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A GLOBAL COORDINATE OF THE TEICHMÜLLER SPACE RELATED TO ASYMPTOTIC JENKINS–STREBEL RAYS

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Abstract. In this paper, we give a parametrization of asymptotic Jenkins–Strebel rays. It is a kind of global coordinates of the Teichmüller space. For any admissible curve family of a surface, the subset of the boundary of the Teichmüller space which is constructed by pinching of the given curve family can be determined. There exists a homeomorphism of the product of the boundary space and several parameter spaces onto the Teichmüller space such that each family of asymptotic Jenkins–Strebel rays is represented by the parameters. The idea is obtained by [MM75].

1. Introduction

The asymptotic behavior of Teichmüller geodesic rays has been studied in the several ways. Ivanov, Lenzhen, and Masur give a condition such that any pair of Teichmüller rays are bounded or divergent when the rays approach the boundary of the Teichmüller space [Mas75, Mas80, Iva01, LM10]. Furthermore, the author proved that under a condition, Jenkins–Strebel rays are asymptote and in general, the limit value of the distance of any pair of Jenkins–Strebel rays is calculated. It is described by the distance of boundary points of the rays in the augmented Teichmüller space and the detour metric in the Gardiner–Masur boundary [Am14a, Am14b], about the detour metric, see [Wal14, Wal12]. It is a limiting case of Minsky's product region theorem [Min96].

We focus on a set of asymptotic Jenkins–Strebel rays. Let $\Sigma = \Sigma_{g,p}$ be a Riemann surface of genus g with p punctures such that $3g - 3 + p > 0$. Let $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ be a set of simple closed curves on Σ which are non-intersecting and non-homotopic each other such that each curve is not homotopic to a point and is not peripheral. We denote the Teichmüller space of Σ by $\mathscr{T} = \mathscr{T}_{g,p}$ and the $(k-1)$-dimensional unit sphere such that all entries are positive by S_+^{k-1}. For any fixed $[S^*, f^*] \in \mathscr{T}$ and $(m_1, \dots, m_k) \in S_+^{k-1}$, we can construct a Jenkins–Strebel ray r emanating from $[S^*, f^*]$ directed by the Jenkins–Strebel differential with the moduli $(\alpha^* m_1, \dots, \alpha^* m_k)$ of associated cylinders where $\alpha^* > 0$ is some constant. The end point of r corresponds to a Riemann surface S_c with nodes. We set $\partial_\Gamma \mathscr{T} = \{[X, g] \mid g\colon S_c \to X \text{ is a quasiconformal mapping}\}$. This is a subset of the augmented Teichmüller space of Σ. Any Riemann surface on a Jenkins–Strebel ray is given by a Riemann surface with nodes and a specific quadratic differential on the noded surface. Indeed, the ordinary Riemann surface without nodes is obtained by cutting off neighborhoods of the nodes and gluing each remaining surface. We establish a mapping of the product of $\partial_\Gamma \mathscr{T}$ and the spaces of parameters (coefficients

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$a_1, \dots, a_k$ of the above differential in a representation near the nodes, length s of the cutting, and arguments $t_1, \dots, t_k$ of twists when gluing) onto the Teichmüller space $\mathscr{T}$. The following is our main theorem in this paper.

Theorem 1.1. *There is a homeomorphism $\hat{\Phi}\colon \partial_\Gamma \mathscr{T} \times S_+^{k-1} \times \mathbf{R}^k \times \mathbf{R} \to \mathscr{T}$ such that the following conditions hold;*

(1) *Let $\hat{\Phi}([X,g], a_1, \dots, a_k, t_1, \dots, t_k, s) = [R,h]$. For any $j = 1, \dots, k$,*

$$\hat{\Phi}([X,g], a_1, \dots, a_k, t_1, \dots, t_j + 2\pi, \dots, t_k, s) = \tau_j([R,h]),$$

where τ_j is a Dehn twist about γ_j.

(2) *Let $[X,g]$ be a point in $\partial_\Gamma \mathscr{T}$. For any $\mathfrak{a} \in S_+^{k-1}$ and $\mathfrak{t} \in \mathbf{R}^k$, the set $r_{\mathfrak{a},\mathfrak{t}} = \{\hat{\Phi}([X,g], \mathfrak{a}, \mathfrak{t}, s) \mid s \in \mathbf{R}_{\geq 0}\}$ is a Jenkins–Strebel ray whose moduli of associated cylinders are represented by a positive scalar multiple of $(m_1, \dots, m_k)$, and it tends to $[X,g]$ as $s \to \infty$. Furthermore, $\{r_{\mathfrak{a},\mathfrak{t}}\}_{\mathfrak{a},\mathfrak{t}}$ are asymptotic each other.*

Corollary 1.2. *The mapping $\hat{\Phi}$ gives a global coordinate of the Teichmüller space $\mathscr{T}$.*

In [Am14a], we find a condition under which two Jenkins–Strebel rays are asymptotic. The condition contributes to the proof of (2) in the above theorem.

As a matter of fact, if $k = 1$, $\hat{\Phi}$ induces a Teichmüller disk, and the mapping is the same as in the main result in [MM75]. We explain the detail in the remark after the proof of Theorem 1.1 in §3.4.

Corollary 1.2 is related to plumbing coordinates, see [Kra90, EM12]. Our coordinate has a high compatibility with Jenkins–Strebel rays. Moreover, we can see some properties for the coordinate in §3.5. They give asymptotic behaviors of the Jenkins–Strebel geodesics and estimations of the Teichmüller distance between two points for the coordinate.

2. Preliminaries

2.1. Teichmüller spaces For Teichmüller spaces, we refer the reader to [IT92]. Let $\Sigma = \Sigma_{g,p}$ be a Riemann surface of genus g with p punctures such that $3g-3+p > 0$. We denote by $\mathscr{T} = \mathscr{T}_{g,p}$ the *Teichmüller space* of Σ. It is described as follows;

$$\mathscr{T} = \{(S,f) \mid f\colon \Sigma \to S \text{ is a quasiconformal mapping}\}/\sim,$$

where $\sim$ is an equivalence relation such that $(S_1, f_1) \sim (S_2, f_2)$ means that there is a conformal mapping $h\colon S_1 \to S_2$ which is homotopic to $f_2 \circ f_1^{-1}$. We write $[S,f]$ as an equivalence class of (S,f), and set a base point of $\mathscr{T}$ by $[\Sigma, \mathrm{id}]$. The Teichmüller space $\mathscr{T}$ has a natural complete distance, called the *Teichmüller distance*, and denote by $d_{\mathscr{T}}$. For any $p_1 = [S_1, f_1],\ p_2 = [S_2, f_2]$,

$$d_{\mathscr{T}}(p_1, p_2) = \frac{1}{2} \log \inf K(h),$$

where the infimum ranges over all quasiconformal mappings $h\colon S_1 \to S_2$ which are homotopic to $f_2 \circ f_1^{-1}$.

2.2. Extremal lengths. Let ρ be a locally L^1-measurable conformal metric on Σ. It is represented by the conformally invariant form $\rho = \rho(z)|dz|$ on any local coordinate z of Σ where $\rho(z) \geq 0$ is a measurable function of z. For any non-zero and non-peripheral simple closed curve γ on Σ, we define the ρ-length of γ and the ρ-area of Σ by

$$l_\rho(\gamma) = \inf_{\gamma' \in [\gamma]} \int_{\gamma'} \rho(z)|dz|, \quad A_\rho = \iint_\Sigma \rho(z)^2\, dx\, dy,$$

respectively, where $[\gamma]$ means the freely homotopy class of γ. The *extremal length* $\mathrm{Ext}_\Sigma(\gamma)$ of γ on Σ is defined by the following;

$$\mathrm{Ext}_\Sigma(\gamma) = \sup_\rho \frac{l_\rho(\gamma)^2}{A_\rho},$$

where ρ ranges over all locally L^1-measurable conformal metrics on Σ such that $0 < A_\rho < \infty$. This definition is equivalent to the following;

$$\mathrm{Ext}_\Sigma(\gamma) = \inf \frac{1}{M(\gamma)},$$

where $M(\gamma)$ ranges over all moduli of annuli on Σ with the core curve γ. For any $[S, f] \in \mathscr{T}$ and any γ on Σ, we define $\mathrm{Ext}_{[S,f]}(\gamma) = \mathrm{Ext}_S(f(\gamma))$.

Kerckhoff shows that the Teichmüller distance is also represented by the ratio of the extremal lengths of simple closed curves.

Theorem 2.1. [Ker80, Kerckhoff's formula for the Teichmüller distance] *For any $p_1 = [S_1, f_1]$, $p_2 = [S_2, f_2] \in \mathscr{T}$, the Teichmüller distance between p_1 and p_2 is represented by*

$$d_{\mathscr{T}}(p_1, p_2) = \frac{1}{2} \log \sup_\gamma \frac{\mathrm{Ext}_{p_2}(\gamma)}{\mathrm{Ext}_{p_1}(\gamma)},$$

where γ ranges over all non-zero and non-peripheral simple closed curves on Σ.

Kerckhoff's formula is useful for finding lower bounds of the Teichmüller distance.

2.3. Quadratic differentials. In this section, we refer to [Str84]. A *holomorphic quadratic differential* φ on Σ is a tensor of the form $\varphi = \varphi(z)\, dz^2$, where z is a local coordinate of Σ and $\varphi(z)$ is a holomorphic function. The holomorphic quadratic differential φ can contain poles at punctures. These differentials have the L^1-norm $\|\varphi\| = \iint_\Sigma |\varphi|$. The orders of poles of φ are at most 1 if and only if $\|\varphi\| < \infty$.

The zeros and punctures of Σ are called *critical points* of φ. There is a local coordinate w of $\Sigma - \{\text{zeros of } \varphi\}$ which satisfies $dw^2 = \varphi(z)\, dz^2$. It is obtained by the integral $w = \int_{z_0}^z \sqrt{\varphi(z)}\, dz$, where z_0, z are not critical points of φ. The coordinate w is called a *natural parameter* of φ. A maximal straight arc $z = \gamma(t)$ which satisfies $\varphi(\gamma(t))(d\gamma(t)/dt)^2 < 0$ is called a *vertical trajectory* of φ. Let C_φ be the set of all critical points and vertical trajectories of φ which join critical points. Any component of $\Sigma - C_\varphi$ is an annulus swept out by simple closed vertical trajectories of φ, or a minimal domain which is generated by recurrent vertical trajectories of φ. We call φ a *Jenkins–Strebel differential* if all components of $\Sigma - C_\varphi$ are annuli. The core curves of these annuli are simple closed curves which are non-intersecting and non-homotopic each other such that each curve is not homotopic to a point and is not peripheral. We call the curve family with such properties by an *admissible curve family.*

2.4. Quadratic differentials with poles of order 2. We also refer to [Str84]. We consider holomorphic quadratic differentials with poles of order 2 at punctures of a surface. Let $\overline{\Sigma}$ be a compact surface after filling the punctures of Σ. Let $x \in \overline{\Sigma}$ be a puncture of Σ. We consider a meromorphic quadratic differential φ on $\overline{\Sigma}$ which has the representation $\varphi = (a^2/z^2 + \cdots)\,dz^2$ such that z is a local coordinate of $\overline{\Sigma}$ near x with $z(x) = 0$ and $a > 0$. The coefficient a is called a *leading coefficient* of φ at x. This φ has a special property which does not work for quadratic differentials without poles of order 2. Let f be a conformal mapping on a neighborhood of x and set $w = f(z)$ such that $f(0) = 0$. Then the representation of φ with respect to w has the same leading coefficient a.

Let D be any component of $\Sigma - C_\varphi$ which has a puncture x. The mapping $w = \exp(1/a \int \sqrt{\varphi(z)}\,dz)$ maps D conformally onto a punctured disk $\{0 < |w| < r\}$. In this situation, the equation $a^2dw^2/w^2 = \varphi(z)\,dz^2$ holds. The vertical trajectories of a^2dw^2/w^2 are circles of the form $\{|w| = r'\}$ for any $0 < r' < r$. The length of each trajectory is $2\pi a$ with respect to the metric $|\varphi(z)|^{\frac{1}{2}}|dz| = a|dw/w|$.

Let $\mathbf{R}_+^k$ be the set of k-tuples of positive real numbers. The following theorem is the existence of quadratic differentials with poles of order 2 for given punctures and leading coefficients.

