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FRACTIONAL DERIVATIVES OF POINTWISE MULTIPLIERS BETWEEN HOLOMORPHIC SPACES

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Abstract. We study the action of fractional differential type operators on the space of pointwise multipliers between holomorphic Triebel–Lizorkin spaces on the unit ball $\mathbf{B}$ of $\mathbf{C}^n$. As an application, we obtain new characterizations and examples of multipliers on Hardy–Sobolev spaces, and we improve some well-known results about the integrals operators I_b and J_b.

1. Introduction and main results

For $0 < p < \infty$, let H^p be the Hardy space in the unit ball $\mathbf{B}$ of $\mathbf{C}^n$. For $\tau \in \mathbf{R}$, denote by $(I+R)^\tau$ the bijective linear fractional differential operator on the space H of holomorphic functions on $\mathbf{B}$, defined on the monomials by $(I+R)^\tau z^\alpha := (1+|\alpha|)^\tau z^\alpha$, where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multiindex of non-negative integers and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For $0 < p < \infty$ and $s \in \mathbf{R}$, the Hardy–Sobolev space H^p_s on $\mathbf{B}$ consists of all the holomorphic functions f on $\mathbf{B}$ such that $(I+R)^s f \in H^p$, that is, $H^p_s := (I+R)^{-s} H^p$.

The Hardy–Sobolev spaces H^p_s for $1 < p < \infty$ and $s < n$ can be described in terms of fractional Cauchy operators, namely, $H^p_s = \mathcal{C}_s[L^p]$ where $\mathcal{C}_s$ is operator with kernel

$$\mathcal{C}_s(z,\zeta) := \frac{d\sigma(\zeta)}{(1 - z\bar{\zeta})^{n-s}},$$

(see [16]). Here $d\sigma$ denotes the normalized surface measure on the unit sphere $\mathbf{S}$ and $\mathcal{C}_s$ is a bijective operator from H^p to H^p_s which plays in some sense the same role that $(I+R)^{-s}$, as we will detail in the forthcoming sections.

One of our motivations arises as a natural question derived from the above representation: Find a description of the functions $\varphi \in L^p(d\sigma)$ for which $\mathcal{C}_s(\varphi)$ is a pointwise multiplier on H^p_s. For the cases $s > n/p$ or $s \le 0$ the answer to this question is known (see for instance [25]). If $s > n/p$, H^p_s is a multiplicative algebra and consequently $\mathcal{C}_s(\varphi)$ is a pointwise multiplier on H^p_s for any $\varphi \in L^p(d\sigma)$. For $s \le 0$, the space of multipliers of H^p_s coincides with H^∞, and $\mathcal{C}_s(\varphi)$ is a pointwise multiplier on H^p_s if and only if $\mathcal{C}_s(\varphi) \in H^\infty$. Our approach to obtain this description for $0 < s < n/p$ is based, among other ingredients, in the behavior of fractional differential operators on the algebra of pointwise multipliers between Hardy–Sobolev spaces. These results, which are interesting by themselves, permits to describe the spaces

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of pointwise mutipliers in terms of trace measures on $\mathbf{S}$ and in terms of Carleson measures on $\mathbf{B}$, which complete the ones given in [25].

A non-negative Borel measure μ on $\mathbf{B}$ is a Carleson measure for H^p_s if $H^p_s \subset L^p(\mu)$. When $s=0$, these measures are characterized by the condition $\mu(T(Q(\zeta,r))) \lesssim r^n$, for any Carleson box $Q(\zeta,r) = \{z \in \mathbf{B}; 1-|z| \leq r, z/|z| \in B(\zeta,r)\}$. In this characterization the Carleson boxes can be substituted by tents. The proof of this result in several variables can be obtained using the methods of the classical theorem in one variable, due to Carleson (see [12] and [13]).

For general $s>0$, we recall, for instance, that if $n-sp<1$, the Carleson measures for the spaces H^p_s can be characterized by a capacitary condition on open sets in $\mathbf{S}$ (see [18]) and for H^2_s and any $s>0$, was obtained (see [29]) a non-capacitary characterization. A non-negative Borel measure μ on $\mathbf{S}$ is a trace measure for H^p_s, $s>0$, if the space of all the boundary values of the functions H^p_s, also denoted by H^p_s, is in $L^p(d\mu)$.

There is a large number of works dealing with the theory of pointwise multipliers acting in pairs of spaces of differentiable functions on $\mathbf{R}^n$. We refer to the book of Maz'ya and Shaposhnikova [21] as a survey of some of the main results in this topic. The fact that we are dealing with spaces of holomorphic functions, where the non-isotropic metrics play a main role, allows us to use different arguments and techniques to the ones used in the real case, which in some situations can not be applied directly to the holomorphic case. One trivial example of this situation is the fact that for $kp>n$, the functions in H^p_k are continuous up to the boundary, whereas for real Sobolev spaces on $\mathbf{B}$, the regularity holds for $kp>2n$.

One second motivation was the study of the integral operators I_b and J_b with holomorphic symbol b defined by

$$I_b(f)(z) = \int_0^1 b(tz)\,(Rf)(tz)\frac{dt}{t}, \quad J_b(f)(z) = \int_0^1 f(tz)\,(Rb)(tz)\frac{dt}{t}.$$

Observe that $I_b(f)(z) + J_b(f)(z) = (bf)(z) - (bf)(0)$. These operators have been thoroughly studied in different settings, and there is a wide literature on the subject. See for instance the classical articles [5], [4] and [20], the recent paper [26] and the references therein.

We study the relationship of these operators with the multipliers, extending some classical results on this topic. Since the techniques used to prove the main theorems are not more involved, we state our results in the general setting of Triebel–Lizorkin spaces which include both the Hardy–Sobolev spaces H^p_s and the Besov spaces B^p_s. Recall that B^p_s consists of all the $f \in H$ such that $(1-|z|^2)^{k-s}(I+R)^k f(z) \in L^p\left((1-|z|^2)^{-1}d\nu(z)\right)$ for some (any) non-negative integer $k>s$. Observe that $B^p_{-1/p}$ is the classical Bergman space $A^p := L^p(d\nu)\cap H$ and therefore $B^p_s = (I+R)^{-s-1/p}A^p$. Here, $d\nu$ denotes the normalized Lebesgue measure on $\mathbf{B}$.

For $0<p,q<\infty$ and a positive Borel measure μ on $\mathbf{B}$, the non-isotropic tent space $T^{p,q}(\mu)$ consists of all measurable functions φ on $\mathbf{B}$ such that

$$\|\varphi\|^p_{T^{p,q}(\mu)} := \int_{\mathbf{S}} \left(\int_{\Gamma_\zeta} |\varphi(w)|^q \frac{d\mu(w)}{(1-|w|^2)^{n+1}} \right)^{p/q} d\sigma(\zeta) < \infty,$$

where $\Gamma_\zeta = \left\{ w \in \mathbf{B} \colon |1-w\bar{\zeta}| < 2(1-|w|^2) \right\}$ is the usual admissible approach region. We extend the above definition to the case $p=q=\infty$ defining $T^{\infty,\infty} := L^\infty$.

Let $0 < p, q < \infty$ and $s \in \mathbf{R}$. The holomorphic Triebel–Lizorkin space $F_s^{p,q}$ consists of all holomorphic functions on $\mathbf{B}$ such that

$$\|f\|_{F_s^{p,q}} := \left\|(1-|z|^2)^{k-s}(I+R)^k f(z)\right\|_{T^{p,q}(\nu)} < \infty,$$

for some (any) non-negative integer $k > s$. Note that if $s < 0$ then we can take $k = 0$.

The same techniques used in [23] to prove that the norms are equivalent for different non-negative integer values of k permit to obtain equivalent norms replacing $(I+R)^k$ by other differential operators, as $(I+R)^\tau$, $\tau > s$ not necessarily integer, or the operators defined in (1.1) and in Remark 2.5. See for instance Chapter 6 in [31] for more equivalent norms on Besov spaces.

Thus, the operator $(I+R)^\tau$ is a bijective operator from $F_s^{p,q}$ to $F_{s-\tau}^{p,q}$. We then have that If $p < \infty$ and $q = 2$, the characterization of H_s^p in terms of the area functions gives that $F_s^{p,2} = H_s^p$ (see for instance [2]). If $p = q < \infty$, then $F_s^{p,p}$ is the Besov space B_s^p. For $p = \infty$, if $s > 0$ the space $B_s^\infty := F_s^{\infty,\infty}$ is the holomorphic Lipschitz-Zygmund space $H \cap \Lambda_s$. The space $F_0^{\infty,\infty}$ is the Bloch space B_0^∞ of holomorphic functions on $\mathbf{B}$ satisfying $\sup_{z\in\mathbf{B}}(1-|z|^2)|Rf(z)| < \infty$.

Given a couple of quasi-normed spaces X and Y of holomorphic functions on $\mathbf{B}$, we denote by $\operatorname{Mult}(X \to Y)$ the space of pointwise multipliers of X to Y. If $X = Y$, then we simply write $\operatorname{Mult}(X)$.

Our first result shows that the differential operator $(I+R)^\tau$, which is a bijective operator on holomorphic Triebel–Lizorkin spaces, is also a bijection between two spaces of multipliers of Triebel–Lizorkin spaces.

