Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 42, 2017, 73–93

THE L^p-NORMS OF THE BEURLING–AHLFORS TRANSFORM ON RADIAL FUNCTIONS

Michał Strzelecki

University of Warsaw, Institute of Mathematics
Banacha 2, 02–097 Warsaw, Poland; m.strzelecki@mimuw.edu.pl

Abstract. We calculate the norms of the operators connected to the action of the Beurling–Ahlfors transform on radial function subspaces introduced by Bañuelos and Janakiraman. In particular, we find the norm of the Beurling–Ahlfors transform acting on radial functions for $p > 2$, extending the results obtained by Bañuelos and Janakiraman, Bañuelos and Osękowski, and Volberg for $1 < p \leq 2$.

1. Introduction and main results

The Beurling–Ahlfors transform is a singular operator defined by

$$Bf(z) = -\frac{1}{\pi}\,\mathrm{p.\,v.}\int_{\mathbf{C}} \frac{f(w)}{(z-w)^2}\,dw,$$

where the integration is with respect to the Lebesgue measure on the complex plane $\mathbf{C}$. It plays an important role in the study of quasiconformal mappings and partial differential equations (see e.g. [1, 13]).

A longstanding conjecture of Iwaniec [13] states that for $1 < p < \infty$,

$$\|B\|_{L^p(\mathbf{C})\to L^p(\mathbf{C})} = p^* - 1,$$

where $p^* = \max\{p, \frac{p}{p-1}\}$. While the lower bound $\|B\|_{L^p(\mathbf{C})\to L^p(\mathbf{C})} \geq p^* - 1$ was already known to Lehto [14], the question about the opposite estimate remains open. Most results rely on the ideas of Burkholder and the Bellman function technique [7, 19, 4, 11, 2, 8], with the current best being $\|B\|_{L^p(\mathbf{C})\to L^p(\mathbf{C})} \leq 1.575(p^* - 1)$ due to Bañuelos and Janakiraman [2] (see also [8] for an asymptotically better estimate as $p \to \infty$).

However, some sharp results are known for the Beurling–Ahlfors transform restricted to the class of radial functions [3, 12, 5, 16, 18, 6]. In this case we have the representation (see [3])

$$BF(z) = \frac{\bar{z}}{z}\big(f(|z|^2) - H_0 f(|z|^2)\big),$$

where $f\colon [0,\infty) \to \mathbf{C}$ is an integrable function, $F(z) = f(|z|^2)$ is the associated radial function, and H_0 is the Hardy operator defined by the formula

$$H_0 f(t) = \frac{1}{t}\int_0^t f(s)\,ds.$$

Bañuelos and Janakiraman [3, Theorem 4.1] (and later, using other techniques, Bañuelos and Osękowski [5, Theorem 5.1], Volberg [18]) proved that for $1 < p \leq 2$ and any radial function $F \in L^p(\mathbf{C})$, we have $\|BF\|_p \leq \frac{1}{p-1}\|F\|_p$. The constant $1/(p-1)$

https://doi.org/10.5186/aasfm.2017.4204
2010 Mathematics Subject Classification: Primary 60G46; Secondary 42B20.
Key words: Beurling–Ahlfors transform, Hardy operator, L^p-norms, martingales, radial functions.

is the best possible and coincides with the constant from Iwaniec's conjecture. As for $p > 2$, Bañuelos and Osękowski [5] observed that $\|BF\|_p \leq \frac{2p}{p-1}\|F\|_p$. This bound is asymptotically sharp (and does not agree with the behavior conjectured in the case of all, not only radial, functions).

In their paper, Bañuelos and Janakiraman [3] went a step further and considered for $m \in \mathbf{N}$ the operators

$$(I - (1+m)H_m)f(t) = f(t) - \frac{1+m}{t^{1+m/2}} \int_0^t f(s)s^{m/2}\,ds, \quad f \in L^1_{\mathrm{loc}}([0,\infty)),$$

which correspond to the action of the Beurling–Ahlfors transform on the radial function subspaces

$$\{F \in L^p(\mathbf{C})\colon F(re^{i\theta}) = f(r)e^{-im\theta}\}.$$

They proved [3, Section 5] that

$$\|H_m\|_{L^p([0,\infty))\to L^p([0,\infty))} = \frac{1}{m/2 + (p-1)/p}, \quad 1 < p < \infty,$$

(with the extremal family $f_\varepsilon(t) = t^{-1/p+\varepsilon}1_{\{t\in(0,1)\}}$), and conjectured [3, Conjecture 1] that the L^p-norm of the operator $I - (1+m)H_m$ is equal to

$$(1+m)\|H_m\|_{L^p([0,\infty))\to L^p([0,\infty))} - 1 = \frac{m/2 + 1/p}{m/2 + (p-1)/p}$$

for $1 < p < 2$. For $p > 2$, this number is smaller than one, and cannot be a candidate for the norm of $I-(1+m)H_m$, since the operator $(1+m)H_m\colon L^p([0,\infty)) \to L^p([0,\infty))$ is not invertible (see Remark 3.4).

In fact, the formula

$$H_m f(t) = \frac{1}{t^{1+m/2}} \int_0^t f(s)s^{m/2}\,ds$$

defines a bounded operator on the space $L^p([0,\infty))$ ($1 < p < \infty$) not only for natural m, but for all $m > -2(p-1)/p$ (see Propositon 3.1). The main goal of this article is to find the L^p-norm of the operator $I - \lambda H_m$ for $1 < p < \infty$, $m > -2(p-1)/p$, and $\lambda \in \mathbf{R}$. The case $\lambda = 1+m$, $m \in \mathbf{N}$, corresponds to the action of the Beurling–Ahlfors transform on radial function subspaces considered by Bañuelos and Janakiraman, but it turns out that Conjecture 1 of [3] does not hold.

For the formulation of the main result we denote $g_{p,m} = m/2 + (p-1)/p$ for $m > -2(p-1)/p$ and $1 < p < \infty$.

Theorem 1.1. *If $1 < p < \infty$, $m > -2(p-1)/p$, and $\lambda \in \mathbf{R}$, then*

$$\|f - \lambda H_m f\|_p \leq C_{p,m,\lambda}\|f\|_p, \quad f \in L^p([0,\infty)), \tag{1.1}$$

where

$$C^p_{p,m,\lambda} = \sup\left\{\frac{(\beta - g_{p,m})|\alpha-\lambda|^p + (g_{p,m}-\alpha)|\beta-\lambda|^p}{(\beta-g_{p,m})|\alpha|^p + (g_{p,m}-\alpha)|\beta|^p}\colon \alpha < g_{p,m} < \beta\right\}.$$

The inequality is sharp. Moreover, the constant $C^p_{p,m=0,\lambda=1}$ is equal to

$$C^p_p := \sup_{\alpha\leq(p-1)/p} \frac{|\alpha-1|^p}{p(1-\alpha)-1+|\alpha|^p} = \begin{cases} \frac{1}{(p-1)^p} & \text{if } 1 < p \leq 2, \\ \frac{(1+|\alpha_p|)^{p-2}}{p-1} & \text{if } p > 2, \end{cases} \tag{1.2}$$

where, for $2 < p < \infty$, $\alpha_p \in \mathbf{R}$ is the unique negative solution to the equation $(p-1)\alpha_p + 2 - p = |\alpha_p|^{p-2}\alpha_p$.

Remark 1.2. Even for $1 < p < 2$, the norm of the operator $I - (1+m)H_m$ is sometimes greater than the conjectured value $(1+m)g_{p,m}^{-1}-1$. E.g. for $p = 3/2$, $m = 1$, $\lambda = 2$, we have $C_{p,m,\lambda}^p \approx 1.81$ (attained in the neighbourhood of $(\alpha, \beta) = (0.4, 5.7)$), whereas $((1+m)g_{p,m}^{-1} - 1)^p = (7/5)^{3/2} \approx 1.66$. In fact, it turns out that for positive integer m (and $\lambda = 1+m$) we always have $C_{p,m,\lambda} > (1+m)g_{p,m}^{-1}-1$ (see Remark 3.8). On the other hand, if $1 < p < 2$, $\lambda = 1 + m$, and m is small, then we can have $C_{p,m,\lambda} = (1 + m)g_{p,m}^{-1} - 1$ (e.g. for $p = 3/2$, $m = 1/4$, $\lambda = 5/4$).

Remark 1.3. Apart from the case $m = 0$, $\lambda = 1$, there are simple formulas for $C_{p,m,\lambda}$ if $\lambda \le 0$ or $p = 2$ (see Section 3.3). Note also that $C_{p,m,\lambda} \ge \max\{|\lambda g_{p,m}^{-1}-1|, 1\}$ (see Lemma 3.3). Moreover, a sufficient and necessary condition for $C_{p,m,\lambda} = |\lambda g_{p,m}^{-1} - 1|$ to hold can be formulated (see the proof of Proposition 3.7 and Section 3.3).

Remark 1.4. Throughout the paper we work with real-valued functions, but Theorem 1.1 also holds (with the same constant) for complex-valued functions (see Lemma 3.9).

The results of Theorem 1.1 are new already for $p > 2$, $m = 0$, and $\lambda = 1$, and give immediately the following extension of results obtained by other authors [3, 5, 18].

Corollary 1.5. *For $1 < p < \infty$ and any complex-valued radial function $F \in L^p(\mathbf{C})$, we have the sharp inequality $\|BF\|_p \le C_p\|F\|_p$.*

The article is organized as follows. A complete and purely analytical proof of inequality (1.1) is contained in Section 3. Section 2 is designed to show a bigger picture. We prove a maximal martingale inequality connected to the special case $m = 0$ and $\lambda = 1$. We also identify the constant $C_{p,0,1}$ and try to explain the main ideas behind the construction of the special functions used in the proofs.

2. Backstage: the martingale inequality

2.1. Motivation and results. For a martingale $f = (f_n)_{n=0}^\infty$ denote its one-sided maximal function by $f_n^* = \sup_{0\le j\le n} f_j$. We also use the notation $f_\infty^* = \sup_{0\le n} f_n$ and $f_\infty = \lim_{n\to\infty} f_n$ (if the limit exist a.s.).

Recall that the L^p-norm, $1 < p < \infty$, of the Hardy operator H_0 is equal to $p/(p-1)$. This number is also the best constant in Doob's inequality: for a martingale $(f_k)_{k=0}^n$ we have $\|f_n^*\|_p \le \frac{p}{p-1}\|f_n\|_p$. It turns out that the martingale inequality can be used to derive the estimate $\|H_0 f\|_p \le \frac{p}{p-1}\|f\|_p$ for nonnegative and nonincreasing functions [10]; a simple rearrangement argument gives then $\|H_0 f\|_p \le \frac{p}{p-1}\|f\|_p$ for all real-valued $f \in L^p([0,\infty))$.