Theorem 2.2. [Str84] *Let $x_1, \ldots, x_k$ be punctures of Σ and fix $(a_1, \ldots, a_k) \in \mathbf{R}_+^k$. There exists a unique holomorphic quadratic differential φ on Σ such that each component of $\Sigma - C_\varphi$ is a punctured disk with each puncture x_j, and all vertical trajectories of φ in each punctured disk are closed and surround the puncture, furthermore, the leading coefficient of φ at x_j is a_j, for any $j = 1, \ldots, k$.*

2.5. Reduced moduli of punctured disks. For reduced moduli, we refer §3.2 of [Str84] or §4.3 of [MM75]. The ordinary moduli are infinite for punctured disks. However, reduced moduli for punctured disks are finite valued quantities and have similar properties as ordinary one. Let $x \in \overline{\Sigma}$ be a puncture of Σ and D be a punctured disk on Σ with the puncture x. Let f be a conformal mapping of $D \cup \{x\}$ onto the unit disk $\{|w| < 1\}$ such that $f(x) = 0$. Let h be another conformal mapping of a neighborhood of x into $\{|z| < r_0\}$ such that $h(x) = 0$. We can regard z as a local coordinate near x. For sufficiently small $r > 0$, we denote by $M(r)$ the modulus of $D - h^{-1}(\{|z| \le r\})$. We can write $w = f \circ h^{-1}(z) = c_1 z + c_2 z^2 + \cdots$ where $c_1 \neq 0$. Since any Möbius transformation on a unit disk which fixes 0 is a rotation, $|c_1|$ is independent on a choice of f. We denote $r_1 = \max_{|z|=r} |f \circ h^{-1}(z)|$ and $r_2 = \min_{|z|=r} |f \circ h^{-1}(z)|$. We can see these settings in Figure 1. By an easy computation, the ratios $r_1/r = \max_{|z|=r} |c_1 + c_2 z + \cdots|$ and $r_2/r = \min_{|z|=r} |c_1 + c_2 z + \cdots|$ both tend to $|c_1|$ as $r \to 0$. By the inclusion relations between annuli $\{r_1 < |w| < 1\}$, $f(D - h^{-1}(\{0 < |z| \le r\}))$, and $\{r_2 < |w| < 1\}$, the inequality

$$\frac{1}{2\pi} \log \frac{1}{r_1} \le M(r) \le \frac{1}{2\pi} \log \frac{1}{r_2}$$

holds and this says that

$$\lim_{r \to 0} \left(M(r) + \frac{1}{2\pi} \log r \right) = \frac{1}{2\pi} \log \frac{1}{|c_1|}.$$

We denote by $\dot{M}(D)$ this value and call it a *reduced modulus* of D with respect to the puncture x and the local coordinate z. Also, if there is the conformal mapping g of $D \cup \{x\}$ onto $\{|w| < r'\}$ such that $g \circ h^{-1}(0) = 0$ and $(dg \circ h^{-1}(z)/dz)(0) = 1$ with respect to the coordinate z, then $\dot{M}(D) = (1/2\pi) \log r'$. Indeed, we consider g/r' as f of the above definition, then $|c_1| = 1/r'$ and $\dot{M}(D) = (1/2\pi) \log(1/|c_1|) = (1/2\pi) \log r'$. For the reduced moduli, this form is slightly useful (we use it in §3.2).

Clearly, for common x and z, if $D \subset D'$ then $\dot{M}(D) \le \dot{M}(D')$.

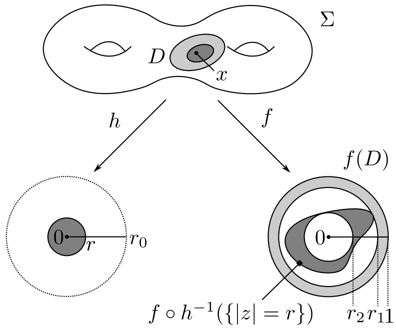

Figure 1. The light gray domain in Σ is D, and the dark gray domain corresponds to $h^{-1}(\{|z| < r\})$ in Σ such that it is included in D for sufficiently small r.

The following theorem is an extremal property for reduced moduli of any quadratic differential comes from Theorem 2.2.

Theorem 2.3. [Str84] *Let $x_1, \ldots, x_k$ be punctures of Σ and fix $(a_1, \ldots, a_k) \in \mathbf{R}_+^k$. By Theorem 2.2, we obtain a holomorphic quadratic differential φ on Σ with the leading coefficients $a_1, \ldots, a_k$ of the representations of φ at $x_1, \ldots, x_k$ respectively. We denote by $\{D_j\}_{j=1,\ldots,k}$ the set of punctured disks with punctures $x_1, \ldots, x_k$ respectively, obtained by $\Sigma - C_\varphi$. Let $\{D'_j\}_{j=1,\ldots,k}$ be any set of non-overlapping punctured disks with punctures $x_1, \ldots, x_k$ respectively. Let $\dot{M}(D_j)$ and $\dot{M}(D'_j)$ be reduced moduli of D_j and D'_j respectively, with x_j and a common local coordinate for any $j = 1, \ldots, k$. Then the inequality*

$$\sum_{j=1}^{k} a_j^2 \dot{M}(D'_j) \leq \sum_{j=1}^{k} a_j^2 \dot{M}(D_j)$$

holds with equality only if $D'_j = D_j$ for any j.

2.6. Teichmüller geodesics. The Teichmüller space has geodesics with respect to the Teichmüller distance. They are given by affine mappings of Riemann surfaces with respect to the natural parameters of the quadratic differentials.

Let $[S, f] \in \mathscr{T}$ and φ be a non-zero holomorphic quadratic differential on S with $\|\varphi\| < \infty$. For any $s \geq 0$, we consider the Beltrami coefficient $\mu_s = \tanh(s)|\varphi|/\varphi$ on S. By μ_s, a new Riemann surface S_s and an extremal quasiconformal mapping (a Teichmüller mapping) $f_s \colon S \to S_s$ are determined. Let $z = x + iy$ be a natural parameter of φ. The Teichmüller mapping f_s is written by $z = x + iy \mapsto z_s = \exp(2s)x + iy$. The holomorphic quadratic differential dz_s^2 on S_s is also determined. We set $r \colon \mathbf{R}_{\geq 0} \to \mathscr{T}$, $r(s) = [S_s, f_s \circ f]$ and call it a *Teichmüller geodesic ray* directed by φ and emanating from $[S, f]$. If φ is Jenkins–Strebel, we call r a *Jenkins–Strebel ray.*

2.7. Augmented Teichmüller spaces. The Teichmüller space has several boundary representations. In particular, an extension of the Teichmüller space called the augmented Teichmüller space consists of not only ordinary Riemann surfaces but also Riemann surfaces with nodes. Any Jenkins–Strebel ray has an end point in the boundary of the augmented Teichmüller space.

A connected Hausdorff space X is called a *Riemann surface with nodes* of genus g with p punctures if X satisfies the following conditions;

(1) Any $x \in X$ has a neighborhood which is homeomorphic to the unit disk $\{|z| < 1\}$, or the set $\{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1| < 1,\ |z_2| < 1,\ z_1 \cdot z_2 = 0\}$. (If x corresponds to the latter case, it is called a *node* of X.)

(2) Let $x_1, \dots, x_k$ be nodes of X, and $X_1, \dots, X_r$ be connected components of $X - \{x_1, \dots, x_k\}$. For any $j = 1, \dots, r$, each X_j is a Riemann surface of genus g_j with p_j punctures such that $2g_j - 2 + p_j > 0$, $n = \sum_{j=1}^{r} p_j - 2k$, and $g = \sum_{j=1}^{r} g_j - r + k + 1$.

If X is a Riemann surface without nodes, it is also contained in this definition.

The *augmented Teichmüller space* $\hat{\mathscr{T}} = \hat{\mathscr{T}}(\Sigma)$ of Σ is defined as follows;

$$\hat{\mathscr{T}} = \{(X, g) \mid X \text{ is a Riemann surface with nodes, } g\colon \Sigma \to X \text{ is a deformation}\}/\sim,$$

where the deformation $g\colon \Sigma \to X$ is a continuous mapping such that it contracts some disjoint loops on Σ to points (the nodes of X) and is a homeomorphism except on the loops. The symbol $\sim$ is an equivalence relation such that $(X_1, g_1) \sim (X_2, g_2)$ means that there is a conformal mapping $h\colon X_1 \to X_2$ such that g_2 is homotopic to $h \circ g_1$. We denote by $[X, g]$ an equivalence class of (X, g).

A topology on $\hat{\mathscr{T}}$ is defined by the following neighborhoods. Let $[X, g] \in \hat{\mathscr{T}}$. For any compact neighborhood V of the set of nodes of X and any $\varepsilon > 0$, a neighborhood $U_{V,\varepsilon} = U_{V,\varepsilon}([X, g])$ of $[X, g]$ is defined by all $[X', g'] \in \hat{\mathscr{T}}$ such that there is a deformation $h\colon X' \to X$ which is a quasiconformal mapping on $h^{-1}(X - V)$ and the dilatation is $1 + \varepsilon$ such that g is homotopic to $h \circ g'$. We can see $\mathscr{T} \subset \hat{\mathscr{T}}$ with the topology, and set the boundary $\partial\mathscr{T} = \hat{\mathscr{T}} - \mathscr{T}$.

Any Jenkins–Strebel ray on $\mathscr{T}$ converges to a point on $\partial\mathscr{T}$. To see this, let r be a Jenkins–Strebel ray directed by φ on S and emanating from $[S, f] \in \mathscr{T}$. The components of $S - C_\varphi$ are corresponding to annuli $A_1, \dots, A_k$ whose core curves are represented by an admissible curve family $\{\gamma_1, \dots, \gamma_k\}$ on S respectively. Let $2\pi a_j$ be the length of each closed vertical trajectory in A_j with respect to the metric $|\varphi(z)|^{\frac{1}{2}}|dz|$. By the natural parameters of φ, there exists a constant $c_j > 0$ such that the coordinate $w = c_j \exp(1/a_j \int \sqrt{\varphi(z)}\, dz)$ determines a conformal mapping of A_j onto the round annulus $\{1 < |w| < \exp(2\pi M_j)\}$, where M_j is the modulus of A_j. We use the same symbol A_j for the round annulus, and cut each A_j into two annuli $A_{j,1} = \{\exp(\pi M_j) < |w| < \exp(2\pi M_j)\}$ and $A_{j,2} = \{1 < |w| < \exp(\pi M_j)\}$, and stretch $A_{j,1}$ to $\{\exp(-\pi M_j) < |w| < 1\}$ by $H(w) = \exp(-2\pi M_j)w$. We also use the same symbol $A_{j,1}$ for the resulting annulus. Now, we obtain two annuli

$$A_{j,1} = \{\exp(-\pi M_j) < |w| < 1\}, \quad A_{j,2} = \{1 < |w| < \exp(\pi M_j)\}$$

for any $j = 1, \dots, k$. By using this representation of S, we can see a geometric construction of the end point of r.

For any $s \geq 0$, we set

$$\begin{aligned} A_{j,1}(s) &= \{\exp(-\pi M_j \exp(2s)) < |w| < 1\}, \\ A_{j,2}(s) &= \{1 < |w| < \exp(\pi M_j \exp(2s))\}. \end{aligned}$$

If $s = 0$, we regard $A_{j,1}(0)$ and $A_{j,2}(0)$ as $A_{j,1}$ and $A_{j,2}$ respectively. The surface S_s corresponding to the point $r(s) = [S_s, f_s \circ f]$ of the ray r is obtained by the annuli $\{A_{j,l}(s)\}$ with the following gluings. In here, we denote the two boundaries of round annuli by adding the characters ∂_i and ∂_o, where ∂_i means a boundary component which has the small radius, and ∂_o means another one. After the action $H_s^{-1}(w) = \exp(2\pi M_j \exp(2s))w$ to $A_{j,1}(s)$, the gluing of $\partial_i A_{j,1}(s)$ and $\partial_o A_{j,2}(s)$ is performed by the identity mapping. Then we obtain $A_j(s) = \{1 < |w| < \exp(2\pi M_j \exp(2s))\}$ for any $j = 1, \dots, k$. Also, for the gluing of $A_1(s), \dots, A_k(s)$, we use the gluing of the neighborhoods of $\partial_i A_j(s)$ and $\partial_o A_{j'}(s)$ as the reverse of the decomposition of $S - C_\varphi$

for any j and j'. The resulting surface is conformally equivalent to S_s. Furthermore, the Teichmüller mapping $f_s\colon S \to S_s$ is represented by $w \mapsto |w|^{\exp(2s)-1}w$ on each $A_{j,l}$.

For $A_{j,1}(s)$ and $A_{j,2}(s)$, as $s \to \infty$, we obtain

$$A_{j,1}(\infty) = \{0 < |w| < 1\}, \quad A_{j,2}(\infty) = \{|w| > 1\},$$

respectively. We use the similar gluing for $A_{j,1}(\infty)$ and $A_{j,2}(\infty)$ as in the case of s, then we obtain a Riemann surface S_∞ with nodes and also a deformation $f_\infty\colon S \to S_\infty$. We define $r(\infty) = [S_\infty, f_\infty \circ f] \in \partial\mathscr{T}$. The ray $r = r(s)$ converges to the point $r(\infty)$ as $s \to \infty$ for the topology of $\hat{\mathscr{T}}$ (for example, see Proposition 4.3 in [HS07]).

Now, we determine the subset of the boundary $\partial\mathscr{T}$ which consists of Riemann surfaces with nodes obtained by pinching the given admissible curve family Γ, it is necessary to state Theorem 1.1. For that purpose, we use the following theorem in [Str84]. Let $S_+^{k-1} = \{(x_1, \dots, x_k) \in \mathbf{R}_+^k \mid \sum_{j=1,\dots,k} x_j^2 = 1\}$.