Theorem 1.1. *Let $0 < p, q < \infty$ and $s, s' \in \mathbf{R}$.*

(i) *If $s' < s$ then $g \in \operatorname{Mult}(F_s^{p,q} \to F_{s'}^{p,q})$ if and only if for some (any) $\tau > s' - s$, $(I+R)^\tau g \in \operatorname{Mult}(F_s^{p,q} \to F_{s'-\tau}^{p,q})$.*

(ii) *If $s = s'$, then $g \in \operatorname{Mult}(F_s^{p,q})$ if and only if $g \in H^\infty$ and for some (any) $\tau > 0$, $(I+R)^\tau g \in \operatorname{Mult}(F_s^{p,q} \to F_{s-\tau}^{p,q})$.*

(iii) *If $s' > s$, $\operatorname{Mult}(F_s^{p,q} \to F_{s'}^{p,q}) = \{0\}$.*

The above theorem permits to describe the space $\operatorname{Mult}(F_s^{p,q} \to F_{s'}^{p,q})$, $s' < s$, in terms of $T^{p,q}$-Carleson measures, that is, in terms of positive Borel measures μ on $\mathbf{B}$ such that $F_s^{p,q} \subset T^{p,q}(\mu)$. Indeed, $g \in \operatorname{Mult}(F_s^{p,q} \to F_{s'}^{p,q})$ if and only if for $\tau > \max\{s', s'-s\}$, $(I+R)^\tau g \in \operatorname{Mult}(F_s^{p,q} \to F_{s'-\tau}^{p,q})$. Taking the norm in $F_{s'-\tau}^{p,q}$ corresponding to $k = 0 > s' - \tau$, we obtain $f(I+R)^\tau g \in F_{s'-\tau}^{p,q}$ if and only if $f \in H \cap T^{p,q}(\mu_g)$, with

$$d\mu_g(z) := |(I+R)^\tau g(z)|^q (1-|z|^2)^{q(\tau-s')} d\nu(z).$$

Thus, assertion (i) in Theorem 1.1 is equivalent to:

(i') For $s' < s$, $g \in \operatorname{Mult}(F_s^{p,q} \to F_{s'}^{p,q})$, if and only if for some (any) $\tau > \max\{s', s'-s\}$ the measure μ_g is a $T^{p,q}$-Carleson measure for $F_s^{p,q}$.

Theorem 1.1(ii) gives $\operatorname{Mult}(F_s^{p,q}) = H^\infty \cap (I+R)^{-\tau} \operatorname{Mult}(F_s^{p,q} \to F_{s-\tau}^{p,q})$. The space $(I+R)^{-\tau} \operatorname{Mult}(F_s^{p,q} \to F_{s-\tau}^{p,q})$ appears in the study of the boundedness of Hankel type operators on Hardy–Sobolev and Besov spaces (see for instance [6], [14] and the references therein).

The following corollary details the main properties of the action of the operator $(I+R)^s$ on spaces of multipliers for the particular case of Hardy–Sobolev spaces:

Corollary 1.2. *Let $0 < p < \infty$ and $s > 0$. The following assertions are equivalent:*

(i) $g \in \operatorname{Mult}(H_s^p)$.

(ii) $g \in H^\infty$ *and for some (any)* $\tau > 0$, $(I+R)^\tau g \in \mathrm{Mult}(H^p_s \to H^p_{s-\tau})$.
(iii) $g \in H^\infty$ *and for some (any)* $\tau > s$ *the measure*

$$d\mu_g(z) = |(I+R)^\tau g(z)|^2 (1-|z|^2)^{2(\tau-s)} d\nu(z)$$

is a $T^{p,2}$*-Carleson measure.*
(iv) $g \in H^\infty$ *and* $(I+R)^s g \in \mathrm{Mult}(H^p_s(\mathbf{S}) \to H^p(\mathbf{S}))$, *that is* $g \in H^\infty$ *and the measure* $d\sigma_g(\zeta) = |(I+R)^s g(\zeta)|^p d\sigma(\zeta)$ *is a trace measure for* H^p_s, *that is,* $H^p_s(\mathbf{S}) \subset L^p(\sigma_g)$.

The next applications of Theorem 1.1 give norm-estimates of the integral operators I_b and J_b, with a holomorphic symbol b already defined.

Since $R\int_0^1 h(tz)\frac{dt}{t} = h(z)$ for any holomorphic function h on $\mathbf{B}$ with $h(0)=0$, we have $RI_b(f) = bRf$ and $RJ_b(f) = fRb$. Thus,

Ib: I_b is bounded from $F^{p,q}_s$ to $F^{p,q}_{s'}$ if and only if $b \in \mathrm{Mult}(F^{p,q}_{s-1} \to F^{p,q}_{s'-1})$.
Jb: J_b is bounded from $F^{p,q}_s$ to $F^{p,q}_{s'}$ if and only if $Rb \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'-1})$.

In this case Theorem 1.1 and Jb give:

Corollary 1.3. *Let* $0 < p,q < \infty$. *If* $s' < s$, *then* J_b *is bounded from* $F^{p,q}_s$ *to* $F^{p,q}_{s'}$ *if and only if* $b \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'})$.

In [5] for $n=1$ and $p \geq 1$, [4] for $n=1$ and $p<1$, and more recently in [26] for any $n \geq 1$ and $p > 0$, it is shown that the operator J_b is bounded on H^p if and only if $b \in BMOA$, that is, if and only if $Rb \in BMOA_{-1}$, where $BMOA_t := (I+R)^{-t} BMOA$ for any $t \in \mathbf{R}$. Recall that $BMOA$ consists of all holomorphic functions f on $\mathbf{B}$ such that $|(I+R)f(z)|^2(1-|z|^2)d\nu(z)$ is a classical Carleson measure, that is, a Carleson measure for some (any) H^p (see for instance Theorem 5.9 in [31]).

Thus, Jb gives $\mathrm{Mult}(H^p \to H^p_{-1}) = BMOA_{-1}$ and by Theorem 1.1 we have:

Corollary 1.4. *If* $0 < p < \infty$ *and* $s' < 0$, *then* $\mathrm{Mult}(H^p \to H^p_{s'}) = BMOA_{s'}$, *that is, the pointwise multipliers from* H^p *to* $H^p_{s'}$ *are the holomorphic functions* g *on* $\mathbf{B}$ *such that* $(I+R)^{s'} g \in BMOA$.

Our main result about the boundedness of the operators I_b and J_b is the following:

Theorem 1.5. *Let* $0 < p,q < \infty$. *Then, we have:*
(i) *If* $s' < s < 1$, *then* I_b *maps boundedly* $F^{p,q}_s$ *to* $F^{p,q}_{s'}$ *if and only if* $b \in B^\infty_{s'-s}$.
(ii) *If* $s' < s = 1$, *then* I_b *maps boundedly* H^p_1 *to* $H^p_{s'}$ *if and only* $b \in BMOA_{s'-1}$.
(iii) *If* $s'-1 < s < 0$, *then* J_b *maps boundedly* $F^{p,q}_s$ *to* $F^{p,q}_{s'}$ *if and only* $b \in B^\infty_{s'-s}$.
(iv) *If* $s' < 1$, *then* J_b *maps boundedly* H^p *to* $H^p_{s'}$ *if and only* $b \in BMOA_{s'}$.

As we stated before, one of our motivations is the description of the space

$$X^p_s := \{\varphi \in L^p(d\sigma) \colon \mathcal{C}_s(\varphi) \in \mathrm{Mult}(H^p_s)\}, \quad p>1,\ 0<s<n/p.$$

The study of this space requires the knowledge of the fractional Cauchy operator $\mathcal{C}_s$ on spaces of pointwise multipliers of Hardy–Sobolev spaces. Since the study of the analogous problem for fractional Bergman operators may be interesting, we give a unified treatment of these problems considering the following integral operators $\mathcal{P}^{N,N+\tau}$.

For $N>0$ and $\tau > -n-N$, let $\mathcal{P}^{N,N+\tau}$ be the integral operator with kernel

$$\mathcal{P}^{N,N+\tau}(z,w) := \frac{d\nu_N(w)}{(1-z\overline{w})^{n+N+\tau}},$$

where $d\nu_N(w) = \frac{\Gamma(n+N)}{n!\Gamma(N)}(1-|w|^2)^{N-1}\, d\nu(w)$.

We extend this definition to the case $N = 0$ by considering the fractional Cauchy type kernel $\mathcal{C}_s(z,\zeta)$. In order to unify notations we write $d\nu_0 := d\sigma$ and $\mathcal{P}^{0,\tau} = \mathcal{C}_{-\tau}$. Observe that $\mathcal{P}^N := \mathcal{P}^{N,N}$ is the orthogonal projection from $L^2(d\nu_N)$ to the weighted Bergman space $H \cap L^2(d\nu_N)$ and that $\mathcal{P}^0 := \mathcal{P}^{0,0}$ is the Cauchy projection $\mathcal{C}$ from $L^2(d\sigma)$ to H^2.

The operator $\mathcal{P}^{N,N+\tau}$ plays a role similar to the differential operator $(I+R)^\tau$ as we will see in Sections 2 and 3. We refer to the book [31] for the main properties of these operators. Observe that if k is a positive integer and R_L^k, $L > 0$, is the differential operator defined by

$$R_L^k = \left(I + \frac{R}{L+k-1}\right) \cdots \left(I + \frac{R}{L}\right), \tag{1.1}$$

it is immediate to check that for $f \in B^1_{-N} \cup H^1$,

$$\mathcal{P}^{N,N+k}(f) = R^k_{n+N}\mathcal{P}^{N,N}(f) = R^k_{n+N} f. \tag{1.2}$$

Hence, $\mathcal{P}^{N,N+k}$ is a bijective operator from $F_s^{p,q}$ to $F_{s-k}^{p,q}$. These results for Bergman spaces can be found for instance in [31].