We consider the following maximal inequality.

Theorem 2.1. *For any martingale $(f_n)_{n=0}^\infty$, we have*

$$\|f_n - f_n^*\|_p \le C_p\|f_n\|_p, \quad 1 < p < \infty, n \ge 0, \tag{2.1}$$

where C_p is defined in (1.2)*. The inequality is sharp.*

The quantity $\|f_n - f_n^*\|_p$ seems natural to study, but the main motivation is the aforementioned link to the Hardy operator (see Section 2.7 for details). Note that this approach is different from that of Bañuelos and Osękowski [5], who used estimates for pure-jump martingales, and the analytical approaches of Bañuelos and Janakiraman [3], and Volberg [18].

Corollary 2.2. *Let $1 < p < \infty$. If $f \in L^p([0,\infty))$ is real-valued and nonincreasing, then*

$$\|f - H_0 f\|_p \le C_p \|f\|_p.$$

Quite unexpectedly, some difficulties arise at the stage of rearrangements. In our setting it is possible that

$$\|g - H_0 g\|_p < \|f - H_0 f\|_p,$$

where g denotes the nonincreasing rearrangement of a real-valued function $f \in L^p([0,\infty))$ (examples can be found with f being a (positive) step function, in which case $H_0 g$, $\|g - H_0 g\|_p$, $\|f - H_0 f\|_p$ can be explicitly calculated). Hence, it seems that Corollary 2.2 does not directly imply Theorem 1.1 (for $m = 0$, $\lambda = 1$). Fortunately, it is possible to use the tools from the proof of the martingale inequality (and adapt them to work not only for $m = 0$, but for $m > -2(p-1)/p$ and all $\lambda \in \mathbf{R}$) to obtain our main result (see Sections 2.7 and 3).

2.2. Method of the proof of Theorem 2.1 and a lower bound for the best constant. We follow Burkholder's approach to the Doob inequality [9, p. 14]: in order to prove inequality (2.1), it suffices to find an appropriate special function (for further reading about maximal martingale inequalities see also [15, Chapter 7]).

Proposition 2.3. *Let $V(x,y) = |x-y|^p - C^p|x|^p$ and suppose that $U\colon \mathbf{R}^2 \to \mathbf{R}$ satisfies the following conditions.*

1. *(Majorization) If $x \le y$, then $V(x,y) \le U(x,y)$.*
2. *(Initial condition) For all $x \in \mathbf{R}$, we have $U(x,x) \le 0$.*
3. *(Maximal condition) If $x \le y, h \in \mathbf{R}$, then*

$$U(x+h, (x+h) \vee y) \le U(x+h, y).$$

4. *(Concavity) For all $y \in \mathbf{R}$, the function $U(\cdot, y)\colon \mathbf{R} \to \mathbf{R}$ is concave.*

Then $\|f_n - f_n^*\|_p \le C\|f_n\|_p$ for any martingale $(f_n)_{n=0}^\infty$ and any $n \ge 0$.

Proof. It suffices to consider the inequality for simple martingales (in which case all expressions below are integrable). Conditions 3 and 4 imply that

$$\begin{aligned}
\mathbf{E}\, U(f_n, f_n^*) &= \mathbf{E}\, U\big(f_{n-1} + (f_n - f_{n-1}), (f_{n-1} + (f_n - f_{n-1})) \vee f_{n-1}^*\big) \\
&\le \mathbf{E}\, U(f_{n-1} + (f_n - f_{n-1}), f_{n-1}^*) \\
&\le \mathbf{E}\, U(f_{n-1}, f_{n-1}^*) + \mathbf{E}(f_n - f_{n-1}) U_{x^+}(f_{n-1}, f_{n-1}^*),
\end{aligned}$$

where U_{x^+} denotes the right derivative. Moreover, $\mathbf{E}(f_n - f_{n-1})U_{x^+}(f_{n-1}, f_{n-1}^*) = 0$ because f is a martingale. Hence, $\mathbf{E}\, U(f_n, f_n^*) \le \mathbf{E}\, U(f_{n-1}, f_{n-1}^*)$. Thus, using Conditions 1 and 2, we arrive at

$$\begin{aligned}
\|f_n - f_n^*\|_p^p - C^p\|f_n\|_p^p &= \mathbf{E}\, V(f_n, f_n^*) \\
&\le \mathbf{E}\, U(f_n, f_n^*) \le \ldots \le \mathbf{E}\, U(f_0, f_0^*) = \mathbf{E}\, U(f_0, f_0) \le 0.
\end{aligned}$$

This ends the proof. □

Remark 2.4. In the above proof it is enough to have $\mathbf{E}(f_n - f_{n-1})U_{x^+}(f_{n-1}, f_{n-1}^*) \le 0$. This inequality holds if f is a nonnegative submartingale and $U_{x^+}(x,y) \le 0$ for $y \ge 0$. This additional assumption is satisfied by the function U which we construct in Section 2.5. In particular, for any martingale $(f_n)_{n=0}^\infty$ also

$$\Big\| |f_n| - \sup_{0 \le j \le n} |f_j| \Big\|_p \le C_p \|f_n\|_p$$

holds, since $(|f_n|)_{n=0}^\infty$ is a nonnegative submartingale whenever $(f_n)_{n=0}^\infty$ is a martingale. This bound is sharp in the case $1 < p \le 2$ (see the example in Sectiion 2.6), but the constant C_p does not seem to be the best possible for $p > 2$.

There is an abstract way of finding a candidate for the function from Proposition 2.3. Namely, let $V(x,y) = |x-y|^p - C^p|x|^p$ and define

$$U^0(x,y) = \sup\{\mathbf{E}\, V(f_\infty, f_\infty^* \vee y) \colon f_0 = x\},$$

where the supremum is taken over the class $\mathcal{M}$ consisting of all simple martingales $f = (f_n)_{n=0}^\infty$ on the probability space $[0,1]$ equipped with the Borel σ-algebra and the Lebesgue measure (the filtration may vary). This approach has one main drawback: the expression defining U^0 is hard to work with. Nonetheless, we can use the function U^0 to extract important information: a lower bound for the constant $C = C(p)$ with which the martingale inequality holds (and on which the function V depends). For explicit examples of extremal martingales see Section 2.6.

Sharpness of (2.1). Let $1 < p < \infty$ be fixed. First note that by the triangle and Doob's inequality the estimate (2.1) holds with some finite constant. Let us denote it by C (of course it may depend on p) and let V, U^0 be the functions defined in the preceding paragraph. Note that, as for now, we do not claim that $U^0 < +\infty$.

Clearly, $U^0(x,y) \ge V(x, x\vee y)$ (since a constant martingale, $f_n \equiv x$, belongs to $\mathcal{M}$), $U^0(x,y) = U^0(x, x\vee y)$ (since $f_0 \le f_\infty^*$), and $U^0(ax, ay) = |a|^p U^0(x,y)$. A "splicing" argument (cf. [15]) gives us concavity of $U^0(\cdot, y)$: if $\lambda \in (0,1)$, $f, g \in \mathcal{M}$, $f_0 = x_1$, and $g_0 = x_2$, then the process defined by $h_0 = \lambda x_1 + (1-\lambda)x_2$ and

$$h_n(\omega) = f_{n-1}(\omega/\lambda) 1_{\{\omega\in[0,\lambda)\}} + g_{n-1}\big((\omega-\lambda)/(1-\lambda)\big) 1_{\{\omega\in[\lambda,1)\}}, \quad n \ge 1,$$

is a simple martingale starting from $x = \lambda x_1 + (1-\lambda)x_2$. Hence

$$U^0(x,y) \ge \mathbf{E}\, V(h_\infty, h_\infty^* \vee y) = \lambda\, \mathbf{E}\, V(f_\infty, f_\infty^*\vee y) + (1-\lambda)\, \mathbf{E}\, V(g_\infty, g_\infty^* \vee y),$$

which after taking the suprema over f and g yields the claim.

Moreover, if $f \in \mathcal{M}$ satisfies $f_0 = y$, then $\mathbf{E}\, V(f_\infty, f_\infty^* \vee y) = \mathbf{E}\, V(f_\infty, f_\infty^*) \le 0$, where the inequality follows from the assumption that the martingale inequality is satisfied with constant C. Therefore $U^0(y,y) \le 0$ for all $y \in \mathbf{R}$, and hence $U^0(x,y) < +\infty$ for any $x, y \in \mathbf{R}$. Indeed, U^0 is concave with respect to the first variable, and a concave function on the real line, which takes values in the set $(-\infty, +\infty]$, and is equal to $+\infty$ at some point, is identically equal to $+\infty$.

We now exploit the function U^0 to get an estimate of the constant C. Fix $\alpha \le (p-1)/p$ and $\delta, t \in (0,1)$. The properties of U^0 imply

$$\begin{aligned} U^0(1,1) &\ge \frac{\delta}{1-\alpha+\delta} U^0(\alpha,1) + \frac{1-\alpha}{1-\alpha+\delta} U^0(1+\delta,1) \\ &\ge \frac{\delta}{1-\alpha+\delta} V(\alpha,1) + \frac{1-\alpha}{1-\alpha+\delta} U^0(1+\delta,1+\delta) \\ &\ge \frac{\delta}{1-\alpha+\delta} V(\alpha,1) + (1-t)\frac{1-\alpha}{1-\alpha+\delta}(1+\delta)^p U^0(1,1) \\ &\quad + t\frac{1-\alpha}{1-\alpha+\delta}(1+\delta)^p V(1,1), \end{aligned}$$

which can be rewritten in the form

$$U^0(1,1)\frac{1-\alpha+\delta-(1-t)(1-\alpha)(1+\delta)^p}{\delta} \ge V(\alpha,1) + t\frac{1-\alpha}{\delta}(1+\delta)^p V(1,1).$$

Now, for $\delta < 1/p$, we put $t = \delta(p - p\alpha - 1)/(1 - \alpha)$ (note that $t \in [0, 1]$, since $\alpha \le (p - 1)/p$), take $\delta \to 0^+$, and arrive at

$$0 \ge V(\alpha, 1) + (p(1 - \alpha) - 1)V(1, 1).$$

Using the definition of the function V we can solve this inequality with respect to C. Taking the supremum over $\alpha \le (p - 1)/p$ yields then

$$C^p \ge \sup_{\alpha \le (p-1)/p} \frac{|\alpha - 1|^p}{p(1 - \alpha) - 1 + |\alpha|^p}$$

(note that $p(1 - \alpha) - 1 + |\alpha|^p$ is strictly positive for $\alpha \ne 1$, since the function $\alpha \mapsto p\alpha - p + 1$ is tangent to the convex function $\alpha \mapsto |\alpha|^p$ at $\alpha = 1$). Hence the best constant with which the martingale inequality is satisfied is not smaller than the right-hand side of the above inequality. □

2.3. Finding the concave majorant. In this subsection we give some informal reasoning, which is helpful in guessing an explicit formula for the function U satisfying the assumptions of Proposition 2.3. As before, we denote $V(x, y) = |x - y|^p - C^p|x|^p$. We look for a function U such that $U(\cdot, y)$ is not only concave, but even affine: let

$$U(x, y) = p(|\alpha y - y|^{p-2}(\alpha y - y) - C^p|\alpha y|^{p-2}\alpha y)(x - \alpha y) + |\alpha y - y|^p - C^p|\alpha y|^p$$

be the tangent to $V(\cdot, y)$ at the point $x = \alpha y$ (for some α, yet to be determined). Note that if V was concave with respect to the first variable, then such a choice of U would automatically guarantee the majorization property (i.e. $V(x, y) \le U(x, y)$). Unfortunately, this is not the case in our setting (cf. Lemma 3.6).