Theorem 2.4. [Str84] *For any admissible curve family $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ on Σ and $\mathfrak{m} = (m_1, \dots, m_k) \in S_+^{k-1}$, there exist $\alpha > 0$ and a Jenkins–Strebel differential φ on Σ whose moduli of the annuli corresponding to Γ are $\alpha\mathfrak{m} = (\alpha m_1, \dots, \alpha m_k)$. Such α is uniquely determined, and φ is also up to a positive scalar multiple.*

Thereafter, we fix an admissible curve family $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ on Σ, $\mathfrak{m} = (m_1, \dots, m_k) \in S_+^{k-1}$, and $[S^*, f^*] \in \mathscr{T}$. These elements play a role of the base setting of Theorem 1.1. We apply the above theorem for $f^*(\Gamma)$ on S^* and $\mathfrak{m}$. There exist $\alpha^* > 0$ and a Jenkins–Strebel differential φ^* on S^* such that the moduli of annuli corresponding to $f^*(\Gamma)$ are $\alpha^*\mathfrak{m}$. Let $L_1, \dots, L_k$ be the lengths of the closed vertical trajectories in the annuli respectively, with respect to the metric $|\varphi^*|^{\frac{1}{2}}$, and set $a_j = L_j/2\pi$. We normalize φ^* to $\varphi^*/\sum_{j=1}^k a_j^2$ and use the same symbol for it, then each corresponding value $a_j^* = a_j/\sqrt{\sum_{j=1}^k a_j^2}$ for φ^* satisfies that $\sum_{j=1}^k a_j^{*2} = 1$. We denote $\mathfrak{a}^* = (a_1^*, \dots, a_k^*) \in S_+^{k-1}$. The Jenkins–Strebel ray directed by φ^* and emanating from $[S^*, f^*]$ determines a deformation f_c of S^* onto a Riemann surface S_c with nodes. We define

$$\partial_\Gamma\mathscr{T} = \{[X, g] \mid g\colon S_c \to X \text{ is a quasiconformal mapping}\}.$$

We can regard $\partial_\Gamma\mathscr{T}$ as the subset of $\partial\mathscr{T}$ via the action of the mapping $f_c \circ f^*\colon \Sigma \to S_c$.

We consider a terminal quadratic differential on S_c introduced by φ^* and f_c. We can choose a constant $c_j > 0$ such that $w = c_j \exp(1/a_j^* \int \sqrt{\varphi^*(z)}\, dz)$ maps each component of $S^* - C_{\varphi^*}$ conformally onto the union of $A_{j,1} = \{\exp(-\pi\alpha^* m_j) < |w| < 1\}$ and $A_{j,2} = \{1 < |w| < \exp(\pi\alpha^* m_j)\}$ with the center line $\{|w| = 1\}$. Then we have the representation $\varphi^*(z)\, dz^2 = a_j^{*2}\, dw^2/w^2$ on $A_{j,l}$. Since the Teichmüller mapping associated to the ray preserves the representation $a_j^{*2}\, dw^2/w^2$ on the stretched $A_{j,l}$, naturally, we can induce the same description $a_j^{*2}\, dw^2/w^2$ on $A_{j,1}(\infty) = \{0 < |w| < 1\}$ and $A_{j,2}(\infty) = \{|w| > 1\}$. Then S_c has a quadratic differential J_c^* which has the representation $(a_j^{*2}/z^2 + \cdots)\, dz^2$ on a neighborhood of each node. On the other hand, we apply Theorem 2.2 to each component of $S_c - \{\text{nodes of } S_c\}$ with assigning a_j^* to each node of S_c. By the uniqueness of the theorem, the resulting differential equals to J_c^*.

2.8. Dehn twists. A Dehn twist about a simple closed curve on a surface is a homeomorphism of the surface onto itself. It acts the Teichmüller space, and the action has the relationship with the parameters of the mapping in Theorem 1.1.

Let $A = \{1 < |z| < r\}$ be a round annulus on the complex plane with the usual orientation. We consider the quasiconformal mapping

$$\tau_\theta(z) = z|z|^{i\frac{\theta}{\log r}}$$

on A. The quasiconformal dilatation $K(\tau_\theta) = (1 + k(\tau_\theta))/(1 - k(\tau_\theta))$ is given by

$$k(\tau_\theta) = \left| \frac{\theta/2\pi M}{2 + i(\theta/2\pi M)} \right|,$$

where $M = (\log r)/2\pi$. If $\theta = 2\pi$, we call $\tau_{2\pi}$ a *Dehn twist* of A. For any $n \in \mathbf{Z}$, it holds $\tau_{2\pi n} = (\tau_{2\pi})^n$. Next, we define Dehn twists on any surface Σ. Let γ be a simple closed curve on Σ which is not homotopic to a point and is not peripheral. We take an annular neighborhood $N_\gamma \subset \Sigma$ of γ. There is a conformal mapping $h\colon N_\gamma \to A$, where A is a round annulus on the complex plane. We can apply $\tau_{2\pi}$ on A and take the conjugate by h, and set the identity mapping on $\Sigma - N_\gamma$, that is, the mapping

$$\tau_\gamma = \begin{cases} h^{-1} \circ \tau_{2\pi} \circ h & \text{in } N_\gamma \\ \mathrm{id} & \text{in } \Sigma - N_\gamma \end{cases}$$

is determined. It depends on the choice of N_γ and h. However, its homotopy class is well-defined. Then, we call (the homotopy class) of τ_γ a *Dehn twist* of Σ about γ. The Dehn twist τ_γ acts the Teichmüller space $\mathscr{T}$. We define $[\tau_\gamma]_*([S, f]) = [S, f \circ \tau_\gamma^{-1}] \in \mathscr{T}$ for any $[S, f] \in \mathscr{T}$.

3. Proof of the main theorem

In this chapter, we prove Theorem 1.1. We also fix the admissible curve family $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ on Σ, $\mathfrak{m} = (m_1, \ldots, m_k) \in S_+^{k-1}$, and $[S^*, f^*] \in \mathscr{T}$ from the previous chapter. We refer to [MM75] for the contents, statements, and proofs in the following sections 3.1, 3.2, and 3.3.

3.1. Fuchsian equivalents, canonical local coordinates. To construct the homeomorphism in Theorem 1.1, we need each criterion for a Fuchsian group and a local coordinate of any Riemann surface with nodes in $\partial_\Gamma \mathscr{T}$.

In §2.7, we construct the Riemann surface S_c with nodes. For each connected component Y of $S_c - \{\text{nodes of } S_c\}$, we denote punctures of Y without the original punctures of S_c as follows, see also Figure 2;

(1) $x_{j,1}, x_{j,2}$: punctures which are obtained by $f_c \circ f^*(\gamma_j)$.
(2) x_j: similar as the above, but the counterpart is in another component.

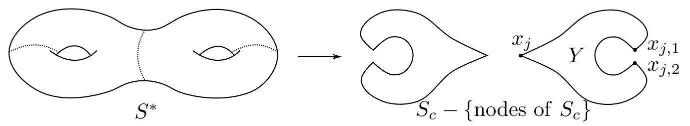

Figure 2. $x_{j,1}, x_{j,2}$ are assigned a pair of punctures corresponding to a node of S_c on the fixed component Y. The counterpart of x_j is not on Y.

We fix $O \in Y$. Let $\alpha_{j,1}, \alpha_{j,2}, \beta_j$ be simple closed curves on Y through O such that $\alpha_{j,1}, \alpha_{j,2}$ are homotopic to $x_{j,1}, x_{j,2}$ respectively, β_j is not homotopic to $\alpha_{j,1}, \alpha_{j,2}$. For x_j, we take similar α_j, β_j.

Let $[X, g] \in \partial_\Gamma \mathscr{T}$, x be a node $g(x_{j,l})$ (or $g(x_j)$) of X for $l = 1, 2$, and Y be a component of $X - \{\text{nodes of } X\}$ associated to x. We define a Fuchsian group Γ_x associated to Y as follows;

(1) The lift of $g(\alpha_{j,l})$ (or $g(\alpha_j)$) corresponds to a translation $L\colon \zeta \mapsto \zeta + 1 \in \Gamma_x$ on the upper half plane $\mathbf{H} = \{\mathrm{Im}(\zeta) > 0\}$.
(2) The lift of $g(\beta_j)$ has an attracting fixed point $\zeta = 1$ on $\mathbf{H}$. (If $g(\beta_j)$ corresponds to a parabolic element, $\zeta = 1$ is a unique fixed point of it. This situation is realized if Y is a 3-punctured sphere.)

The Fuchsian group Γ_x is uniquely determined by the component Y and the puncture x. It is called a *Fuchsian equivalent*.

Let $\pi\colon \mathbf{H} \to Y(= \mathbf{H}/\Gamma_x)$ be a natural projection. The image $N_x = \pi(\{\mathrm{Im}(\zeta) > 1\})$ is conformally equivalent to a punctured disk on Y by Shimizu's lemma. For any $\zeta \in \mathbf{H}$, we set $z = P(\zeta) = \exp(2\pi i\zeta)$. The mapping P maps $\mathbf{H}/\langle L\rangle$ conformally onto $D(1)$, where $D(r) = \{0 < |z| < r\}$ for any $r > 0$. The composition $P \circ \pi^{-1}$ of N_x onto $D(\exp(-2\pi))$ is a conformal mapping. Furthermore, let $\mathscr{D}$ be any punctured disk on Y with the puncture x. If we regard $\pi^{-1}(\mathscr{D})$ as a connected component which intersects $\{\mathrm{Im}(\zeta) > 1\}$, then the composition $P \circ \pi^{-1}$ of $\mathscr{D}$ into $D(1)$ is a conformal embedding. We use z as a coordinate of N_x via $P \circ \pi^{-1}$, and call it a *canonical local coordinate*.

We decide the names of punctures in each component of $S_c - \{\text{nodes of } S_c\}$ in the first argument of this section. In addition to, we re-denote by $x_{j,1}, x_{j,2}$ the pair of punctures which are determined by $f_c \circ f^*(\gamma_j)$ and the one of them is in one component of $S_c - \{\text{nodes of } S_c\}$, another puncture is in the other component (the case of (2) in the first discussion). Therefore, we can say that $x_{j,1}, x_{j,2}$ are punctures of $S_c - \{\text{nodes of } S_c\}$ which are determined by $f_c \circ f^*(\gamma_j)$ for any $j = 1, \dots, k$.

Let $[X, g] \in \partial_\Gamma \mathscr{T}$. For any $\mathfrak{a} = (a_1, \dots, a_k) \in S_+^{k-1}$, we use Theorem 2.2 then there exists a quadratic differential $J_c\,dz^2$ on X such that for any $g(x_{j,l})$, $J_c dz^2 = (a_j^2/z^2 + \cdots)dz^2$ in any local coordinate z with $z(g(x_{j,l})) = 0$. Let $X'_{j,l}$ be a component of $X - C_{J_c}$ which corresponds to $g(x_{j,l})$. We notice that C_{J_c} contains all nodes of X, so $X'_{j,l}$ does not contain $g(x_{j,l})$. We use z as the canonical local coordinate near $g(x_{j,l})$ and define the conformal mapping $w = p_{j,l}(z) = c'_{j,l} \exp(1/a_j \int \sqrt{J_c(z)}\,dz)$, where the constant $c'_{j,l}$ satisfies $(dw/dz)(0) = 1$. Then, there exists $r_{j,l} = r_{j,l}([X, g], \mathfrak{a}) > 0$ such that $p_{j,l}(X'_{j,l}) = D(r_{j,l})$ for any $j = 1, \dots, k$ and $l = 1, 2$, see Figure 3. Incidentally, by the argument in §2.5, we have $\dot{M}(X'_{j,l}) = (1/2\pi) \log r_{j,l}$.

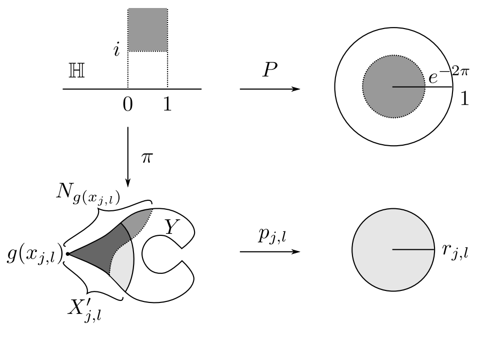

Figure 3. The light gray domain corresponds to $X'_{j,l}$, the dark one is $N_{g(x_{j,l})}$, and the more dark one in Y means their intersection.

3.2. The continuity of $J_c\,dz^2$. To obtain the continuity of the mapping in Theorem 1.1, we would like to show the following proposition. The following proofs of the proposition and lemmas are similar as of [MM75], but with little difficultly by the existence of $\mathfrak{a}$.