The following theorem is the version of Theorem 1.1 for the operators $\mathcal{P}^{N,N+\tau}$.

Theorem 1.6. *Let $0 < p, q < \infty$ and $s, s' \in \mathbf{R}$ satisfying $s' \le s$. Let $N \ge 0$ such that $F_s^{p,q} \subset B^1_{-N} \cup H^1$.*

(i) *If $s' < s$ then $g \in \mathrm{Mult}(F_s^{p,q} \to F_{s'}^{p,q})$ if and only if for some (any) $\tau > s' - s$, $\mathcal{P}^{N,N+\tau} g \in \mathrm{Mult}(F_s^{p,q} \to F_{s'-\tau}^{p,q})$.*

(ii) *If $s = s'$, then $g \in \mathrm{Mult}(F_s^{p,q})$ if and only if $g \in H^\infty$ and for some (any) $\tau > 0$, $\mathcal{P}^{N,N+\tau} g \in \mathrm{Mult}(F_s^{p,q} \to F_{s-\tau}^{p,q})$.*

The proof of the above theorems for integer values of τ follows from some basic properties of the pointwise multipliers of $F_s^{p,q}$ and the Leibnitz's formula. The general case needs also some adequate Taylor expansions with precise estimates of the error term. As a consequence of this result we will obtain the following description of the space X_s^p, that is, the space of functions $\varphi \in L^p(d\sigma)$ such that $\mathcal{C}_s(\varphi) \in \mathrm{Mult}(H_s^p)$.

Theorem 1.7. *Let $1 < p < \infty$ and $0 < s < n/p$. Let $\varphi \in L^p$. Then, $\mathcal{C}_s(\varphi)$ is a pointwise multiplier for H_s^p if and only if $\mathcal{C}_s(\varphi) \in H^\infty$ and the measure $|\mathcal{C}(\varphi)|^p d\sigma$ is a trace measure for H_s^p.*

For $p > 1$ and $0 < s < n$, consider the non-isotropic Riesz potential space

$$K_s(L^p) := \left\{ \int_{\mathbf{S}} \frac{\varphi(\zeta)}{|1-\eta\bar{\zeta}|^{n-s}}\, d\sigma(\zeta) \colon \varphi \in L^p(d\sigma) \right\}.$$

A trace measure for $K_s(L^p)$ is a positive Borel measure μ on $\mathbf{S}$ such that $\|K_s(\varphi)\|_{L^p(\mu)} \lesssim \|\varphi\|_{L^p(d\sigma)}$. When $0 < s < n/p$, these measures can be characterized in terms of non-isotropic Riesz capacities (see [1] and [17]).

The following results provide examples of pointwise multipliers for H_s^p.

Theorem 1.8. *Let $p > 1$, $0 < s < n/p$ and let $\psi \in L^p(\sigma)$ such that $|\psi|^p d\sigma$ is a trace measure for $K_s(L^p)$. Then, $\mathcal{C}_s(\psi) \in \mathrm{Mult}(H_s^p)$ if and only if $\mathcal{C}_s(\psi) \in H^\infty$.*

The next theorem gives other examples of functions in X_s^p.

Theorem 1.9. *If $0 < p < \infty$ and $0 < s < n/p$, then $H_s^{n/s} \cap H^\infty \subset \mathrm{Mult}(H_s^p)$. In particular, if $p > 1$, $\varphi \in L^{n/s}(d\sigma)$ and $\mathcal{C}_s(\varphi) \in H^\infty$, then $\varphi \in X_s^p$.*

This is a generalization of a result of [11], where the authors prove this result for a positive integer $s < n/p$.

The paper is organized as follows. Section 2 is devoted to state some basic properties of the multipliers of Triebel–Lizorkin spaces, which permit us to prove Theorems 1.1 and 1.6 for integer values of τ. In Section 3, we prove our main theorems. In Subsection 3.1 we prove Theorem 1.1 and Corollary 1.2. We prove Theorem 1.6 in Subsections 3.3 and 3.4. Both theorems use properties of the integral operator, whose study is postponed to the subsection 3.5. In Subsection 3.2 we give some applications of Theorem 1.1 and in particular we prove Theorem 1.5. Subsection 3.6 is devoted to the study of the spaces X_s^p. In this subsection we prove Theorems 1.7, 1.8 and 1.9.

Throughout the paper, the notation $f(z) \lesssim g(z)$ means that there exists $C > 0$, which does not depend on z, f and g, such that $f(z) \le Cg(z)$. We write $f(z) \approx g(z)$ when $f(z) \lesssim g(z)$ and $g(z) \lesssim f(z)$.

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2. Multipliers and proof of Theorem 1.1 for integer values of τ

In this section we state some properties of the space of pointwise multipliers from $F_s^{p,q}$ to $F_{s'}^{p,q}$, $s' \le s$. We will need the following proposition.

Proposition 2.1. *Let $0 < p, q < \infty$ and $s \in \mathbf{R}$.*

(i) *For any $s \ge t$, $F_s^{p,q} \subset F_t^{p,q} \subset B_{t-n/p}^{\infty}$.*
(ii) *If $0 < p < u < \infty$ and $s - n/p \ge t - n/u$, $F_s^{p,q} \subset F_t^{u,v}$ for any $v > 0$.*
(iii) *Let $0 < u \le p$. If either $s = t$ and $q \le v$ or $s > t$, then $F_s^{p,q} \subset F_t^{u,v}$.*
(iv) *If $0 < u \le \min\{p, q\}$ and $s - n/p \le t - n/u$, then $B_t^u \subset F_s^{p,q}$.*
(v) *If either $0 < p \le 1$ and $s = n/p$ or $s > n/p$, then $F_s^{p,q} \subset H^\infty$.*
(vi) *If $0 < p, q < \infty$, $s_0, s_1 \in \mathbf{R}$, $0 < \theta < 1$ and $s = (1-\theta)s_0 + \theta s_1$, then*

$$(F_{s_0}^{p,q}, F_{s_1}^{p,q})_\theta = F_s^{p,q}.$$

Here $(X, Y)_\theta$ denotes the intermediate space between X and Y obtained by the complex interpolation method.

Proof. The above assertions for Besov and Hardy–Sobolev spaces can be found for instance in [10], [8], [30] and [31]. A proof of (i), (ii) and (iv) can be found for instance in Theorem 4.1 in [24]. Part (iii) for $p = u$, $s = t$ and $q \le v$ is also proved in Theorem 4.1 in [24]. For $u < p$, $s = t$ and $q = v$ this follows from Hölder's inequality. Combining the above cases we prove (iii) for $u \le p$, $s = t$ and $q \le v$. If $s > t$, we apply (ii) with $u_* > p$ satisfying $t - n/u_* = s - n/p$, to obtain $F_s^{p,q} \subset F_t^{u_*,v} \subset F_t^{u,v}$. The last embedding follows from (iii) for $u < u_*$, $s = t$ and $q = v$.

Assertion (v) for $s > n/p$ is a consequence of (i) and the fact that if $r > 0$, then the Lipschitz space B_r^∞ is in H^∞. The proof of the case $0 < p \le 1$ and $s = n/p$ for Besov and Hardy–Sobolev spaces can be found in Theorem 1.4 in [8]. The general case is proved in Theorem 4.3 in [24].

Assertion (vi) for the case $p = q$ is proved in Theorem 1.3 in [9]. An independent proof for $1 < p < \infty$ and $1 \le q < \infty$ can be found for instance in Corollary 3.4 in [23]. We does not know an explicit reference for the cases $0 < p \ne q$ and $n \ge 1$, but the same proof for the case $p = q$ given in [9], based in the representation formula $f = \mathcal{P}^N(f)$ and in norm-estimates of the operators $\mathcal{P}^{N,N+\tau}$, can be adapted to prove the interpolation result for the case $p \ne q$. $\square$

The following corollary gives conditions on p, q, s so that the operator $\mathcal{P}^{N,N+\tau}$ is well defined on $F^{p,q}_s$.

Corollary 2.2. *The spaces B^1_{-N} and H^1 satisfy the following relations:*

(i) *The space $B^1_{-N} \cup H^1$ coincides with H^1 if $N = 0$ and with B^1_{-N} if $N > 0$. Therefore, $\mathcal{P}^{N,N+\tau}$ is well defined in $B^1_{-N} \cup H^1$.*

(ii) *The embedding $F^{p,q}_s \subset B^1_{-N}$, $N > 0$ holds in the following cases:*
  (a) *$p < 1$ and $s - n/p \geq -N - n$,*
  (b) *$p \geq 1$ and $s > -N$,*
  (c) *$p = 1$, $s = -N$ and $q \leq 1$.*

(iii) *$F^{p,q}_s \subset H^1$ holds in the following cases:*
  (a) *$p < 1$ and $s - n/p \geq -n$,*
  (b) *$p \geq 1$ and $s > 0$,*
  (c) *$p = 1$, $s = 0$ and $q \leq 2$.*

Proof. These results are a consequence of Proposition 2.1 and the equalities $B^1_{-N} = F^{1,1}_{-N}$ and $H^1 = F^{1,2}_0$. More precisely: Part (i) for $N = 0$ follows from Proposition 2.1(iii). The case $N > 0$ is a consequence of Proposition 2.1(iii), with $s = 0 > t = -N$. In both cases (ii) and (iii), part (a) is a consequence of Proposition 2.1(ii), parts (b) and (c) are consequences of Proposition 2.1(iii). □

Now we use these results to prove some properties of the pointwise multipliers of the Triebel–Lizorkin spaces. Some of these results for Hardy–Sobolev and Besov spaces are well known. However, we do not know explicit references for all the considered cases. So, we include a briefly proof of all of them.