The maximal condition states that $U(x + h, x + h) \le U(x + h, y)$ for $x + h > y$, and implies $U_y(x, x) \le 0$. Let us assume that $U_y(x, x) = 0$. Some calculations reveal that this condition is equivalent to

$$C^p = |\alpha^{-1} - 1|^{p-2}(\alpha^{-1} - 1)\big(\big((p - 1)(1 - \alpha)\big)^{-1} - 1\big)$$

(provided that $\alpha \notin \{0, 1\}$). Taking such C we arrive at

$$U(x, y) = -\frac{|1 - \alpha|^{p-2}}{p - 1}|y|^{p-2}y(px - y(p - 1)).$$

Note that for this choice the initial condition (i.e. $U(x, x) \le 0$) is also satisfied.

Moreover, the preceding subsection suggests that for the right choice of α we should have

$$|\alpha^{-1} - 1|^{p-2}(\alpha^{-1} - 1)\big(\big((p - 1)(1 - \alpha)\big)^{-1} - 1\big) = \sup_{\alpha' \le (p-1)/p} \frac{|\alpha' - 1|^p}{p(1 - \alpha') - 1 + |\alpha'|^p}.$$

We identify the correct values of $\alpha = \alpha(p)$ and $C = C(p)$ in some technical lemmas in the next section. This is relatively easy in the case $1 < p \le 2$, where one can simply take $\alpha(p) = (p - 1)/p$.

In Section 2.5 we check that for these choices the function U is indeed the majorant of V. We prove the martingale inequality (2.1) in Section 2.6.

2.4. Technical lemmas. The first three results are needed to identify the value of the optimal constant in the martingale inequality (2.1).

Lemma 2.5. *For each $p \in (2, \infty)$, there exists exactly one number $\alpha_p \le (p-1)/p$ such that*

$$\frac{|\alpha_p - 1|^p}{p(1 - \alpha_p) - 1 + |\alpha_p|^p} = \sup_{\alpha \le (p-1)/p} \frac{|\alpha - 1|^p}{p(1 - \alpha) - 1 + |\alpha|^p} = \sup_{\alpha \ne 1} \frac{|\alpha - 1|^p}{p(1 - \alpha) - 1 + |\alpha|^p}.$$

Moreover, $(p-1)\alpha_p + 2 - p = |\alpha_p|^{p-2}\alpha_p$ *and* $\alpha_p < -(p-1)^{1/(p-2)} < 0$.

Proof. Recall that $p(1-\alpha) - 1 + |\alpha|^p$ is strictly positive for $\alpha \neq 1$ (since the function $\alpha \mapsto p\alpha - p + 1$ is tangent to the convex function $\alpha \mapsto |\alpha|^p$ at $\alpha = 1$). Moreover, $\lim_{\alpha\to 1} |\alpha-1|^p/(p(1-\alpha) - 1 + |\alpha|^p) = 0$. Hence the function

$$h(\alpha) = \frac{|\alpha-1|^p}{p(1-\alpha)-1+|\alpha|^p} 1_{\{\alpha\neq 1\}}$$

is continuous. Its derivative (for $\alpha \neq 1$) is equal to

$$h'(\alpha) = \frac{p|\alpha-1|^{p-2}(\alpha-1)(p(1-\alpha)-1+|\alpha|^p) - |\alpha-1|^p(-p+p|\alpha|^{p-2}\alpha)}{(p(1-\alpha)-1+|\alpha|^p)^2},$$

which is nonpositive if and only if

$$(\alpha-1)\big((p(1-\alpha)-1+|\alpha|^p) - (\alpha-1)(-1+|\alpha|^{p-2}\alpha)\big) \le 0,$$

which we can simplify to

$$(\alpha-1)\big(|\alpha|^{p-2}\alpha - (p-1)\alpha - 2 + p\big) \le 0 \tag{2.2}$$

The function $\alpha \mapsto (p-1)\alpha + 2 - p$ is linear and tangent (at $\alpha = 1$) to the function $\alpha \mapsto |\alpha|^{p-2}\alpha$, which is strictly concave on $(-\infty, 0]$ and strictly convex on $[0,\infty)$. Therefore, the equation $(p-1)\alpha + 2 - p = |\alpha|^{p-2}\alpha$ has exactly one negative solution, which we denote by α_p. Moreover, the inequality (2.2) holds if and only if $\alpha \in [\alpha_p, 1]$. Hence, the function h is increasing on $(-\infty, \alpha_p]$, decreasing on $[\alpha_p, 1]$, and increasing on $[1,\infty)$. The observation that $\lim_{\alpha\to\pm\infty} h(\alpha) = 1$ ends the proof of the first part of the lemma.

Moreover, the inequality (2.2) does hold for $\alpha = -(p-1)^{1/(p-2)}$ and hence $\alpha_p < -(p-1)^{1/(p-2)}$. $\square$

Lemma 2.6. *Let α_p be the number defined for $p > 2$ in Lemma 2.5. Then*

$$\begin{aligned}|\alpha_p^{-1} - 1|^{p-2}(\alpha_p^{-1}-1)\Big(\big((p-1)(1-\alpha_p)\big)^{-1} - 1\Big) &= \frac{|\alpha_p - 1|^p}{p(1-\alpha_p) - 1 + |\alpha_p|^p} \\ &= \frac{(1+|\alpha_p|)^{p-2}}{p-1} > 1.\end{aligned}$$

Proof. We have

$$\big((p-1)(1-\alpha_p)\big)^{-1} - 1 = \frac{(p-1)\alpha_p + 2 - p}{(p-1)(1-\alpha_p)} = \frac{|\alpha_p|^{p-2}\alpha_p}{(p-1)(1-\alpha_p)}$$

and hence the first and third expressions are equal. Also

$$p(1-\alpha_p) - 1 + |\alpha_p|^p = p(1-\alpha_p) - 1 + \alpha_p((p-1)\alpha_p + 2 - p) = (p-1)(\alpha_p - 1)^2$$

and hence the second and third expressions are equal. Finally, the inequality follows directly from the estimate for α_p from the preceding lemma. $\square$

Lemma 2.7. *Let $p \in (1,2)$. If we denote $\alpha_p = (p-1)/p$, then*

$$\begin{aligned}\frac{1}{(p-1)^p} = \frac{|\alpha_p-1|^p}{p(1-\alpha_p)-1+|\alpha_p|^p} &= \sup_{\alpha\le(p-1)/p} \frac{|\alpha-1|^p}{p(1-\alpha)-1+|\alpha|^p} \\ &= |\alpha_p^{-1}-1|^{p-2}(\alpha_p^{-1}-1)\Big(\big((p-1)(1-\alpha_p)\big)^{-1} - 1\Big).\end{aligned}$$

The proof is less involved than in the case $p > 2$. Therefore we leave the details of checking that the function

$$\alpha \mapsto \frac{|\alpha - 1|^p}{p(1-\alpha) - 1 + |\alpha|^p}, \quad \alpha \in (-\infty, (p-1)/p],$$

attains its maximum at $\alpha = (p-1)/p$ to the readers. Let us only remark that in contrast to the case $p > 2$, for $1 < p < 2$ we have

$$\sup_{\alpha \neq 1} \frac{|\alpha - 1|^p}{p(1-\alpha) - 1 + |\alpha|^p} = \infty.$$

We also need the following technical lemma.

Lemma 2.8. *If $p \in (1, 2)$, then*

$$p^{p-2} \geq (p-1)^{p-1},$$
$$(p+1)^{p-1} \geq (2p-1)(p-1)^{p-1}.$$

Proof. Both inequalities are satisfied in the limit for $p \to 1^+$ and $p \to 2^-$. To prove the first, we notice that the difference of the logarithms of both sides is a concave function since

$$\big((p-2)\ln(p) - (p-1)\ln(p-1)\big)'' = \frac{p-2}{(p-1)p^2} \leq 0,$$

for $p \in (1, 2)$.

In order to prove the second inequality, we substitute $s = p-1$, divide both sides by $(2s+1)s^s$, take the logarithm of both sides, and arrive at the following equivalent formulation of the assertion:

$$s \ln(1 + 2/s) - \ln(2s+1) \geq 0, \quad s \in (0, 1).$$

The left-hand side is a concave function, since for $s \in (0, 1)$,

$$\big(s \ln(1 + 2/s) - \ln(2s+1)\big)'' = \frac{4(s^3 - 1)}{s(s+2)^2(2s+1)^2} \leq 0.$$

Hence the assertion of the lemma holds. □

2.5. The special function. Define the constant C_p by the formula

$$C_p^p = \sup_{\alpha \leq (p-1)/p} \frac{|\alpha - 1|^p}{p(1-\alpha) - 1 + |\alpha|^p} = \begin{cases} \frac{1}{(p-1)^p} & \text{if } 1 < p \leq 2, \\ \frac{(1+|\alpha_p|)^{p-2}}{p-1} & \text{if } p > 2. \end{cases}$$

Here $\alpha_p \in \mathbf{R}$, $2 < p < \infty$, is the unique negative solution to the equation $(p-1)\alpha_p + 2 - p = |\alpha_p|^{p-2}\alpha_p$ (see Lemma 2.5). We also denote $\alpha_p = (p-1)/p$ for $1 < p \leq 2$.

We introduce the special functions

$$\begin{aligned} V(x, y) &= |x - y|^p - C_p^p |x|^p, \\ U(x, y) &= -\frac{|1 - \alpha_p|^{p-2}}{p-1} |y|^{p-2} y (px - (p-1)y) \\ &= \begin{cases} -\frac{1}{(p-1)p^{p-2}} |y|^{p-2} y (px - (p-1)y) & \text{if } 1 < p \leq 2, \\ -C_p^p |y|^{p-2} y (px - (p-1)y) & \text{if } p > 2. \end{cases} \end{aligned}$$

The following proposition is the core of the proof of the martingale inequality and the main result (in the case $m = 0$, $\lambda = 1$). Note that the assertion is stronger than the majorization condition from Proposition 2.3.

