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RUBEL'S PROBLEM ON BOUNDED ANALYTIC FUNCTIONS

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Abstract. The paper shows that for any G_δ set F of Lebesgue measure zero on the unit circle T there exists a function $f \in H^\infty$ such that the radial limits of f exist at each point of T and vanish precisely on F. This solves a problem proposed by Rubel in 1973.

1. Introduction

Let Δ and T respectively be the open unit disc and its boundary circle in the complex plane $\mathbf{C}$. As usual, we denote by H^∞ the space of all bounded analytic functions in Δ. It is well known that every $f \in H^\infty$ has radial limits $f(e^{i\theta})$ a. e. on T. A point $e^{i\theta} \in T$ is called a Fatou point for $f \in H^\infty$ if $f(e^{i\theta})$ exists. Below we assume that any function $f \in H^\infty$ is defined also a.e. on T by its radial limits $f(e^{i\theta})$.

The main purpose of this paper is to give an affirmative solution to Rubel's Problem 5.29 published in the well-known research problem collection of Hayman [1] on the materials of "Symposium on complex analysis" held in 1973 at the University of Kent, Canterbury. The formulation of the problem is the following.

Problem 5.29. (See [1, p. 168]) Let F be a G_δ of measure zero on T. Then does there exist an $f \in H^\infty$, $f \neq 0$, such that $f = 0$ on F and every point of T is a Fatou point of f?

The following minor modification of the problem asks a slightly more precise question.

Modified Problem 5.29. Let F be a G_δ of measure zero on T. Then does there exist an $f \in H^\infty$ such that $f = 0$ *precisely* on F and every point of T is a Fatou point of f?

Problem 5.29 has remained open since it was proposed. The following theorem completely solves both Problem 5.29 and its modification.

Theorem 1. *Let F be a G_δ of measure zero on T. Then there exists a non-vanishing $f \in H^\infty$ (even $\Re f > 0$ on Δ) such that $f = 0$ precisely on F and every point of T is a Fatou point of f.*

Note that Theorem 1 in a sense is an extension of Fatou's following classical interpolation theorem of 1906: *If F is closed and of measure zero on T, then there exists an element in the disc algebra which vanishes precisely on F* (see, for example, [2, p. 80]).

Now assume that for some set $F \subset T$ there exists an $f \in H^\infty$ such that $f = 0$ precisely on F and every point of T is a Fatou point of f. Then F is G_δ since it is

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the zero set on T of the function f which belongs to the first Baire class on T. Also, by the classical boundary uniqueness theorem, F is of measure zero on T. Thus, Theorem 1 can be formulated also as the following "if and only if" result.

Corollary 1. *Let $F \subset T$. There exists an $f \in H^\infty$ such that $f = 0$ precisely on F and every point of T is a Fatou point of f if and only if F is a G_δ of measure zero on T.*

As a corollary of (the proof of) Theorem 1 we also have the following description of the peak sets for those elements of H^∞ for which all points of T are Fatou points.

Corollary 2. *Let F be a G_δ of measure zero on T. Then there exists a $\lambda \in H^\infty$ such that:*

(a) *All points of T are Fatou points of λ;*
(b) *$\lambda = 1$ on F; and*
(c) *$|\lambda| < 1$ on $\overline{U} \setminus F$.*

As above the converse implication is obvious and Corollary 2 in fact is the complete description of peak sets for those elements of H^∞ for which all point of T are Fatou points.

The following lemma is due to Kolesnikov (see Lemma 2 in [3]).

Lemma. *Let G be an open subset on T and let $F \subset G$ be a set of measure zero on T. For any $\epsilon > 0$ there exists an open set $O,\ F \subset O \subset G$, and a function $g \in H^\infty$ such that:*

1) *$|g(z)| < 2,\ 0 < \Re g(z) < 1$ for $z \in \Delta$;*
2) *the function g has a finite radial limit $g(\zeta)$ at each point $\zeta \in T$;*
3) *at the points $\zeta \in O$ the function g is analytic and $\Re g(\zeta) = 1$;*
4) *$|g(z)| \leq \epsilon$ on every radius R_{ζ_0} with end-point at $\zeta_0 \in T \setminus G$.*

We use this lemma in our proof of Theorem 1 (we repeat some relevant arguments from [3] for the sake of completeness).

The main result of the paper [3] is the following theorem of Kolesnikov, which solves the classical problem on the description of the sets of nonexistence of radial limits of bounded analytic functions.

Theorem (Kolesnikov). *Let $E \subset T$. There exists an $f \in H^\infty$ such that the radial limits of f exist exactly on the set $T \setminus E$ if and only if E is a $G_{\delta\sigma}$ of measure zero.*

The necessity part of this theorem is a well-known elementary result, while the sufficiency part uses the above lemma and Carathéodory's general theorem on the boundary correspondence under the conformal mappings (involving the concept of a prime end).

In conclusion of the present paper, however, we completely eliminate Carathéodory's theorem from the proof of Kolesnikov's theorem. The main ingredient of this simplified proof is Kolesnikov's lemma (of course), but we just apply Theorem 1, which makes the presentation shorter.

2. Proofs

Proof of Theorem 1. We denote by m the Lebesgue measure on T. As a G_δ of (Lebesgue) measure zero, the set F is an intersection of open sets G_k on T such that

$m(G_k) < 1/2^k$, $k = 1, 2, \ldots$. We assume $G_{k+1} \subset G_k$ (otherwise replace each G_k by $\bigcap_{j=1}^{k} G_j$).