Proposition 3.1. *The quadratic differential $J_c\,dz^2$ is continuous on $\partial_\Gamma \mathscr{T} \times S_+^{k-1}$.*

Let $[X_n, g_n] \in \partial_\Gamma \mathscr{T}, \mathfrak{a}_n = ((a_1)_n, \dots, (a_k)_n) \in S_+^{k-1}$ be arbitrary sequences which converge to $[X, g] \in \partial_\Gamma \mathscr{T}, \mathfrak{a} = (a_1, \dots, a_k) \in S_+^{k-1}$ respectively. For each n, by Theorem 2.2, there exists a quadratic differential $(J_c)_n\, dz^2$ corresponding to X_n and $\mathfrak{a}_n$. Similarly, $J_c\, dz^2$ corresponding to X and $\mathfrak{a}$ is determined. To show that $(J_c)_n dz^2$ converges to $J_c\, dz^2$ as $n \to \infty$, we use the following lemmas. We notice that for the name of the projection of $\mathbf{H}$ onto $\mathbf{H}/\Gamma_x$ for any Fuchsian equivalent Γ_x, we use the common character π.

Lemma 3.2. *There exists $R_1 = R_1(\mathfrak{a}) > 0$ such that $r_{j,l,n} = r_{j,l}([X_n, g_n], \mathfrak{a}_n) \geq R_1$ for any j, l, and sufficiently large n.*

This lemma says that the radius $r_{j,l,n}$ which is determined in the previous section, is not degenerated when n tends to ∞ for any j, l. Moreover, the lower bound of $r_{j,l,n}$ depends only on $\mathfrak{a}$.

Proof. Let $\{D_{j,l,n}\}$ be a family of mutually disjoint punctured disks on X_n such that each $D_{j,l,n}$ has the puncture $x_n = g_n(x_{j,l})$. The mapping $P \circ \pi^{-1}$ of $D_{j,l,n}$ into $D(1)$ is conformal. We use the canonical local coordinate z of $D_{j,l,n}$ as in §3.1, then $P \circ \pi^{-1}$ is an identity mapping near x_n, we have $P \circ \pi^{-1}(0) = 0$ and $(d(P \circ \pi^{-1})(z)/dz)(0) = 1$. We combine this fact with the properties of reduced moduli (see §2.5), we have $\dot{M}(D_{j,l,n}) \leq \dot{M}(D(1)) = 0$. Furthermore, let $X'_{j,l,n}$ be a component of $X_n - C_{(J_c)_n}$ which corresponds to x_n, and $p_{j,l,n}$ be the conformal mapping of $X'_{j,l,n}$ onto $D(r_{j,l,n})$ in the last paragraph of §3.1. We also have $(1/2\pi)\log r_{j,l,n} = \dot{M}(X'_{j,l,n}) \leq \dot{M}(D(1)) = 0$ and $r_{j,l,n} \leq 1$. Let $(a_{j,1})_n = (a_{j,2})_n$ be both $(a_j)_n$, and use Theorem 2.3, then

$$\sum_{j,l} (a_{j,l})_n^2 \dot{M}(D_{j,l,n}) \leq \sum_{j,l} (a_{j,l})_n^2 \dot{M}(X'_{j,l,n}) = \frac{1}{2\pi} \sum_{j,l} (a_{j,l})_n^2 \log r_{j,l,n} \leq 0.$$

If we take $N_{x_n} \subset D_{j,l,n}$, then $-1 = \dot{M}(N_{x_n}) \leq \dot{M}(D_{j,l,n}) \leq 0$, and

$$-2 = -2\sum_j (a_j)_n^2 = -\sum_{j,l} (a_{j,l})_n^2 = \sum_{j,l} (a_{j,l})_n^2 \dot{M}(N_{x_n}) \leq \frac{1}{2\pi} \sum_{j,l} (a_{j,l})_n^2 \log r_{j,l,n} \leq 0.$$

Then we obtain $-4\pi \leq \sum_{j,l} (a_{j,l})_n^2 \log r_{j,l,n} \leq 0$. By $\log r_{j,l,n} \leq 0$ for any j, l, n, the inequality $-4\pi \leq (a_j)_n^2 \log r_{j,l,n}$ holds. Let $a = \min_j a_j$. By the assumption, $\mathfrak{a}_n$ converges to $\mathfrak{a}$, so we can take $a/2 \leq a_j/2 < (a_j)_n$ for any j and sufficiently large n. Therefore, we can set a bound $R_1 = R_1(\mathfrak{a}) = \exp(-16\pi/a^2) > 0$ such that $R_1 < r_{j,l,n} \leq 1$. □

For any $R < \exp(-2\pi)$, we denote $N_x(R) = \{p \in N_x |\ |z(p)| < R\}$. We notice that $P \circ \pi^{-1}(N_x(R)) = D(R)$.

Lemma 3.3. *There exists $R_2 = R_2(\mathfrak{a}) > 0$ such that $N_{x_n}(R_2) \subset X'_{j,l,n}$ for any j, l, and sufficiently large n.*

The lemma says that we can ensure a small radius R_2 in N_{x_n} such that the whole of $N_{x_n}(R_2)$ can be mapped conformally into $D(r_{j,l,n})$ by $p_{j,l,n}$, and the radius is not degenerated when n tends to ∞ for any j, l. In Figure 3, the above process is to reduce the dark domain in Y such that it is contained the light domain.

Proof. We consider the conformal mapping $F(w) = P \circ \pi^{-1} \circ p_{j,l,n}^{-1}(w)$ of $D(R_1)$ into $D(1)$ such that $F(0) = 0$. We can see $(dF(w)/dw)(0) = 1$ by the definitions of $p_{j,l,n}$, and the canonical local coordinate. Therefore, we can apply Koebe's 1/4-theorem, and conclude that $D(R_1/4) \subset F(D(R_1))$. Let $R_2 = \min(R_1/4, \exp(-2\pi)) =$

$R_1/4$ then $D(R_2) \subset F(D(R_1))$. On the other hand, $p_{j,l,n}^{-1}(D(R_1)) \subset X'_{j,l,n}$ by Lemma 3.2. By combining the above two, we have $D(R_2) \subset F(D(R_1)) \subset P \circ \pi^{-1}(X'_{j,l,n})$ and hence $N_{x_n}(R_2) \subset X'_{j,l,n}$. $\square$

In particular, $R_2 = R_1/4 = \exp(-16\pi/a^2)/4$ is continuous when $\mathfrak{a}$ varies continuously.

Lemma 3.4. *We take any $0 < r \leq R_2$ and set $X_n(r) = X_n - \bigcup_{j,l} N_{x_n}(r)$. There exists $M = M(\mathfrak{a}, r) > 0$ such that $\iint_{X_n(r)} |(J_c)_n| \leq M$ for sufficiently large n.*

It says that the differential $(J_c)_n dz^2$ is locally uniformly bounded for fixed $\mathfrak{a}$.

Proof. We suffice to consider only in $X'_{j,l,n}(r) = X'_{j,l,n} \cap X_n(r)$. By Lemma 3.3, we have that $X'_{j,l,n}(r) \neq \emptyset$ for any j, l and sufficiently large n. We apply Koebe's 1/4-theorem to the conformal mapping $p_{j,l,n} \circ \pi \circ P^{-1}$ of $D(r)$ into $D(r_{j,l,n})$ so that $D(r/4) \subset p_{j,l,n}(N_{x_n}(r)) \subset D(r_{j,l,n})$. We obtain

$$\iint_{X'_{j,l,n}(r)} |(J_c)_n| \leq 2\pi (a_j)_n^2 \log \frac{r_{j,l,n}}{\frac{r}{4}} < 2\pi \log \frac{4}{r},$$

that is, the upper bound comes from the area of the annulus $\{r/4 < |w| < r_{j,l,n}\}$ in $D(r_{j,l,n})$ with respect to the metric $(a_j)_n^2 |dw^2/w^2|$, and by $(a_j)_n < 1$, $r_{j,l,n} \leq 1$. We can take M as the multiple of $2k$ of the right hand side (the number $2k$ comes from the all combinations of $j = 1, \ldots, k$ and $l = 1, 2$). $\square$

Remark. Maybe we cannot take the constants R_1, R_2, and M which are independent on $\mathfrak{a}$ in Lemma 3.2, 3.3, and 3.4 respectively, see §23.4 in [Str84].

Now, we obtain the proof of Proposition 3.1.

Proof of Proposition 3.1. We recall the assumption written in the below of the statement of Proposition 3.1. We can choose each components Y_n of $X_n - \{\text{nodes of } X_n\}$ and Y of $X - \{\text{nodes of } X\}$ such that the corresponding Fuchsian equivalents Γ_{x_n} of Y_n converges to Γ_x of Y for each fixed puncture. Let $(\widetilde{J_c})_n$ be a holomorphic quadratic differential on $\mathbf{H}$ which is given by lifting of $(J_c)_n dz^2$ on Y_n. Since the sequence $\{(\widetilde{J_c})_n\}$ is normal by Lemma 3.4, then we can choose a subsequence if necessary such that $\{(\widetilde{J_c})_n\}$ converges locally uniformly to a holomorphic quadratic differential $(\widetilde{J_c})_\infty$ on $\mathbf{H}$. By Lemma 3.2, each $r_{j,l,n}$ does not degenerate as $n \to \infty$. This means that $(\widetilde{J_c})_\infty \neq 0$. We use Γ_x and project $(\widetilde{J_c})_\infty$ to $H\,dz^2$ on $Y = \mathbf{H}/\Gamma_x$. We regard the puncture of Y corresponding to $L\colon \zeta \mapsto \zeta + 1 \in \Gamma_x$ as $x = g(x_{j,l})$. For any $\varepsilon > 0$, let σ be a non-critical vertical trajectory of $(\widetilde{J_c})_\infty$ on $\mathbf{H}$ whose length is greater than $2\pi a_j + \varepsilon$ with respect to the metric $|(\widetilde{J_c})_\infty|^{\frac{1}{2}}$. By the uniformity of $\{(\widetilde{J_c})_n\}$, there exists a sequence $\{\sigma_n\}$ such that each σ_n is a non-critical vertical trajectory of $(\widetilde{J_c})_n$ whose length is greater than $2\pi(a_j)_n$ with respect to the metric $|(\widetilde{J_c})_n|^{\frac{1}{2}}$ for sufficiently large n, and σ_n converges to σ. Since such σ_n is projected to a closed trajectory of $(J_c)_n\,dz^2$ on Y_n, it satisfies $L(\sigma_n) \cap \sigma_n \neq \emptyset$ and then σ also satisfies $L(\sigma) \cap \sigma \neq \emptyset$. The projection of σ is a closed trajectory of Hdz^2 on Y such that it surrounds x. We conclude that any non-critical vertical trajectory of Hdz^2 on Y whose length is greater than or equal to $2\pi a_j$ with respect to the metric $|H|^{\frac{1}{2}}|dz|$ is closed. On the other hand, by the convergence of $(\widetilde{J_c})_n$, clearly we can see that Hdz^2 has the expansion $(a_j^2/z^2 + \cdots)dz^2$ near x. We carry out the above method for any components Y_n and Y, that is, for any $j = 1, \ldots, k$ and $l = 1, 2$.

By the uniqueness of the quadratic differential with poles of order 2 which has the closed vertical trajectories and the specified leading coefficients near the punctures of Theorem 2.2, Hdz^2 equals to $J_c dz^2$. □

3.3. The construction of Φ. In this section, we construct a mapping Φ of $\partial_\Gamma \mathscr{T} \times S_+^{k-1} \times (\mathbf{R}/2\pi\mathbf{Z})^k \times \mathbf{R}$ onto $\mathscr{T}/\langle \tau_1, \dots, \tau_k\rangle$ before obtaining $\hat{\Phi}$ in Theorem 1.1.

We recall the construction of S_c, see §2.7. For the fixed admissible curve family Γ on Σ, $\mathfrak{m} \in S_+^{k-1}$, and $[S^*, f^*] \in \mathscr{T}$, there exist $\alpha^* > 0$ and a Jenkins–Strebel differential φ^* on S^* with the moduli $\alpha^*\mathfrak{m}$ for the annuli corresponding to $f^*(\Gamma)$. The Riemann surface S_c with nodes corresponds to the end point of the Jenkins–Strebel ray directed by φ^* and emanating from $[S^*, f^*]$. It is constructed by $A_{j,1}(\infty) = \{0 < |z| < 1\}$, $A_{j,2}(\infty) = \{|z| > 1\}$ with the appropriate gluing. The differential φ^* induces a quadratic differential $J_c^*\, dz^2$ on S_c which has the representation $a_j^{*2}\, dw^2/w^2$ on $A_{j,l}(\infty)$ for any $j = 1, \dots, k$ and $l = 1, 2$. In other words, the leading coefficients of $J_c^*\, dz^2$ is $\mathfrak{a}^*$ at the nodes. For each component $(S_c)'_{j,l}$ of $S_c - C_{J_c^*}$, we can decide a conformal mapping $p^*_{j,l}$ of $(S_c)'_{j,l}$ onto $A_{j,l}(\infty)$ by the construction of J_c^* in the last paragraph of §2.7. We notice that it is not equal to $p_{j,l}$ in §3.1.

