{\widetilde{E},E}(t)) \le d_{T(\widetilde{E})}(s,t)$$

for all s and t in $T(\widetilde{E})$.

When E is finite. Let E be a finite set (as usual, 0, 1, and ∞ belong to E). Its complement $E^c = \Omega$ is the Riemann sphere with punctures at the points of E. Since $T(E)$ and the classical Teichmüller space $Teich(\Omega)$ are quotients of $M(\mathbf{C})$ by the same equivalence relation, $T(E)$ can be naturally identified with $Teich(\Omega)$. It is given in Example 3.1 in [15]. For the reader's convenience, we include this discussion. Let $\theta\colon T(E) \to Teich(\Omega)$ be the map defined by setting $\theta(P_E(\mu))$ equal to the Teichmüller class of the restriction of w^μ to Ω (where, as usual, μ is in $M(\mathbf{C})$). It is clear that θ is a well-defined map of $T(E)$ into $Teich(\Omega)$. We claim that θ is injective. For, suppose that the restrictions of w^μ and w^ν to Ω are in the same Teichmüller class. Then, there is a conformal map h of $w^\mu(\Omega)$ onto $w^\nu(\Omega)$ such that $(w^\nu)^{-1} \circ h \circ w^\mu$ is isotopic to the identity rel E. This map h is the identity, for it is obviously a Möbius transformation and it fixes 0, 1, and ∞ because w^μ and w^ν are normalized. Therefore, w^μ and w^ν are E-equivalent, and so θ is injective. Also, θ is surjective, since the restriction map $\mu \mapsto \mu|\Omega$ from $M(\mathbf{C})$ to $M(\Omega)$ is bijective and $\theta(P_E(\mu)) = \Phi(\mu|\Omega)$ for all μ in $M(\mathbf{C})$, where $\Phi\colon M(\Omega) \to Teich(\Omega)$ is the standard projection. This also shows that θ is biholomorphic, since P_E and Φ induce the complex structures of $T(E)$ and $Teich(\Omega)$. Under this indentification $d_{T(E)}$ becomes the (classical) Teichmüller metric for $Teich(\Omega)$. Furthermore, the norm of ℓ_μ is simply the norm of the linear functional that μ induces on the Banach space of integrable holomorphic functions on Ω. For standard facts on classical Teichmüller spaces, the reader is refereed to the books [9, 11, 19].

We need the following form of Teichmüller contraction for $T(E)$ when E is a finite set. (Recall that when E is finite, $T(E)$ is naturally identified with the classical Teichmüller space $Teich(\widehat{\mathbf{C}} \setminus E)$.)

Theorem 1.13. *Let $\mu \in M(\mathbf{C})$, and $P_E(\mu) = \tau$ in $T(E)$. Let μ_0 be an extremal in $P_E(\mu)$. Set $k_0 = \|\mu_0\|_\infty$, $k = \|\mu\|_\infty$, $K_0 = (1+k_0)/(1-k_0)$, and $K = (1+k)/(1-k)$. Then*

$$\frac{1}{K_0} - \frac{1}{K} \leq \frac{2}{1-k^2}\Big(k - \|\ell\|_{T(E)}\Big) \leq K - K_0.$$

See Theorem 2 in [5].

Remark 1.14. Earle proved this result for $Teich(X)$ where X is any hyperbolic Riemann surface. He used the Reich–Strebel inequalities to obtain his result. We need the special case when $X = \widehat{\mathbf{C}} \setminus E$ and $E = \{0, 1, \infty, \zeta_1, \cdots, \zeta_n\}$, $n \geq 1$.

Approximations by finite subsets. Let E be infinite and let $E_1, E_2, \cdots, E_n, \cdots$ be a sequence of finite subsets of E such that $\{0, 1, \infty\} \subset E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$ and $\bigcup_{n=1}^\infty E_n$ is dense in E. Let 0 be the basepoint of $T(E)$, and for each $n \geq 1$, let π_n be the forgetful map p_{E,E_n} from $T(E)$ to $T(E_n)$. For any τ in $T(E)$ and $n \geq 1$ let $\tau_n = \pi_n(\tau)$. In particular, $0_n = \pi_n(0)$ is the basepoint of $T(E_n)$ for all $n \geq 1$. By Proposition 1.12, we have

$$d_{T(E_n)}(0_n, \tau_n) \leq d_{T(E_{n+1})}(0_{n+1}, \tau_{n+1}) \leq d_{T(E)}(0, \tau)$$

for all τ in $T(E)$ and $n \geq 1$.

The following two facts will be important in our paper. For proofs, see [15] and [16].

Proposition 1.15. *For each τ in $T(E)$ the increasing sequence $\{d_{T(E_n)}(0_n, \tau_n)\}$ converges to $d_{T(E)}(0, \tau)$.*

Proposition 1.16. *Let the infinite closed set E and the finite subsets E_n, $n \geq 1$, be as above, and let μ belong to $L^\infty(\mathbf{C})$. The sequence $\{\|\ell_\mu\|_{T(E_n)}\}$ is increasing and converges to $\|\ell_\mu\|_{T(E)}$.*

We will also need the following theorem. This appears in Earle's paper [5].

Theorem 1.17. *Let V be a complex Banach space and $g\colon \Delta \to V$ be a holomorphic map with $g(0) = 0$ and $\|g(t)\| \leq 1$, $\forall t \in \Delta$. Fix $t \in \Delta \setminus \{0\}$. If either of the inequalities $\|g'(0)\| \leq 1$ or $\|g(t)\| \leq |t|$ is strict, then both are strict and*

$$\rho_\Delta\left(\frac{\|g(t)\|}{|t|}, \|g'(0)\|\right) \leq \rho_\Delta(0, t).$$

2. Statements of the main results

For classical Teichmüller spaces, the principle of Teichmüller contraction was proved in [8]. A sharp form of Teichmüller contraction was proved by Earle in [5]. Gardiner's result was extended to the generalized Teichmüller space $T(E)$ in [16], which proved a $\delta - \epsilon$ form of Teichmüller contraction. Our first result extends Earle's form of Teichmüller contraction to $T(E)$; this sharpens and improves the $\delta - \epsilon$ inequalities in [16].

Theorem I. *Let $\mu \in M(\mathbf{C})$, and $P_E(\mu) = \tau$ in $T(E)$. Let μ_0 be an extremal in the E-equivalence of μ. Set $k_0 = \|\mu_0\|_\infty$, $k = \|\mu\|_\infty$, $K_0 = (1 + k_0)/(1 - k_0)$, and $K = (1 + k)/(1 - k)$. Then*

$$\frac{1}{K_0} - \frac{1}{K} \leq \frac{2}{1 - k^2}\Big(k - \|\ell\|_{T(E)}\Big) \leq K - K_0.$$

Our next result is on holomorphic isometries from Δ into $T(E)$. This extends Theorem 5 in [6] to $T(E)$.