Proposition 2.9. *For $1 < p < \infty$ and any $x, y \in \mathbf{R}$, we have $V(x,y) \le U(x,y)$.*

Proof for $2 < p < \infty$. The inequality is satisfied for $y = 0$ and $x = y$, so by homogeneity it is enough to consider it for $y = 1$ and $x \neq 1$. We can rewrite it as

$$C_p^p(|x|^p - px + p - 1) \ge |x-1|^p.$$

For $x \neq 1$ the left-hand side is positive (the function $x \mapsto px - p + 1$ is tangent to the convex function $x \mapsto |x|^p$ at $x = 1$), and therefore we conclude that the assertion is equivalent to

$$C_p^p \ge \sup_{x \neq 1} \frac{|x-1|^p}{p(1-x) - 1 + |x|^p},$$

which is true by Lemma 2.5. □

Remark 2.10. The above proof stresses the fact that C_p is chosen exactly so, that the statement is true, but we can also use a slightly different approach. Again, it is enough to consider $y = 1$. The function $V(\cdot,1)$ is continuously differentiable, its second derivative exists in all but two points, and moreover $V_{xx}(x,1) = 0$ if and only if $|x-1|^{p-2} = C_p^p|x|^{p-2}$ or equivalently $x = 1/(1 \pm C_p^{p/(p-2)})$. Hence the function $V(\cdot,1)$ is concave on the interval $(-\infty, a]$, convex on the interval $[a,b]$, and again concave on the interval $[b,\infty)$, where $a = 1/(1 - C_p^{p/(p-2)})$, $b = 1/(1 + C_p^{p/(p-2)})$. Moreover $U(\cdot,1)$ is the tangent to $V(\cdot,1)$ at the points α_p and 1. This implies the inequality $V(x,1) \le U(x,1)$ for $x \in \mathbf{R}$ (see Lemma 3.6 below).

We turn to the case $1 < p < 2$ (for $p = 2$ the assertion is trivial). The argument is similar to that above, but slightly more complicated.

Proof for $1 < p < 2$. The inequality is satisfied for $y = 0$, so by homogeneity it is enough to consider it for $y = 1$. We can rewrite it as (recall that $\alpha_p = (p-1)/p$)

$$C_p^p\left(|x|^p - p\frac{\alpha_p^{p-1}}{1-\alpha_p}(x - \alpha_p)\right) \ge |x-1|^p.$$

It is easy to see that left-hand side is strictly positive (the global minimum is attained for $x = \alpha_p(1-\alpha_p)^{-1/(p-1)}$, for which the expression in the brackets on the left-hand side is equal to

$$|\alpha_p|^p(1-\alpha_p)^{-p/(p-1)}\big(1 - p + p(1-\alpha_p)^{1/(p-1)}\big),$$

which is positive by the first inequality from Lemma 2.8). Therefore we conclude that the assertion is equivalent to the inequality

$$C_p^p \ge \frac{|x-1|^p}{|x|^p - p\frac{\alpha_p^{p-1}}{1-\alpha_p}(x-\alpha_p)} \tag{2.3}$$

holding for every $x \in \mathbf{R}$. We denote the right-hand side of the above inequality by $R(x)$. A calculation shows that $R'(x)$ is positive if and only if

$$\begin{aligned} &p|x-1|^{p-2}(x-1)\left(|x|^p - p\frac{\alpha_p^{p-1}}{1-\alpha_p}(x-\alpha_p)\right) - |x-1|^p\left(p|x|^{p-2}x - p\frac{\alpha_p^{p-1}}{1-\alpha_p}\right) \\ &= p|x-1|^{p-2}(x-1)\big(T(x) - S(x)\big) \ge 0, \end{aligned}$$

where we have denoted

$$S(x) = \frac{\alpha_p^{p-1}}{1-\alpha_p}\big((p-1)x - p\alpha_p + 1\big) = p\alpha_p^{p-1}\big((p-1)x - p + 2\big),$$
$$T(x) = |x|^{p-2}x.$$

The function T is convex on $(-\infty, 0)$ and concave on $(0,\infty)$, since $1 < p < 2$. Moreover, $S(\alpha_p) = T(\alpha_p)$ and

$$S'(\alpha_p) = p(p-1)\alpha_p^{p-1} < (p-1)\alpha_p^{p-2} = T'(\alpha_p).$$

We conclude that the equation $S(x) = T(x)$ has three solutions: $x_1 < 0$, $x_2 = \alpha_p$, and $x_3 > \alpha_p$. Moreover, $x_3 \geq (p+1)/p > 1$ since $S((p+1)/p) \leq T((p+1)/p)$ by the second inequality from Lemma 2.8.

Therefore, the function R is decreasing on each of the intervals $(-\infty, x_1)$, $(\alpha_p, 1)$, (x_3,∞), and increasing on (x_1, α_p) and $(1, x_3)$. Since $R(\alpha_p) = (1/\alpha_p - 1)^p = C_p^p$ and $\lim_{x\to-\infty} R(x) = 1 \leq C_p^p$, in order to prove (2.3) it is enough to check that $C_p^p \geq R(x_3)$. But $S(x_3) = T(x_3) = x_3^{p-1}$, so

$$|x_3|^p - p\frac{\alpha_p^{p-1}}{1-\alpha_p}(x_3 - \alpha_p) = x_3 S(x_3) - p\frac{\alpha_p^{p-1}}{1-\alpha_p}(x_3-\alpha_p) = \frac{\alpha_p^{p-1}}{1-\alpha_p}(p-1)(x_3-1)^2$$

and consequently

$$R(x_3) = \frac{1-\alpha_p}{\alpha_p^{p-1}} \cdot \frac{(x_3-1)^{p-2}}{(p-1)} = \frac{p^{p-2}}{(p-1)^p}(x_3-1)^{p-2}.$$

Hence, $C_p^p \geq R(x_3)$ is equivalent to $1 \leq p(x_3 - 1)$ (recall that $p - 2 < 0$). Since we already know that $x_3 \geq (p+1)/p$, the proof is finished. □

2.6. Proof of the martingale inequality. In order to prove the martingale inequality, we just gather the results of the preceding sections.

Proof of inequality (2.1). For $p \in (1,\infty)$, the functions V and U defined in Section 2.5 satisfy all assumptions of Proposition 2.3. Indeed, the majorization property follows from Proposition 2.9. The initial condition is satisfied, since $U(x, x) = -|x|^p|1-\alpha_p|^{p-2}/(p-1) \leq 0$. If $y \leq 0 \leq x+h$, then $U(x+h, x+h) \leq 0 \leq U(x+h, y)$. If on the other hand $y < x + h < 0$ or $0 < y < x + h$, then there exists $\xi \in (y, x + h)$ such that

$$\begin{aligned} U(x+h, x+h) - U(x+h, y) &= U_y(x+h, \xi)(x+h-y) \\ &= p|1-\alpha_p|^{p-2}|\xi|^{p-2}(\xi - x - h)(x+h-y) \leq 0. \end{aligned}$$

This implies the maximal condition. Finally, U is clearly concave with respect to the first variable. Hence the martingale inequality (2.1) holds by Proposition 2.3. □

Moreover, from the abstract argument in Subsection 2.2 we already know that the constant C_p is optimal. Let us however give explicit extremal examples here.

Sharpness of (2.1). Fix $1 < p < \infty$, $\alpha \in (-\infty, (p-1)/p) \setminus \{0\}$ and $s \in (0, 1)$, and let $\beta = \beta(s)$ be given by the relation $s\alpha + (1-s)\beta = 1$. Observe that if s is sufficiently small (depending on p, α), then $(1-s)\beta^p > 1$ (indeed, the inequality $(1-s\alpha)^p > (1-s)^{p-1}$ can be verified by comparison of derivatives at $s = 0$). We consider a martingale $(f_n)_{n=0}^{\infty}$ such that $f_0 = 1$, and such that conditioned on the event $\{(f_n, f_n^*) = (x, x)\}$, one of the following events occurs:

(1) With probability s we have $f_{n+1} = \alpha x$ and the martingale stops, i.e. $f_{n+1} = f_{n+2} = \ldots$; note that in this case $f^*_{n+1} = f_n = x$.
(2) With probability $1-s$, we have $f_{n+1} = \beta x$ and the evolution continues according to our rules. In this case $f^*_{n+1} = f_{n+1}$.

Note that f_n takes values in the set $\{\alpha, \alpha\beta, \alpha\beta^2, \ldots, \alpha\beta^{n-1}, \beta^n\}$. Moreover, $\mathbf{P}(f_n = \alpha\beta^k) = s(1-s)^k$ for $k \in \{0, \ldots, n-1\}$, $\mathbf{P}(f_n = \beta^n) = (1-s)^n$, and

$$\mathbf{E}\,|f_n|^p 1_{\{f_n \neq \beta^n\}} = \sum_{k=0}^{n-1} s(1-s)^k (|\alpha|\beta^k)^p = s|\alpha|^p \frac{1-(1-s)^n\beta^{np}}{1-(1-s)\beta^p}.$$

Also, $f_n - f^*_n = (1-1/\alpha)f_n$ if $f_n \neq \beta^n$, and $f_n - f^*_n = 0$ if $f_n = \beta^n$. Hence, the p-th power of the constant with which the martingale inequality (2.1) holds has to be equal at least

$$\begin{aligned}\lim_{s\to 0^+}\lim_{n\to\infty} \frac{\|f_n - f^*_n\|_p^p}{\|f_n\|_p^p} &= \lim_{s\to 0^+}\lim_{n\to\infty} \frac{|1-1/\alpha|^p s|\alpha|^p \frac{1-(1-s)^n\beta^{np}}{1-(1-s)\beta^p}}{s|\alpha|^p \frac{1-(1-s)^n\beta^{np}}{1-(1-s)\beta^p} + (1-s)^n\beta^{np}} \\ &= \lim_{s\to 0^+} \frac{|\alpha-1|^p s}{s|\alpha|^p + (1-s)\beta^p - 1} = \frac{|\alpha-1|^p}{|\alpha|^p - p\alpha + p - 1},\end{aligned}$$

where we used $(1-s)\beta^p > 1$ in the second last equality, and $\beta = (1-s\alpha)/(1-s)$ in the last equality. To obtain the sharpness of (2.1) we take $\alpha \to (p-1)/p^-$ in the case $1 < p \leq 2$ or take $\alpha = \alpha_p$ (see Lemma 2.5) in the case $p > 2$. □

Remark 2.11. In the above example f_n converges a.s. to a random variable f_∞, but $\mathbf{E}\,|f_\infty|^p = +\infty$. In the case $1 < p \leq 2$ one can consider $\alpha > (p-1)/p$ instead of $\alpha < (p-1)/p$ to obtain an example in which $\mathbf{E}\,|f_\infty|^p < \infty$.