Proposition 2.3. *Let $0 < p, q < \infty$ and $s, s' \in \mathbf{R}$. Then, we have:*

(i) *If either $p > 1$ and $s > n/p$ or $0 < p \leq 1$ and $s \geq n/p$, then $\mathrm{Mult}(F^{p,q}_s) = F^{p,q}_s$.*

(ii) *If $s' < s$, then $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'}) \subset B^\infty_{s'-s} \cap F^{p,q}_{s'}$. In the particular case where $s' < s < 0$, we have $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'}) = B^\infty_{s'-s}$*

(iii) *$\mathrm{Mult}(F^{p,q}_s) \subset H^\infty \cap F^{p,q}_s$. If $s < 0$, then $\mathrm{Mult}(F^{p,q}_s) = H^\infty$.*

(iv) *If $s' \leq s$ and $\tau > 0$, then $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'}) \subset \mathrm{Mult}(F^{p,q}_{s-\tau} \to F^{p,q}_{s'-\tau})$.*

(v) *If $s' > s$, then $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'}) = \{0\}$.*

Proof. Assertion (i) follows easily from the fact that $F^{p,q}_s \subset H^\infty$ (see Proposition 2.1(v)). Indeed, for a non-negative integer j, a real $t \geq 0$ and h a holomorphic function on $\mathbf{B}$, let $D_{j,t}(h)(z) := (1 - |z|^2)^{j-t}(I + R)^j h(z)$.

As we have observed in the introduction, the norm of $h \in F^{p,q}_s$ is equivalent to the norm of $(2I + R)^m h$ in $F^{p,q}_{s-m}$ for any non-negative integer m. Hence, for a positive integer $k > s$, we have $\|fg\|_{F^{p,q}_s} \approx \|(1 - |z|^2)^{2k-s}(2I + R)^{2k}(fg)(z)\|_{T^{p,q}}$. Since $(2I + R)(fg) = g(I + R)f + f(I + R)g$, Leibnitz's formula and the fact that for $s > n/p$, $F^{p,q}_s \subset H^\infty \subset B^\infty_0$, give

$$\begin{aligned}\|fg\|_{F^{p,q}_s} &\lesssim \sum_{j=0}^{k} \|D_{j,0}(f)\|_{L^\infty} \|D_{2k-j,s} g\|_{T^{p,q}} + \sum_{j=k+1}^{2k} \|D_{2k-j,0} g\|_{L^\infty} \|D_{j,s}(f)\|_{T^{p,q}} \\ &\lesssim \|f\|_{H^\infty} \|g\|_{F^{p,q}_s} + \|g\|_{H^\infty} \|f\|_{F^{p,q}_s} \lesssim \|f\|_{F^{p,q}_s} \|g\|_{F^{p,q}_s},\end{aligned}$$

which proves that $\mathrm{Mult}(F^{p,q}_s) = F^{p,q}_s$.

Let us prove (ii). Since $F^{p,q}_s$ contains the constant functions, then it is clear that $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'}) \subset F^{p,q}_{s'}$. If $s \geq n/p$, then by Proposition 2.1(i), we have $F^{p,q}_{s'} \subset B^\infty_{s'-n/p} \subset B^\infty_{s'-s}$.

Now we prove the case $s < n/p$. By Proposition 2.1((iv) and (i)), $B^u_{s-n/p+n/u} \subset F^{p,q}_s$, $u < \min\{p,q\}$, and $F^{p,q}_{s'} \subset B^\infty_{s'-n/p}$. Thus,

$$\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'}) \subset \mathrm{Mult}(B^u_{s-n/p+n/u} \to B^\infty_{s'-n/p}).$$

Fixed $z \in \mathbf{B}$, let $f_z(w) = \frac{(1-|z|^2)^{n+N}}{(1-w\bar{z})^{n+N}}$. By Proposition 1.4.10 in [27], for N large enough we have that $\|f_z\|_{B^u_{s-n/p+n/u}} \approx (1-|z|^2)^{n/p-s}$. Therefore, for $g \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'})$,

$$\begin{aligned}(1-|z|^2)^{n/p-s'}|g(z)| &\le \|(1-|w|^2)^{n/p-s'}g(w)f_z(w)\|_{L^\infty} \approx \|gf_z\|_{B^\infty_{s'-n/p}} \\ &\lesssim \|g\|_{\mathrm{Mult}(B^u_{s-n/p+n/u}\to B^\infty_{s'-n/p})}(1-|z|^2)^{n/p-s},\end{aligned} \tag{2.3}$$

which gives

$$\|g\|_{B^\infty_{s'-s}} = \sup_{z\in\mathbf{B}}(1-|z|^2)^{s-s'}|g(z)| \lesssim \|g\|_{\mathrm{Mult}(F^{p,q}_s\to F^{p,q}_{s'})}.$$

Hence $g \in B^\infty_{s'-s} \cap F^{p,q}_{s'}$. If, now, $s' < s < 0$ and $g \in B^\infty_{s'-s}$, we have

$$\begin{aligned}\|gf\|_{F^{p,q}_{s'}} &\approx \|g(z)f(z)(1-|z|^2)^{-s'}\|_{T^{p,q}} \lesssim \|g\|_{B^\infty_{s'-s}}\|f(z)(1-|z|^2)^{-s}\|_{T^{p,q}} \\ &\approx \|g\|_{B^\infty_{s'-s}}\|f\|_{F^{p,q}_s},\end{aligned} \tag{2.4}$$

which gives $\|g\|_{\mathrm{Mult}(F^{p,q}_s\to F^{p,q}_{s'})} \lesssim \|g\|_{B^\infty_{s'-s}}$.

Assertion (iii) is a consequence of the fact that the point evaluation is bounded on $F^{p,q}_s$. Indeed, let $g \in \mathrm{Mult}(F^{p,q}_s)$. For any $f \in F^{p,q}_s$ and $z \in \mathbf{B}$, then

$$|(gf)(z)| \le \sup_{\substack{h\in F^{p,q}_s\\ h\neq 0}} \frac{|h(z)|}{\|h\|_{F^{p,q}_s}}\|gf\|_{F^{p,q}_s} \le \sup_{\substack{h\in F^{p,q}_s\\ h\neq 0}} \frac{|h(z)|}{\|h\|_{F^{p,q}_s}}\|g\|_{\mathrm{Mult}(F^{p,q}_s)}\|f\|_{F^{p,q}_s},$$

which gives $\|g\|_\infty \le \|g\|_{\mathrm{Mult}(F^{p,q}_s)}$. The same arguments used to prove (2.4) show that if $s < 0$, then the converse inequality holds. Thus (iii) is proved.

In order to prove (iv), note that assertions (ii) and (iii) show that $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'}) \subset \mathrm{Mult}(F^{p,q}_{-s_0} \to F^{p,q}_{s'-s-s_0})$ for any $s_0 > 0$. Therefore (iv) follows from the above mentioned interpolation result in Proposition 2.1.

Now we prove (v). Note that if $s' \le n/p$, (2.3) gives $|g(z)| \lesssim (1-|z|^2)^{s'-s}$, and by the maximum modulus principe $g = 0$. The case $s' > n/p$ can be reduced to the above case. Indeed, by part (ii), Proposition 2.1(i) and part (ii), we have

$$\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'}) \subset F^{p,q}_{s'} \subset B^\infty_{s'-n/p} \subset H^\infty = \mathrm{Mult}(F^{p,q}_{-1}).$$

Hence, by Proposition 2.1(vi), for $0 < \theta < 1$ such that $(1-\theta)s' - \theta < n/p$, we obtain $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'}) \subset \mathrm{Mult}(F^{p,q}_{(1-\theta)s-\theta} \to F^{p,q}_{(1-\theta)s'-\theta}) = 0$. □

We conclude this section proving Theorem 1.1 for the particular case where τ is a positive integer and that will be used as a tool to prove Theorem 1.1.

Proposition 2.4. *Let $0 < p, q < \infty$. We then have:*

- (i) *Let $s' \le s$. If $g \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'})$, then for any positive integer k, $(I+R)^k g \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'-k})$.*
- (ii) *Let $s' < s$. If $(I+R)^k g \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'-k})$ for some positive integer k, then $g \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'})$.*
- (iii) *$g \in \mathrm{Mult}(F^{p,q}_s)$ if and only if $g \in H^\infty$ and $(I+R)^k g \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s-k})$ for some (any) positive integer k.*

Proof. Throughout the proof we use the previous mentioned fact that $f \in F^{p,q}_s$ if and only if $(I+R)^m f \in F^{p,q}_{s-m}$. By iteration, it is enough to prove the assertions for the case $k=1$. Let $f \in F^{p,q}_s$. Since $g \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'}) \subset \mathrm{Mult}(F^{p,q}_{s-1} \to F^{p,q}_{s'-1})$ (see (iv) in Proposition 2.3), we have $f(I+R)g = (I+R)(gf) - gRf \in F^{p,q}_{s'-1}$. This proves (i).