Theorem II. *Let $f\colon \Delta \to T(E)$ be holomorphic and let $t_1 \in \Delta$. Suppose either that*

(1) *$d_{T(E)}(f(t_1), f(t_2)) = \rho_\Delta(t_1, t_2)$ for some $t_2 \in \Delta \setminus \{t_1\}$,*

or

(2) *$F_{T(E)}(f(t_1),\, f'(t_1)) = \dfrac{1}{1 - |t_1|^2}$, then f is a holomorphic isometry.*

In [5], Earle proved a form of Schwarz's lemma for classical Teichmüller spaces. Our next result extends that theorem to $T(E)$. Let $f\colon \Delta \to T(E)$ be a basepoint preserving holomorphic map and set

$$k_0(t) = \|\mu\|_\infty \quad \text{if } t \in \Delta,\ f(t) = P_E(\mu), \text{ and } \mu \text{ is extremal}\ .$$

Theorem III. *Let $f\colon \Delta \to T(E)$ be a basepoint preserving holomorphic map. Fix any t in $\Delta \setminus \{0\}$. If either of the inequalities $\|f'(0)\|_{T(E)} \leq 1$ or $k_0(t) \leq |t|$ is strict, then both are strict and*

$$\rho_\Delta\left(\frac{k_0(t)}{|t|}, \|f'(0)\|_{T(E)}\right) \leq 2\rho_\Delta(0, t).$$

Our final theorem is on complex geodesics and unique extremality in $T(E)$. It extends Theorem 6 in [6] to $T(E)$. We first need two definitions.

Definition 2.1. A geodesic segment J in $T(E)$ is the image of an injective continuous map $\alpha\colon [0,1]\to T(E)$ such that

$$d_{T(E)}(\alpha(x_0),\alpha(x_2)) = d_{T(E)}(\alpha(x_0),\alpha(x_1)) + d_{T(E)}(\alpha(x_1),\alpha(x_2))$$

whenever $0\le x_1\le x_2\le x_3\le 1$. The points $f(0)$ and $f(1)$ are called the endpoints of J. We say that the geodesic segment J *joins* the points τ_1 and τ_2 in $T(E)$ if they are the endpoints of J.

Definition 2.2. A Beltrami coefficient μ in $M(\mathbf{C})$ is called *uniquely extremal* if $P_E(\nu)\ne P_E(\mu)$ whenever $\nu\in M(\mathbf{C})$, $\nu\ne\mu$, and $\|\nu\|_\infty\le\|\mu\|_\infty$.

It is obvious that every "uniquely extremal" μ is extremal.

Theorem IV. *Let $\mu_0\in M(\mathbf{C})$, $\mu_0\ne 0$ and μ_0 be extremal in its E-equivalence class. Then the following four statements are equivalent:*

(1) *The Beltrami coefficient μ_0 is uniquely extremal and $|\mu_0| = \|\mu_0\|_\infty$ a.e.*
(2) *There exists only one geodesic segment joining $P_E(0)$ and $P_E(\mu_0)$.*
(3) *There exists only one holomorphic isometry $f\colon\Delta\to T(E)$ such that $f(0) = P_E(0)$ and $f(\|\mu_0\|_\infty) = P_E(\mu_0)$.*
(4) *There exists only one holomorphic map $g\colon\Delta\to M(\mathbf{C})$ such that $g(0) = 0$ and $P_E(g(\|\mu_0\|_\infty)) = P_E(\mu_0)$.*

Remark 2.3. Recall from §1 that when E is finite, $T(E)$ is naturally identified with the classical Teichmüller space $Teich(\widehat{\mathbf{C}}\setminus E)$, and so, $T(E)$ is finite-dimensional. In that case all the above theorems are well-known; see [5], §7 and §8 in [6], and also §9.3 and §9.5 in [13]. Therefore, for the rest of our paper, the blanket assumption will be that E is an infinite closed set and that 0, 1, and ∞ belong to E.

3. Proof of Theorem I

Let $\tau\in T(E)$, $P_E(\mu=\tau$, and μ_0 be extremal in the E-equivalence class of μ. So we have $P_E(\mu) = P_E(\mu_0)$ and $\|\mu_0\|_\infty\le\|\mu\|_\infty$. Let $k = \|\mu\|_\infty$ and $k_0 = \|\mu_0\|_\infty$. Also, let

$$K = \frac{1+k}{1-k} \quad\text{and}\quad K_0 = \frac{1+k_0}{1-k_0}.$$

We follow the construction given immediately after Theorem 1.13 (in §1). Let $\tau_n = \pi_n(\tau)$, and let $\mu_0(n)$ be extremal in its E_n-equivalence class. Let $k_0(n) = \|\mu_0(n)\|_\infty$ and let

$$K_0(n) = \frac{1+k_0(n)}{1-k_0(n)}.$$

Since $T(E_n)$ is identified with the classical Teichmüller space $Teich(\widehat{\mathbf{C}}\setminus E_n)$, by Theorem 1.13, the following is true for all n:

$$\frac{1}{K_0(n)} - \frac{1}{K} \le \frac{2}{1-k^2}\Big(k - \|\ell_\mu\|_{T(E_n)}\Big) \le K - K_0(n). \tag{3.1}$$

Since $\mu_0(n)$ is extremal in its E_n-equivalence class, we have $d_{T(E_n)}(0_n,\tau_n) = \|\mu_0(n)\|_\infty = k_0(n)$. Also, since μ_0 is extremal in its E-equivalence class, we have $d_{T(E)}(0,\tau) = \|\mu_0\|_\infty = k_0$. By Propositions 1.14 and 1.15, we have

$$\lim_{n\to\infty} K_0(n) = K_0 \quad\text{and}\quad \lim_{n\to\infty}\|\ell_\mu\|_{T(E_n)} = \|\ell_\mu\|_{T(E)}.$$

Taking limits in Equation (3.1), we obtain

$$\frac{1}{K_0} - \frac{1}{K} \le \frac{2}{1-k^2}\Big(k - \|\ell_\mu\|_{T(E)}\Big) \le K - K_0. \tag{3.2}$$

□

The following two corollaries will be very useful in our paper.

Corollary 3.1. (Hamilton–Krushkal–Reich–Strebel extremality condition for $T(E)$) *A Beltrami coefficient μ is extremal in its E-equivalence class if and only if it is infinitesimally extremal in its E-equivalence class.*

The proof is obvious.

We follow the same notations as in Theorem I. Let $P_E(\mu) = \tau$, and let μ_0 be extremal in its E-equivalence class. Let $k = \|\mu\|_\infty$ and $k_0 = \|\mu_0\|_\infty$.

