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ZERO DISTRIBUTION OF DIRICHLET *L*-FUNCTIONS

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Abstract. In this paper, we point out that there exists a positive percentage of the simple a-points of L_χ for a complex number a. Furthermore, we establish some inequalities about the number of distinct zeros of Dirichlet L-functions employing value distribution theory and other analytic tools.

1. Introduction and main results

Take a positive integer k and let χ be a Dirichlet character modulo k associated with the Dirichlet L-function

$$L_\chi(s) = L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{1}$$

The series L_χ converges absolutely and uniformly in the region $\mathrm{Re}(s) \geq 1+\varepsilon$, for any $\varepsilon > 0$. It therefore represents a holomorphic function on the half-plane $\mathrm{Re}(s) > 1$, which further extends to a meromorphic function in the complex plane $\mathbf{C}$. In particular, for the principal character $\chi = 1$, we get back the Riemann zeta function $\zeta(s)$. For $a \in \mathbf{C}$, the zeros of $L_\chi - a$ which we denote by $\rho_a = \beta_a + i\gamma_a$, are called the a-points of L_χ, and their distribution has long been an object of study (see [17, 18, 20]).

The function L_χ has only real zeros in the half plane $\mathrm{Re(s)} < 0$, these zeros are called the trivial zeros. If $\chi(-1) = 1$, the trivial zeros of L_χ are $s = -2n$ for all non-negative integers n. If $\chi(-1) = -1$, the trivial zeros of L_χ are $s = -2n-1$ for all non-negative integers n. Beside the trivial zeros of L_χ, there are infinitely many non-trivial zeros lying in the strip $0 < \mathrm{Re(s)} < 1$. For $a \neq 0$, it can be shown that there is always a a-point in some neighbourhood of any trivial zero of L_χ with sufficiently large negative real part, and with finitely many exceptions there are no other in the left half-plane, thus the number of these a-points having real part in $[-R, 0]$ is asymptotically $\frac{1}{2}R$. The remaining a-points all lie in a strip $0 < \mathrm{Re(s)} < \mathrm{A}$, where A depends on a, and we call these non-trivial a-points (see [16, 19]). For a positive number T, let $N_\chi^a(T)$ denote the number of non-trivial a-points $\rho_a = \beta_a + i\gamma_a$ of L_χ with $|\gamma_a| \leq T$. We have the following formula

$$N_\chi^a(T) = \frac{T}{\pi}\log T + c_\chi T + O(\log T), \tag{2}$$

where c_χ is a constant depending on a and χ (see [19, p. 145]).

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Note that all trivial zeros of $L_\chi - a$ lied on negative real axis are simple, it is nature to ask whether all or almost all of a-points are simple. Mathematicians have done a lof of work for $a = 0$ successively (see [3]–[9], [14]). For $a \neq 0$, Garunkštis and Steduing (see [10]) have recently shown that for every a an infinite number of a-points of $\zeta(s)$ are simple. In addition, Selberg has stated that at least 50% of non-trivial a-points of $\zeta(s)$ are simple and lie to the left of the line $\mathrm{Re(s)} = \frac{1}{2}$ under the assumption that the Riemann Hypothesis is true. Meanwhile, Selberg posed a conjecture that for a given $a \neq 0$, three quarters of the a-points are to the left of the line $\mathrm{Re(s)} = \frac{1}{2}$ (see [18]). Let $N^a\big(\sigma_1, \sigma_2, T\big)$ $\big(N^a(\sigma_1, T)\big)$ denote the number of a-points $\rho_a = \beta_a + i\gamma_a$ of $\zeta(s)$ with $\sigma_1 < \beta_a < \sigma_2$ $\big(\beta_a > \sigma_1\big)$ and $|\gamma_a| \leq T$, then we can restate Selberg conjecture as follows.

Conjecture 1.1. (Selberg's a-points Conjecture) *For all $a \neq 0$ we have*

$$N^a\left(0, \frac{1}{2}, T\right) = \left(\frac{3}{4} + o(1)\right)\frac{T}{\pi}\log T. \tag{3}$$

Recently, Gomek, Lester and Milinovich asserted that if Selberg's a-points Conjecture is true and enough zeros of $\zeta(s)$ are on the line $\mathrm{Re(s)} = \frac{1}{2}$, then there is a positive proportion of the simple a-points of $\zeta(s)$, and the analogous results are true for any Dirichlet L-function L_χ with a primitive character χ (see [11]). However, their theorems leave open question of the simplicity of a-points to the right of $\mathrm{Re(s)} = \frac{1}{2}$, this together with Selberg's a-points Conjecture implies that

$$N^a\left(\frac{1}{2}, T\right) = \left(\frac{1}{4} + o(1)\right)\frac{T}{\pi}\log T.$$

They pointed out that it would be interesting if one could show that there is a positive proportion of the simple a-points of $\zeta(s)$ to the right of $\mathrm{Re(s)} = \frac{1}{2}$. In this paper, we prove unconditionally that there exist a positive percentage of the simple a-points of L_χ. We primarily focus on the Riemann zeta function $\zeta(s)$, and our results extend to other Dirichlet L-function L_χ with a fixed character χ as well.