The way of constructing the mapping Φ is that, roughly speaking, we cut neighborhoods near nodes of a Riemann surface with nodes in $\partial_\Gamma \mathscr{T}$, glue the resulting surfaces using given parameters, and obtain an element of $\mathscr{T}/\langle \tau_1, \dots, \tau_k\rangle$. So we must decide a criterion when gluing.

We denote by $x = x_{j,l}$ the puncture of $(S_c)'_{j,l}$ corresponding to a node of S_c. Let Γ_x be the Fuchsian equivalent of $(S_c)'_{j,l}$ and x. The natural projection $\pi = \pi_x \colon \mathbf{H} \to \mathbf{H}/\Gamma_x \supset (S_c)'_{j,l}$ is determined by Γ_x. We fix a point $\xi_{j,l} \in D(1)$ such that it satisfies the following conditions;

(1) The absolute value holds $|\xi_{j,l}| < R_2(\mathfrak{a}^*)$, so $\pi \circ P^{-1}(\xi_{j,l})$ is in $N_x(R_2(\mathfrak{a}^*)) \subset (S_c)'_{j,l}$ (recall Lemma 3.3).
(2) The point $\pi \circ P^{-1}(\xi_{j,l})$ is sent on the positive real axis in $A_{j,l}(\infty)$ by $p^*_{j,l}$.

For any $[X, g] \in \partial_\Gamma \mathscr{T}$ and $\mathfrak{a} \in S_+^{k-1}$, let $J_c = J_c([X, g], \mathfrak{a})$ be a quadratic differential on X such that it has each leading coefficient a_j at the puncture $x = g(x_{j,l})$, and $X'_{j,l}$ be the component of $X - C_{J_c}$ associated to x. Let $\pi = \pi_x$ be the natural projection determined by the Fuchsian equivalent of $X'_{j,l}$ and x. By the above $\xi_{j,l}$, a point $\pi \circ P^{-1}(\xi_{j,l} \cdot (R_2(\mathfrak{a})/R_2(\mathfrak{a}^*)))$ is in $N_x(R_2(\mathfrak{a})) \subset X'_{j,l}$ by Lemma 3.3. We set

$$q_{j,1}(\zeta) = e^{i\theta_{j,1}} p_{j,1}(\zeta)/r_{j,1},\ q_{j,2}(\zeta) = e^{i\theta_{j,2}} r_{j,2}/p_{j,2}(\zeta),$$

where the conformal mapping $p_{j,l} \colon X'_{j,l} \to D(r_{j,l})$ is introduced in §3.1, and we take $\theta_{j,l}$ such that $q_{j,l} \circ \pi \circ P^{-1}(\xi_{j,l} \cdot (R_2(\mathfrak{a})/R_2(\mathfrak{a}^*))) > 0$. Then we have the conformal mappings $q_{j,1} \colon X'_{j,1} \to A_{j,1}(\infty)$, $q_{j,2} \colon X'_{j,2} \to A_{j,2}(\infty)$.

We notice that J_c is continuous when $[X, g]$ and $\mathfrak{a}$ vary continuously by Lemma 3.1, so $p_{j,l}$, $r_{j,l}$ are continuous. In addition to, $R_2(\mathfrak{a})$ is also continuous so that $\theta_{j,l}$ and $q_{j,l}$ are.

Now, we define a mapping

$$\Phi \colon \partial_\Gamma \mathscr{T} \times S_+^{k-1} \times (\mathbf{R}/2\pi\mathbf{Z})^k \times \mathbf{R} \to \mathscr{T}/\langle \tau_1, \dots, \tau_k\rangle,$$

where each τ_j is the Dehn twist of Σ about γ_j, that is τ_{γ_j}. We fix $[X, g] \in \partial_\Gamma \mathscr{T}$, $\mathfrak{a} = (a_1, \dots, a_k) \in S_+^{k-1}$, $\mathfrak{t} = (t_1, \dots, t_k) \in (\mathbf{R}/2\pi\mathbf{Z})^k$, and $s \in \mathbf{R}$. Let denote $\omega = ([X, g], \mathfrak{a}, \mathfrak{t}, s)$. First, there is a quadratic differential $J_c dz^2$ on X given by Theorem 2.2 for $[X, g]$ and $\mathfrak{a}$. By each mapping $q_{j,l}$ in the above discussion, each component $X'_{j,l}$ of $X - C_{J_c}$ is mapped onto $A_{j,l}(\infty)$. We remove the set $\{|\zeta| \le \exp(-\pi m_j \exp(2s))\}$

from $A_{j,1}(\infty)$, and $\{|\zeta| \geq \exp(\pi m_j \exp(2s))\}$ from $A_{j,2}(\infty)$. We denote the resulting two annuli by $A_{j,l}(s)$ and apply the stretch and the rotation

$$h_j \colon \zeta \mapsto \exp(2\pi m_j \exp(2s) + it_j)\zeta$$

to $A_{j,1}(s)$. We also denote the resulting surface by $A_{j,1}(s)$. Now, we have

$$A_{j,1}(s) = \{\exp(\pi m_j \exp(2s)) < |\zeta| < \exp(2\pi m_j \exp(2s))\},$$
$$A_{j,2}(s) = \{1 < |\zeta| < \exp(\pi m_j \exp(2s))\}.$$

We glue the boundaries $\partial_i A_{j,1}(s)$ and $\partial_o A_{j,2}(s)$ by the identity mapping so that obtain $A_j(s) = \{1 < |\zeta| < \exp(2\pi m_j \exp(2s))\}$ whose modulus is $m_j e^{2s}$. Next, we glue $A_1(s), \dots, A_k(s)$ each other by the original gluing mappings of X after each action of h_j^{-1} in a small tubular neighborhood in $\partial_o A_j(s)$. Then, we obtain the resulting Riemann surface and denote it by S_ω. The associated homeomorphism $g_c \colon X - \{\text{nodes of } X\} \to S_\omega - \Gamma_\omega$ is also determined, where Γ_ω is some admissible curve family on S_ω. For the composition $f_\omega = g_c \circ g \circ f_c \circ f \colon \Sigma - \Gamma \to S_\omega - \Gamma_\omega$, since $f(\Gamma)$ is degenerated by f_c, we can find a homeomorphism of Σ onto S_ω which is homotopic to f_ω up to the Dehn twists $\tau_1, \dots, \tau_k$ of Σ. We denote it by the same symbol f_ω, and set $\Phi(\omega) = [S_\omega, f_\omega] \in \mathscr{T}/\langle \tau_1, \dots, \tau_k \rangle$.

In particular, by the above construction, we have

$$\Phi([S_c, \mathrm{id}], \mathfrak{a}^*, 0, \dots, 0, (\log \alpha^*)/2) = [S^*, f^*].$$

In fact, the setting of each criterion $\xi_{j,l}$ in this section, is to give the equation.

3.4. Complete the proof of Theorem 1.1. To prove Theorem 1.1, we perform by the following 3-steps.

Step 1. Φ is bijection. We construct the inverse Φ^{-1}, that is, we decide each element of $\partial_\Gamma \mathscr{T}$, S_+^{k-1}, $(\mathbf{R}/(2\pi\mathbf{Z}))^k$, and $\mathbf{R}$ for any $[S, f] \in \mathscr{T}/\langle \tau_1, \dots, \tau_k \rangle$. By Theorem 2.4, there exist $\alpha > 0$ and a Jenkins–Strebel differential φ on S associated to the admissible curve family $f(\Gamma)$ with moduli $\alpha\mathfrak{m}$. We set $s(\alpha) = (\log \alpha)/2 \in \mathbf{R}$. The Jenkins–Strebel ray r directed by φ and emanating from $[S, f]$ determines the end point $r(\infty) \in \partial_\Gamma \mathscr{T}$, and denote it by $[X, g]$. As in §2.7, let $\mathfrak{a}(\varphi) = \{a_1(\varphi), \dots, a_k(\varphi)\}$ be the lengths of the closed vertical trajectories homotopic to each of $f(\Gamma)$ divided by 2π, with respect to the metric $|\varphi|^{\frac{1}{2}}$. We normalize φ so that $\mathfrak{a}(\varphi) \in S_+^{k-1}$.

The conformal mapping $w = \varrho_{j,l}(z) = c_{j,l} \exp(1/a_j(\varphi) \int \sqrt{\varphi(z)}\, dz)$ where $c_{j,l} > 0$ and z is any local coordinate of S, maps each component of $S - C_\varphi - f(\Gamma)$ onto a round annulus $A_{j,l}$. We stretch these annuli, then the Riemann surface X with nodes is obtained by $A_{j,1}(\infty) = \{0 < |w| < 1\}$ and $A_{j,2}(\infty) = \{|w| > 1\}$ with the same gluing as in the case of S. The quadratic differential φ has the representation $a_j(\varphi)^2 dw^2/w^2$ in $A_{j,l}$, and so in $A_{j,l}(\infty)$. Hence we have a quadratic differential J_c on X with the leading coefficients $\mathfrak{a}(\varphi)$, it is also obtained by Theorem 2.2. By the above discussion, we can regard $\varrho_{j,l}$ as a conformal mapping of each component $X'_{j,l}$ of $X - C_{J_c}$ onto $A_{j,l}(\infty)$. Compare the last paragraph of §2.7. Now, we can see the difference between the mapping $\varrho_{j,l}$ which is determined from S along the ray r, and the mapping $q_{j,l}$ which is determined by the mapping $p_{j,l}$ and the criterion $\xi_{j,l}$ (see §3.3). The composition $q_{j,l} \circ \varrho_{j,l}^{-1}(1)$ is in the unit circle, and let $\vartheta_{j,l} \in \mathbf{R}/(2\pi\mathbf{Z})$ be an argument of it. Then each $t_j = \vartheta_{j,2} - \vartheta_{j,1}$ modulo 2π is well defined in $\mathbf{R}/(2\pi\mathbf{Z})$. We set $\mathfrak{t}(\varphi) = (t_1, \dots, t_k) \in (\mathbf{R}/(2\pi\mathbf{Z}))^k$.

We need to confirm that α, φ, and $r(\infty)$ do not depend on the actions of $\tau_1, \dots, \tau_k$ before set $\Phi^{-1}([S, f]) = (r(\infty), \mathfrak{a}(\varphi), \mathfrak{t}(\varphi), s(\alpha))$. Let γ be one of $\Gamma = \{\gamma_1, \dots, \gamma_k\}$

and we compare the elements as above for $[S,f]$ and $[\tau_\gamma]_*([S,f])$ respectively. In general, a number $\alpha > 0$ and a Jenkins–Strebel differential φ in Theorem 2.4 are determined by a base Riemann surface S, an admissible curve family Γ, and numbers $\mathfrak{m} = (m_1, \ldots, m_k)$. We recall the action of the mapping class group to the Teichmüller space, that is $[\tau_\gamma]_*([S,f]) = [S, f \circ \tau_\gamma^{-1}]$. Then we use a common S. We consider $f(\Gamma)$ and $f \circ \tau_\gamma^{-1}(\Gamma)$. We see that $\tau_\gamma^{-1}(\gamma_j) = \gamma_j$ for any $j = 1, \ldots, k$, because γ is one of the admissible curve family Γ whose curves are disjoint and are not homotopic each other, then each γ_j is invariant by the Dehn twists about any $\gamma_{j'}$ so we have $f(\Gamma) = f \circ \tau_\gamma^{-1}(\Gamma)$. Since $\mathfrak{m}$ is fixed, then the resulting α and φ on S are common elements. Therefore, $\mathfrak{a}(\varphi), \mathfrak{t}(\varphi), s(\alpha)$ are also common values. Next, we consider Jenkins–Strebel rays r and r' directed by the common φ and emanating from $[S,f]$ and $[\tau_\gamma]_*([S,f])$ respectively. We can write the rays $r(s) = [S_s, f_s \circ f]$ and $r'(s) = [S_s, f_s \circ f \circ \tau_\gamma^{-1}]$ by the construction of the Teichmüller geodesic rays with the common S and φ (see §2.6). Clearly, it holds $r'(s) = [\tau_\gamma]_*(r(s))$. There is well-known fact that the equations $\tau_{f(\gamma)} = f \circ \tau_\gamma \circ f^{-1}$ and $\tau_{f_s \circ f(\gamma)} = f_s \circ \tau_{f(\gamma)} \circ f_s^{-1}$ hold, where these equalities mean as the homotopy equivalent. Then

$$d_{\mathscr{T}}(r(s), r'(s)) \leq \frac{1}{2} \log K(\tau_{f_s \circ f(\gamma)}).$$

By the definition of twists in §2.8, the dilatation $K(\tau_{f_s \circ f(\gamma)}) = (1+k_s)/(1-k_s)$ is determined by

$$k_s = \left| \frac{\frac{2\pi}{2\pi\alpha m e^{2s}}}{2 + i\frac{2\pi}{2\pi\alpha m e^{2s}}} \right|,$$

where αm is a modulus of the annulus corresponding to $f(\gamma)$ determined by φ such that m is a corresponding entry of $\mathfrak{m}$. The modulus of the annulus corresponding to $f_s \circ f(\gamma)$ is $\alpha m e^{2s}$. As $s \to \infty$, we conclude that r and r' are asymptotic, then their end points coincide, that is $r(\infty)$ by the main theorem of [Am14a].