Let us prove (ii) and (iii). Assume that $(I+R)g \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'-1})$. We want to prove that if $s' < s$ then $g \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'})$, and that if $s'=s$ and $g \in H^\infty$, then $g \in \mathrm{Mult}(F^{p,q}_s)$. In order to prove these results, it is sufficient to show that $(I+R)^m(gf) \in F^{p,q}_{s'-m}$ for a positive integer $m > s$. By Leibnitz's formula we have $(I+R)^m(gf) = \sum_{j=0}^m c_j (I+R)^j g R^{m-j} f$. If $s' \le s$ and $j \ge 1$, then assertion (i) and Proposition 2.3 give

$$(I+R)^j g \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'-j}) \subset \mathrm{Mult}(F^{p,q}_{s+j-m} \to F^{p,q}_{s'-m})$$

and thus $(I+R)^j g R^{m-j} f \in F^{p,q}_{s'-m}$. If $s' \le s$ and $j=0$, then $(I+R)g \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'-1}) \subset B^\infty_{s'-1-s}$. Therefore, $g \in B^\infty_{s'-s} = \mathrm{Mult}(F^{p,q}_{s-m} \to F^{p,q}_{s'-m})$, and $gR^m f \in F^{p,q}_{s'-m}$.

Finally, $s'=s$ and $j=0$, then $g \in H^\infty = \mathrm{Mult}(F^{p,q}_{s-m})$ and $gR^m f \in F^{p,q}_{s-m}$. The necessary condition in (iii) is a consequence of (iii) in Proposition 2.3 and the above part (i). □

Remark 2.5. The results in the above proposition have been stated in terms of the bijective operator $(I+R)^k$, but the same proof shows that they are valid for operators $(\lambda_1 I + R)\cdots(\lambda_k I + R)$, $\lambda_1, \cdots, \lambda_k \in \mathbf{C}$, and in particular for the operators R^k_{n+N}. Since $\mathcal{P}^{N,N+k}(g) = R^k_{n+N} g$, this gives a proof of Theorem 1.6 for integer values of τ.

3. Proof of main results

3.1. Proof of Theorem 1.1. In this section we will begin with the following proposition, which will be used in the proofs of both Theorems 1.1 and 1.6. We postpone the proof of this result to the end of this section.

Definition 3.1. For $\phi \in L^1[0,1]$, let Φ be the operator on the space of holomorphic functions H defined by

$$\Phi(h)(z) = \int_0^1 \phi(t) h(tz)\, dt.$$

Proposition 3.2. *Let $\phi \in L^1[0,1]$ such that $|\phi(t)| \lesssim (1-t)^{\kappa-1}$ for some $\kappa > 0$ and $t_0 < t < 1$. For $0 < p, q < \infty$ and $s' < s$, we have:*

(i) *The operator Φ maps $F^{p,q}_s$ to $F^{p,q}_{s+\kappa}$.*
(ii) *If $s' + \kappa < s$, then Φ maps $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'})$ to $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'+\kappa})$.*

Proof of Theorem 1.1. Using the integral expression of $(I+R)^{-\kappa}$ for $\kappa > 0$ (see for instance [3])

$$(I+R)^{-\kappa} f(z) = \frac{1}{\Gamma(\kappa)} \int_0^1 \left(\log \frac{1}{t}\right)^{\kappa-1} f(tz)\, dt,$$

for a positive integer $l > \tau$ we have

$$(I+R)^{\tau} g(z) = \frac{1}{\Gamma(l-\tau)} \int_0^1 \left(\log \frac{1}{t}\right)^{l-\tau-1} (I+R)^l g(tz)\, dt.$$

Let us prove the first assertion (i) in Theorem 1.1. If $\tau > s' - s$ and $g \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'})$, Proposition 2.4(i) gives $(I+R)^l g \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'-l})$. By

Proposition 3.2(ii) with $\phi(t) = \frac{1}{\Gamma(l-\tau)}\left(\log\frac{1}{t}\right)^{l-\tau-1}$ and $\kappa = l-\tau < l+s-s'$, we obtain $(I+R)^\tau g \in \mathrm{Mult}(F_s^{p,q} \to F_{s'-\tau}^{p,q})$.

Conversely, if $(I+R)^{\tau_0} g \in \mathrm{Mult}(F_s^{p,q} \to F_{s'-\tau_0}^{p,q})$ for some $\tau_0 > s'-s$, the same argument shows that $(I+R)^\tau g = (I+R)^{\tau-\tau_0}(I+R)^{\tau_0} g \in \mathrm{Mult}(F_s^{p,q} \to F_{s'-\tau}^{p,q})$ for any $\tau > s'-s$, and in particular for any non-negative integer $k > s'-s$. This together with Proposition 2.4 finishes the proof of (i).

Let us prove assertion (ii). By Proposition 2.4(iii), $g \in \mathrm{Mult}(F_s^{p,q})$ if and only if $g \in H^\infty$ and $(I+R)^k g \in \mathrm{Mult}(F_s^{p,q} \to F_{s-k}^{p,q})$ for some positive integer k. By part (i), $(I+R)^k g \in \mathrm{Mult}(F_s^{p,q} \to F_{s-k}^{p,q})$ if and only if $(I+R)^\tau g \in \mathrm{Mult}(F_s^{p,q} \to F_{s-\tau}^{p,q})$ for any $\tau > 0$, which concludes the proof.

Assertion (iii) was proved in Proposition 2.3(v). □

Remark 3.3. If $s' < s$, Theorem 1.1 shows that if $\tau > s'-s$, then $(I+R)^\tau$ is a bijective operator from $\mathrm{Mult}(F_s^{p,q} \to F_{s'}^{p,q})$ to $\mathrm{Mult}(F_s^{p,q} \to F_{s'-\tau}^{p,q})$. If $\tau = s-s' > 0$, then $\mathrm{Mult}(F_s^{p,q}) = H^\infty \cap (I+R)^{-\tau}\,\mathrm{Mult}(F_s^{p,q} \to F_{s-\tau}^{p,q})$. In the case $\tau > s-s' > 0$, we have

$$(3.5) \qquad (I+R)^{-\tau}\,\mathrm{Mult}(F_s^{p,q} \to F_{s'}^{p,q}) \subset B_{s'-s+\tau}^\infty \cap \mathrm{Mult}(F_s^{p,q}).$$

Indeed, if $g \in \mathrm{Mult}(F_s^{p,q} \to F_{s'}^{p,q})$, then, by Proposition 2.3(ii), $g \in B_{s'-s}^\infty$ and therefore $(I+R)^{-\tau} g \in B_{s'-s+\tau}^\infty \subset H^\infty$ as $s'-s+\tau > 0$. Since $F_{s'}^{p,q} \subset F_{s-\tau}^{p,q}$, $g \in \mathrm{Mult}(F_s^{p,q} \to F_{s-\tau}^{p,q})$ and $(I+R)^{-\tau} g \in H^\infty \cap (I+R)^{-\tau}\,\mathrm{Mult}(F_s^{p,q} \to F_{s-\tau}^{p,q}) = \mathrm{Mult}(F_s^{p,q})$.

Observe that the embedding (3.5) in general is not exhaustive. For instance, if $\tau > -s' > 0$, $\mathrm{Mult}(H^p \to H_{s'}^p) = BMOA_{s'}$ (see Corollary 1.4), $B_{s'+\tau}^\infty \subset H^\infty = \mathrm{Mult}(H^p)$ and $(I+R)^{-\tau} BMOA_{s'} = BMOA_{s'+\tau} \subsetneq B_{s'+\tau}^\infty$.

Proof of Corollary 1.2. The equivalence between assertions (i) and (ii) is a consequence of Theorem 1.1 and the fact that $H_s^p = F_s^{p,2}$. From the definition of $H_{s'}^p = F_{s'}^{p,2}$ with $k = 0 > s'$, we observe that for $\tau > s$, $(I+R)^\tau g \in \mathrm{Mult}(H_s^p \to H_{s-\tau}^p)$ if and only $d\mu_g(z) = |(I+R)^\tau g(z)|^2 (1-|z|^2)^{\tau-s} d\nu(z)$ is a $T^{p,2}$-Carleson measure. This gives the equivalence between assertions (ii) with $\tau > s$ and (iii).

Analogously, note that $(I+R)^s g \in \mathrm{Mult}(H_s^p \to H^p)$ if and only $d\sigma_g = |(I+R)^s g|^2 d\sigma$ is a trace measure for H_s^p, which gives the equivalence between assertions (ii) with $\tau = s$ and (iv). □

3.2. Consequences of Theorem 1.1. We start this section with the proof of Theorem 1.5.

Proof of Theorem 1.5. (i) Assume $s' < s < 1$. Then, by property Ib, the operator I_b is bounded from $F_s^{p,q}$ to $F_{s'}^{p,q}$ if and only if $b \in \mathrm{Mult}(F_{s-1}^{p,q} \to F_{s'-1}^{p,q})$. By Proposition 2.3(ii) this is equivalent to $g \in B_{s'-s}^\infty$.

(ii) Assume $s' < s = 1$. Then, by Ib, I_b is bounded from H_1^p to $H_{s'}^p$ if and only if $b \in \mathrm{Mult}(H^p \to H_{s'-1}^p) = BMOA_{s'-1}$. The last equality is a consequence of Corollary 1.4.