Theorem 1.2. *For $a \in \mathbf{C}$, there exist a positive percentage of the simple a-points of $\zeta(s)$ except for at most two values.*

We will use Nevanlinna theory to show Theorem 1.2. Additionally, the reviewer presents a method of improving our result (see appendix).

Denote the number of a-points of L_χ in the disc $|s| \leq T$ by $n\left(T, \frac{1}{L_\chi - a}\right)$. Note that there are about T trivial zeros in the disk of radius T centered at origin, it follows that

$$n\left(T, \frac{1}{L_\chi - a}\right) = N_\chi^a(T) + O(T).$$

The Nevanlinna's valence function $N\left(r, \frac{1}{L_\chi - a}\right)$ of $L_\chi - a$ for zeros (counting multiplicities) is defined by

$$N\left(r, \frac{1}{L_\chi - a}\right) = \int_0^r \frac{n\big(t, \frac{1}{L_\chi - a}\big) - n\big(0, \frac{1}{L_\chi - a}\big)}{t}\,dt + n\left(0, \frac{1}{L_\chi - a}\right)\log r, \tag{4}$$

it follows easily from (2) and (4) that

$$N\left(r, \frac{1}{L_\chi - a}\right) = \frac{r}{\pi}\log r + O(r). \tag{5}$$

Similarly, let $\overline{N}_{\chi}^{a}(T)$ denote the number of non-trivial distinct a-points $\rho_a = \beta_a + i\gamma_a$ of L_χ with $|\gamma_a| \leq T$, then the number $\overline{n}\left(T, \frac{1}{L_\chi - a}\right)$ of distinct a-points of L_χ in the disc $|s| \leq T$ satisfies the estimate

$$\overline{n}\left(T, \frac{1}{L_\chi - a}\right) = \overline{N}_{\chi}^{a}(T) + O(T),$$

the Nevanlinna's valence function $\overline{N}\left(r, \frac{1}{L_\chi - a}\right)$ of $L_\chi - a$ for distinct zeros (ignoring multiplicities) is defined as similar to (4). If all or almost all of a-points are simple, then the Nevanlinna's valence function of $L_\chi - a$ for distinct zeros approximates the estimate

$$\overline{N}\left(r, \frac{1}{L_\chi - a}\right) = \frac{r}{\pi} \log r + O(r). \tag{6}$$

A further discussion on the number of distinct a-points, we obtain the following theorem.

Theorem 1.3. *For any nonzero complex number a, we have*

$$\overline{N}\left(r, \frac{1}{\zeta(s) - a}\right) + \overline{N}\left(r, \frac{1}{\zeta(s)}\right) \geq \frac{r}{\pi} \log r + O(r).$$

The analogous results are true for any Dirichlet L-function L_χ. Moreover, note that

$$\overline{N}\left(r, \frac{1}{L_\chi}\right) = \int_1^r \frac{\overline{N}_\chi(t)}{t}\, dt + O(r), \tag{7}$$

hence,

$$\liminf_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{L_\chi}\right)}{r \log r} = \liminf_{r\to\infty} \frac{\int_1^r \frac{\overline{N}_\chi(t)}{t}\, dt + O(r)}{r \log r} = \liminf_{r\to\infty} \frac{\frac{\overline{N}_\chi(r)}{r}}{\log r + 1},$$

by simple calculation, we have

$$\liminf_{r\to\infty} \frac{\overline{N}_\chi(r)}{r \log r} = \liminf_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{L_\chi}\right)}{r \log r}.$$

Therefore, Theorem 1.3 yields immediately the following fact.

Corollary 1.4. *For any nonzero complex number a, we have*

$$\overline{N}_{\chi}^{a}(r) + \overline{N}_{\chi}^{0}(r) \geq \{1 + o(1)\} \frac{r}{\pi} \log r. \tag{8}$$

In 1998, under the assumption of the Riemann Hypothesis and the Generalized Lindelöf Hypothesis, Conrey, Ghosh and Gonek (see [7]) proved that more than 84.56% of the zeros of the Riemann zeta function are distinct. Recently, Bui and Heath-Brown (see [4]) improved the result, they showed that at least 84.665% of the zeros of the Riemann zeta-function are distinct, assuming the Riemann Hypothesis. Moreover, for any Dirichlet L-function (both primitive and imprimitive) Bauer [2] has shown that the proportion of simple zeros is at least 0.356. In this paper, we first find out the family of L_χ with the distinct Dirichlet characters modulo k are linearly independent, although it is a simple consequence of orthogonality of characters, it would be very helpful in exploring a new research method about estimating the number of distinct zeros of the family of L_χ with the distinct Dirichlet characters modulo k.