Step 2. Φ is continuous. We consider the continuity of Φ. We set sequences and a point in $\mathscr{T}$, namely, $p_n = [S_n, f_n] = \Phi([X_n, g_n], \mathfrak{a}_n, \mathfrak{t}_n, s_n)$, $p'_n = [S'_n, f'_n] = \Phi([X_n, g_n], \mathfrak{a}_n, \mathfrak{t}, s)$, and $p = \Phi([X, g], \mathfrak{a}, \mathfrak{t}, s)$, and let $([X_n, g_n], \mathfrak{a}_n, \mathfrak{t}_n, s_n)$ converges to $([X, g], \mathfrak{a}, \mathfrak{t}, s)$. For the triangle inequality $d(p_n, p) \leq d(p_n, p'_n) + d(p'_n, p)$, we suffice to show that the right hand side converges to 0 as $n \to \infty$, where d is the Teichmüller distance on $\mathscr{T}/\langle \tau_1, \ldots, \tau_k \rangle$ by the natural projection.

For $d(p_n, p'_n)$, the Riemann surfaces S_n and S'_n are determined by a common quadratic differential $(J_c)_n \, dz^2$ on X_n with the leading coefficients $\mathfrak{a}_n$ for any n. Let φ_n and φ'_n be the Jenkins–Strebel differentials on S_n and S'_n respectively, obtained by the construction of Φ. This means that we can construct a quasiconformal mapping of S_n onto S'_n with a suitable homotopy class as the mapping such that each component of $S_n - C_{\varphi_n}$ is mapped one of $S'_n - C_{\varphi'_n}$. Let $\mathfrak{a}_n = ((a_1)_n, \ldots, (a_k)_n)$, $\mathfrak{t}_n = ((t_1)_n, \ldots, (t_k)_n)$, and $\mathfrak{t} = (t_1, \ldots, t_k)$. The components corresponding to $f_n(\gamma_j)$ and $f'_n(\gamma_j)$ are annuli, and also by the construction of Φ, they are represented by parallelograms

$$P_j(0, (h_j)_n + i(t_j)_n(a_j)_n, (h_j)_n + i((t_j)_n(a_j)_n + (l_j)_n), i(l_j)_n),$$
$$P'_j(0, (h'_j)_n + it_j(a_j)_n, (h'_j)_n + i(t_j(a_j)_n + (l_j)_n), i(l_j)_n)$$

respectively, where $(h_j)_n = 2\pi(a_j)_n m_j e^{2s_n}$, $(h'_j)_n = 2\pi(a_j)_n m_j e^{2s}$, and $(l_j)_n = 2\pi(a_j)_n$ for any $j = 1, \ldots, k$ and any n. In fact, $(h_j)_n$ and $(h'_j)_n$ are the heights, $(l_j)_n$ is the circumference of the annuli with respect to φ_n and φ'_n. In particular, we can take these parallelograms such that $|(t_j)_n - t_j| \leq \pi$ for any j and sufficiently large n because now we consider the quotient space $(\mathbf{R}/2\pi\mathbf{Z})^k$ which contains $\mathfrak{t}_n$ and $\mathfrak{t}$. There is each natural affine mapping of P_j onto P'_j for any $j = 1, \ldots, k$, that is

$$x + iy \mapsto \frac{(h'_j)_n}{(h_j)_n} x + i\left(\frac{(t_j - (t_j)_n)(a_j)_n}{(h_j)_n} x + y\right).$$

So we obtain a quasiconformal mapping of S_n onto S'_n by combining the above mappings for all $j = 1, \ldots, k$. We calculate its dilatation, hence we have

$$d(p_n, p'_n) \leq \frac{1}{2} \log \max_{j=1,\ldots,k} \frac{1 + (k_j)_n}{1 - (k_j)_n},$$

where

$$(k_j)_n = \left|\frac{\frac{(t_j-(t_j)_n)(a_j)_n}{(h_j)_n} + i\left(1 - \frac{(h'_j)_n}{(h_j)_n}\right)}{\frac{(t_j-(t_j)_n)(a_j)_n}{(h_j)_n} + i\left(1 + \frac{(h'_j)_n}{(h_j)_n}\right)}\right| = \left|\frac{\frac{t_j-(t_j)_n}{2\pi} + im_j\left(e^{2s_n} - e^{2s}\right)}{\frac{t_j-(t_j)_n}{2\pi} + im_j\left(e^{2s_n} + e^{2s}\right)}\right|.$$

Clearly $d(p_n, p'_n) \to 0$ as $n \to \infty$.

For $d(p'_n, p)$, by Proposition 3.1, a differential $(J_c)_n dz^2$ obtained by $[X_n, g_n], \mathfrak{a}_n$ converges to a differential $J_c dz^2$ corresponding to $[X, g], \mathfrak{a}$. Then for the common $\mathfrak{t}$ and s, the sequence p'_n converges to p, so we have $d(p'_n, p) \to 0$ as $n \to \infty$. By these argument, we conclude that Φ is continuous.

Step 3. The lift $\hat{\Phi}$ of Φ. Finally, we consider a lift of Φ. Since the set $\partial_\Gamma \mathscr{T} \times S_+^{k-1} \times \mathbf{R}^k \times \mathbf{R}$ and the Teichmüller space $\mathscr{T}$ are simply connected, there exists the bijective and continuous lift $\hat{\Phi}\colon \partial_\Gamma \mathscr{T} \times S_+^{k-1} \times \mathbf{R}^k \times \mathbf{R} \to \mathscr{T}$ of Φ such that

$$\hat{\Phi}([S_c, \mathrm{id}], a_1^*, \ldots, a_k^*, 0, \ldots, 0, (\log \alpha^*)/2) = [S^*, f^*].$$

The complex dimension of $\partial_\Gamma \mathscr{T}$ is $3g - 3 + p - k$. Therefore the real dimension of $\partial_\Gamma \mathscr{T} \times S_+^{k-1} \times \mathbf{R}^k \times \mathbf{R}$ and $\mathscr{T}$ are both $6g - 6 + 2p$. We can use Brouwer's theorem hence $\hat{\Phi}$ is a homeomorphism.

For any j, the equation

$$\hat{\Phi}([X, g], \mathfrak{a}, t_1, \ldots, t_j + 2\pi, \ldots, t_k, s) = \tau_j \circ \hat{\Phi}([X, g], \mathfrak{a}, t_1, \ldots, t_j, \ldots, t_k, s)$$

clearly holds. For any $s \geq 0$ and other fixed parameters, $\hat{\Phi}([X, g], \mathfrak{a}, \mathfrak{t}, s)$ is a Jenkins–Strebel ray directed by φ and emanating from $\hat{\Phi}([X, g], \mathfrak{a}, \mathfrak{t}, 0)$ where φ is determined by Theorem 2.4 with the corresponding Riemann surface of $\hat{\Phi}([X, g], \mathfrak{a}, \mathfrak{t}, 0)$ and $\mathfrak{m}$. For any $\mathfrak{a}, \mathfrak{t}$, these ray have the same end point $[X, g]$ and positive scalar multiples of the common moduli $\mathfrak{m}$ of the annuli corresponding to each φ. By Theorem in [Am14a], they are asymptotic each other. We complete the proof of Theorem 1.1. $\square$

Remark. We consider the case of $k = 1$ and denote the curve γ_1 by γ. The mapping $\Phi\colon \partial_\gamma \mathscr{T} \times \{1\} \times (\mathbf{R}/2\pi\mathbf{Z}) \times \mathbf{R} \to \mathscr{T}/\langle \tau_\gamma \rangle$ is related to a Teichmüller disk. For simplicity, we drop $\{1\}$. In this situation, $\mathfrak{m} = 1$ and α^* is a modulus of an annulus $S^* - C_{\varphi^*}$. We set $\Psi\colon (\mathbf{R}/2\pi\mathbf{Z}) \times \mathbf{R} \to \mathbf{D}/\langle T \rangle$ as

$$\Psi(t, s) = \frac{e^{2s} - \alpha^* + i\dfrac{t}{2\pi}}{e^{2s} + \alpha^* + i\dfrac{t}{2\pi}},$$

where T is a biholomorphic automorphism of $\mathbf{D} = \{|z| < 1\}$ such that

$$T(z) = \frac{(i - 2\alpha^*)z - i}{iz - (i + 2\alpha^*)}.$$

For any $z \in \mathbf{D}$, let $F_z\colon S^* \to S_z$ be a Teichmüller mapping constructed by the Beltrami coefficient $z|\varphi^*|/\varphi^*$, and

$$D[\varphi^*] = \{[S_z, F_z \circ f^*] \mid z \in \mathbf{D}\}$$

be the Teichmüller disk of the origin $[S^*, f^*]$. The mapping $F(z) = [S_z, F_z \circ f^*]$ of $\mathbf{D}$ onto $D[\varphi^*]$ is a biholomorphic mapping such that $F(0) = [S^*, f^*]$. The action $[\tau_\gamma]_*$ is a biholomorphic isometry of $D[\varphi^*]$ onto itself. The mapping T is a pull-back of $[\tau_\gamma^{-1}]_*$ by F. The composition $\Phi \circ (\mathrm{id}_{\partial_\gamma \mathscr{T}}, \Psi^{-1})\colon \partial_\gamma \mathscr{T} \times \mathbf{D}/\langle T\rangle \to \mathscr{T}/\langle \tau_\gamma\rangle$ satisfies that $\{[X, g]\} \times \mathbf{D}/\langle T\rangle$ is mapped to $D[\varphi]/\langle\tau_\gamma\rangle$ where $D[\varphi]$ is a Teichmüller disk by some Jenkins–Strebel differential φ whose boundary contains $[X, g]$, for any $[X, g] \in \partial_\gamma \mathscr{T}$. The lift of the composition $\Phi \circ (\mathrm{id}_{\partial_\gamma \mathscr{T}}, \Psi^{-1})\colon \partial_\gamma \mathscr{T} \times \mathbf{D}/\langle T\rangle \to \mathscr{T}/\langle\tau_\gamma\rangle$ such that $([S_c, \mathrm{id}], 0)$ is mapped to $[S^*, f^*]$ coincides to the mapping of the main result in [MM75].

For any admissible curve family $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$, we apply the above Masur and Marden's theorem to $\mathscr{T}$ iteratively, then obtain a homeomorphism of $\partial_\Gamma \mathscr{T}$ and parameter spaces onto $\mathscr{T}$. That is, first we construct a homeomorphism $\hat{\Phi}_1\colon \partial_{\gamma_1} \mathscr{T} \times \mathbf{D} \to \mathscr{T}$, and the set $\partial_{\gamma_1} \mathscr{T}$ is represented by the Teichmüller space $\mathscr{T}(\Sigma \backslash \{\gamma_1\})$ or a product of Teichmüller spaces comes from the components of $\mathscr{T}(\Sigma\backslash\{\gamma_1\})$, next we apply the theorem to it and combine with $\hat{\Phi}_1$, we have a homeomorphism $\hat{\Phi}_2\colon \partial_{\{\gamma_1,\gamma_2\}} \mathscr{T} \times \mathbf{D}^2 \to \mathscr{T}$. We continue the process, finally a homeomorphism $\hat{\Phi}_k\colon \partial_\Gamma \mathscr{T} \times \mathbf{D}^k \to \mathscr{T}$ is obtained. However, in the domain of $\hat{\Phi}_k$, each unit disk $\mathbf{D}$ is not correspond to any Teichmüller disk on the whole $\mathscr{T}$ without the first $\mathbf{D}$ by $\hat{\Phi}_1$. It is difficult to consider various geometric properties of $\mathscr{T}$ by $\hat{\Phi}_k$ ($k \geq 2$). On the other hand, we can see some properties for our mapping $\hat{\Phi}$ in the following section.

3.5. Some properties of $\hat{\Phi}$. We already know the asymptotic behavior of two Jenkins–Strebel rays, see [Am14b] and [Am14a]. In this time, we describe it by using $\hat{\Phi}$. The following corollary is obtained directly by Theorem 1.1 and the main result in [Am14b].

Corollary 3.5. *For any $[X, g], [X', g'] \in \partial_\Gamma \mathscr{T}$, $\mathfrak{a}, \mathfrak{a}' \in S_+^{k-1}$, and $\mathfrak{t}, \mathfrak{t}' \in \mathbf{R}^k$, the following equation holds;*

$$\lim_{s\to\infty} d_{\mathscr{T}}(\hat{\Phi}([X, g], \mathfrak{a}, \mathfrak{t}, s), \hat{\Phi}([X', g'], \mathfrak{a}', \mathfrak{t}', s + \lambda)) = \max\{d_{\partial_\Gamma \mathscr{T}}([X, g], [X', g']), |\lambda|\},$$

where $d_{\partial_\Gamma \mathscr{T}}$ is the Teichmüller distance in $\partial_\Gamma \mathscr{T}$ and λ is a constant.