Corollary 3.2. *If either k_0 or $\|\ell_\mu\|_{T(E)}$ is less than k then both are less than k. Moreover,*

$$\rho_\Delta\left(\frac{k_0}{k}, \frac{\|\ell_\mu\|_{T(E)}}{k}\right) \le \rho_\Delta(0,k). \tag{3.3}$$

The proof is straightforward. See, for example, the proof of Corollary 1 in [5].

4. Proof of Theorem II

Let $\mathcal{L}\colon L^\infty(\mathbf{C}) \to A(E)^*$ be the linear map that takes μ in $L^\infty(\mathbf{C})$ to the functional ℓ_μ defined as

$$\mathcal{L}(\mu)(\phi) = \ell_\mu(\phi) = \iint_{\mathbf{C}} \mu\phi\, dx\, dy, \quad \text{for } \phi \in A(E).$$

By Proposition 1.4, we can assume, without loss of generality that $t_1 = 0$ and that $f(0) = 0$. We use the same notation 0 for the basepoints in Δ, $M(\mathbf{C})$, and $T(E)$. By Proposition 1.6, there exists a holomorphic map $\widehat{f}\colon \Delta \to M(\mathbf{C})$ such that $\widehat{f}(0) = 0$ and $P_E \circ \widehat{f} = f$.

Let us assume there is $t_2 \in \Delta \setminus \{0\}$ such that $d_{T(E)}(0, f(t_2)) = \rho_\Delta(0, t_2)$. We have

$$\rho_\Delta(0,t_2) = d_{T(E)}(0, f(t_2)) \le \rho_M(0, \widehat{f}(t_2)) \le \rho_\Delta(0,t_2)$$

where $d_{T(E)}(0, f(t_2)) \le \rho_M(0, \widehat{f}(t_2))$ because $P_E\colon M(\mathbf{C}) \to T(E)$ is holomorphic, $P_E(0) = 0$ and $P_E(\widehat{f}(t_2)) = f(t_2)$; and also, $\rho_M(0, \widehat{f}(t_2)) \le \rho_\Delta(0, t_2)$ because $\widehat{f}\colon \Delta \to M(\mathbf{C})$ is holomorphic and $\widehat{f}(0) = 0$. From the above inequality, we get

$$d_{T(E)}(0, f(t_2)) = \rho_M(0, \widehat{f}(t_2)) = \rho_\Delta(0, t_2).$$

Hence $\widehat{f}(t_2)$ is extremal and $\|\widehat{f}(t_2)\|_\infty = |t_2|$.

Let $g\colon \Delta \to A(E)^*$ be defined as $g = \mathcal{L} \circ \widehat{f}$; then $\|g(t)\| \le \|\widehat{f}(t)\|_\infty < 1$, for all t in Δ. For all $\mu \in L^\infty(\mathbf{C})$, we have $\|\ell_\mu\| \le \|\mu\|_\infty$. We also have $g(0) = 0$ since $\widehat{f}(0) = 0$ and $\ell_0 = 0$. So we can apply Schwarz's Lemma to both g and $\widehat{f}$, and since $\widehat{f}(t_2)$ is extremal, it will be infinitesimally extremal by Corollary 3.1. Hence we have

$$\|g(t_2)\| = \|\ell_{\widehat{f}(t_2)}\| = \|\widehat{f}(t_2)\|_\infty = |t_2|.$$

This is the case of equality in Schwarz's lemma, and hence we get

$$\|g'(0)\| = \|\widehat{f}'(0)\|_\infty = 1.$$

From the definition of $\mathcal{L}$ we see that $\mathcal{L}(\mu) = 0$ if and only if $P'_E(0)\mu = 0$. Using chain rule we obtain

$$\|\ell_\mu\| = \inf\{\|\nu\|_\infty\colon \ell_\mu = \ell_\nu\} = \inf\{\|\nu\|_\infty\colon {P_E}'(0)\mu = {P_E}'(0)\nu\}.$$

Hence we get

$$\|\ell_{\widehat{f}'(0)}\| = \inf\{\|\nu\|_\infty \colon P'_E(0)\nu = P'_E(0)\widehat{f}'(0)\} = \inf\{\|\nu\|_\infty \colon P'_E(0)\nu = f'(0))\},$$

which gives

$$1 = \|\widehat{f}'(0)\|_\infty = \|\ell_{\widehat{f}'(0)}\| = F_{T(E)}(0, f'(0)).$$

Since we assumed $t_1 = 0$ and $f(0) = 0$, we obtain

$$F_{T(E)}(f(t_1), f'(t_1)) = \frac{1}{1 - |t_1|^2}.$$

This proves $1 \Rightarrow 2$.

Now let us assume 2, that is there is a $t_1 \in \Delta$ such that

$$F_{T(E)}(f(t_1), f'(t_1)) = \frac{1}{1 - |t_1|^2}.$$

Again without loss of generality we assume $t_1 = 0$ and $f(0) = 0$. With our assumption we thus have $f\colon \Delta \to T(E)$ is a holomorphic map, and $f(0) = 0$ and $F_{T(E)}(0, f'(0)) = 1$.

Consider the holomorphic map $\widehat{f}\colon \Delta \to M(\mathbf{C})$ such that $\widehat{f}(0) = 0$ and $P_E \circ \widehat{f} = f$. Using Schwarz's lemma as before we observe that

$$1 = F_{T(E)}(0, f'(0)) \le \|\widehat{f}'(0)\|_\infty \le 1.$$

This implies that $\|\widehat{f}'(0)\|_\infty = 1$.

Again let $g = \mathcal{L} \circ \widehat{f}$ that is $g(t) = \ell_{\widehat{f}(t)}$. We get $\|g'(0)\| = \|\widehat{f}'(0)\|_\infty = 1$. This is the case of equality in Schwarz's lemma, and hence we get

$$\|g(t)\| = \|\widehat{f}(t)\|_\infty = |t| \quad \text{for all } t \in \Delta.$$

So for all t in Δ, $\widehat{f}(t)$ is extremal and $\|\widehat{f}(t)\|_\infty = |t|$. We see that for all t in Δ the following is true because of extremality and the last equation:

$$d_{T(E)}(0, f(t)) = d_{T(E)}(P_E(0), P_E(\widehat{f}(t))) = \rho_M(0, \widehat{f}(t)) = \rho_\Delta(0, t).$$

Since $t_1 = 0$ and $f(0) = 0$ we get $d_{T(E)}(f(0), f(t)) = \rho_\Delta(0, t)$, and so,

$$d_{T(E)}(f(t_1), f(t)) = \rho_\Delta(t_1, t)$$

for all t in Δ. So $2 \Rightarrow 1$ trivially, and actually does imply something stronger.