In the sequel, let $\chi_1, \chi_2, \cdots, \chi_{\varphi(k)}$ be the distinct Dirichlet characters modulo k, where χ_1 is the principal character and $\varphi(k)$ is the Euler's φ-function which counts the number of prime residue classes (mod k), and abbreviate

$$L_i = L_{\chi_i}, \quad i = 1, \ldots, \varphi(k).$$

Note that

$$\chi_1 + \cdots + \chi_{\varphi(k)} = \varphi(k)\delta_k, \tag{9}$$

where

$$\delta_k(n) = \begin{cases} 1, & n \equiv 1 \pmod{k}, \\ 0, & n \not\equiv 1 \pmod{k}. \end{cases} \tag{10}$$

Then we obtain an important functional equation

$$L_1 + \cdots + L_{\varphi(k)} = L_0, \tag{11}$$

where L_0 is defined by

$$L_0(s) = \varphi(k)L(s, \delta_k) = \varphi(k)\sum_{n=1}^{\infty} \frac{\delta_k(n)}{n^s}, \tag{12}$$

which extends to a meromorphic function in $\mathbf{C}$ by the equation (11) such that $s = 1$ is unique simple pole of L_0.

Moreover, $L_1, \cdots, L_{\varphi(k)}$ have no common trivial zeros. In fact, we only need to consider the case $k \geq 3$. Note that L_i has either trivial zeros $0, -2, -4, \cdots$ corresponding to $\chi_i(-1) = 1$ or trivial zeros $-1, -3, -5, \cdots$ when $\chi_i(-1) = -1$ (see [16, p. 116]). Hence it is sufficient to deduce that there exists at least one $i \in \{2, \ldots, \varphi(k)\}$ such that $\chi_i(-1) = -1$ since $\chi_1(-1) = 1$, which easily follows from the equation (9). This fact proves a part answer to the following question:

Conjecture 1.5. *For any positive integer k, $L_1, L_2, \cdots, L_{\varphi(k)}$ have no common zeros.*

Besides, for any positive integer k, the Dirichlet L-functions $L_1, \ldots, L_{\varphi(k)}$ of modulo k are linearly independent, we will give a simple proof of this proposition in section 3. Even more important, we get the following theorem by using this proposition. In the sequel, we define

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}_{(2}\left(r, \frac{1}{f}\right) + \cdots + \overline{N}_{(k}\left(r, \frac{1}{f}\right),$$

where $\overline{N}_{(k}\left(r, \frac{1}{f}\right)$ is the valence function of f for zeros with multiplicity at least k, in which the multiplicity is not counted.

Theorem 1.6. *If Conjecture 1.5 is true, then for any positive integer $k \geq 3$, we have*

$$\sum_{i=0}^{\varphi(k)} N_{\varphi(k)-1}\left(r, \frac{1}{L_i}\right) \geq \frac{r}{\pi}\log r + O(r) \tag{13}$$

and

$$N_{\frac{1}{2}\varphi(k)(\varphi(k)-1)}\left(r, \frac{1}{L_0L_1\cdots L_{\varphi(k)}}\right) \geq \frac{r}{\pi}\log r + O(r). \tag{14}$$

Corollary 1.7. *If Conjecture 1.5 is true, then for any positive integer $k \geq 3$, we have*

$$\sum_{i=0}^{\varphi(k)} \overline{N}\left(r, \frac{1}{L_i}\right) \geq \frac{c_1}{\pi} r \log r + O(r) \tag{15}$$

and

$$\overline{N}\left(r, \frac{1}{L_0 L_1 \cdots L_{\varphi(k)}}\right) \geq \frac{c_2}{\pi} r \log r + O(r), \tag{16}$$

where $c_1 = \frac{1}{\varphi(k)-1}$, $c_2 = \frac{2}{\varphi(k)(\varphi(k)-1)}$.

More generally, if we cancel the assumption on Conjecture 1.5, we also can get the following results.

Theorem 1.8. *For any positive integer $k \geq 3$, the following estimates is valid for arbitrary nonzero complex number a,*

$$N_{\varphi(k)}\left(r, \frac{1}{L_0 - a}\right) + \sum_{i=1}^{\varphi(k)} N_{\varphi(k)}\left(r, \frac{1}{L_i}\right) \geq \frac{r}{\pi} \log r + O(r) \tag{17}$$

and

$$N_{\frac{1}{2}\varphi(k)(\varphi(k)+1)}\left(r, \frac{1}{(L_0 - a) L_1 \cdots L_{\varphi(k)}}\right) \geq \frac{r}{\pi} \log r + O(r). \tag{18}$$

Corollary 1.9. *For any positive integer $k \geq 3$, the following estimates is valid for arbitrary nonzero complex number a,*

$$\overline{N}\left(r, \frac{1}{L_0 - a}\right) + \sum_{i=1}^{\varphi(k)} \overline{N}\left(r, \frac{1}{L_i}\right) \geq \frac{c_1}{\pi} r \log r + O(r) \tag{19}$$

and

$$\overline{N}\left(r, \frac{1}{(L_0 - a) L_1 \cdots L_{\varphi(k)}}\right) \geq \frac{c_2}{\pi} r \log r + O(r), \tag{20}$$

where $c_1 = \frac{1}{\varphi(k)}$, $c_2 = \frac{2}{\varphi(k)(\varphi(k)+1)}$.

Specially, for $k = 3$ we have the following corollary.