2.7. Relation to Theorem 1.1 for $m = 0$. As announced before, the martingale inequality implies the main result for $m = 0$ and $\lambda = 1$ in the special case of nonincreasing functions.

Proof of Corollary 2.2. First note that a standard approximation arguments yields a continuous time version of (2.1): for any martingale $(X_t)_{t\geq 0}$ with right-continuous trajectories, we have

$$\|X_t - \sup_{0\leq s\leq t} X_s\|_p \leq C_p \|X_t\|_p.$$

Let $f \in L^p([0,1])$ be a nonincreasing function. On the probability space $[0,1]$, equipped with the Lebesgue measure, consider the filtration

$$\mathcal{F}_t = \sigma\big([0, 1-t], \mathcal{B}([1-t, 1])\big), \quad t \in [0,1],$$

(i.e. the σ-algebra $\mathcal{F}_t$ is generated by the set $[0, 1-t]$ and all Borel subsets of the interval $[1-t, 1]$) and the martingale $X_t = \mathbf{E}(f|\mathcal{F}_t)$, $t \in [0,1]$. Using the definition of the filtration, we get the explicit formula

$$X_t(\omega) = \begin{cases} \dfrac{1}{1-t}\displaystyle\int_0^{1-t} f(s)\,ds & \text{if } \omega \in [0, 1-t), \\ f(\omega) & \text{if } \omega \in [1-t, 1]. \end{cases}$$

In particular, the martingale is right-continuous. Hence,

$$
\begin{aligned}
C_p^p\|f\|_{L^p([0,1])}^p &= C_p^p\,\mathbf{E}\,|X_1|^p \geq \mathbf{E}\,|X_1 - \sup_{0\leq t\leq 1} X_t|^p \\
&= \int_0^1 \Big|f(\omega) - \sup_{0\leq t\leq 1}\Big\{1_{\{\omega<1-t\}}\frac{1}{1-t}\int_0^{1-t} f(s)\,ds + 1_{\{\omega\geq 1-t\}}f(\omega)\Big\}\Big|^p\,d\omega \\
&= \int_0^1 \big|f(\omega) - \frac{1}{\omega}\int_0^{\omega} f(s)\,ds\big|^p\,d\omega = \|f - H_0 f\|_{L^p([0,1])}^p,
\end{aligned}
$$

where the second last equality follows from the fact that f is nonincreasing. A rescaling argument (see proof of inequality (1.1) in Section 3) yields $C_p\|f\|_{L^p([0,\infty))} \geq \|f - H_0 f\|_{L^p([0,\infty))}$ for nonincreasing functions $f \in L^p([0,\infty))$. □

Let us now briefly sketch of the proof of the main result for $m = 0$, $\lambda = 1$, and not necessarily nonincreasing functions. The key idea is to use the tools from the proof of the martingale inequality proof, i.e. the special functions V and U from Section 2.5, rather than to apply it directly. The following lemma is needed (its proof is based on integration by parts and we shall prove a more general result later, cf. Lemma 3.5).

Lemma 2.12. *If $1 < p < \infty$ and $f\colon [0,1] \to \mathbf{R}$ is continuous, then*

$$
(p-1)\int_0^1 |H_0 f(t)|^p\,dt \leq p\int_0^1 |H_0 f(t)|^{p-2} H_0 f(t) f(t)\,dt.
$$

Instead of looking at the values of V, U in the points $(f_n, \sup_{k\leq n} f_k)$, where $(f_n)_{n=0}^{\infty}$ is a martingale, we consider their values in the points $(f(t), H_0 f(t))$, where $f\colon [0,1] \to \mathbf{R}$ is a continuous function. We use Proposition 2.9 with $x = f(t)$ and $y = H_0 f(t)$, integrate over $t \in [0,1]$, and apply Lemma 2.12, arriving at

$$
\begin{aligned}
&\int_0^1 |f(t) - H_0 f(t)|^p\,dt - C_p^p\int_0^1 |f(t)|^p\,dt \\
&\leq -\frac{|1-\alpha_p|^{p-2}}{p-1}\int_0^1 |H_0 f(t)|^{p-2} H_0 f(t)\big(pf(t) - (p-1)H_0 f(t)\big)\,dt \leq 0.
\end{aligned}
$$

Standard approximation and a simple scaling argument gives $\|f - H_0 f\|_{L^p([0,\infty))} \leq C_p\|f\|_{L^p([0,\infty))}$. As for sharpness, it is enough to consider the functions

$$
f_\alpha(t) = 1_{\{t\in[0,1)\}} + \alpha t^{\alpha-1} 1_{\{t\in[1,\infty)\}}, \quad \alpha < (p-1)/p.
$$

It turns out that this approach can be adapted to work in a more general setting. This is done with all details in the next section.

3. Proof of Theorem 1.1

3.1. Notation, preliminary results. For $1 < p < \infty$ and $m > -2(p-1)/p$, we denote

$$
g_{p,m} = \frac{m}{2} + \frac{p-1}{p}.
$$

Clearly, $g_{p,m}$ is a positive number. Moreover, we have the following slight extension of [3, Proposition 5.1].

Proposition 3.1. *For $1 < p < \infty$ and $m > -2(p-1)/p$, the formula*

$$
H_m f(t) = \frac{1}{t^{1+m/2}}\int_0^t f(s) s^{m/2}\,ds
$$

defines a bounded operator on the space $L^p([0,\infty))$. Moreover,

$$\|H_m\|_{L^p([0,\infty))\to L^p([0,\infty))} = g_{p,m}^{-1}.$$

Proof. If $f \in L^p([0,\infty))$ is bounded, then the function $H_m f$ is well defined (since $m/2 > -1$), and the Minkowski integral inequality yields

$$\begin{aligned}
\|H_m f\|_p &= \Big(\int_0^\infty \Big|\frac{1}{t^{1+m/2}}\int_0^t f(s)s^{m/2}\,ds\Big|^p\,dt\Big)^{1/p} \\
&= \Big(\int_0^\infty \Big|\int_0^1 f(ut)u^{m/2}\,du\Big|^p\,dt\Big)^{1/p} \le \int_0^1 \Big(\int_0^\infty |f(ut)|^p\,dt\Big)^{1/p} u^{m/2}\,du \\
&= \int_0^1 u^{m/2-1/p}\|f\|_p\,du = g_{p,m}^{-1}\|f\|_p.
\end{aligned}$$

A standard density argument implies the claim. The family $t \mapsto t^{-1/p+\varepsilon}1_{\{t\in[0,1]\}}$ extremizes the norm as $\varepsilon \to 0^+$. □

We define the set $\Omega_{p,m}$, the function $c_{p,m,\lambda}\colon \Omega_{p,m} \to \mathbf{R}$, and the constant $C_{p,m,\lambda}$ by the formulas:

$$\begin{aligned}
\Omega_{p,m} &= \{(\alpha,\beta) \in \mathbf{R}^2 \colon \alpha < g_{p,m} < \beta\}, \\
c_{p,m,\lambda}(\alpha,\beta) &= \Big(\frac{(\beta - g_{p,m})|\alpha-\lambda|^p + (g_{p,m}-\alpha)|\beta-\lambda|^p}{(\beta-g_{p,m})|\alpha|^p + (g_{p,m}-\alpha)|\beta|^p}\Big)^{1/p}, \quad (\alpha,\beta) \in \Omega_{p,m}, \\
C_{p,m,\lambda} &= \sup\{c_{p,m,\lambda}(\alpha,\beta)\colon (\alpha,\beta) \in \Omega_{p,m}\}.
\end{aligned}$$

Recall that our aim is to show that $C_{p,m,\lambda} = \|I - \lambda H_m\|_{L^p([0,\infty))\to L^p([0,\infty))}$. The first lemma shows that $C_{p,m,\lambda}$ is a lower bound for the norm of the operator $I - \lambda H_m$. It also serves as a proof that $C_{p,m,\lambda}$ is finite.

Lemma 3.2. *If $1 < p < \infty$, $m > -2(p-1)/p$, $\lambda \in \mathbf{R}$, then*

$$\|I - \lambda H_m\|_{L^p([0,\infty))\to L^p([0,\infty))} \ge C_{p,m,\lambda}.$$

Proof. Fix $1 < p < \infty$ and $m > -2(p-1)/p$. For $\alpha < g_{p,m} < \beta$, consider the function

$$f_{\alpha,\beta}(t) = \beta t^{\beta - g_{p,m} - 1/p}1_{\{t\in[0,1)\}} + \alpha t^{\alpha - g_{p,m}-1/p}1_{\{t\in[1,\infty)\}}, \tag{3.1}$$

which clearly belongs to the space $L^p([0,\infty))$. We have

$$H_m f_{\alpha,\beta}(t) = t^{\beta-g_{p,m}-1/p}1_{\{t\in[0,1)\}} + t^{\alpha-g_{p,m}-1/p}1_{\{t\in[1,\infty)\}}$$

and

$$\begin{aligned}
\|f_{\alpha,\beta} - \lambda H_m f_{\alpha,\beta}\|_p^p &= \frac{|\beta-\lambda|^p}{p(\beta-g_{p,m})} - \frac{|\alpha-\lambda|^p}{p(\alpha-g_{p,m})}, \\
\|f_{\alpha,\beta}\|_p^p &= \frac{|\beta|^p}{p(\beta-g_{p,m})} - \frac{|\alpha|^p}{p(\alpha-g_{p,m})}.
\end{aligned}$$

Thus we see that $\|I - \lambda H_m\|_{L^p([0,\infty))\to L^p([0,\infty))} \ge C_{p,m,\lambda}$. □

The next lemma summarizes further observations about the constant $C_{p,m,\lambda}$.

Lemma 3.3. *If $1 < p < \infty$, $m > -2(p-1)/p$, and $\lambda \in \mathbf{R}$, then*

$$C_{p,m,\lambda} \ge \max\{|1 - \lambda g_{p,m}^{-1}|, 1\}.$$

Also, if the above inequality is strict, then the supremum in the definition of $C_{p,m,\lambda}$ is attained at some point of the set $\Omega_{p,m}$. Moreover, $C_{p,m,\lambda} > 1$ unless $\lambda = 0$ or $p = 2$.