(iii) Assume $s'-1 < s < 0$. Then, by Jb, the operator J_b is bounded from $F_s^{p,q}$ to $F_{s'}^{p,q}$ if and only if $Rb \in \mathrm{Mult}(F_s^{p,q} \to F_{s'-1}^{p,q}) = B_{s'-s-1}^\infty$, which is equivalent to $b \in B_{s'-s}^\infty$

(iv) Assume $s' < 1$. Then, by Jb the operator J_b is bounded from H^p to $H_{s'}^p$ if and only if $Rb \in \mathrm{Mult}(H^p \to H_{s'-1}^p) = BMOA_{s'-1}$, which is equivalent to $b \in BMOA_{s'}$.

This ends the proof. □

Corollary 1.4 also gives a characterization of $BMOA_s$ in terms of Carleson measures.

Corollary 3.4. *If $s \in \mathbf{R}$, then the following assertions are equivalent:*

(i) $g \in BMOA_s$

(ii) *For some (any) $\tau > s$, the measure $|(I+R)^\tau g(z)|^2(1-|z|^2)^{2(\tau-s)-1}d\nu(z)$ is a Carleson measure.*

Proof. For $s < 0$, $g \in BMOA_s = \mathrm{Mult}(H^2 \to H^2_s)$ if and only if the measure $|g(z)|^2(1-|z|^2)^{-2s-1}d\nu(z)$ is a Carleson measure for H^2 and thus for any H^p. Therefore, using that $g \in BMOA_s$ if and only if $(I+R)^\tau g \in BMOA_{s-\tau}$, we finish the proof. □

As a consequence of Theorem 1.1, we obtain the following characterization of the multipliers from $F^{p,q}_s$ to $F^{p,q}_{s'}$ that generalizes Proposition 2.3(i).

Corollary 3.5. *Let $0 < p, q < \infty$ and $s' \le s$. If either $p > 1$ and $s > n/p$ or $0 < p \le 1$ and $s \ge n/p$, then $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'}) = F^{p,q}_{s'}$.*

Proof. By Proposition 2.3(i), we have $\mathrm{Mult}(F^{p,q}_s) = F^{p,q}_s$. Thus, if $\tau = s - s' > 0$, Theorem 1.1 gives

$$F^{p,q}_{s'} = (I+R)^{s-s'}F^{p,q}_s = (I+R)^{s-s'}\,\mathrm{Mult}(F^{p,q}_s) \subset \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'}) \subset F^{p,q}_{s'},$$

which concludes the proof. □

3.3. Preliminaries for the proof of Theorem 1.6. We start recalling some properties of the operators $\mathcal{P}^{N,N+\tau}$. For any positive integer k and any $f \in B^1_{-N} \cup H^1$, (1.2) states that $\mathcal{P}^{N,N+k}(f) = R^k_{n+N}f$. Consequently, $\mathcal{P}^{N,N+k}$ is the restriccion to $B^1_{-N} \cup H^1$ of a bijective operator from $F^{p,q}_s$ to $F^{p,q}_{s-k}$ for any $0 < p, q < \infty$ and $s \in \mathbf{R}$.

Its inverse is given by $\mathcal{P}^{N+k,N} \colon B^1_{-N-k} \cup F^{1,2}_{-k} \to B^1_{-N} \cup H^1$. Indeed, by Fubini's Theorem, $\mathcal{P}^{N+k,N}(\mathcal{P}^{N,N+k}(f)) = \mathcal{P}^N(f\mathcal{P}^{N+k}(1)) = \mathcal{P}^N(f) = f$. In fact, the bijectivity of the operator $\mathcal{P}^{N,N+\tau} \colon B^1_{-N} \cup H^1 \to B^1_{-N-\tau} \cup F^{1,2}_{-\tau}$ holds for $\tau > -n-N$. As above, if $\tau \ge 0$, its inverse is $\mathcal{P}^{N+\tau,N}$. For the general case $\tau > -n-N$, the inverse can be given by $f \to \mathcal{P}^{N+k+\tau,N}(R^k_{n+N+\tau}f)$ for any non-negative integer $k > -\tau$. Indeed,

$$\mathcal{P}^{N+k+\tau,N}(R^k_{n+N+\tau}\mathcal{P}^{N,N+\tau}(f)) = \mathcal{P}^{N+k+\tau,N}(\mathcal{P}^{N,N+k+\tau}(f)) = f. \tag{3.6}$$

The following lemma shows that the operator $\mathcal{P}^{N,N+\tau}$ can be extended to the space of holomorphic functions as an operator of the type introduced in Definition 3.1.

Definition 3.6. For $N \ge 0$ and $0 < \lambda < n+N$, let $\Phi^{N,N-\lambda}$ be the operator on $H(\mathbf{B})$ defined by

$$\Phi^{N,N-\lambda}(f)(z) = \frac{1}{\beta(\lambda, n+N-\lambda)}\int_0^1 (1-t)^{\lambda-1}t^{n+N-\lambda-1}f(tz)\,dt.$$

Recall that the β function is defined by $\beta(N,M) = \frac{\Gamma(N)\Gamma(M)}{\Gamma(N+M)} = \int_0^1 (1-t)^{N-1}t^{M-1}dt$.

Lemma 3.7. *Let $N \ge 0$, $\lambda < n+N$ and $f \in H(\overline{\mathbf{B}})$.*

(i) *For any non-negative integer k,*

$$\mathcal{P}^{N,N-\lambda}(f) = \mathcal{P}^{N+k,N-\lambda}(R^k_{n+N}f) = R^k_{n+N}\mathcal{P}^{N+k,N-\lambda}(f).$$

(ii) *If $\lambda > 0$, then $\mathcal{P}^{N,N-\lambda}(f)(z) = \Phi^{N,N-\lambda}(f)(z)$.*

(iii) *For any positive integer* $k > -\lambda$,

$$\mathcal{P}^{N,N-\lambda}(f)(z) = \Phi^{N+k,N-\lambda}(R^k_{n+N} f)(z).$$

Proof. It is enough to prove the result for monomials $f(w) = w^\alpha$. Using the Taylor expansion of $(1-z\overline{w})^{-n-N+\lambda}$ and the orthogonality in $L^2(d\nu)$ of the monomials w^α, we obtain

$$\mathcal{P}^{N,N-\lambda}(w^\alpha)(z) = \frac{1}{n\beta(N,n)} \frac{\Gamma(n+N-\lambda+|\alpha|)}{\alpha!\Gamma(n+N-\lambda)} z^\alpha \int_{\mathbf{B}} |w^\alpha|^2 (1-|w|^2)^{N-1}\, d\nu(w).$$

Now, integration in polar coordinates and the fact that $\|\zeta^\alpha\|^2_{L^2(d\sigma)} = \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!}$ give

$$\int_{\mathbf{B}} |w^\alpha|^2 (1-|w|^2)^{N-1}\, d\nu(w) = \beta(N, n+|\alpha|) \frac{n!\alpha!}{(n-1+|\alpha|)!} = \frac{n!\alpha!\Gamma(N)}{\Gamma(n+N+|\alpha|)}.$$

Combining these results we obtain

$$\mathcal{P}^{N,N-\lambda}(w^\alpha)(z) = \frac{\Gamma(n+N)\Gamma(n+N-\lambda+|\alpha|)}{\Gamma(n+N-\lambda)\Gamma(n+N+|\alpha|)} z^\alpha =: A_{N,N-\lambda} z^\alpha.$$

Analogous arguments show that the above formula remains valid for $N = 0$.

Assertion (i) is a consequence of $R^m_{n+N} z^\alpha = \frac{\Gamma(n+N)\Gamma(n+N+k+|\alpha|)}{\Gamma(n+N+k)\Gamma(n+N+|\alpha|)} z^\alpha$, which gives the equality $A_{N,N-\lambda} z^\alpha = A_{N+m,N-\lambda} R^m_{n+N} z^\alpha$. If $\lambda > 0$, then $A_{N,N-\lambda} = \frac{\beta(\lambda, n+N-\lambda+|\alpha|)}{\beta(\lambda, n+N-\lambda)}$. Hence

$$A_{N,N-\lambda} z^\alpha = \frac{1}{\beta(\lambda, n+N-\lambda)} \int_0^1 (1-t)^{\lambda-1} t^{n+N-\lambda-1} (tz)^\alpha\, dt,$$

which proves part (ii). Assertion (iii) follows from (i) and (ii). □

Remark 3.8. Observe that, combining (3.6), Lemma 3.7 and Proposition 3.2, we obtain that if $F^{p,q}_s \subset B^1_{-N} \cup H^1$, then $\mathcal{P}^{N,N+\tau}$ is a bijective operator from $F^{p,q}_s$ to $F^{p,q}_{s-\tau}$ for any $\tau > -n-N$. In particular, if $1 < p < \infty$ and $0 < s < n$, $\mathcal{C}_s$ is a bijective operator from H^p to H^p_s.

3.4. Proof of Theorem 1.6. The proof is similar to the one of Theorem 1.1. By Lemma 3.7(iii), $\mathcal{P}^{N,N+\tau}(g) = \Phi^{N+k,N+\tau}(R^k_{n+N} g)$ for any non-negative integer $k > \tau$. Since R^k_{n+N} is a bounded operator from $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'})$ to $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'-k})$ (see Remark 2.5), Proposition 3.2(ii) gives that $\mathcal{P}^{N,N+\tau}$ maps $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'})$ to $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'-\tau})$.

Conversely, for any non-negative integer $k > -\tau$, the inverse of $\mathcal{P}^{N,N+\tau}$ is

$$\mathcal{P}^{N+\tau+k,N} \circ R^k_{n+N+\tau} = \Phi^{N+\tau+k,N} \circ R^k_{n+N+\tau}$$

(see (3.6)). Hence, the same argument used above shows that this inverse maps $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'-\tau})$ to $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'})$. Thus, the operator $\mathcal{P}^{N,N+\tau}$ is bijective from $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'})$ to $\mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'-\tau})$. This concludes the proof of part (i).