Corollary 1.10. *If Conjecture 1.5 is true, we have*

$$\overline{N}\left(r, \frac{1}{L_0 L_1 L_2}\right) \geq \frac{r}{\pi} \log r + O(r). \tag{21}$$

Corollary 1.11. *For arbitrary nonzero complex number a, we have*

$$\overline{N}\left(r, \frac{1}{(L_0 - a) L_1 L_2}\right) \geq \frac{r}{3\pi} \log r + O(r). \tag{22}$$

Finally, we point out that the inequalities (15) (16),(19) and (20) also yield the similar results as the inequality (8).

In addition, it must be pointed out that Bauer [2] has given a better bound for the distinct zeros of Dirichlet L-function. In his study, bounds on the number of simple zeros of the derivatives of a function are used to give bounds on the number of distinct zeros of the function, and the main inequality comes from [8]. However, Farmer in [8] showed that, if the lower bounds for the proportion of simple zeros of the derivatives of a function were actually equalities, then the lower bound given by this inequality is sharp. So it is hard to improve the results in this way. Unlike this, we establish inequalities about distinct zeros of Dirichlet L-function by use of complex analytic tools, it is more easily to understand, especially for some researchers don't know number theory well. Moreover, our method is valid for general L-functions if they are linearly independent. Specially, Corollary 1.7 and 1.9 are an approach to the question that we pose in the begin of this thesis, because we think $c_1 = c_2 = \varphi(k)+1$ which imply that all or almost all of zeros of L_χ are simple immediately, it would be interesting if one could show that.

2. Preliminaries

The Riemann zeta-function is a function of a complex variable $s = \sigma + it$, which is given by (see [16])

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

for $\sigma > 1$, where the product is taken over all prime numbers p, and has a meromorphic continuation to the whole complex plane with a simple pole at $s = 1$. It satisfies the functional equation

$$\xi(s) = \xi(1-s),$$

where the entire function $\xi(s)$ is defined by

$$\xi(s) = \frac{1}{2} s(1-s)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

After a little computation, we have

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s). \tag{23}$$

Recall the definiton of Dirichlet L-function (see [16] or [19]). Given a Dirichlet character χ mod k (i.e., a group homomorphism from the group of prime residue classes modulo k to $\mathbf{C}^*$, extended to $\mathbf{Z}$ by setting $\chi(n) = 0$ for all n which are not coprime with k), for $\sigma > 1$, the associated Dirichlet L-function is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

By analytic continuation, $L(s,\chi)$ extends to a meromorphic function in the complex plane with a single pole at $s = 1$, if χ is a principal character (i.e., $\chi(n) = 1$ for all n coprime with some k). The zeta function $\zeta(s)$ may be regarded as the Dirichlet L-function to the principal character χ_0 mod 1. Furthermore, if χ is a principal character of mod k, then

$$L(s,\chi) = \zeta(s)\prod_{p|k}\left(1 - \frac{1}{p^s}\right). \tag{24}$$

A character that is not induced by a character of smaller modulus is said to be primitive, but principal characters are not considered as primitive. If χ (mod k) is induced by a primitive character χ^* mod k^*, then

$$L(s,\chi) = L(s,\chi^*)\prod_{p|k,\ p\nmid k^*}\left(1 - \frac{\chi^*(p)}{p^s}\right). \tag{25}$$

If χ (mod k) is a primitive character, then $L(s,\chi)$ satisties the functional equation

$$\Big(\frac{k}{\pi}\Big)^{(s+\delta)/2}\Gamma\Big(\frac{s+\delta}{2}\Big)L(s,\chi)=\frac{\tau(\chi)}{i^{\delta}\sqrt{k}}\Big(\frac{k}{\pi}\Big)^{(1+\delta-s)/2}\Gamma\Big(\frac{1+\delta-s}{2}\Big)L(1-s,\overline{\chi}),$$

where $\delta=\dfrac{1}{2}(1-\chi(-1))$ and

$$\tau(\chi)=\sum_{a\ \mathrm{mod}\ k}\chi(a)e^{\frac{2\pi ia}{k}}$$

is the Gauss sum attached to χ. After a little computation, we have

$$L(s,\chi)=\begin{cases}\tau(\chi)k^{-s}2^s\pi^{s-1}\sin\dfrac{\pi s}{2}\Gamma(1-s)L(1-s,\overline{\chi}), & \delta=0;\\ -i\tau(\chi)k^{-s}2^s\pi^{s-1}\cos\dfrac{\pi s}{2}\Gamma(1-s)L(1-s,\overline{\chi}), & \delta=1.\end{cases}\tag{26}$$

Moreover, the family of L_χ with the distinct Dirichlet characters modulo k satisfy the following proposition.