Proof. Throughout the proof we consider only $(\alpha,\beta)\in\Omega_{p,m}$. For $\alpha\neq 0$, we can write the function $c^p_{p,m,\lambda}$ as a convex combination:

$$c^p_{p,m,\lambda}(\alpha,\beta)=w_1(\alpha,\beta)\cdot|1-\lambda/\alpha|^p+w_2(\alpha,\beta)\cdot|1-\lambda/\beta|^p, \tag{3.2}$$

where

$$\begin{aligned} w_1(\alpha,\beta)&=\frac{(\beta-g_{p,m})|\alpha|^p}{(\beta-g_{p,m})|\alpha|^p+(g_{p,m}-\alpha)|\beta|^p},\\ w_2(\alpha,\beta)&=\frac{(g_{p,m}-\alpha)|\beta|^p}{(\beta-g_{p,m})|\alpha|^p+(g_{p,m}-\alpha)|\beta|^p}. \end{aligned}$$

Using (3.2) we see that

$$\lim_{\alpha,\beta} c^p_{p,m,\lambda}(\alpha,\beta)=\begin{cases}1 & \text{if } \alpha\to-\infty \text{ and } \beta\to\infty,\\ |1-\lambda g_{p,m}^{-1}|^p & \text{if } \alpha\to g^-_{p,m} \text{ and } \beta\to g^+_{p,m},\end{cases}$$

which implies the first part of the assertion.

We now claim, that if $(\alpha,\beta)\to\partial\Omega_{p,m}$ or $\alpha^2+\beta^2\to\infty$, then

$$\limsup_{\alpha,\beta} c^p_{p,m,\lambda}(\alpha,\beta)\le\max\{1,|1-\lambda g_{p,m}^{-1}|^p\}. \tag{3.3}$$

It follows from (3.2) that (3.3) holds if $\alpha\to-\infty$ and $\beta\to g^+_{p,m}$, or $\alpha\to g^-_{p,m}$ and $\beta\to\infty$. If on the other hand $\alpha\to-\infty$ or $\alpha\to g^-_{p,m}$, and $\beta\to\beta_\infty\in(g_{p,m},\infty)$, then $w_2(\alpha,\beta)\to 0$, and consequently

$$\lim_{\alpha,\beta} c^p_{p,m,\lambda}(\alpha,\beta)\in\{1,|1-\lambda g_{p,m}^{-1}|\}.$$

Similarly, (3.3) also holds if $\alpha\to\alpha_\infty\in(-\infty,g_{p,m})\setminus\{0\}$ and $\beta\to g^+_{p,m}$ or $\beta\to\infty$ (because then $w_1(\alpha,\beta)\to 0$). Finally, if $\alpha\to 0$ and $\beta\to g^+_{p,m}$ or $\beta\to\infty$, then

$$\begin{aligned}\limsup_{\alpha,\beta} c^p_{p,m,\lambda}(\alpha,\beta)&\le\lim_{\alpha,\beta}\frac{(\beta-g_{p,m})|\alpha-\lambda|^p+(g_{p,m}-\alpha)|\beta-\lambda|^p}{(g_{p,m}-\alpha)|\beta|^p}\\ &=\max\{1,|1-\lambda g_{p,m}^{-1}|\}.\end{aligned}$$

These observations imply that if the supremum in the definition of $C_{p,m,\lambda}$ is strictly greater than $\max\{1,|1-\lambda g_{p,m}^{-1}|\}$, then it is attained at some point of the set $\Omega_{p,m}$.

The last part of the assertion clearly holds if $\lambda<0$. Assume henceforth that $\lambda>0$ and $p\neq 2$. Choose $A>0$ and $B>\max\{\lambda-g_{p,m},0\}$ so that the inequality

$$A^{p-1}B>(g_{p,m}+A)(B+g_{p,m})^{p-1}$$

is satisfied (i.e. pick A sufficiently large if $p>2$ or B sufficiently large if $1<p<2$). Since

$$\begin{aligned}(A+\lambda)^p-A^p&\ge p\lambda A^{p-1},\\ p\lambda(B+g_{p,m})^{p-1}&\ge(g_{p,m}+B)^p-(g_{p,m}+B-\lambda)^p,\end{aligned}$$

such a choice of A,B implies that

$$((A+\lambda)^p-A^p)B>(g_{p,m}+A)((g_{p,m}+B)^p-(g_{p,m}+B-\lambda)^p),$$

which is equivalent to $c_{p,m,\lambda}(-A,g_{p,m}+B)>1$. $\square$

Remark 3.4. We can see that $\|I-\lambda H_m\|_{L^p([0,\infty))\to L^p([0,\infty))} \geq 1$ in another way. Indeed, for $f_n(t) = 1_{\{t\in[n,n+1)\}}$ have $\|f_n\|_p = 1$, but

$$\begin{aligned}
&\|H_m f_n\|_p^p \\
&= \int_0^\infty \Big|\frac{t^{1+m/2}-n^{1+m/2}}{(1+m/2)t^{1+m/2}}1_{\{t\in[n,n+1)\}} + \frac{(n+1)^{1+m/2}-n^{1+m/2}}{(1+m/2)t^{1+m/2}}1_{\{t\in[n+1,\infty)\}}\Big|^p\,dt \\
&\leq \int_n^\infty \Big(\frac{(n+1)^{1+m/2}-n^{1+m/2}}{(1+m/2)t^{1+m/2}}\Big)^p\,dt \\
&= \frac{((n+1)^{1+m/2}-n^{1+m/2})^p}{(1+m/2)^p(p+pm/2-1)n^{p+pm/2-1}} \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

Hence the operator $\lambda H_m\colon L^p([0,\infty)) \to L^p([0,\infty))$ is not invertible, and consequently we cannot have $\|I-\lambda H_m\|_{L^p([0,\infty))\to L^p([0,\infty))} < 1$.

3.2. Key tools. The first result generalizes Lemma 2.12 presented above without proof.

Lemma 3.5. *If $1<p<\infty$, $m>-2(p-1)/p$, and $f\colon[0,1]\to\mathbf{R}$ is continuous, then*

$$(p(1+m/2)-1)\int_0^1 |H_m f(t)|^p\,dt \leq p\int_0^1 |H_m f(t)|^{p-2}H_m f(t)f(t)\,dt.$$

Proof. Define $F(t) = \int_0^t f(s)s^{m/2}\,ds$. Since f is continuous, we have $F'(t) = f(t)t^{m/2}$ (in particular, $H_m f(t) = F(t)/t^{1+m/2} \to f(0)/(1+m/2)$ as $t\to 0^+$). Hence integration by parts yields

$$\begin{aligned}
&(p(1+m/2)-1)\int_0^1 |H_m f(t)|^p dt = (p(1+m/2)-1)\int_0^1 t^{-p(1+m/2)}|F(t)|^p\,dt \\
&= \big[-t^{-p(1+m/2)+1}|F(t)|^p\big]_0^1 + p\int_0^1 t^{-p(1+m/2)+1}|F(t)|^{p-2}F(t)f(t)t^{m/2}\,dt \\
&= -|F(1)|^p + \lim_{t\to 0^+} t|F(t)/t^{1+m/2}|^p + p\int_0^1 |H_m f(t)|^{p-2}H_m f(t)f(t)\,dt \\
&= -|F(1)|^p + p\int_0^1 |H_m f(t)|^{p-2}H_m f(t)f(t)\,dt.
\end{aligned}$$

This implies the assertion of the lemma. □

Moreover, the following elementary lemma is useful for us.

Lemma 3.6. *Suppose that $v\colon\mathbf{R}\to\mathbf{R}$ is continuously differentiable and strictly concave on $(-\infty,a)$, strictly convex on (a,b), and strictly concave on (b,∞) for some $a,b\in\mathbf{R}$. Let $u\colon\mathbf{R}\to\mathbf{R}$ be an affine function tangent to v at two points. Then $v(x)\leq u(x)$ for $x\in\mathbf{R}$.*

Proof. Denote $c=u'(x)$, $x\in\mathbf{R}$. There exist $\alpha<\beta$, such that $v(\alpha)=u(\alpha)$, $v(\beta)=u(\beta)$, and $v'(\alpha)=v'(\beta)=c$. By Rolle's theorem applied to the function $v-u$, there exists $\gamma\in(\alpha,\beta)$ such that $v'(\gamma)=c$. Since the function v' attains every value at most thrice, we conclude that $\alpha\in(-\infty,a]$, $\gamma\in(a,b)$, and $\beta\in[b,\infty)$. The assertion follows from the assumption about concavity (respectively convexity) of v on those intervals. □

Finally, we have the following analog and generalization of Proposition 2.9.

Proposition 3.7. *If $1 < p < \infty$, $p \neq 2$, $m > -2(p-1)/p$, and $\lambda > 0$, then there exists a positive constant $D_{p,m,\lambda}$, such that the inequality*

$$|x - \lambda y|^p - C^p_{p,m,\lambda}|x|^p \leq -D_{p,m,\lambda}|y|^{p-2}y(x - g_{p,m}y)$$

holds for all $x, y \in \mathbf{R}$.

Proof. We shall consider two cases. As will be clear from the proof (and the following results) they correspond to the situation when the supremum in the definition of $C_{p,m,\lambda}$ is equal to $|\lambda g^{-1}_{p,m} - 1|$, and the situation when the supremum in definition of $C_{p,m,\lambda}$ is attained in the interior of the set $\Omega_{p,m}$ (cf. Lemma 3.3).

For the first case, we assume that $\lambda > 2g_{p,m}$ and the inequality

$$|x - \lambda|^p - (\lambda g^{-1}_{p,m} - 1)^p|x|^p \leq -p(\lambda - g_{p,m})^{p-1}\lambda g^{-1}_{p,m}(x - g_{p,m}) \tag{3.4}$$

holds for all $x \in \mathbf{R}$. Define

$$\begin{aligned} V(x,y) &= |x - \lambda y|^p - (\lambda g^{-1}_{p,m} - 1)^p|x|^p, \\ U(x,y) &= -p(\lambda - g_{p,m})^{p-1}\lambda g^{-1}_{p,m}|y|^{p-2}y(x - g_{p,m}y). \end{aligned}$$

The inequality $V(x,y) \leq U(x,y)$ holds for $y = 0$ (since $\lambda g^{-1}_{p,m} - 1 > 1$) and for $y = 1$ (due to (3.4)), and hence by homogeneity for all $x, y \in \mathbf{R}$. Hence the assertion is satisfied with $D_{p,m,\lambda} = p(\lambda - g_{p,m})^{p-1}\lambda g^{-1}_{p,m} > 0$ (and with $\lambda g^{-1}_{p,m} - 1$, which is not greater than $C_{p,m,\lambda}$, in the place of $C_{p,m,\lambda}$). This finishes the proof in the first case.