Finally, we will show that for all t, t' in Δ, $d_{T(E)}(f(t), f(t')) = \rho_\Delta(t, t')$. If $t_1 = t'$ we have nothing to prove, so let us assume $t_1 \ne t'$. We have already seen that any $t \in \Delta$ could have been chosen as t_1 and hence we can simply assume $t = t_1$ and we thus get

$$\rho_\Delta(t, t') = \rho_\Delta(t_1, t') = d_{T(E)}(f(t_1), f(t')) = d_{T(E)}(f(t), f(t'))$$

which proves that f is a holomorphic isometry. □

We note the following corollary, whose proof is obvious.

Corollary 4.1. *Let $f\colon \Delta \to T(E)$ be a holomorphic map with $f(0) = P_E(0)$. Let $t \in \Delta \setminus \{0\}$. Define $k_0(t) = \|\nu\|_\infty$ where $f(t) = P_E(\nu)$ and ν is extremal in its E-equivalence class. We also know that $\|f'(0)\|_{T(E)} = F_{T(E)}(0, f'(0))$. Then $k_0(t) = |t|$ if and only if $\|f'(0)\|_{T(E)} = 1$.*

5. Proof of Theorem III

Let $f\colon \Delta \to T(E)$ be a baepoint preserving holomorphic map; by Proposition 1.6, there exists a holomorphic map $\widehat{f}\colon \Delta \to M(\mathbf{C})$, such that $\widehat{f}(0) = 0$ and $f = P_E \circ \widehat{f}$. Let V_0 be the Banach space of all tangent vectors at the basepoint of $T(E)$. We also know that ${P_E}'(0)$ takes the tangent vectors ν in the tangent space at the basepoint of $M(\mathbf{C})$ (which is $L^\infty(\mathbf{C})$) to the functional ℓ_ν. So ${P_E}'(0) \equiv \mathcal{L}$. Let $g = \mathcal{L} \circ \widehat{f}$ such that $g(t) = \ell_{\widehat{f}(t)}$; then $g\colon \Delta \to V_0$ is holomorphic and

$$f'(0) = (P_E \circ \widehat{f})'(0) = P_E'(0)(\widehat{f}'(0)) = \mathcal{L}(\widehat{f}'(0)) = \ell_{\widehat{f}'(0)} = g'(0)$$

since $\mathcal{L}$ is linear. Let $t \in \Delta \setminus \{0\}$ be fixed and one of the following inequalities $\|f'(0)\| \le 1$ and $k_0(t) \le |t|$ be strict, then both are strict by Corollary 4.1. So we get

$$\|g'(0)\| = \|f'(0)\| < 1$$

and so by Theorem 1.16 we get $\|g(t)\| < |t|$ and hence $\|\ell_{\widehat{f}(t)}\| < |t|$ or $\|\ell_{\widehat{f}(t)}\|_{T(E)} < |t|$. By the same theorem we also get

$$\rho_\Delta\left(\frac{\|\ell_{\widehat{f}(t)}\|_{T(E)}}{|t|}, \|f'(0)\|_{T(E)}\right) \le \rho_\Delta(0,t). \tag{5.1}$$

If $\|\ell_{\widehat{f}(t)}\|_{T(E)} = \|\widehat{f}(t)\|_\infty$, then by Corollary 3.1, $\widehat{f}(t)$ is extremal and $k_0(t) = \|\ell_{\widehat{f}(t)}\|_{T(E)}$ and so by (5.1) we get

$$\rho_\Delta\left(\frac{k_0(t)}{|t|}, \|f'(0)\|_{T(E)}\right) \le \rho_\Delta(0,t). \tag{5.2}$$

Suppose $\|\ell_{\widehat{f}(t)}\|_{T(E)} < \|\widehat{f}(t)\|_\infty$. Let $r = \frac{\|\widehat{f}(t)\|_\infty}{|t|}$. Let $k = \|\mu\|_\infty$ and $k_0 = \|\mu_0\|_\infty$. By Corollary 3.2 we have

$$\rho_\Delta\left(\frac{k_0}{k}, \frac{\|\ell_\mu\|_{T(E)}}{k}\right) \le \rho_\Delta(0,k). \tag{5.3}$$

So for $\mu = \widehat{f}(t)$, $k = r|t|$ and $k_0(t) = k_0$, we have

$$\rho_\Delta\left(\frac{k_0(t)}{r|t|}, \frac{\|\ell_{\widehat{f}(t)}\|_{T(E)}}{r|t|}\right) \le \rho_\Delta(0,r|t|). \tag{5.4}$$

Let us consider the map $\alpha\colon \Delta \to \Delta$ given by $\alpha(z) = rz$; then α is holomorphic and $\alpha(0) = 0$. Let

$$\frac{k_0(t)}{r|t|} = a \quad \text{and} \quad \frac{\|\ell_{\widehat{f}(t)}\|_{T(E)}}{r|t|} = b.$$

Then, a, $b \in \Delta$ and by Schwarz's lemma we get $\rho_\Delta(ar,br) \le \rho_\Delta(a,b)$. This gives

$$\rho_\Delta\left(\frac{k_0(t)}{|t|}, \frac{\|\ell_{\widehat{f}(t)}\|_{T(E)}}{|t|}\right) \le \rho_\Delta\left(\frac{k_0(t)}{r|t|}, \frac{\|\ell_{\widehat{f}(t)}\|_{T(E)}}{r|t|}\right) \tag{5.5}$$

We also have

$$\rho_\Delta(0,r|t|) = \rho_\Delta(\alpha(0),\alpha|t|) \le \rho_\Delta(0,|t|) = \rho_\Delta(0,t). \tag{5.6}$$

Combining (5.4), (5.5), and (5.6), we get

$$\rho_\Delta\left(\frac{k_0(t)}{|t|}, \frac{\|\ell_{\widehat{f}(t)}\|_{T(E)}}{|t|}\right) \le \rho_\Delta(0,t). \tag{5.7}$$

Combining (5.1) and (5.7), and using the triangle inequality, we obtain

$$\rho_\Delta\left(\frac{k_0(t)}{|t|}, \|f'(0)\|_{T(E)}\right) \le 2\rho_\Delta(0,t). \qquad \square$$

6. Proof of Theorem IV

Step 1. (2) implies (3). Let f_1 and f_2 be two holomorphic isometries from Δ into $T(E)$, such that $f_1(0) = f_2(0) = P_E(0)$ and $f_1(\|\mu_0\|_\infty) = f_2(\|\mu_0\|_\infty) = P_E(\mu_0)$. By (2) there is only one geodesic segment joining 0 and $P_E(\|\mu_0\|_\infty)$. Therefore, the image of the line segment $[0, \|\mu_0\|_\infty]$ is pointwise the same under both f_1 and f_2. This implies that the holomorphic mapping $f_1 - f_2$ is identically zero on the line segment $[0, \|\mu_0\|_\infty]$, and so $f_1 - f_2$ is identically zero on Δ.