Proposition 2.1. *For any positive integer k, the Dirichlet L-functions $L_1,\ldots,L_{\varphi(k)}$ of modulo k are linearly independent.*

For convenience of the reader who might not be familiar with Nevanlinna theory, we list here the notations and results from Nevanlinna theory, which will be used in the proof (see [21] or [13]). The Nevanlinna characteristic function $T(r,f)$ of a nonconstant meromorphic function f is defined by

$$T(r,f)=N(r,f)+m(r,f).$$

The definitions of the proximity function $m(r,f)$ and the valence function $N(r,f)$ of f for poles are defined respectively by

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta,$$
$$N(r,f)=\int_0^r\frac{n(t,f)-n(0,f)}{t}\,dt+n(0,f)\log r,$$

where $\log^+x$ =max$\{\log x,0\}$ for all $x\ge 0$, $n(t,f)$ denotes the number of poles of f in the disc $|z|\le t$, counting multiplicities. Recall the following known result (see [21] or [13]).

(i) The arithmetic properties of $T(r,f)$ and $m(r,f)$:

$$T(r,fg)\le T(r,f)+T(r,g),\quad T(r,f+g)\le T(r,f)+T(r,g)+O(1).$$

The same inequalities holds for $m(r,f)$.

(ii) The Nevanlinna's first fundamental theorem: $T(r,f)=T\left(r,\dfrac{1}{f}\right)+O(1)$.

(iii) The logarithmic derivative lemma: $m\left(r,\dfrac{f'}{f}\right)=O(\log r)$, if the order

$$\rho(f):=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}$$

of f is finite.

(iv) The Nevanlinna's second fundamental theorem:

$$(q-2)T(r,f)\le\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{f-a_j}\right)+S(r,f),$$

where $a_1, a_2, \cdots, a_q$ are distinct complex values in $\bar{\mathbf{C}}$ and $S(r, f)$ denotes a quantity satisfying the following condition

$$S(r, f) = O\{\log r + \log T(r, f)\}$$

for all r outside possibly a set of finite Lebesgue measure. If f is of finite order, then $S(r, f) = O(\log r)$.

In the end of this section, we will state a theorem due to Hu and Yang (see [15, p. 295]) which will play an important role in our paper.

Theorem 2.2. *Assume that $f_0, f_1, f_2 \cdots, f_n (n \geq 2)$ are entire functions in $\mathbf{C}$ without common zeros and not all constants such that*

$$f_1 + f_2 + \cdots + f_n = f_0. \tag{27}$$

If $f_1, f_2 \cdots, f_n$ are linearly independent, then for $r_0 < r < \rho < R$, we have

$$m(r) < \sum_{i=0}^{n} N_\omega\left(r, \frac{1}{f_i}\right) + l \log\left\{\frac{\rho m(R)}{r(\rho - r)}\right\} + O(1),$$

$$m(r) < N_l\left(r, \frac{1}{f_0 f_1 \cdots f_n}\right) + l \log\left\{\frac{\rho m(R)}{r(\rho - r)}\right\} + O(1),$$

where $l = \frac{n(n-1)}{2}$, $\omega = n - 1$ are respectively the index and the Wronskian degree of the family $\{f_1, f_2, \cdots f_n\}$, and

$$m(r) = \frac{1}{2\pi}\int_0^{2\pi} \log\left(|f_0|^2 + |f_1|^2 + \cdots + |f_n|^2\right)^{\frac{1}{2}} d\theta.$$

3. Proof of Proposition 2.1

Firstly, we prove that characters $\chi_1, \ldots, \chi_{\varphi(k)}$ are linearly independent. For $k = 1$ or 2, the result is valid clearly. We consider the case $k \geq 3$ in the following. Assume, to the contrary, that $\chi_1, \cdots, \chi_{\varphi(k)}$ are linearly dependent, then there exist constants a_j for $j = 1, 2, \cdots, \varphi(k)$ (at least one of them is not zero), such that

$$\sum_{j=1}^{\varphi(k)} a_j \chi_j(n) = 0$$

holds for any integer n. The orthogonality of characters shows that if χ_1 and χ_2 are two character modulo k, then

$$\sum_{n \pmod{k}} \chi_1(n)\overline{\chi}_2(n) = \begin{cases} \varphi(k), & \text{if } \chi_1 = \chi_2, \\ 0, & \text{if not.} \end{cases}$$

Hence, we easily deduce that $a_1 = a_2 = \cdots = a_{\varphi(k)} = 0$, which contradicts with our assumption. Thus, we obtain that $\chi_1, \chi_2, \cdots, \chi_{\varphi(k)}$ are linearly independent.

Finally, if there exist constants b_j $(j = 1, 2, \cdots, \varphi(k))$ satisfying

$$\sum_{j=1}^{\varphi(k)} b_j L_j = 0,$$

that is, for $\mathrm{Re}(s) > 1$

$$\sum_{n=1}^{\infty} \frac{B_n}{n^s} = 0,$$

where

$$B_n = b_1\chi_1(n) + b_2\chi_2(n) + \cdots + b_{\varphi(k)}\chi_{\varphi(k)}(n),$$

it follows from the classic uniqueness theorem of Derichlet series (cf. [12, Theorem 6, p. 6]) that $B_n = 0$ for all $n \geq 1$, and hence $b_j = 0$ for all $j \in \{1, \ldots, \varphi(k)\}$ since $\chi_1, \chi_2, \cdots, \chi_{\varphi(k)}$ are linearly independent. Thus, we complete the proof.