Let us now consider the second case: we have either $\lambda \in (0, 2g_{p,m}]$, or we have $\lambda > 2g_{p,m}$, but the inequality (3.4) does not hold for all $x \in \mathbf{R}$. We claim that the supremum in the definition of the constant $C_{p,m,\lambda}$ is attained at some point of the set $\Omega_{p,m}$. Indeed, if $\lambda \in (0, 2g_{p,m}]$, then $\max\{|1 - \lambda g^{-1}_{p,m}|, 1\} = 1$ and Lemma 3.3 implies the claim. If on the other hand $\lambda > 2g_{p,m}$ and the inequality (3.4) does not hold for every $x \in \mathbf{R}$, then there exists some $x_0 \in \mathbf{R}$, such that

$$|x_0 - \lambda|^p - (\lambda g^{-1}_{p,m} - 1)^p|x_0|^p > -p(\lambda - g_{p,m})^{p-1}\lambda g^{-1}_{p,m}(x_0 - g_{p,m}). \tag{3.5}$$

Of course we cannot have $x_0 = g_{p,m}$. Suppose first, that $x_0 > g_{p,m}$. Since

$$\lim_{x \to g_{p,m}} \frac{|x - \lambda|^p - (\lambda g^{-1}_{p,m} - 1)^p|x|^p}{x - g_{p,m}} = -p(\lambda - g_{p,m})^{p-1}\lambda g^{-1}_{p,m},$$

we conclude from (3.5), that for some $(\alpha, \beta) \in \Omega_{p,m}$ we have

$$|\beta - \lambda|^p - (\lambda g^{-1}_{p,m} - 1)^p|\beta|^p > \frac{|\alpha - \lambda|^p - (\lambda g^{-1}_{p,m} - 1)^p|\alpha|^p}{\alpha - g_{p,m}}(\beta - g_{p,m})$$

(it suffices to take $\beta = x_0$ and α smaller than, but close to $g_{p,m}$) or equivalently

$$\frac{(\beta - g_{p,m})|\alpha - \lambda|^p + (g_{p,m} - \alpha)|\beta - \lambda|^p}{(\beta - g_{p,m})|\alpha|^p + (g_{p,m} - \alpha)|\beta|^p} > (\lambda g^{-1}_{p,m} - 1)^p.$$

We arrive at the same conclusion, if $x_0 < g_{p,m}$ (it suffices to take $\alpha = x_0$ and β greater than, but close to $g_{p,m}$). This finishes the proof of the claim: in the second case we always have

$$C^p_{p,m,\lambda} = c^p_{p,m,\lambda}(\alpha_0, \beta_0),$$

for some point $(\alpha_0, \beta_0) \in \Omega_{p,m}$ (of course α_0, β_0 may depend of p, m, and λ; uniqueness is not important to us).

After denoting

$$K(\alpha,\beta) = (\beta - g_{p,m})|\alpha-\lambda|^p + (g_{p,m}-\alpha)|\beta-\lambda|^p,$$
$$L(\alpha,\beta) = (\beta - g_{p,m})|\alpha|^p + (g_{p,m}-\alpha)|\beta|^p,$$

we can rewrite the condition $\frac{\partial}{\partial\alpha} c^p_{p,m,\lambda}(\alpha_0,\beta_0) = 0$ as

$$\begin{aligned} &\big(p(\beta_0 - g_{p,m})|\alpha_0-\lambda|^{p-2}(\alpha_0-\lambda) - |\beta_0-\lambda|^p\big)\cdot L(\alpha_0,\beta_0) \\ &- K(\alpha_0,\beta_0)\cdot\big(p(\beta_0-g_{p,m})|\alpha_0|^{p-2}\alpha_0 - |\beta_0|^p\big) = 0. \end{aligned} \tag{3.6}$$

The condition $\frac{\partial}{\partial\beta} c^p_{p,m,\lambda}(\alpha_0,\beta_0) = 0$ implies a similar equation.

Define now

$$V(x,y) = |x-\lambda y|^p - C^p_{p,m,\lambda}|x|^p,$$
$$U(x,y) = \frac{V(\beta_0,1)-V(\alpha_0,1)}{\beta_0-\alpha_0}|y|^{p-2}y(x-g_{p,m}y).$$

Using the fact that $C^p_{p,m,\lambda} = K(\alpha_0,\beta_0)/L(\alpha_0,\beta_0)$, we see that

$$V(\alpha_0,1) = \frac{(g_{p,m}-\alpha_0)\big(|\alpha_0-\lambda|^p|\beta_0|^p - |\alpha_0|^p|\beta_0-\lambda|^p\big)}{L(\alpha_0,\beta_0)},$$
$$V(\beta_0,1) = \frac{(\beta_0-g_{p,m})\big(|\alpha_0|^p|\beta_0-\lambda|^p - |\alpha_0-\lambda|^p|\beta_0|^p\big)}{L(\alpha_0,\beta_0)},$$

and consequently

$$\frac{V(\beta_0,1)-V(\alpha_0,1)}{\beta_0-\alpha_0} = \frac{|\alpha_0|^p|\beta_0-\lambda|^p - |\alpha_0-\lambda|^p|\beta_0|^p}{L(\alpha_0,\beta_0)}. \tag{3.7}$$

Hence $V(\alpha_0,1) = U(\alpha_0,1)$ and $V(\beta_0,1) = U(\beta_0,1)$.

On the other hand (we use (3.6) in the second equality, and the definitions of K and L in the third),

$$\begin{aligned} V_x(\alpha_0,1) &= \frac{p|\alpha_0-\lambda|^{p-2}(\alpha_0-\lambda)L(\alpha_0,\beta_0) - K(\alpha_0,\beta_0)p|\alpha_0|^{p-2}\alpha_0}{L(\alpha_0,\beta_0)} \\ &= \frac{L(\alpha_0,\beta_0)|\beta_0-\lambda|^p - K(\alpha_0,\beta_0)|\beta_0|^p}{L(\alpha_0,\beta_0)(\beta_0-g_{p,m})} = \frac{|\alpha_0|^p|\beta_0-\lambda|^p - |\alpha_0-\lambda|^p|\beta_0|^p}{L(\alpha_0,\beta_0)}. \end{aligned}$$

By (3.7), we conclude that $V_x(\alpha_0,1) = U_x(\alpha_0,1)$. Similarly, $V_x(\beta_0,1) = U_x(\beta_0,1)$ follows from $\frac{\partial}{\partial\beta} c^p_{p,m,\lambda}(\alpha_0,\beta_0) = 0$.

Therefore, $U(\cdot,1)\colon x \mapsto U(x,1)$ is tangent to $V(\cdot,1)\colon x\mapsto V(x,1)$ at $x=\alpha_0$ and $x=\beta_0$. Hence Lemma 3.6 implies that $V(x,1)\le U(x,1)$ for any $x\in\mathbf{R}$, and by homogeneity also $V(x,y)\le U(x,y)$ for any $x,y\in\mathbf{R}$ (for $y=0$ the inequality holds, since $C_{p,m,\lambda}>1$).

Finally, let us notice that

$$V_x(\beta_0,1) = p(|\beta_0-\lambda|^{p-2}(\beta_0-\lambda) - C^p_{p,m,\lambda}\beta_0^{p-1}) \le p\beta_0^{p-1}(1-C^p_{p,m,\lambda}) < 0$$

(we used the fact that $\beta_0>0$, $\lambda>0$, and $C_{p,m,\lambda}>1$) and hence the assertion is satisfied with

$$D_{p,m,\lambda} := -U_x(\beta_0,1) = -V_x(\beta_0,1) > 0. \qquad \square$$

Remark 3.8. The above proof together with the results of Section 3.3 can be used to see that if m is a positive integer then we have $C_{p,m,\lambda} > \lambda g_{p,m}^{-1} - 1$ for $1 < p < 2$ and $\lambda = 1 + m$. This follows from the fact that for such m, p, and λ the inequality (3.4) does not hold. Indeed, if $m \geq 2$, then

$$\sup_{x \in \mathbf{R}} -(\lambda g_{p,m}^{-1} - 1)^p |x|^p + p(\lambda - g_{p,m})^{p-1} \lambda g_{p,m}^{-1}(x - g_{p,m}) > 0,$$

since the supremum is attained at $x = g_{p,m}\big(\lambda/(\lambda - g_{p,m})\big)^{1/(p-1)}$ and we need to check that $p(\lambda - g_{p,m})^{1/(p-1)} < (p-1)\lambda^{1/(p-1)}$ or equivalently $p^{p-1}(m/2 + 1/p) < (p-1)^{p-1}(1+m)$. It is enough to verify this inequality for $m = 2$ and in order to do this one can proceed like in the proof of Lemma 2.8. In the case $m = 1$ (and $\lambda = 2$) we claim that

$$|x - \lambda|^p - (\lambda g_{p,m}^{-1} - 1)^p |x|^p + p(\lambda - g_{p,m})^{p-1} \lambda g_{p,m}^{-1}(x - g_{p,m}) > 0$$

for $x = g_{p,m} + 2p g_{p,m}(\lambda - g_{p,m})$ (note that $x > \lambda$). For such a choice of x the above inequality is equivalent to

$$(3p - 3)^p - (p + 3)^p + 4p^2 > 0,$$

which is true since we have equality for $p \in \{1, 2\}$, and one can numerically check that the second derivative of the function on the left-hand side is negative on the interval $(1, 2)$. Thus we are done.

3.3. Proof of the main result. We start with the following observation (cf. [17]).

Lemma 3.9. *Let $T\colon L^p([0,\infty)) \to L^p([0,\infty))$ be a linear operator which maps real-valued functions to real-valued functions. If the inequality $\|Tf\|_p \leq C\|f\|_p$ holds for any real-valued function $f \in L^p([0,\infty))$, then it also holds (with the same constant) for any complex-valued function $f \in L^p([0,\infty))$.*

Proof. Suppose that $f = u + iv \in L^p([0,\infty))$, where u, v are real-valued. Let G_1, G_2 be independent Gaussian random variables with mean zero and variance one. Using the fact that for $a_1, a_2 \in \mathbf{R}$ the random variable $a_1 G_1 + a_2 G_2$ has the same distribution as $\sqrt{a_1^2 + a_2^2}\, G_1$, we arrive at

$$\begin{aligned}\|Tf\|_p^p\, \mathbf{E}\,|G_1|^p &= \int_0^\infty |T(u)^2 + T(v)^2|^{p/2}\, \mathbf{E}\,|G_1|^p\, dt = \int_0^\infty \mathbf{E}\,|T(u)G_1 + T(v)G_2|^p\, dt \\ &= \mathbf{E} \int_0^\infty |T(uG_1 + vG_2)|^p\, dt \leq C^p\, \mathbf{E} \int_0^\infty |uG_1 + vG_2|^p\, dt \\ &= C^p \|f\|_p^p\, \mathbf{E}\,|G_1|^p\end{aligned}$$

(we have suppressed the dependence of the functions on the argument $t \in [0,\infty)$ in the notation). This finishes the proof of the lemma. □

Henceforth we assume without loss of generality that all functions are real-valued. The proof of Theorem 1.1 is divided into three parts.