Let us prove assertion (ii). By Remark 2.5, $g \in \mathrm{Mult}(F^{p,q}_s)$ if and only if $g \in H^\infty$ and $R^k_{n+N} g = \mathcal{P}^{N,N+k}(g) \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s-k})$ for some non-negative integer k. By part (i), $\mathcal{P}^{N,N+k}(g) \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s-k})$ if and only if $\mathcal{P}^{N,N+\tau}(g) \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s-\tau})$ for any $\tau > 0$, which concludes the proof. □

3.5. Proof of Proposition 3.2. In order to prove Proposition 3.2 we will need the following Taylor formulas.

Lemma 3.9. *Let $0 < t < 1$ and $z, w \in \mathbf{B}$. For any positive integers m and l,*

$$\frac{1}{(1-z\overline{w})^{n+l}} = \sum_{j=0}^{m} \frac{(1-t)^j}{j!} \frac{d^j}{dt^j} \frac{1}{(1-tz\overline{w})^{n+l}} + R_1^{n+l-1} \frac{(1-t)^{m+1}(z\overline{w})^{m+1}}{(1-z\overline{w})(1-tz\overline{w})^{m+1}}.$$

Proof. If $|\lambda| < 1$, for any $0 < t < 1$ we have

$$\frac{1}{1-\lambda} = \sum_{j=0}^{m} \frac{(1-t)^j}{j!} \frac{d^j}{dt^j} \frac{1}{1-t\lambda} + \frac{(1-t)^{m+1}\lambda^{m+1}}{(1-\lambda)(1-t\lambda)^{m+1}}.$$

Choosing $\lambda = z\overline{w}$ and applying the operator R_1^{n+l-1} (with respect to the variable z) to the terms in the above identity we obtain the result. □

Lemma 3.10. *Let m be a positive integer and assume that $f \in B^1_{-N}$ for some $N > 0$. Then, for $0 \le t < 1$,*

$$f(z) = \sum_{j=0}^{m} \frac{(1-t)^j}{j!} \sum_{|\alpha|=j} \binom{j}{\alpha} z^\alpha \partial^\alpha f(tz) + E_m(f)(z,t),$$

where $\partial^\alpha = \frac{\partial^j f}{\partial w^\alpha}$ and the function $E_m(f)(z,t)$ satisfies

$$|E_m(f)(z,t)| \lesssim (1-t)^{m+1} \int_{\mathbf{B}} \frac{|R^k_{n+N} f(w)|(1-|w|^2)^{k+N-1}}{|1-z\overline{w}|^{n+N}|1-tz\overline{w}|^{m+1}}\, d\nu(w),$$

for any non-negative integer k.

Proof. Since $1-|w|^2 \le 2|1-z\overline{w}|$, it is enough to prove the result for integer values of N. Using the representation formula $f = \mathcal{P}^N(f)$ and Lemma 3.9 we obtain

$$f(z) = \sum_{j=0}^{m} \frac{(1-t)^j}{j!} \frac{d^j f(tz)}{d\,t^j} + \int_{\mathbf{B}} f(w) R_1^{n+N-1} \frac{(1-t)^{m+1}(z\overline{w})^{m+1}}{(1-z\overline{w})(1-tz\overline{w})^{m+1}}\, d\nu_N(w).$$

Using the integration by parts formula

$$\int_{\mathbf{B}} \varphi\, d\nu_N = \int_{\mathbf{B}} R^k_{n+N}\, \varphi\, d\nu_{N+k}, \quad \varphi \in \mathcal{C}^k(\overline{\mathbf{B}}),$$

which follows easily from $\displaystyle\sum_{j=1}^{N} \int_{\mathbf{B}} \frac{\partial}{\partial w_j} \left(w_j \varphi(w)(1-|w|^2)^N\right) d\nu(w) = 0$, we obtain

$$\begin{aligned} f(z) = &\sum_{j=0}^{m} \frac{(1-t)^j}{j!} \sum_{|\alpha|=j} \binom{j}{\alpha} z^\alpha \frac{\partial^j f}{\partial w^\alpha}(tz) \\ &+ c \int_{\mathbf{B}} R^k_{n+N} f(w)(1-|w|^2)^{N+k-1} R_1^{n+N-1} \frac{(1-t)^{m+1}(z\overline{w})^{m+1}}{(1-z\overline{w})(1-tz\overline{w})^{m+1}}\, d\nu(w). \end{aligned}$$

This formula together with the fact that $|1-z\overline{w}| \lesssim |1-tz\overline{w}|$ prove the result. □

Proof of Proposition 3.2. In order to prove (i), we will show that if $f \in F^{p,q}_s$ and a non-negative integer $l > s+\kappa$, then

$$(I+R)^l \Phi(f)(z) = \int_0^1 \phi(t)(I+R)^l f(tz)\, dt \in F^{p,q}_{s+\kappa-l}.$$

If $h := (I+R)^l f$, this is equivalent to prove that $\|(1-|z|^2)^{l-s-\kappa}\Phi(h)(z)\|_{T^{p,q}} \lesssim \|(1-|z|^2)^{l-s}\, h(z)\|_{T^{p,q}}$. Since for $0 < t_0 < 1$, $\Phi_0(h)(z) := \int_0^{t_0} \phi(t)h(tz)\, dt$ is holomorphic on a neighborhood of $\overline{\mathbf{B}}$, for any k, $(I+R)^k f$ is also holomorphic on a neighborhood of $\overline{\mathbf{B}}$ and, consequently, $\Phi_0(h)$ is in $F^{p,q}_{s'+\kappa}$. Hence, it is enough to prove the result for the function $\Phi_1(h)(z) := \int_{t_0}^1 \phi(t)h(tz)\,dt$.

Let N be large enough so that $F^{p,q}_{s-l} \subset B^1_{-N}$. Then,

$$|h(tz)| = \left|\mathcal{P}^N(h)(tz)\right| \lesssim \int_{\mathbf{B}} \frac{|h(w)|}{|1-tz\overline{w}|^{n+N}}\,d\nu_N(w).$$

Since $|1-tz\overline{w}| \approx 1-t+|1-z\overline{w}|$, Fubini's theorem gives

$$|\Phi_1(h)(z)| \lesssim \int_{t_0}^1 (1-t)^{\kappa-1}|h(tz)|\,dt \lesssim \int_{\mathbf{B}} \frac{|h(w)|}{|1-z\overline{w}|^{n+N-\kappa}}\,d\nu_N(w),$$

provided $n+N-\kappa>0$. Combining these results, we obtain that

$$(1-|z|^2)^{l-s-\kappa}\int_{t_0}^1 (1-t)^{\kappa-1}|h(tz)|\,dt \lesssim \mathcal{B}^{N-l+s,l-s-\kappa}\left((1-|w|^2)^{l-s}h(w)\right)(z),$$

where $\mathcal{B}^{N,M}$ denotes the Berezin type integral operator with kernel

$$\mathcal{B}^{N,M}(z,w) := \frac{(1-|w|^2)^{N-1}(1-|z|^2)^M}{|1-z\overline{w}|^{n+N+M}}\,d\nu(w).$$

By Proposition 2.8 in [23], if $1<p,q<\infty$ and $N,M>0$, the operator $\mathcal{B}^{N,M}$ is bounded on $T^{p,q}$. Thus, in this case we have

$$\left\|(1-|z|^2)^{l-s-\kappa}\int_{t_0}^1 (1-t)^{\kappa-1}|h(tz)|\,dt\right\|_{T^{p,q}} \lesssim \|(1-|w|^2)^{l-s}h(w)\|_{T^{p,q}}.$$

The proof of the cases $0<p\le 1$ or $0<q\le 1$ can be reduced to the above case, choosing $0<\theta<\min\{p,q\}$ and N large enough, and using the facts that $\|\varphi\|_{T^{p,q}} = \|\varphi^\theta\|^{1/\theta}_{T^{p/\theta,q/\theta}}$ and

$$\mathcal{B}^{N,M}((1-|w|^2)^{l-s}|h(w)|)^\theta \lesssim \mathcal{B}^{(n+N)\theta-n,(n+M)\theta-n}\left((1-|w|^2)^{(l-s)\theta}|h(w)|^\theta\right)$$

(see Corollary 5.3 in [7]).

Now we prove (ii). Let $g \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'})$ with $s'<s$, and $s'+\kappa<s$. As above $\Phi_0(g) \in H(\overline{\mathbf{B}}) \subset \mathrm{Mult}(F^{p,q}_t)$ for any $t\in\mathbf{R}$. Thus, it is enough to show that $\Phi_1(g) \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'+\kappa})$. Since $s'+\kappa<s$, by Proposition 2.4, this is equivalent to prove that for a positive integer $l>s>s'+\kappa$

$$(I+R)^l\Phi_1(g)(z) = \int_{t_0}^1 \phi(t)(I+R)^l g(tz)\,dt = \Phi_1((I+R)^l g)(z) \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'+\kappa-l}),$$

that is, we want to prove that $f\Phi_1((I+R)^l g) \in F^{p,q}_{s'+\kappa-l}$ for any $f\in F^{p,q}_s$. By Taylor expansion in Lemma 3.10, this result will be true if for $f\in F^{p,q}_s$ the functions

$$F_\alpha(z) := \int_{t_0}^1 \phi(t)(1-t)^j\partial^\alpha f(tz)(I+R)^l g(tz)\,dt, \quad |\alpha|=j\le m, \quad \text{and}$$

$$E(z) := \int_{t_0}^1 \phi(t)E_m(f)(z,t)(I+R)^l g(tz)\,dt$$

are in $F^{p,q}_{s'+\kappa-l}$ for some m.