4. Proof of Theorem 1.2

Let $n_{1)}(r, \frac{1}{\zeta - a})$ denote the number of the simple zeros of $\zeta - a$, and $N_{1)}(r, \frac{1}{\zeta - a})$ represent the corresponding Nevanlinna's valence function. Assume, to the contrary, that there exist $a_j (j = 1, 2, \cdots, q, q \geq 3)$ such that $N_{1)}(r, \frac{1}{\zeta - a_j}) = o(T(r, \zeta))$, by the second fundament theorem of Nevanlinna, we have

$$\begin{aligned}
qT(r,\zeta) &\leq \overline{N}\left(r, \frac{1}{\zeta - a}\right) + \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{\zeta - a_j}\right) + N(r,\zeta) + O(\log r) \\
&\leq \overline{N}\left(r, \frac{1}{\zeta - a}\right) + \sum_{j=1}^{q} \overline{N}_2\left(r, \frac{1}{\zeta - a_j}\right) + O(\log r) \\
&\leq \frac{q+2}{2} T(r,\zeta) + O(\log r),
\end{aligned}$$

which contradicts with $q \geq 3$. Therefore, it yields the result of theorem 1.2 immediately.

5. Proof of Theorem 1.6

Since $L(s, \chi)$ is a meromorphic function in the complex plane with a single pole at $s = 1$ if χ is a principal character, and $L(s, \chi)$ is an entire function for non-principal character of modulo $k (\geq 3)$, then we obtain entire functions as follows.

$$f_i(s) = (s-1)L_i(s), \quad i = 0, 1, \cdots, \varphi(k).$$

For positive integer $l_1 = \varphi(k) - 1$ and $l_2 = \frac{\varphi(k)(\varphi(k)-1)}{2}$, we have

$$\begin{aligned}
N_{l_1}\left(r, \frac{1}{f_i}\right) &\leq N_{l_1}\left(r, \frac{1}{s-1}\right) + N_{l_1}\left(r, \frac{1}{L_i}\right) \\
&\leq N_{l_1}\left(r, \frac{1}{L_i}\right) + O(\log r) \leq l_1 \overline{N}\left(r, \frac{1}{L_i}\right) + O(\log r),
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
N_{l_2}\left(r, \frac{1}{f_0 f_1 \cdots f_{\varphi(k)}}\right) &\leq N_{l_2}\left(r, \frac{1}{(s-1)^{\varphi(k)+1}}\right) + N_{l_2}\left(r, \frac{1}{L_0 L_1 \cdots L_{\varphi(k)}}\right) \\
&\leq N_{l_2}\left(r, \frac{1}{L_0 L_1 \cdots L_{\varphi(k)}}\right) + O(\log r) \\
&\leq l_2 \overline{N}\left(r, \frac{1}{L_0 L_1 \cdots L_{\varphi(k)}}\right) + O(\log r).
\end{aligned} \tag{29}$$

Hence, under the condition of Theorem 1.6, by using Proposition 2.1 and Theorem 2.2 we only need to estimate lower bounds of $m(r)$, which now satisfies the following relations

$$\begin{aligned} m(r) &= \frac{1}{2\pi}\int_0^{2\pi} \log(|f_0(re^{i\theta})|^2 + |f_1(re^{i\theta})|^2 + \cdots + |f_{\varphi(k)}(re^{i\theta})|^2)^{\frac{1}{2}}\, d\theta \\ &\geq \frac{1}{2\pi}\int_0^{2\pi} \log\left(\sqrt{\varphi(k)}\right)\left(\left|f_1(re^{i\theta})\cdots f_{\varphi(k)}(re^{i\theta})\right|\right)^{\frac{1}{\varphi(k)}}\, d\theta \\ &= \frac{1}{2\varphi(k)\pi}\left[\int_0^{2\pi} \log|f_1(re^{i\theta})|\, d\theta + \cdots + \int_0^{2\pi}\log|f_{\varphi(k)}(re^{i\theta})|\, d\theta\right] + O(1). \end{aligned}$$

Next we estimate the integral on f_i for $i = 1, 2, \cdots \varphi(k)$. Let $L(s,\chi)$ be a primitive Dirichlet L-function (mod k) (non-principal). By Jensen's formula

$$\frac{1}{2\pi}\int_0^{2\pi} \log|L(re^{i\theta},\chi)|\, d\theta = \int_0^r \frac{n_\chi(t)}{t} + \log|L(0,\chi)|.$$

Write ρ_χ for a zero of $L(s,\chi)$ and $\mathcal{R}(T) = [0,1]\times[-T,T] \subset \mathbf{C}$. It is well-known that (for fixed k)

$$\sum_{\rho_\chi \in \mathcal{R}(T)} 1 = \frac{T}{\pi}\log T + O(T).$$

Note that there are about r trivial zeros in the disk of radius r centered at origin, it follows that

$$n_\chi(r) = \frac{r}{\pi}\log r + O(r).$$

Therefore, we conclude that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|L(re^{i\theta},\chi)|\, d\theta = \frac{r}{\pi}\log r + O(r). \tag{30}$$