Step 2. (1) implies (4). Let μ_0 be extremal and $|\mu_0| = \|\mu_0\|_\infty$ a.e. Let $g\colon \Delta \to M(\mathbf{C})$ be a holomorphic map with $g(0) = 0$ and $P_E(g(\|\mu_0\|_\infty)) = P_E(\|\mu_0\|_\infty)$. By Schwarz's lemma, $\|g(\|\mu_0\|_\infty)\|_\infty \le \|\mu_0\|_\infty$. Since μ_0 is uniquely extremal, we have $g(\|\mu_0\|_\infty) = \mu_0$. Consider a function f in $\overline{M(\mathbf{C})}$ (the closure of $M(\mathbf{C})$ in $L^\infty(\mathbf{C})$), with $|f(z)| = 1$ a.e. Let h be another function in $\overline{M(\mathbf{C})}$ such that $h(z) \ne 0$ in $\mathbf{C} \setminus Z_h$ where $Z_h = \{z \in \mathbf{C}\colon h(z) = 0\}$ and $m(Z_h) = 0$, where m denotes the usual Lebesgue measure. Let $E_f = \{z \in \mathbf{C}\colon |f(z)| \ne 1\}$. By our assumption, $m(E_f) = 0$. Consider the function $f_t(z) = f(z) + th(z)$. Let $F_h = \{f_t, t \in \overline{\Delta}\}$. Suppose $F_h \subset \overline{M(\mathbf{C})}$. For any $t \in \Delta$ define $H_t = \{z \in \mathbf{C}\colon |f_t(z)| > 1\}$. Let $f(z) = e^{i\theta(z)}$, $h(z) = |h(z)|e^{i\phi(z)}$ and $l(z) = \phi(z) - \theta(z)$. Also, $t = |t|e^{i\psi}$. Then we have

$$|f_t(z)| = \sqrt{1 + |t|^2(|h(z)|)^2 + 2|t|(|h(z)|)\cos(l(z)+\psi)}.$$

If $f_t \in \overline{M(\mathbf{C})}$, then $m(H_t) = 0$, and if $z \in \mathbf{C} \setminus H_t$, then

$$1 + |t|^2(|h(z)|)^2 + 2|t|(|h(z)|)\cos(l(z)+\psi) \le 1.$$

But $|t|^2(|h(z)|)^2 + 2|t|(|h(z)|)\cos(l(z)+\psi) \le 0$. This implies

$$-\cos l(z)\cos\psi - \sin l(z)\sin\psi \ge \frac{|t||h(z)|}{2}.$$

Consider the functions f_1, f_i, f_{-1} and f_{-i}. Let $G = E_f \cup Z_h \cup H_1 \cup H_i \cup H_{-1} \cup H_{-i}$. By our assumption, $m(G) = 0$ and if $z \in \mathbf{C} \setminus G$, then $h(z) \ne 0$, and

$$-\cos l(z) \ge \frac{|h(z)|}{2}, \quad -\sin l(z) \ge \frac{|h(z)|}{2}, \quad \cos l(z) \ge \frac{|h(z)|}{2}, \quad \sin l(z) \ge \frac{|h(z)|}{2}.$$

This is not possible. Therefore, at least one of the following functions f_1, f_{-1}, f_i, or f_{-i} does not belong to $\overline{M(\mathbf{C})}$. This implies that f is a complex extreme point of $\overline{M(\mathbf{C})}$. Let $\lambda = \frac{\mu_0}{\|\mu_0\|_\infty}$. Since $|\mu_0| = \|\mu_0\|_\infty$ a.e. we have $|\lambda| = 1$ a.e. Therefore λ is a complex extreme point for $\overline{M(\mathbf{C})}$.

Now define $h\colon \Delta \to M(\mathbf{C})$ as,

$$h(t) = \begin{cases} \frac{g(t)}{t} & \text{if } t \ne 0, \\ g'(0) & \text{if } t = 0. \end{cases}$$

Then h is holomorphic and $h(\|\mu_0\|_\infty) = \lambda$. By the strong maximum modulus principle (see Proposition 6.19 in [4]) we have $h(t) = \lambda$. This implies

$$g(t) = t\lambda = \frac{t\mu_0}{\|\mu_0\|_\infty}.$$

Since μ_0 is uniquely extremal, g is uniquely determined, and we are done.

Step 3. (4) implies (3). Let $f\colon \Delta \to T(E)$ be a holomorphic isometry such that $f(0) = P_E(0)$ and $f(\|\mu_0\|_\infty) = P_E(\mu_0)$. Consider the holomorphic map $\widehat{f}\colon \Delta \to M(\mathbf{C})$ such that $\widehat{f}(0) = 0$ and $P_E \circ \widehat{f} = f$. Then $P_E(\widehat{f}(\|\mu_0\|_\infty)) = P_E(\mu_0)$. By the uniqueness condition in (4), we have

$$\widehat{f}(t) = \frac{t\mu_0}{\|\mu_0\|_\infty}, \quad t \in \Delta.$$

This implies

$$f(t) = P_E\left(\frac{t\mu_0}{\|\mu_0\|_\infty}\right), \quad t \in \Delta.$$

So f is uniquely determined.

Step 4. (3) implies (1). We first show that if (3) holds, then $|\mu_0| = \|\mu_0\|_\infty$ a.e. Let $r \in (0,1)$ and $Z_r = \{z \in \mathbf{C}\colon |\mu_0(z)| < r\|\mu_0\|_\infty\}$ we need to show that $m(Z_r) = 0$. Let χ_r be the characteristic function of Z_r. Let $\phi \in A(E)$, where $A(E)$ is the closed subspace of $L^1(\mathbf{C})$ consisting of maps holomorphic in E^c. Define functions $f_1\colon \Delta \to T(E)$ and $f_r\colon \Delta \to T(E)$ by

$$f_1(t) = P_E\left(\frac{t\mu_0}{\|\mu_0\|_\infty}\right)$$

and

$$f_r(t) = P_E\left(\frac{t\mu_0}{\|\mu_0\|_\infty} + \frac{1-r}{2}t\left(t - \|\mu_0\|_\infty\left(\chi_r\frac{|\phi|}{\phi}\right)\right)\right).$$

These maps are holomorphic and we also have $f_1(0) = f_r(0) = 0$ and $f_1(\|\mu_0\|_\infty) = f_r(\|\mu_0\|_\infty) = P_E(\mu_0)$. They are also isometries since $\rho_\Delta(0, \|\mu_0\|_\infty) = d_{T(E)}(0, P_E(\mu_0))$. So, by (3) they coincide and we obtain

$$0 = f_1'(0) - f_r'(0) = \frac{1-r}{2}\|\mu_0\|_\infty P_E'(0)\left(\chi_r\frac{|\phi|}{\phi}\right).$$