Proof of inequality (1.1) *for $\lambda > 0$ and $p \neq 2$.* Fix $1 < p < \infty$, $p \neq 2$, $m > 2(p-1)/p$, $\lambda \in \mathbf{R}$, and denote

$$\begin{aligned}V(x,y) &= |x - \lambda y|^p - C_{p,m,\lambda}^p |x|^p, \\ U(x,y) &= -D_{p,m,\lambda}|y|^{p-2} y(x - g_{p,m} y),\end{aligned}$$

where $D_{p,m,\lambda}$ is the positive number from Proposition 3.7.

Let $f\colon [0,1] \to \mathbf{R}$ be a continuous function. By Proposition 3.7, for every $t \in [0,1]$ we have $V(f(t), H_m f(t)) \le U(f(t), H_m f(t))$. After integrating over the interval $[0,1]$ and applying Lemma 3.5, we arrive at

$$\begin{aligned}&\int_0^1 |f(t) - \lambda H_m f(t)|^p\,dt - C_{p,m,\lambda}^p \int_0^1 |f(t)|^p\,dt\\&\le -D_{p,m,\lambda} \int_0^1 |H_m f(t)|^{p-2} H_m f(t)\big(f(t) - g_{p,m} H_m f(t)\big)\,dt \le 0.\end{aligned}$$

A standard approximation argument gives $\|f - \lambda H_m f\|_{L^p([0,1])} \le C_{p,m,\lambda}\|f\|_{L^p([0,1])}$ for $f \in L^p([0,1])$. If $f \in L^p([0,\infty))$, then

$$\begin{aligned}\int_0^n |f(t) - \lambda H_m f(t)|^p\,dt &= n\int_0^1 |f(nt) - \lambda H_m(f(n\cdot))(t)|^p\,dt\\&\le nC_{p,m,\lambda}^p \int_0^1 |f(nt)|^p\,dt = C_{p,m,\lambda}^p \int_0^n |f(t)|^p\,dt,\end{aligned}$$

and it suffices to take $n \to \infty$ to arrive at $\|f - \lambda H_m f\|_{L^p([0,\infty))} \le C_{p,m,\lambda}\|f\|_{L^p([0,\infty))}$. This ends the proof of inequality (1.1). □

Proof of inequality (1.1) *for* $\lambda > 0$ *and* $p = 2$. By an argument similar to that for $p \ne 2$, we show that $\|I - \lambda H_m\|_{L^2([0,\infty))\to L^2([0,\infty))} \le \lambda g_{p=2,m}^{-1} - 1$ for $\lambda > 2g_{p=2,m} = 1+m$. Indeed, we only need to use the functions

$$\begin{aligned}V(x,y) &= (x - \lambda y)^2 - (\lambda g_{p=2,m}^{-1} - 1)^2 x^2,\\U(x,y) &= -2(\lambda - g_{p=2,m})\lambda g_{p=2,m}^{-1} y(x - g_{p=2,m} y),\end{aligned}$$

for which checking the majorization is straightforward ($x \mapsto U(x,1)$ is the tangent to the concave function $x \mapsto V(x,1)$ at $x = g_{p=2,m}$). We conclude that $\|I - \lambda H_m\|_{L^2([0,\infty))\to L^2([0,\infty))} = \lambda g_{p=2,m}^{-1} - 1 = C_{p=2,m,\lambda}$ for $\lambda > 1 + m$.

Moreover, the L^2-norm of the operator $I - \lambda H_m$ is clearly a convex function of the variable λ:

$$\begin{aligned}&\|I - (s\lambda_1 + (1-s)\lambda_2)H_m\|_{L^2([0,\infty))\to L^2([0,\infty))}\\&= \|s(I - \lambda_1 H_m) + (1-s)(I - \lambda_2 H_m)\|_{L^2([0,\infty))\to L^2([0,\infty))}\\&\le s\|I - \lambda_1 H_m\|_{L^2([0,\infty))\to L^2([0,\infty))} + (1-s)\|I - \lambda_2 H_m\|_{L^2([0,\infty))\to L^2([0,\infty))}.\end{aligned}$$

Since this norm is equal to 1 for $\lambda = 0$, tends to 1 as $\lambda \to (1+m)^+$, and is always at least 1 (by Lemmas 3.2 and 3.3, or Remark 3.4), we conclude that $\|I - \lambda H_m\|_{L^2([0,\infty))\to L^2([0,\infty))} = 1$ for $\lambda \in [0, 1+m]$. □

Remark 3.10. The operator $I - (1+m)H_m$ is an isometry on $L^2([0,\infty))$ (since the Beurling–Ahlfors transform is an L^2-isometry; see [3, Theorem 1.1]).

Proof of inequality (1.1) *for* $\lambda \le 0$. The triangle inequality and Lemma 3.3 yield

$$\|I - \lambda H_m\|_{L^p([0,\infty))\to L^p([0,\infty))} \le |\lambda| g_{p,m}^{-1} + 1 \le C_{p,m,\lambda}.$$

Moreover, the opposite inequality is also true by Lemma 3.2, so we in fact have equalities above. □

Sharpness of inequality (1.1) follows from Lemma 3.2. In order to complete the proof of Theorem 1.1, we have to explain why $C^p_{p,m=0,\lambda=1}$ is equal to

$$C^p_p = \sup\{c_{p,m=0,\lambda=1}(\alpha,1)\colon \alpha < (p-1)/p\} = \begin{cases} \frac{1}{(p-1)^p} & \text{if } 1 < p \le 2, \\ \frac{(1+|\alpha_p|)^{p-2}}{p-1} & \text{if } p > 2 \end{cases}$$

(see Section 2 for the definition of α_p and details).

Rather than to check directly that the supremum in the definition of $C_{p,0,1}$ is attained in the point $(\alpha_p, 1)$ we refer to results obtained in Section 2. For $m = 0$ and $\lambda = 1$, the above proof of inequality (1.1), can be repeated with the functions V, U being defined like in Subsection 2.5 (of course, instead of using Proposition 3.7, we use Proposition 2.9). This gives us $\|f - H_0 f\|_{L^p([0,\infty))} \le C_p \|f\|_{L^p([0,\infty))}$, but since clearly $C_p \le C_{p,0,1}$, and the constant $C_{p,0,1}$ is best possible in this inequality, we conclude that $C_p = C_{p,0,1}$.

Acknowledgments. I thank Adam Osękowski for suggesting the study of the Hardy operator by the means of maximal martingale inequalities, as well as for his invaluable help, and many fruitful and inspiring discussions. I also thank Tomasz Tkocz, whose nicely written bachelor thesis helped me to understand the connection between Doob's inequality and the Hardy operator, when I was first learning about them.

References

[1] Astala, K.: Area distortion of quasiconformal mappings. - Acta Math. 173:1, 1994, 37–60.

[2] Bañuelos, R., and P. Janakiraman: L^p-bounds for the Beurling–Ahlfors transform. - Trans. Amer. Math. Soc. 360:7, 2008, 3603–3612.

[3] Bañuelos, R., and P. Janakiraman: On the weak-type constant of the Beurling–Ahlfors transform. - Michigan Math. J. 58:2, 2009, 459–477.

[4] Bañuelos, R., and P. J. Méndez-Hernández: Space-time Brownian motion and the Beurling–Ahlfors transform. - Indiana Univ. Math. J. 52:4, 2003, 981–990.

[5] Bañuelos, R., and A. Osękowski: Sharp inequalities for the Beurling–Ahlfors transform on radial functions. - Duke Math. J. 162:2, 2013, 417–434.

[6] Bañuelos, R., and A. Osękowski: On the operator Λ^* and the Beurling–Ahlfors transform on radial functions. - Michigan Math. J. 63:1, 2014, 213–221.

[7] Bañuelos, R., and G. Wang: Sharp inequalities for martingales with applications to the Beurling–Ahlfors and Riesz transforms. - Duke Math. J. 80:3, 1995, 575–600.

[8] Borichev, A., P. Janakiraman, and A. Volberg: Subordination by conformal martingales in L^p and zeros of Laguerre polynomials. - Duke Math. J. 162:5, 2013, 889–924.

[9] Burkholder, D. L.: Explorations in martingale theory and its applications. - In: École d'Été de Probabilités de Saint-Flour XIX—1989, Lecture Notes in Math. 1464, Springer, Berlin, 1991, 1–66.

[10] Chatterji, S. D.: Some comments on the maximal inequality in martingale theory. - In: Measure theory, Oberwolfach 1979 (Proc. Conf., Oberwolfach, 1979), Lecture Notes in Math. 794, Springer, Berlin, 1980, 361–364.

[11] Dragičević, O., and A. Volberg: Bellman function, Littlewood–Paley estimates and asymptotics for the Ahlfors–Beurling operator in $L^p(\mathbf{C})$. - Indiana Univ. Math. J. 54:4, 2005, 971–995.

[12] Gill, J. T.: On the Beurling–Ahlfors transform's weak-type constant. - Michigan Math. J. 59:2, 2010, 353–363.

[13] Iwaniec, T.: Extremal inequalities in Sobolev spaces and quasiconformal mappings. - Z. Anal. Anwendungen 1:6, 1982, 1–16.

[14] Lehto, O.: Remarks on the integrability of the derivatives of quasiconformal mappings. - Ann. Acad. Sci. Fenn. Ser. A I Math. 371, 1965, 1–8.

[15] Osękowski, A.: Sharp martingale and semimartingale inequalities. - Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) 72, Birkhäuser/Springer Basel AG, Basel, 2012.

[16] Osękowski, A.: Sharp logarithmic bounds for Beurling–Ahlfors operator restricted to the class of radial functions. - Mediterr. J. Math. 10:4, 2013, 1883–1894.

[17] Pełczyński, A.: Norms of classical operators in function spaces. - In: Colloquium in honor of Laurent Schwartz, Vol. 1 (Palaiseau, 1983), Astérisque, 131, 1985, 137–162.

[18] Volberg, A.: Ahlfors–Beurling operator on radial functions - Preprint, arXiv:1203.2291, 2012.

[19] Vol′berg, A., and F. Nazarov: Heat extension of the Beurling operator and estimates for its norm. - Algebra i Analiz 15:4, 2003, 142–158.

Received 4 March 2016 • Accepted 29 April 2016