The imprimitive case can be handled in the same way and for the principal character one uses a simple modification of this argument. In addition,

$$\begin{aligned} \left|\int_0^{2\pi} \log|re^{i\theta}-1|\, d\theta\right| &\leq \int_0^{2\pi} \left|\log|re^{i\theta}-1|\right|\, d\theta \\ &\leq \int_0^{2\pi} \log^+|re^{i\theta}-1|\, d\theta + \int_0^{2\pi}\log^+\frac{1}{|re^{i\theta}-1|}\, d\theta \\ &\leq 2\pi T(r, s-1) + 2\pi T\left(r, \frac{1}{s-1}\right) = 4\pi\log r + O(1). \end{aligned}$$

Thus, we obtain

$$\int_0^{2\pi} \log|re^{i\theta}-1|\, d\theta = O(\log r), \tag{31}$$

and hence

$$\begin{aligned} \int_0^{2\pi} \log|f_i(re^{i\theta})|\, d\theta &= \int_0^{2\pi}\log|re^{i\theta}-1|\, d\theta + \int_0^{2\pi}\log|L_i(re^{i\theta})|\, d\theta \\ &= \frac{r}{\pi}\log r + O(r) \end{aligned} \tag{32}$$

for $i = 1, 2, \cdots, \varphi(k)$, it then follows that

$$m(r) \geq \frac{r}{\pi}\log r + O(r). \tag{33}$$

On another hand, we have the following inequality (see [15, p. 295]),

$$m(r) \le \varphi(k)T(r) + O(1), \quad \text{where } T(r) = \max_{1\le j\le \varphi(k)} T\left(r, \frac{f_j}{f_0}\right). \tag{34}$$

Furthermore, by the first fundamental theorem of Nevanlinna, we have

$$\begin{aligned} T\left(r, \frac{f_j}{f_0}\right) &= T\left(r, \frac{L_j}{L_0}\right) \le T(r, L_j) + T(r, L_0) + O(1) \\ &\le T(r, L_j) + \sum_{i=1}^{\varphi(k)} T(r, L_i) + O(1). \end{aligned} \tag{35}$$

Using first fundamental theorem of Nevanlinna again, we deduce from (24) that

$$\begin{aligned} T(r, L_1) &\le T(r,\zeta) + \sum_{p|k} T(r, p^s) + O(1) \le T(r,\zeta) + \sum_{p|k} T(r, e^{s\log p}) + O(1) \\ &\le T(r,\zeta) + \sum_{p|k} T(r\log p, e^z) + O(1) \le \frac{r}{\pi}\log r + O(r), \end{aligned} \tag{36}$$

where we used well-known facts

$$T(r,\zeta) = \frac{r}{\pi}\log r + O(r)$$

and

$$T(r, e^s) = \frac{r}{\pi} + O(1).$$

If χ_j $(j = 2, \cdots, \varphi(k))$ is the primitive character modulo k, then (see [19, p. 150])

$$T(r, L_j) = \frac{r}{\pi}\log r + O(r). \tag{37}$$

If χ_j $(j = 2, \cdots, \varphi(k))$ is not the primitive character modulo k, then in the same way as estimating $T(r, L_1)$, we can obtain

$$T(r, L_j) \le \frac{r}{\pi}\log r + O(r) \tag{38}$$

from (25) and (37). Combining these facts with (34), (35), it yields that

$$m(r) \le \varphi(k)(\varphi(k)+1)\frac{r}{\pi}\log r + O(r). \tag{39}$$

Therefore, by using Theorem 2.2, Theorem 1.6 follows from (28), (29), (33) and (39).

6. Proof of Theorem 1.8

For any complex number $a \ne 0$, set

$$f_1 = (s-1)L_1, \cdots, f_{\varphi(k)} = (s-1)L_{\varphi(k)}, \quad f_{\varphi(k)+1} = -a(s-1).$$

It is easy to show that entire functions $f_1, \cdots, f_{\varphi(k)}, f_{\varphi(k)+1}$ have no common zeros. Next we show that $f_1, \cdots, f_{\varphi(k)+1}$ are linearly independent. Assume, to the contrary, that $f_1, \cdots, f_{\varphi(k)+1}$ are linearly dependent, then there exist constants b_j for $j = 1, \cdots, \varphi(k)+1$ (at least one of them is not zero), such that

$$b_1 f_1 + \cdots + b_{\varphi(k)} f_{\varphi(k)} + b_{\varphi(k)+1} f_{\varphi(k)+1} = 0,$$

it then follows that

$$b_1 L_1 + \cdots + b_{\varphi(k)} L_{\varphi(k)} - b_{\varphi(k)+1} a = 0, \tag{40}$$

this implies that $b_{\varphi(k)+1} \neq 0$, otherwise, $b_1 = \cdots = b_{\varphi(k)+1} = 0$ since $L_1, \cdots, L_{\varphi(k)}$ are linearly independent, which is a contradiction. Moreover, we can restate (40) for Re(s) > 1 as follows,