This implies

$$P_E'(0)\left(\chi_r\frac{|\phi|}{\phi}\right) = 0.$$

Since $P_E'(0)(\mu) = \ell_\mu$, we get

$$\ell_{\left(\chi_r\frac{|\phi|}{\phi}\right)} = 0.$$

In particular,

$$\ell_{\left(\chi_r\frac{|\phi|}{\phi}\right)}(\phi) = 0.$$

So,

$$\iint_{Z_r} |\phi|\,dx\,dy = 0.$$

This shows that $m(Z_r) = 0$ since ϕ is an arbitrary function in $A(E)$. Let $Z = \bigcup_{r\in\mathbf{Q}\cap(0,1)} Z_r$, then $m(Z) = 0$. This shows that $|\mu_0| = \|\mu_0\|_\infty$ a.e. For any (normalized) quasiconformal homeomorphism h of $\widehat{\mathbf{C}}$, we define its Beltrami coefficient as

$$\mu_h = \frac{h_{\bar{z}}}{h_z}.$$

If h and j are two quasiconformal homeomorphisms, we have the composition formula

$$\mu_{h\circ j} = \frac{\mu_j + (\mu_h \circ j)\alpha_j}{1 + \bar{\mu_j}(\mu_h \circ j)\alpha_j}$$

where

$$\alpha_j = \frac{|j_z|^2}{(j_z)^2}.$$

If $\nu \in M(\mathbf{C})$, then by w^ν we mean the unique normalized quasiconformal homeomorphism with Beltrami coefficient ν a.e.

Let $\nu \in M(\mathbf{C})$ such that $\|\nu\|_\infty \le \|\mu_0\|_\infty$ and $P_E(\nu) = P_E(\mu_0)$. Since μ_0 is extremal, it follows that ν is also extremal and $\|\nu\|_\infty = \|\mu_0\|_\infty$. Hence $f(t) = P_E\left(\frac{t\nu}{\|\nu\|_\infty}\right)$ is a holomorphic isometry. So, by (3) we obtain

$$P_E\left(\frac{t\nu}{\|\nu\|_\infty}\right) = P_E\left(\frac{t\mu_0}{\|\mu_0\|_\infty}\right).$$

Since ν is extremal, by (3) we obtain $|\nu| = \|\nu\|_\infty = \|\mu_0\|_\infty$ a.e. Also, $P_E(s\nu) = P_E(s\mu_0)$, for any s in $(0,1)$.

So $(w^{s\mu_0})^{-1} \circ w^{s\nu}$ is isotopic to the identity *rel* E. This implies $w^{\mu_0} \circ (w^{s\mu_0})^{-1} \circ w^{s\nu}$ is isotopic to w^{μ_0} *rel* E. This implies $(w^{\mu_0})^{-1} \circ w^\lambda$ is isotopic to the identity *rel* E, where

$$w^\lambda = w^{\mu_0} \circ (w^{s\mu_0})^{-1} \circ w^{s\nu}.$$

This implies $P_E(\lambda) = P_E(\mu_0)$. Now let $h = w^{\mu_0} \circ (w^{s\mu_0})^{-1}$ and $j = w^{s\mu_0}$ such that $h \circ j = w^{\mu_0}$. By the formula for composition of quasiconformal mappings, we get

$$|\mu_h \circ j| = \frac{|\mu_0|(1-s)}{1 - s|\mu_0|^2}.$$

We know that $|\mu_0| = \|\mu_0\|_\infty$ a.e. Let $\|\mu_0\|_\infty = k$ and $sk = k'$. We get

$$|\mu_h \circ j| = \frac{k - k'}{1 - kk'} = k'' \quad \text{a.e.}$$

Since j is quasiconformal and therefore absolutely continuous, it follows that $|\mu_h| = k''$ a.e. Now let us consider $h = w^{\mu_0} \circ (w^{s\mu_0})^{-1}$ and $j = w^{s\nu}$ so that $h \circ j = w^\lambda$. By similar calculations we obtain

$$\lambda = \frac{s\nu + (\mu_h \circ j)\alpha_j}{1 + s\bar{\nu}(\mu_h \circ j)\alpha_j}.$$

Since $|s\nu| = k'$ a.e. and $|\mu_h \circ j| = k''$ a.e. and $|\alpha_j| = 1$ we write $s\nu = k'e^{i\theta}$ a.e. and $(\mu_h \circ j)\alpha_j = k''e^{i\phi}$ a.e. Hence

$$\lambda = e^{i\theta} \frac{k' + k''e^{il}}{1 + k'k''e^{il}}$$

where $l = \phi - \theta$. Therefore, $|\lambda| = |\frac{k'+k''e^{il}}{1+k'k''e^{il}}|$. Next, note that

$$\frac{k' + k''e^{il}}{1 + k'k''e^{il}} \le \frac{k' + k''}{1 + k'k''} \iff \frac{(k' + k''\cos l)^2 + k''^2 \sin^2 l}{(1 + k'k''\cos l)^2 + k'^2k''^2 \sin^2 l} \le \frac{k'^2 + 2k'k'' + k''^2}{1 + 2k'k'' + k'^2k''^2}$$
$$\iff (1 - k'^2)(1 - k''^2)(1 - \cos l) \ge 0.$$

The last inequality is true since $k' < 1$, $k'' < 1$ and $\cos l \le 1$. So we get

$$|\lambda| \le \frac{k' + k''}{1 + k'k''} = k.$$

Since $P_E(\mu_0) = P_E(\lambda)$ and μ_0 is extremal, it follows that λ is extremal. Hence $|\lambda| = k$ a.e. This implies

$$\left|\frac{k' + k''e^{il}}{1 + k'k''e^{il}}\right| = \frac{k' + k''e^{il}}{1 + k'k''e^{il}}.$$

This implies

$$(1-k'^2)(1-k''^2)(1-\cos l)=0.$$

Since $k'<1$ and $k''<1$, this holds if and only if $\cos l=1$, i.e. $\cos(\phi-\theta)=1$. This implies $s\nu=k'e^{i\theta}$ a.e. and $(\mu_h\circ j)\alpha_j=k''e^{i\phi}$ a.e. have the same arguments and can be rewritten as $s\nu=k'e^{i\theta}$ a.e. and $(\mu_h\circ j)\alpha_j=k''e^{i\theta}$ a.e.

We can write $(\mu_h\circ j)\alpha_j=m\cdot s\nu$ where $m=\frac{k'}{k''}>0$, so

$$\lambda=\nu\frac{s+ms}{1+ms^2k^2}.$$

This shows that λ is a positive multiple of ν. Let us write (for simplicity) $\lambda=p\nu$ where $p>0$. So, $\|\lambda\|_\infty=p\|\nu\|_\infty$. But we have $\|\lambda\|_\infty=\|\nu\|_\infty=\|\mu_0\|_\infty=k>0$. So $p=1$ and hence $\lambda=\nu$ a.e. Hence

$$w^\nu=w^\lambda=w^{\mu_0}\circ(w^{s\mu_0})^{-1}\circ w^{s\nu}\ \text{ a.e.}\implies w^\nu\circ(w^{s\nu})^{-1}=w^{\mu_0}\circ(w^{s\mu_0})^{-1}\ \text{ a.e.}$$

Since $s\in(0,1)$ is arbitrary, letting $s\to0$, we observe $w^\nu=w^{\mu_0}$ *a.e* and hence $\nu=\mu_0$ *a.e.*. This proves that μ_0 is uniquely extremal.