By Propositions 2.4(i) and 2.3(iv) we have

$$(I+R)^l g \in \mathrm{Mult}(F^{p,q}_s \to F^{p,q}_{s'-l}) \subset \mathrm{Mult}(F^{p,q}_{s-j} \to F^{p,q}_{s'-l-j}).$$

As $f\in F^{p,q}_s$ then $\partial^\alpha f \in F^{p,q}_{s-j}$ which gives $(\partial^\alpha f)\cdot(I+R)^l g \in F^{p,q}_{s'-j-l}$. Hence, part (i) (with $\tilde{\phi}(t) = (1-t)^j\phi(t)$ and $\tilde{\kappa}=\kappa+j$) shows that $F_\alpha \in F^{p,q}_{s'+\kappa-l}$.

In order to prove that $E \in F^{p,q}_{s'+\kappa-l}$, we will show that for $k > s$,

$$\|(1-|z|^2)^{l-s'-\kappa}E(z)\|_{T^{p,q}} \lesssim \|(1-|z|^2)^{k-s}(I+R)^k f(z)\|_{T^{p,q}}.$$

Since $s' < s$, $(I+R)^l g \in B^\infty_{s'-s-l}$ and hence $g \in B^\infty_{s'-s}$. Thus, for $m \geq l+s-s'-\kappa$, from

$$1-t|z| \approx 1-t+1-|z| \lesssim |1-tz\overline{w}| \approx 1-t+|1-z\overline{w}|,$$

we obtain

$$(1-t)^{m+\kappa}|(I+R)^l g(tz)| \lesssim \|g\|_{B^\infty_{s'-s}} \frac{(1-t)^{m+\kappa}}{(1-t|z|)^{l+s-s'}} \leq \|g\|_{B^\infty_{s'-s}} |1-tz\overline{w}|^{m+\kappa+s'-s-l}.$$

This estimate together Lemma 3.10 with $N > s$ give

$$\begin{aligned}
&(1-|z|^2)^{l-s'-\kappa}E(z) \lesssim (1-|z|^2)^{l-s'-\kappa}\int_{t_0}^1 (1-t)^{\kappa-1}|E_m(f)(z,t)(I+R)^l g(tz)|\,dt \\
&\lesssim \|g\|_{B^\infty_{s'-s}} \int_{t_0}^1 \int_{\mathbf{B}} \frac{|R^k_{n+N}f(w)|(1-|w|^2)^{k+N-1}(1-|z|^2)^{l-s'-\kappa}}{|1-z\overline{w}|^{n+N}(1-t+|1-z\overline{w}|)^{l+s-s'-\kappa+1}}\,d\nu(w)\,dt \\
&\lesssim \|g\|_{B^\infty_{s'-s}} \int_{\mathbf{B}} \frac{|R^k_{n+N}f(w)|(1-|w|^2)^{k+N-1}(1-|z|^2)^{l-s'-\kappa}}{|1-z\overline{w}|^{n+N+l+s-s'-\kappa}}\,d\nu(w) \\
&= \|g\|_{B^\infty_{s'-s}} \mathcal{B}^{N+s,l-s'-\kappa}\left((1-|w|^2)^{k-s}|(I+R)^k f(w)|\right).
\end{aligned}$$

Now, as we stated before, the integral operator $\mathcal{B}^{N+s,l-s'-\kappa}$ is bounded on $T^{p,q}$ and $\|g\|_{B^\infty_{s'-s}}\|(1-|w|^2)^{k-s}|(I+R)^k f(w)|\|_{T^{p,q}} \approx \|g\|_{B^\infty_{s'-s}}\|f\|_{F^{p,q}_s}$, which concludes the proof. □

3.6. The space X^p_s. In this section we prove the last results about the space X^p_s stated in the Introduction.

Proof of Theorem 1.7. For $0 < s < n$, the operator $\mathcal{C}_s = \mathcal{P}^{0,-s}$ is a bijective operator from H^p to H^p_s, whose inverse is $\mathcal{P}^{n-s,0} \circ R^n_{n-s}$ (see (3.6)). By Proposition 2.4 (see also Remark 2.5), $g \in \mathrm{Mult}(H^p_s)$ if and only if $g \in H^\infty$ and $R^n_{n-s}g \in \mathrm{Mult}(H^p_s \to H^p_{s-n})$ and, by Theorem 1.6(i), this is equivalent to $g \in H^\infty$ and $\mathcal{P}^{n-s,0}(R^n_{n-s}g) \in \mathrm{Mult}(H^p_s \to H^p)$, or equivalently to $g \in H^\infty$ and the measure $|\mathcal{P}^{n-s,0}(R^n_{n-s}g)|^p d\sigma$ is a trace measure for H^p_s.

Thus, if $h \in H^p$, we have $\mathcal{C}_s(h) \in \mathrm{Mult}(H^p_s)$ if and only if $\mathcal{C}_s(h) \in H^\infty$ and the measure $|h|^p d\sigma$ is a trace measure for H^p_s. Since $\mathcal{C}_s(\varphi) = \mathcal{C}_s(\mathcal{C}(\varphi))$ for any $\varphi \in L^p(d\sigma)$, the above result applied to $h = \mathcal{C}(\varphi)$ concludes the proof. □

Proof of Theorem 1.8. Recall that a nonnegative weight w on $\mathbf{S}$ is in the Muckenhoupt class A_p if

$$[w]_p := \sup_B \left(\frac{1}{\sigma(B)}\int_B w\,d\sigma\right)\left(\frac{1}{\sigma(B)}\int_B w^{-1/(p-1)}\,d\sigma\right)^{p-1} < \infty,$$

where $B = B(\zeta, r) = \{\eta \in \mathbf{S}\,;\, |1-\zeta\overline{\eta}| < r\}$, $\zeta \in \mathbf{S}$ and $r > 0$, denotes a non-isotropic ball on $\mathbf{S}$.

In the proof of this theorem we will use Lemma 3.1 in [22] which states the following: Assume that $|\psi|^p\,d\sigma$ is a trace measure for the space $K_s(L^p)$. If $g \in L^p(\sigma)$ satisfies that for any w in the Muckenhoupt class A_p,

$$\int_{\mathbf{S}} |g|^p w\,d\sigma \leq C(n,p,[w]_p)\int_{\mathbf{S}} |\psi|^p w\,d\sigma,$$

then the measure $|g|^p d\sigma$ is also a trace measure for $K_s(L^p)$.

Since the Cauchy transform is a bounded operator on $L^p(w)$, we have that for any $w \in A_p$,

$$\int_{\mathbf{S}} |\mathcal{C}(\psi)|^p w d\sigma \le C(n,p,[w]_p) \int_{\mathbf{S}} |\psi|^p w d\sigma.$$

(See, for instance, [28, Corollary of Theorem 2, p. 205]). Hence, the above mentioned lemma gives that $d\mu = |\mathcal{C}(\psi)|^p d\sigma$ is also a trace measure for $K_s(L^p)$.

Moreover, for any $f \in H^p_s$ there exists $\varphi \in L^p$, satisfying that $f = \mathcal{C}_s(\varphi)$ and

$$\|K_s(\varphi)\|_{K_s(L^p)} = \|\varphi\|_p \lesssim \|f\|_{H^p_s}.$$

But $|f| \le K_s(|\varphi|)$, and consequently $d\mu$ is also a trace measure for H^p_s.

Finally $\mathcal{C}_s(\psi) = \mathcal{C}_s(\mathcal{C}(\psi))$ and we conclude the proof. □

In [15] there are obtained more detailed examples of pointwise multipliers for H^p_s.

Proof of Theorem 1.9. Let us prove that $H^{n/s}_s \cap H^\infty \subset \mathrm{Mult}(H^p_s)$. By Corollary 1.2, $g \in \mathrm{Mult}(H^p_s)$ if and only if $g \in H^\infty$ and $(I+R)^s g \in \mathrm{Mult}(H^p_s \to H^p)$. Thus, it is enough to prove that $(I+R)^s H^{n/s}_s = H^{n/s} \subset \mathrm{Mult}(H^p_s \to H^p)$.

If $0 < p < \infty$, $0 < s < n/p$, $h \in H^{n/s}$ and $f \in H^p_s$, then the embeddding $H^p_s \subset H^q$, $s - n/p = -n/q$ (see Section 5 in [10] or Proposition 2.1) and Hölder's inequality with exponent q/p give

$$\|hf\|_{H^p} \le \|h\|_{H^{n/s}} \|f\|_{H^q} \lesssim \|h\|_{H^{n/s}} \|f\|_{H^p_s},$$

which proves the result.

If $p > 1$, the operator $\mathcal{C}_s$ is bounded from $L^{n/s}$ to $H^{n/s}_s$. Hence, if $\varphi \in L^{n/s}$ and $\mathcal{C}_s(\varphi) \in H^\infty$, then $\mathcal{C}_s(\varphi) \in H^\infty \cap H^{n/s}_s$, which ends the proof. □

We remark that in [15] there are obtained more detailed examples of pointwise multipliers for H^p_s.

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