Proof. Let $[S, f] = \hat{\Phi}([X, g], \mathfrak{a}, \mathfrak{t}, s)$ and $[S', f'] = \hat{\Phi}([X', g'], \mathfrak{a}', \mathfrak{t}', s + \lambda)$. The associated Jenkins–Strebel differentials on S and S' have annuli corresponding to the core curves $f(\gamma_j)$ and $f'(\gamma_j)$ respectively, for any $j = 1, \ldots, k$. Hence, such annuli are homotopic each other by the mapping $f' \circ f^{-1}$. (We call this condition by "similar" in [Am14b].) By the construction of $\hat{\Phi}$, we use the common $\mathfrak{m} = (m_1, \ldots, m_k)$ for moduli of such annuli, in detail they are $m_1 e^{2s}, \ldots, m_k e^{2s}$ and $m_1 e^{2(s+\lambda)}, \ldots, m_k e^{2(s+\lambda)}$, respectively. We have

$$\begin{aligned}
&\lim_{s\to\infty} d_{\mathscr{T}}([S,f],[S',f']) \\
&= \max\left\{ d_{\partial_\Gamma \mathscr{T}}([X,g],[X',g']), \frac{1}{2}\log \max_{j=1,\dots,k}\left\{\frac{m_j e^{2(s+\lambda)}}{m_j e^{2s}}, \frac{m_j e^{2s}}{m_j e^{2(s+\lambda)}}\right\}\right\} \\
&= \max\{d_{\partial_\Gamma \mathscr{T}}([X,g],[X',g']), |\lambda|\},
\end{aligned}$$

by the main theorem in [Am14b]. $\square$

Next, we check several estimations of the Teichmüller distance between two points when each element of $\hat{\Phi}$ varies. We denote the elements by $[X,g],[X',g'] \in \partial_\Gamma\mathscr{T}$, $\mathfrak{a} = (a_1,\dots,a_k), \mathfrak{a}' = (a'_1,\dots,a'_k) \in S^{k-1}_+$, $\mathfrak{t} = (t_1,\dots,t_k), \mathfrak{t}' = (t'_1,\dots,t'_k) \in \mathbf{R}^k$, and $s, s' \in \mathbf{R}$.

Proposition 3.6. *We have*

$$d_{\mathscr{T}}(\hat{\Phi}([X,g],\mathfrak{a},\mathfrak{t},s), \hat{\Phi}([X,g],\mathfrak{a},\mathfrak{t},s')) = |s-s'|, \tag{1}$$

$$d_{\mathscr{T}}(\hat{\Phi}([X,g],\mathfrak{a},\mathfrak{t},s), \hat{\Phi}([X,g],\mathfrak{a},\mathfrak{t}',s)) \le \frac{1}{2}\log \max_j K(\tau_{t_j - t'_j}), \tag{2}$$

where $K(\tau_\theta)$ is defined in §2.8. There exists a constant $c(\mathfrak{a},\mathfrak{a}')$ and we fix any $s_0 > c(\mathfrak{a},\mathfrak{a}')$. Then, there exists a constant $C(\mathfrak{a},\mathfrak{a}',s_0)$ such that the following inequality

$$d_{\mathscr{T}}(\hat{\Phi}([X,g],\mathfrak{a},\mathfrak{t},s), \hat{\Phi}([X,g],\mathfrak{a}',\mathfrak{t},s)) \le C(\mathfrak{a},\mathfrak{a}',s_0) \tag{3}$$

holds for any $s > s_0$. Moreover, the right hand side of the above inequalities (2),(3) tend to 0 as $s\to\infty, s_0\to\infty$ respectively. On the lower estimate, we have

$$d_{\mathscr{T}}(\hat{\Phi}([X,g],\mathfrak{a},\mathfrak{t},s), \hat{\Phi}([X',g'],\mathfrak{a}',\mathfrak{t}',s')) \ge |s-s'| - \frac{1}{2}\log k. \tag{4}$$

Proof. The first two equality and inequality are easy. If only parameter s varies, the image of $\hat{\Phi}$ lies a Jenkins–Strebel geodesic line on $\mathscr{T}$, then we obtain (1). Next, for a parameter $\mathfrak{t}$, let $[R,h] = \hat{\Phi}([X,g],\mathfrak{a},\mathfrak{t},s)$ and $[R',h'] = \hat{\Phi}([X,g],\mathfrak{a},\mathfrak{t}',s)$. By the construction of Φ and $\hat{\Phi}$, there is a quasiconformal mapping of the Riemann surface R onto R' such that any annular subset $A_j(s)$ of R is mapped onto one of R' as the twist $\tau_{t'_j - t_j}$, and it is homotopic to $h'\circ h^{-1}$. By $K(\tau_{t'_j-t_j}) = K(\tau_{t_j - t'_j})$, we conclude that the inequality (2) holds.

Now, we consider about (3). Let $\mathrm{id}_X\colon X\to X$ be the identity mapping. We use the canonical local coordinate z of X (see §3.1) and write $id_X(z) = z$. Let $p_{j,l}$ and $p'_{j,l}$ be the conformal mappings on each components $X - C_{J_c}$ and $X - C_{J'_c}$ described in also §3.1 such that the differentials J_c and J'_c on X are determined by $\mathfrak{a}$ and $\mathfrak{a}'$ respectively. We recall $q_{j,l}$ in §3.3, and let $w = q_{j,l}(z)$ and $w' = q'_{j,l}(z)$ onto $A_{j,l}(\infty)$. On the sets $A_{j,1}(\infty) = \{0<|w|<1\}$, $A_{j,2}(\infty) = \{|w|>1\}$, for the simplicity, we regard them as $D(1) = \{0<|w|<1\}$ by acting the mapping $w\mapsto 1/w$ for $A_{j,2}(\infty)$. We have

$$\begin{aligned}
\left|\frac{dw'}{dw}(0)\right| &= \left|\frac{dw'}{dz}(0)\right| \Big/ \left|\frac{dw}{dz}(0)\right| = \left|\frac{e^{i(\pm\theta'_{j,l})}}{r_{j,l}(\mathfrak{a}')}\frac{dp'_{j,l}(z)}{dz}(0)\right| \Big/ \left|\frac{e^{i(\pm\theta_{j,l})}}{r_{j,l}(\mathfrak{a})}\frac{dp_{j,l}(z)}{dz}(0)\right| \\
&= \frac{r_{j,l}(\mathfrak{a})}{r_{j,l}(\mathfrak{a}')}
\end{aligned}$$

by $(dp_{j,l}(z)/dz)(0) = (dp'_{j,l}(z)/dz)(0) = 1$. We use the estimation of $r_{j,l}(\mathfrak{a})$ in Lemma 3.2, then

$$R_1(\mathfrak{a}) \leq \frac{r_{j,l}(\mathfrak{a})}{r_{j,l}(\mathfrak{a}')} \leq \frac{1}{R_1(\mathfrak{a}')}.$$

We consider the conformal mapping $I_{j,l} = q'_{j,l} \circ id_X \circ (q_{j,l})^{-1}$ of $A_{j,l}(\infty) = D(1)$ into $\mathbf{C}$. It is described by $I_{j,l}(w) = c_1 w + \psi(w)$ where $c_1 \neq 0$ and $\psi(w) = c_n w^n + \cdots$ is a convergent power series where $c_n \neq 0$, $n \geq 2$. It has the absolute value of the derivative $|c_1| = r_{j,l}(\mathfrak{a})/r_{j,l}(\mathfrak{a}')$ at the origin. Then we have $R_1(\mathfrak{a}) \leq |c_1| \leq 1/R_1(\mathfrak{a}')$. Let

$$c = c(\mathfrak{a}, \mathfrak{a}') = \frac{1}{2} \log \left(-\frac{2 \log(R_2(\mathfrak{a}) R_1(\mathfrak{a}'))}{\pi m} \right),$$

where $m = \min_j m_j$. We take $s_0 > c$ and fix it, and also take any $s > s_0$. Let $R_0 = \exp(-\pi m \exp(2s_0))$ and we have $R_0 < (R_2(\mathfrak{a})R_1(\mathfrak{a}'))^2$. Now, we consider the following continuous mapping $F_{j,l}$ of $D(1)$ onto the image of $I_{j,l}$.

$$F_{j,l}(w) = \begin{cases} w & (0 < |w| \leq R_0) \\ c_1^{2\left(1 - \frac{\log |w|}{\log R_0}\right)} w & (R_0 \leq |w| \leq R_0^{\frac{1}{2}}) \\ c_1 w + \phi(w)\psi(w) & (R_0^{\frac{1}{2}} \leq |w| \leq 2R_0^{\frac{1}{2}}) \\ I_{j,l}(w) & (2R_0^{\frac{1}{2}} \leq |w| < 1) \end{cases}$$

where $\phi(w) = (|w| - R_0^{\frac{1}{2}})/R_0^{\frac{1}{2}} \in [0,1]$. Let $[S, f] = \hat{\Phi}([X, g], \mathfrak{a}, \mathfrak{t}, s)$ and $[S', f'] = \hat{\Phi}([X, g], \mathfrak{a}', \mathfrak{t}, s)$. Our strategy is that, we construct a quasiconformal mapping of S onto S' which is homotopic to $f' \circ f^{-1}$ to estimate $d_{\mathscr{T}}([S, f], [S', f'])$. By the construction of $\hat{\Phi}$, S and S' are assembled by

$$A_{j,l}(s) = \{\exp(-\pi m_j \exp(2s)) < |w| < 1\}$$

for any $j = 1, \ldots, k$ and $l = 1, 2$ with the gluings come from J_c and J'_c respectively. Hence, we show that the mapping $F_{j,l}$ is quasiconformal, and modify it to the mapping of $A_{j,l}(s)$ which is still quasiconformal. Finally we compose the mappings of all $j = 1, \ldots, k$ and $l = 1, 2$, so that it becomes as a mapping of S onto S'.

We see that $F_{j,l}$ is the appropriate mapping. We apply Koebe's 1/4-theorem to $I_{j,l}$ and conclude that the image $I_{j,l}(D(1))$ contains $D(R_1(\mathfrak{a})/4) = D(R_2(\mathfrak{a}))$, and the inequality $R_2(\mathfrak{a}) > (R_2(\mathfrak{a})R_1(\mathfrak{a}'))^2 > R_0$ holds. This yields that the image of the punctured disk $D(R_0)$ by $F_{j,l}$ (it equals to itself) is contained in the image of $D(1)$ by $I_{j,l}$. The mapping satisfies that $F_{j,l}(w) = c_1 w$ on $\{|w| = R_0^{\frac{1}{2}}\}$. We can check that the image of $\{R_0 \leq |w| \leq R_0^{\frac{1}{2}}\}$ by $F_{j,l}$ does not invert, and it is contained in $D(R_2(\mathfrak{a}))$, that is, $R_0 < |c_1| R_0^{\frac{1}{2}} < R_2(\mathfrak{a})$. Certainly, we can see that

$$R_0 < R_2(\mathfrak{a})R_1(\mathfrak{a}')R_0^{\frac{1}{2}} < R_1(\mathfrak{a})R_0^{\frac{1}{2}} \leq |c_1| R_0^{\frac{1}{2}} < |c_1| R_2(\mathfrak{a}) R_1(\mathfrak{a}') \leq R_2(\mathfrak{a}).$$

We confirm that the mapping $F_{j,l}$ is a quasiconformal mapping. Clearly, the mappings on $\{0 < |w| \leq R_0\}$ and $\{2R_0^{\frac{1}{2}} \leq |w| < 1\}$ are conformal. The mapping on $\{R_0 \leq |w| \leq R_0^{\frac{1}{2}}\}$ is a quasiconformal mapping. Indeed, if the mapping is transformed by the multiple of R_0, then we have the mapping

$$w \mapsto |w|^{\frac{\log |c_1|}{\log R_0^{-\frac{1}{2}}}} \cdot |w|^{i \frac{\arg c_1}{\log R_0^{-\frac{1}{2}}}} w$$

of $\{1 \le |w| \le R_0^{-\frac{1}{2}}\}$ onto $\{1 \le |w| \le |c_1|R_0^{-\frac{1}{2}}\}$. It consists of the twist of $\{1 \le |w| \le R_0^{-\frac{1}{2}}\}$ of the angle $\arg c_1$ (see §2.8), and the expansion to $\{1 \le |w| \le |c_1|R_0^{-\frac{1}{2}}\}$. The absolute value of the Beltrami coefficient of the mapping is the following;

$$\left|\frac{\log c_1}{\log R_0 - \log c_1}\right| < 1. \tag{5}$$

Since the mapping does not invert that we see as above, it is smaller than 1.