Step 5. (1) implies (2). Let μ_0 be uniquely extremal, and $|\mu_0|=\|\mu_0\|_\infty=k$ a.e. Let $\alpha\colon[0,1]\to T(E)$ be an injective continuous map, defined by $\alpha(t)=P_E(t\mu)$, so that $\alpha([0,1])$ is a geodesic segment joining $P_E(0)$ and $P_E(\mu_0)$. We want to show this is the only geodesic segment joining $P_E(0)$ and $P_E(\mu_0)$.

Let us assume that there is another injective continuous map $\beta\colon[0,1]\to T(E)$, such that $\beta([0,1])$ is another geodesic segment joining $P_E(0)$ and $P_E(\mu_0)$. Let $\nu\in M(\mathbf{C})$ be a point such that $P_E(\nu)\in\beta([0,1])\setminus\alpha([0,1])$. Let ν_0 be extremal in the *E-equivalence* class of ν. Since $P_E(\nu_0)$ is an interior point of the geodesic segment we see that

$$d_{T(E)}(P_E(0),P_E(\nu_0))\le d_{T(E)}(P_E(0),P_E(\mu_0)). \tag{6.1}$$

Since $|\mu_0|=k$ a.e. and ν_0 is extremal, we see that $|\mu_0|\ge|\nu_0|$ a.e. Consider the mapping $w^\eta=w^{\mu_0}\circ(w^{\nu_0})^{-1}$, so that $w^\eta\circ w^{\nu_0}=w^{\mu_0}$.

Let η_0 be the extremal in the *E-equivalence* class of η. Observe that $w^{\eta_0}\circ w^{\nu_0}=w^{\widetilde\mu}$ for some $\widetilde\mu$ such that $P_E(\widetilde\mu)=P_E(\mu_0)$. So we get

$$|\eta\circ w^{\nu_0}|=\left|\frac{\mu_0-\nu_0}{1-\overline{\nu_0}\mu_0}\right|$$

and

$$|\eta_0\circ w^{\nu_0}|=\left|\frac{\widetilde\mu-\nu_0}{1-\overline{\nu_0}\widetilde\mu}\right|.$$

Let $\|\widetilde\mu\|_\infty=n$ and $\|\nu_0\|_\infty=l$. Since μ_0 and ν_0 are both extremal, we get $l<k\le n$.

Now consider the map

$$f(z)=\frac{z-a}{1-\bar az},\quad a\in\Delta.$$

This map is holomorphic in Δ and $f(a)=0$. So, if $1>\delta_1>\delta_2>a$, then $a\in\overline{B_{\delta_2}(0)}\subset B_{\delta_1}(0)$, where $B_\delta(0)=\{z\in\Delta\colon|z-a|<\delta\}$. Since f is a Möbius transformation, by maximum modulus principle,

$$\delta_1>\delta_2\iff\sup_{z\in B_{\delta_1}(0)}|f(z)|=\sup_{z\in\partial B_{\delta_1}(0)}|f(z)|>\sup_{z\in\partial B_{\delta_2}(0)}|f(z)|=\sup_{z\in B_{\delta_2}(0)}|f(z)|$$

and

$$\delta_1=\delta_2\iff\sup_{z\in B_{\delta_1}(0)}|f(z)|=\sup_{z\in\partial B_{\delta_1}(0)}|f(z)|=\sup_{z\in\partial B_{\delta_2}(0)}|f(z)|=\sup_{z\in B_{\delta_2}(0)}|f(z)|.$$

Applying this to our problem we see that for all possible values of ν_0, since $\|\widetilde{\mu}\|_\infty = n$ and $n \geq k$, we have

$$\sup_{\widetilde{\mu}} \Big| \frac{\widetilde{\mu} - \nu_0}{1 - \overline{\nu_0}\widetilde{\mu}} \Big| \geq \sup_{\mu_0} \Big| \frac{\mu_0 - \nu_0}{1 - \overline{\nu_0}\mu_0} \Big|.$$

So

$$\sup_{\nu_0} \sup_{\widetilde{\mu}} \Big| \frac{\widetilde{\mu} - \nu_0}{1 - \overline{\nu_0}\widetilde{\mu}} \Big| \geq \sup_{\nu_0} \sup_{\mu_0} \Big| \frac{\mu_0 - \nu_0}{1 - \overline{\nu_0}\mu_0} \Big|.$$

This implies that $\|\eta_0\|_\infty \geq \|\eta\|_\infty$. Since $\|\eta_0\|_\infty \leq \|\eta\|_\infty$, we conclude that $\|\eta_0\|_\infty = \|\eta\|_\infty$. By the above discussion we have $n = k$, that is $\|\mu_0\|_\infty = \|\widetilde{\mu}\|_\infty$.

We conclude that $\widetilde{\mu}$ is extremal, and since μ_0 is uniquely extremal, $\widetilde{\mu} = \mu_0$. So

$$w^{\eta_0} = w^{\mu_0} \circ (w^{\nu_0})^{-1}.$$

This gives us

$$d_{T(E)}(P_E(0), P_E(\eta_0)) = d_{T(E)}(P_E(\nu_0), P_E(\mu_0)). \tag{6.2}$$

Since $P_E(0)$, $P_E(\nu_0)$ and $P_E(\mu_0)$ are on a geodesic segment, we have

$$d_{T(E)}(P_E(0), P_E(\nu_0)) + d_{T(E)}(P_E(\nu_0), P_E(\mu_0)) = d_{T(E)}(P_E(0), P_E(\mu_0)).$$

Using Equation (6.2) we get

$$d_{T(E)}(P_E(0), P_E(\nu_0)) + d_{T(E)}(P_E(0), P_E(\eta_0)) = d_{T(E)}(P_E(0), P_E(\mu_0)).$$

Since μ_0, ν_0, and η_0 are extremal in their respective equivalence classes, we get

$$\rho_\Delta(0, \|\nu_0\|_\infty) + \rho_\Delta(0, \|\eta_0\|_\infty) = \rho_\Delta(0, \|\mu_0\|_\infty). \tag{6.3}$$

This implies

$$\|\eta_0\|_\infty = \frac{\|\mu_0\|_\infty - \|\nu_0\|_\infty}{1 - \|\nu_0\|_\infty \|\mu_0\|_\infty}. \tag{6.4}$$