$$b_1 + \cdots + b_{\varphi(k)} - b_{\varphi(k)+1}a + \sum_{n=2}^{\infty} \frac{b_1\chi(n) + \cdots + b_{\varphi(k)}\chi_{\varphi(k)}(n)}{n^s} = 0,$$

it follows from the classic uniqueness theorem of Dirichlet series that

$$b_1 + \cdots + b_{\varphi(k)} - b_{\varphi(k)+1}a = 0$$

and

$$b_1\chi(n) + \cdots + b_{\varphi(k)}\chi_{\varphi(k)}(n) = 0$$

for all $n \geq 2$. Specially, let $n = k+1$, it yields that $b_1 + \cdots + b_{\varphi(k)} = 0$, thus, $b_{\varphi(k)+1} = 0$, is a contradiction. Therefore, $f_1, \cdots, f_{\varphi(k)+1}$ are linearly independent. By analogy with the estimation of $m(r)$ in the proof of the Theorem 1.6, it then yields the results of Theorem 1.8 by Theorem 2.2.

7. Proof of Theorem 1.3

For arbitrary nonzero constant a, considering functions $f_1(s) = (s-1)\zeta(s)$ and $f_2(s) = -a(s-1)$, then entire functions f_1 and f_2 have no common zeros and they are linearly independently. Set $f_0 = f_1 + f_2 = (s-1)(\zeta(s) - a)$, using Theorem 2.2, and by analogy with the estimation of $m(r)$ in the proof of Theorem 1.6, it then yields the results of the Theorem 1.3.

Certainly, it also can be seen as a simple application of Nevanlinna's second fundamental theory.

8. Appendix

The following result is given by the reviewer.

Theorem 8.1. *Let $a_1, a_2, a_3 \in \mathbf{C}$ be distinct then for at least one of a_1, a_2, a_3 the proportion of a_j-points of $\zeta(s)$ which are simple exceeds $\frac{1}{3} - \varepsilon$.*

Proof. Let $a_1, a_2, a_3 \in \mathbf{C}$ be distinct and defined $F_{a_j}(s) = \zeta(s) - a_j$. We denote the zeros of F_{a_j} by ρ_{a_j} and write $m(\rho_{a_j})$ for the multiplicity of ρ_{a_j}.

Define the rectangle $\mathcal{R}(T) = [0,1] \times [0,T] \subset \mathbf{C}$ and define

$$N_a(T) = \sum_{\rho_a \in \mathcal{R}(T)} 1 = \frac{T}{2\pi} \log T + O(T),$$

where the last estimate is well known and can be find in Levinson [17]. Let

$$M_{a_j}(T) = \sum^{*}_{\substack{\rho_{a_j} \in \mathcal{R}(T) \\ m(\rho_{a_j}) \geq 2}} m(\rho_{a_j}),$$

where the sum $\sum^*$ is over distinct zeros of F_{a_j}. Thus, the number of simple zeros of F_{a_j} in $\mathcal{R}(T)$, which we denote by $N_{\text{simple},a_j}(T)$, is given by

$$N_{\text{simple},a_j}(T) = N_{a_j}(T) - M_{a_j}(T).$$

Also notice $F'_{a_j}(s) = \zeta'(s)$, so for each ρ_{a_j} such that $m(\rho_{a_j}) \geq 2$ we have $\zeta'(\rho_{a_j}) = 0$. Writing $m'(\rho')$ for the multiplicity of a zero ρ' of $\zeta'(s)$, we see that for an a-point, with $m(\rho_a) \geq 2$ we have $m(\rho_a) = m'(\rho_a) + 1$. Notice for $a_1 \neq a_2$, an a_1-point cannot also be an a_2-point (since $\zeta(\rho_{a_j}) = a_j, j = 1, 2)$. Thus,

$$\sum_{j=1}^{3} M_{a_j}(T) = \sum_{j=1}^{3} \sideset{}{^*}\sum_{\substack{\rho_{a_j} \in \mathcal{R}(T) \\ m(\rho_{a_j}) \geq 2}} m'(\rho_{a_j}) + 1 \leq 2 \sideset{}{^*}\sum_{\rho' \in \mathcal{R}(T)} m'(\rho') = 2 \sum_{\rho' \in \mathcal{R}(T)} 1$$

where the summation is over zeros ρ' of $\zeta'(s)$ and each zero is counted according to its multiplicity (which is the usual convention). Since

$$\sum_{\rho' \in \mathcal{R}(T)} 1 = \frac{T}{2\pi} \log T + O(T) = N_a(T) + O(T)$$

for any $a \in \mathbf{C}$, we deduce that for any $\varepsilon > 0$ we must have

$$M_{a_j}(T) \leq \left(\frac{2}{3} + \varepsilon \right) N_{a_j}(T)$$

for one $j = 1$, $j = 2$ or $j = 3$. So that

$$N_{\mathrm{simple}, a_j}(T) \geq \left(\frac{1}{3} - \varepsilon \right) N_{a_j}(T)$$

for $j = 1$, $j = 2$ or $j = 3$. □

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