So, we aim at the mapping $c_1 w + \phi(w)\psi(w)$. We denote it by $h(w)$ and compute its partial derivatives. We have

$$h_w(w) = c_1 + \frac{1}{2R_0^{\frac{1}{2}}} w^{-\frac{1}{2}}\bar{w}^{\frac{1}{2}}\psi(w) + \phi(w)\frac{d\psi(w)}{dw}, \quad h_{\bar{w}}(w) = \frac{1}{2R_0^{\frac{1}{2}}} w^{\frac{1}{2}}\bar{w}^{-\frac{1}{2}}\psi(w),$$

and

$$|h_w(w)| \ge |c_1| - \frac{1}{2R_0^{\frac{1}{2}}}|\psi(w)| - \left|\frac{d\psi(w)}{dw}\right|, \quad |h_{\bar{w}}(w)| = \frac{1}{2R_0^{\frac{1}{2}}}|\psi(w)|.$$

We would like to obtain a lower bound of $|h_w(w)| - |h_{\bar{w}}(w)|$ apart from 0 and an upper bound of the ratio $|h_{\bar{w}}(w)/h_w(w)|$ apart from 1. Let $\psi(w) = w^{n-1}\omega(w)$ and $\omega(w) = c_n w + \cdots$. The derivative is $d\psi(w)/dw = (n-1)w^{n-2}\omega(w) + w^{n-1}d\omega(w)/dw$. We apply Koebe's distortion theorem for conformal mappings to $\omega(w)$, then

$$\left|\frac{\omega(w)}{c_n}\right| \le \frac{|w|}{(1-|w|)^2}, \quad \left|\frac{1}{c_n}\frac{d\omega(w)}{dw}\right| \le \frac{1+|w|}{(1-|w|)^3}.$$

On the other hand, we apply de Branges's theorem to $I(w)/c_1$, then $|c_n/c_1| \le n$. Let $R = 2R_0^{\frac{1}{2}}$, and we notice that the domain of h is contained in $|w| \le R$. We combine the above inequalities, and have

$$\begin{aligned}
|\psi(w)| &\le \frac{n|c_1||w|^n}{(1-|w|)^2} \le \frac{n|c_1|R^n}{(1-R)^2}, \\
\left|\frac{d\psi(w)}{dw}\right| &\le (n-1)|w|^{n-2}|\omega(w)| + |w|^{n-1}\left|\frac{d\omega(w)}{dw}\right| \\
&\le \frac{n(n-1)|c_1||w|^{n-1}}{(1-|w|)^2} + \frac{n|c_1||w|^{n-1}(1+|w|)}{(1-|w|)^3} \\
&= \frac{|c_1||w|^{n-1}(n^2 + (2n-n^2)|w|)}{(1-|w|)^3} \le \frac{n^2|c_1|R^{n-1}}{(1-R)^3}
\end{aligned}$$

when $n \ge 2$.

Claim 3.7. *If $R < \exp(-1)$, then $nR^n \le 2R^2$ and $n^2R^{n-1} \le 4R$ for any $n \ge 2$.*

Proof. The function xR^x $(x > 0)$ has a maximum value at $x = -1/(\log R) < 1$ and is monotone decreasing in $x > -1/(\log R)$, then we have $nR^n \le 2R^2$. Similarly we consider the function x^2R^{x-1} $(x > 0)$. It has a maximum value at $x = -2/(\log R) < 2$ and is also monotone decreasing in $x > -2/(\log R)$, then we have $n^2R^{n-1} \le 4R$. □

By a rough estimation, $R = 2R_0^{\frac{1}{2}} < 2R_2(\mathfrak{a})R_1(\mathfrak{a}') < \exp(-32\pi)/2 < \exp(-1)$, it fills the requirement of the claim. Therefore, the estimates of the absolute values of ψ and its derivative are

$$|\psi(w)| \le \frac{2|c_1|R^2}{(1-R)^2}, \quad \left|\frac{d\psi(w)}{dw}\right| \le \frac{4|c_1|R}{(1-R)^3}.$$

We give a lower estimation of $|h_w(w)| - |h_{\bar{w}}(w)|$ as follows;

$$\begin{aligned} |h_w(w)| - |h_{\bar{w}}(w)| &\geq |c_1| - \frac{2}{R}\frac{2|c_1|R^2}{(1-R)^2} - \frac{4|c_1|R}{(1-R)^3} \\ &= |c_1|\left(1 - \frac{4R}{(1-R)^2} - \frac{4R}{(1-R)^3}\right) \\ &= |c_1|\frac{1-11R+7R^2-R^3}{(1-R)^3} > 0. \end{aligned}$$

The first positive zero of the right hand side is about 0.096788. Of course, our R is smaller than such value. Then $|h_w|^2 - |h_{\bar{w}}|^2 > 0$, it says that h is a locally C^1-diffeomorphism. Moreover, since the domain of h, that is, $\{R_0^{\frac{1}{2}} \leq |w| \leq 2R_0^{\frac{1}{2}}\}$ is a compact set, h is proper, so it is a covering mapping. Since h is a rotation on $\{|w| = R_0^{\frac{1}{2}}\}$, the covering transformation group of h is $\mathbf{Z}/\mathbf{Z} = 1$. We conclude that h is a C^1-diffeomorphism. We calculate the dilatation

$$\begin{aligned} \left|\frac{h_{\bar{w}}}{h_w}\right| &\leq \frac{2|c_1|R}{(1-R)^2}\frac{1}{|c_1|\left(1 - \frac{2R}{(1-R)^2} - \frac{4R}{(1-R)^3}\right)} \\ &= \frac{2R(1-R)}{(1-R)^3 - 2R(1-R) - 4R} = \frac{2R(1-R)}{1-9R+5R^2-R^3} < 1. \end{aligned} \tag{6}$$

The right hand side equals to 1 if R is also about 0.096788. Consequently, h is a quasiconformal mapping, then the entire mapping $F_{j,l}$ is also quasiconformal for any $j = 1, \ldots, k$ and $l = 1, 2$.

Remark. The modified part of $F_{j,l}$ from $I_{j,l}$ does not contact the critical sets C_{J_c} and $C_{J_c'}$. That is, the image of $\{0 < |w| \leq 2R_0^{\frac{1}{2}}\}$ by $F_{j,l}$ is contained in $D(1)$. Because, if $|z| = 2R_0^{\frac{1}{2}} = R$,

$$\begin{aligned} |I_{j,l}(w)| = |h(w)| \leq |c_1||w| + |\psi(w)| &\leq \frac{R}{R_1(\mathfrak{a}')} + \frac{2R^2}{R_1(\mathfrak{a}')(1-R)^2} \\ &= \frac{R}{R_1(\mathfrak{a}')}\left\{1 + \frac{2R}{(1-R)^2}\right\} < 2R_2(\mathfrak{a})\left\{1 + \frac{2R}{(1-R)^2}\right\} \\ &< \frac{e^{-16\pi}}{2}\left\{1 + \frac{2R}{(1-R)^2}\right\} < 1. \end{aligned}$$

We construct a quasiconformal mapping of S onto S' which is homotopic to $f' \circ f^{-1}$. We recall that S and S' are assembled by $A_{j,l}(s) = \{\exp(-\pi m_j \exp(2s)) < |w| < 1\}$. We notice that $\exp(-\pi m_j \exp(2s)) < R_0$ for any j and l. Let $H_{j,l}$ be a mapping of $A_{j,l}(s)$ onto $A_{j,l}(\infty) = D(1)$ which enlarges the sub annulus $\{\exp(-\pi m_j \exp(2s)) < |w| < R_0\}$ to $\{0 < |w| < R_0\}$. Therefore, we can define the composition

$$H_{j,l}^{-1} \circ F_{j,l} \circ H_{j,l}(w) = \begin{cases} w & (\exp(-\pi m_j \exp(2s)) < |w| \leq R_0) \\ F_{j,l}(w) & (R_0 \leq |w| < 1) \end{cases}$$

becomes as a mapping F of S onto S' after combining the mappings of all $j = 1, \ldots, k$ and $l = 1, 2$. We confirm that F is homotopic to $f' \circ f^{-1}$. We recall that the original mapping $I_{j,l}$ corresponding to the identity mapping of X. By the construction of $\hat{\Phi}$, the mapping $I_{j,l}$ leads to the same homotopy class of $f' \circ f^{-1}$ in the outside of neighborhoods of core curves $f(\gamma_j)$ and $f'(\gamma_j)$ on S and S' respectively. Therefore, we only consider $F_{j,l}(z)$ in $\{R_0 \le |z| \le R_0^{\frac{1}{2}}\}$ and $\{R_0^{\frac{1}{2}} \le |z| \le 2R_0^{\frac{1}{2}}\}$. In $\{R_0 \le |z| \le R_0^{\frac{1}{2}}\}$, $F_{j,l}$ induces the twist of $\arg c_1$ (and an expansion). Let $c_{j,l}$ be as c_1 in $A_{j,l}(\infty)$. We can choice each argument of c_1 such that $|\arg c_{j,1} + \arg c_{j,2}| \le \pi$. In $\{R_0^{\frac{1}{2}} \le |z| \le 2R_0^{\frac{1}{2}}\}$, if the half twist ($\theta = \pi$) appears, it has a large dilatation. Indeed, its dilatation $k(\pi)$ is

$$k(\pi) = \left| \frac{1}{2\log 2/\pi + i} \right| \approx 0.914886.$$

However, the mapping $F_{j,l}$ has the dilatation at most about 2.187543×10^{-44}. (The right hand side of (6) as $R = \exp(-32\pi)/2$.) We can say that the mapping F cannot change the homotopy class of $f' \circ f^{-1}$. Furthermore, for the gluing of each $A_{j,l}(s)$ of S and S', it is used a common $\mathfrak{t}$. Therefore, we finish to prove that the mapping F is homotopic to $f' \circ f^{-1}$.

We have

$$d_{\mathscr{T}}([R,h],[R',h']) \le \frac{1}{2}\log K(F) = \frac{1}{2}\log \max_{j,l}\left\{ \frac{1+(k_1)_{j,l}}{1-(k_1)_{j,l}}, \frac{1+(k_2)_{j,l}}{1-(k_2)_{j,l}} \right\},$$

where $(k_1)_{j,l}$, $(k_2)_{j,l}$ are (5), (6) for each $j = 1,\dots,k$ and $l = 1,2$ respectively. We set the constant $C(\mathfrak{a},\mathfrak{a}',s_0)$ as the right hand side of the above inequality, and it tends to 0 as $s_0 \to \infty$ (so $s \to \infty$) since (5), (6) tend to 0.

The last inequality is again easy. We set $[S,f] = \hat{\Phi}([X,g],\mathfrak{a},\mathfrak{t},s)$, $[S',f'] = \hat{\Phi}([X',g'],\mathfrak{a}',\mathfrak{t}',s')$, and φ is the holomorphic quadratic differential on S determined by X, $\mathfrak{a}$, and the construction of $\hat{\Phi}$. Let ρ be a conformal metric on R equals to $|\varphi|^{\frac{1}{2}}$. By the definition of the extremal lengths in §2.2, we have

$$\frac{a_j^2}{\sum_{j=1}^k a_j^2 m_j e^{2s}} = \frac{(2\pi a_j)^2}{\sum_{j=1}^k (2\pi a_j)^2 m_j e^{2s}} = \frac{l_\rho(f(\gamma_j))^2}{A_\rho} \le \mathrm{Ext}_{[S,f]}(\gamma_j) \le \frac{1}{m_j e^{2s}}.$$

For $[S',f']$, the similar inequality holds. By Kerckhoff's formula of the Teichmüller distance (Theorem 2.1),

$$\begin{aligned} d_{\mathscr{T}}([S,f],[S',f']) &\ge \frac{1}{2}\log\frac{\mathrm{Ext}_{[S',f']}(\gamma_j)}{\mathrm{Ext}_{[S,f]}(\gamma_j)} \ge \frac{1}{2}\log\left(\frac{a_j'^2}{\sum_{j=1}^k a_j'^2 m_j e^{2s'}}\cdot m_j e^{2s}\right) \\ &= s - s' + \frac{1}{2}\log\frac{a_j'^2 m_j}{\sum_{j=1}^k a_j'^2 m_j}. \end{aligned}$$

This is satisfied for any $j = 1,\dots,k$, then the following holds;

$$d_{\mathscr{T}}([S,f],[S',f']) \ge s - s' + \frac{1}{2}\log\frac{\max_j a_j'^2 m_j}{\sum_{j=1}^k a_j'^2 m_j}.$$

An easy method gives that the logarithmic part of the above inequality equals or greater than $1/k$. By the symmetry of the distance, the similar inequality that changes $s - s'$ to $s' - s$ holds. Consequently, we obtain (4). □

3.6. Questions.

(1) What is an upper estimate of $d_{\mathscr{T}}(\hat{\Phi}([X,g],\mathfrak{a},\mathfrak{t},s), \hat{\Phi}([X',g'],\mathfrak{a},\mathfrak{t},s))$?
(2) What is a lower estimate of the Teichmüller distance between any two points depending on $d([X,g],[X',g'])$? (Maybe, it will be required that parameters s, s' are sufficiently large.)

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