We apply the Lemma for F and G_k, and for $\epsilon = 1/2^k$. Thus, we have the open sets O_k on T, $F \subset O_k \subset G_k$, and the functions $g_k \in H^\infty$ such that for each k:

(i) $|g_k(z)| < 2,\ 0 < \Re g_k(z) < 1$ for $z \in \Delta$;

(ii) the function g_k has a finite radial limit $g_k(\zeta)$ at each point $\zeta \in T$;

(iii) at the points $\zeta \in O_k$ the function g_k is analytic and $\Re g_k(\zeta) = 1$;

(iv) $|g_k(z)| \le \epsilon_k$ on every radius R_{ζ_0} with end-point at $\zeta_0 \in T \setminus G_k$.

Since by (i) each g_k is bounded by 2 and by (iv) the radial limits of g_k on $T \setminus G_k$ are bounded by $1/2^k$, by the Cauchy integral representation of the function g_k we have

$$\begin{aligned} |g_k(z)| = \left| \frac{1}{2\pi i} \int_T \frac{g_k(\zeta)}{\zeta - z}\, d\zeta \right| &\le \frac{1}{2\pi} \int_{T \setminus G_k} \frac{1/2^k}{|\zeta - z|} |d\zeta| + \frac{1}{2\pi} \int_{G_k} \frac{2}{|\zeta - z|} |d\zeta| \\ &\le \frac{1/2^k}{1 - |z|} + \frac{2}{2\pi(1 - |z|)} (1/2^k). \end{aligned}$$

This estimate clearly implies that the series $\sum_{k=1}^{\infty} g_k(z) = h(z)$ converges uniformly on compact subsets of Δ to an analytic function h on Δ (cf. [3]).

Since by (i) we have $\Re g_k(z) > 0$ for $z \in \Delta$, we also have $\Re h(z) > 0$ for $z \in \Delta$. By (iii) we have $\Re g_k(\zeta) = 1$ on O_k and thus the radial limit of $\Re h(z)$ is $+\infty$ at each point of F.

Now let $\zeta_0 \in T \setminus F$. Then $\zeta_0 \in G_k$ only for finite many values of k, and by (iv), for all large enough k we have $|g_k(z)| \le 1/2^k$ on the radius R_{ζ_0} (with end-point at ζ_0). Thus the series $\sum_{k=1}^{\infty} g_k(z) = h(z)$ converges uniformly on the radius R_{ζ_0}. Also, by (ii) each g_k has a finite radial limit at ζ_0 and thus h has a finite radial limit at ζ_0 (cf. [3]).

The radial limit properties of the function $1 + h$ are evident from above; we also note that $\Re(1 + h(z)) > 1$ for $z \in \Delta$. In particular, $1 + h$ has finite and nonzero radial limits everywhere on the set $T \setminus F$. The analytic function $f = 1/(1 + h)$ is bounded by 1 and has finite and nonzero radial limits everywhere on $T \setminus F$. Obviously f is zero free on Δ and moreover $\Re f > 0$ on Δ. Since the radial limit of $\Re h(z)$ is $+\infty$ at each point of F, the radial limit of f is zero at each point of F.

Theorem 1 is proved. □

Proof of Corollary 2. Let h be the function from the previous proof. To complete the proof one can simply take $\lambda = h/(1 + h)$. This function clearly satisfies all the requirements of Corollary 2. □

Finally we simplify the proof of Kolesnikov's theorem by showing that it does not need to use prime ends at all. Instead we apply an elementary (known) argument.

Simplified proof of Kolesnikov's theorem. Let $E = \cup_{n=1}^{\infty} E_n$, where each E_n is a G_δ of measure zero as in Kolesnikov's theorem. By Theorem 1 for each E_n we have a function $f_n \in H^\infty$ with a positive real part, such that $f_n = 0$ precisely on E_n and every point of T is a Fatou point of f_n. Since $\Re f_n(z) > 0$ one can find an analytic function $\log f_n(z) = \log |f_n(z)| + i \arg f_n(z)$ such that $|\arg f_n(z)| < \pi/2$ on Δ. We have that $\log |f_n(z)| \to -\infty$ as $z \in \Delta$ approaches radially to any point of E_n and $\log |f_n(z)|$ has finite radial limits at each point of $T \setminus E_n$. Obviously the radial limit of the bounded analytic function

$$\varphi_n(z) = e^{i \log f_n(z)} = e^{-\arg f_n(z)} [\cos(\log |f_n(z)|) + i \sin(\log |f_n(z)|)]$$

exists for each $\zeta \in T \setminus E_n$ and for no $\zeta \in E_n$. Moreover, on the radii terminating on E_n the oscillation of φ_n is uniformly large and exceeds $e^{-\pi/2}$ (we use this property below).

The bounded analytic function $f(z) = \sum_{n=1}^{\infty} 1000^{-n}\varphi_n(z)$ has all desired properties. At each $\zeta \in T \setminus E$ it has a radial limit since each φ_n does and the series converges uniformly.

It remains to show that f does not have radial limits on E. If $\zeta_0 \in E$, let E_m be the set with the smallest index m such that $\zeta_0 \in E_m$. The partial sum $\sum_{n=1}^{m-1} 1000^{-n}\varphi_n(z)$ has a finite radial limit at ζ_0. But at ζ_0 the oscillation of the radial limit of the term $1000^{-m}\varphi_m(z)$ is not less than $1000^{-m}e^{-\pi/2}$ and the reminder series $\sum_{n=m+1}^{\infty} 1000^{-n}\varphi_n(z)$ does not exceed $1000^{-m}\,\frac{e^{\pi/2}}{999}$. Thus at ζ_0 the oscillation of the radial limit of f is larger than some positive number (say, $0.5e^{-\pi/2}1000^{-m}$). The proof is over. $\square$

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