Since we have

$$w^{\eta_0} = w^{\mu_0} \circ (w^{\nu_0})^{-1},$$

we obtain

$$|\eta_0 \circ w^{\nu_0}| = \Big| \frac{\mu_0 - \nu_0}{1 - \bar{\nu_0}\mu_0} \Big|. \tag{6.5}$$

Let $\nu_0 = s\mu_0$, $s = |s|e^{i\phi}$ and $\mu_0 = ke^{i\theta}$ and $|s| < 1$. By Equation (6.4) we get

$$\|\eta_0\|_\infty = k \frac{1 - \sup|s|}{1 - \sup|s|k^2}. \tag{6.6}$$

By Equation (6.5) we get

$$|\eta_0 \circ w^{\nu_0}| = k \Big| \frac{1 - |s|e^{i(\phi-\theta)}}{1 - |s|k^2 e^{i(\theta-\phi)}} \Big|. \tag{6.7}$$

Setting $\omega = \phi - \theta$, we rewrite this as

$$|\eta_0 \circ w^{\nu_0}| = k \Big| \frac{1 - |s|e^{i\omega}}{1 - |s|k^2 e^{-i\omega}} \Big|. \tag{6.8}$$

It is easy to see that

$$\begin{aligned}
&\Big|\frac{1-|s|e^{i\omega}}{1-|s|k^2e^{-i\omega}}\Big| \geq \frac{1-|s|}{1-|s|k^2}\\
&\iff \frac{(1-|s|\cos\omega)^2+|s|^2\sin^2\omega}{(1-|s|k^2\cos\omega)^2+|s|^2k^4\sin^2\omega} \geq \frac{(1-|s|)^2}{(1-|s|k^2)^2}\\
&\iff (1-k^2)(1-|s|^2k^2)(1-\cos\omega) \geq 0.
\end{aligned}$$

The last inequality is true since $k<1$, $s<1$ and $\cos\omega \leq 1$. So Equation (6.7) gives

$$|\eta_0 \circ w^{\nu_0}| \geq k\frac{1-|s|}{1-|s|k^2}.$$

Hence

$$\|\eta_0\|_\infty \geq k\frac{1-|s|}{1-|s|k^2}. \tag{6.9}$$

It is easy to see from Equations (6.6) and (6.9) that

$$k\frac{1-\sup|s|}{1-\sup|s|k^2} \geq k\frac{1-|s|}{1-|s|k^2} \implies |s| = \sup|s| := S.$$

From Equations (6.6) and (6.8) we get

$$k\frac{1-S}{1-Sk^2} = \|\eta_0\|_\infty \geq |\eta_0 \circ w^{\nu_0}| = k\Big|\frac{1-Se^{i\omega}}{1-Sk^2e^{i\omega}}\Big| \geq k\frac{1-S}{1-Sk^2}.$$

This gives

$$k\Big|\frac{1-Se^{i\omega}}{1-Sk^2e^{i\omega}}\Big| = k\frac{1-S}{1-Sk^2}.$$

This is true if and only if

$$(1-k^2)(1-S^2k^2)(1-\cos\omega) = 0.$$

That can happen only when $\cos\omega = \cos(\phi-\theta) = 1$, which means ν_0 and μ_0 have the same arguments and hence we can write $\nu_0 = S\mu_0, 1 > S > 0$. But that contradicts our assumption. So we conclude that there is only one geodesic segment joining $P_E(0)$ and $P_E(\mu_0)$, which completes the proof. □

Acknowledgement. I want to thank the referee for her/his careful reading and valuable suggestions. I also want to thank Professor Yunping Jiang who read an earlier draft of this paper and gave several helpful comments.

References

[1] Ahlfors, L. V.: Lectures on quasiconformal mappings. Second edition. - Univ. Lecture Ser. 38, Amer. Math. Soc., 2006.

[2] Ahlfors, L. V., and L. Bers: Riemann's mapping theorem for variable metrics. - Ann. of Math. 72, 1960, 385–404.

[3] Chae, S. B.: Holomorphy and calculus in normed spaces. - Marcel Dekker, New York, 1985.

[4] Dineen, S.: The Schwarz lemma. - Oxford Math. Monogr., Oxford Univ. Press, Oxford, 1990.

[5] Earle, C. J.: Schwarz's lemma and Teichmüller contraction. - Contemp. Math. 311, 2002, 79–85.

[6] Earle, C. J., I. Kra, and S. L. Krushkal: Holomorphic motions and Teichmüller spaces. - Trans. Amer. Math. Soc. 343:2, 1994, 927–948.

[7] Earle, C. J., and S. Mitra: Variation of moduli under holomorphic motions. - Contemp. Math. 256, 2000, 39–67.

[8] Gardiner, F. P.: - On Teichmüller contraction. - Proc. Amer. Math. Soc. 118, 1993, 865–875.

[9] Gardiner, F. P., and N. Lakic: Quasiconformal Teichmüller theory. - Math. Surveys Monogr. 76, Amer. Math. Soc., Providence, 1993.

[10] Harris, L. A.: Schwarz–Pick systems of pseudometrics for domains in normed linear spaces. - In: Advances in holomorphy, North-Holland Math. Studies 34, North-Holland, Amsterdam, 1979, 345–406.

[11] Hubbard, J. H.: Teichmüller theory and applications to geometry, topology and dynamics. Volume I. Teichmüller theory. - Matrix Editions, Ithaca, NY, 2006.

[12] Jiang, Y., and S. Mitra: Douady–Earle section, holomorphic motions, and some applications. - Contemp. Math. 575, 2012, 219–251.

[13] Lehto, O.: Univalent functions and Teichmüller spaces. - Grad. Texts in Math. 109, Springer-Verlag, Berlin, 1987.

[14] Lieb, G. S.: Holomorphic motions and Teichmüller space. - Ph.D. dissertation, Cornell University, 1990.

[15] Mitra, S.: Teichmüller spaces and holomorphic motions. - J. Anal. Math. 81, 2000, 1–33.

[16] Mitra, S.: Teichmüller contraction in the Teichmüller space of a closed set in the sphere. - Israel J. Math. 125, 2001, 45–51.

[17] Mitra, S.: On extensions of holomorphic motions – a survey. - In: Geometry of Riemann surfaces, edited by Gardiner, González-Diez and Kourouniotis, London Math. Soc. Lecture Note Ser. 368, 2010, 283–308.

[18] Mitra, S., and H. Shiga: Extensions of holomorphic motions and holomorphic families of Möbius groups. - Osaka J. Math. 47, 2010, 1167–1187.

[19] Nag, S.: The complex analytic theory of Teichmüller spaces. - John Wiley and Sons, New York, 1988.

[20] Rudin, W.: Functional analysis. Second edition. - McGraw-Hill, New York, 1992.

[21] Slodkowski, Z.: Holomorphic motions and polynomial hulls. - Proc. Amer. Math. Soc. 111, 1991, 347–355.

Received 4 October 2016 • Accepted 23 December 2016