Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 41, 2016, 693–697

NOTES ON ADMISSIBLE MEASURES IN ONE DIMENSION

Shiliang Zhao

Fudan University, Department of Mathematics
220 Handan Road, Shanghai 200433; P. R. China; zhaoshiliang2013@gmail.com

Abstract. μ is called a p-admissible measure in one dimension if it is a doubling measure that supports a $(1,p)$-Poincaré inequality. In this note, we estimate the range of p that $(1,p)$-Poincaré inequality holds on $(\mathbf{R}, |\cdot|, \mu)$ where $|\cdot|$ is the Euclidean metric.

1. Introduction

Let μ be a measure on $\mathbf{R}^n$. We call a measure doubling if there is a constant c such that for every $B = B(x,r)$ centered at x with radius r, the following holds,

$$\mu(B(x,2r)) \le c\mu(B(x,r)). \tag{1.1}$$

As a result, there exist constants $C_\mu, \nu > 0$, that depend only on the constant c in above inequality such that

$$\left(\frac{r}{R}\right)^\nu \le C_\mu \frac{\mu(B(y,r))}{\mu(B(x,R))}, \tag{1.2}$$

whenever $0 < r < R < \infty$, $x \in \mathbf{R}^n$, and $y \in B(x,R)$.

We say that μ admits a $(1,p)$-Poincaré inequality if there exists $C > 0$ such that the following holds

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \le Cr \left(\frac{1}{\mu(B)} \int_B |\nabla u|^p \, d\mu\right)^{1/p} \tag{1.3}$$

for all ball B and locally Lipschitz function u on B. Here and in what follows, $u_B = (\mu(B))^{-1} \int_B u \, d\mu$.

A measure μ on $\mathbf{R}^n$ is called p-admissible with $p \ge 1$ if it satisfies (1.1) and (1.3). We denote by C_p the infimum of constants C such that (1.3) holds.

We recall that a nonnegative locally integrable function w on $\mathbf{R}^n$ is called a Muckenhoupt A_p-weight for $p \ge 1$ if for some $C > 0$ and every ball $B \subset \mathbf{R}^n$,

$$\frac{1}{|B|} \int_B w \, dx \le \begin{cases} C \left(\frac{1}{|B|} \int_B w^{\frac{1}{1-p}} \, dx\right)^{1-p} & \text{if } p > 1, \\ C \operatorname{ess\,inf}_B w & \text{if } p = 1, \end{cases} \tag{1.4}$$

where $|B|$ is the Lebesgue measure of B. We write $w \in A_p$ and denote by $M_{p,w}$ the infimum of C on the right side of (1.4).

---

doi:10.5186/aasfm.2016.4140
2010 Mathematics Subject Classification: Primary 26D10; Secondary 46E35.
Key words: p-admissible measures, Poincaré inequality.
This work is done during the author visiting Macquarie University. The author thanks Professor Duong for his generous help. The author also thanks the referee for his/her careful reading of the manuscript and for his/her constructive and detailed comments. The author is supported by the China Scholarship Council (Grant No. 201406100171).

It is well known that Muckenhoupt A_p weights have the open ended property ([2]). This result has numerous applications. And the range of p that $w \in A_p$ has been studied in [1], [6] ,[7] and [9]. On the other hand, according to [4], the Poincaré inequality also has the open ended property. See [3] for the self-improving property of Poincaré inequalities which also has lots of applications. In this note, we get an estimate of the range in a special case, i.e., the p-admissible measures in one dimension.

For every p-admissible measure $\mu = w\,dx$, set

$$\begin{aligned} I_{A_p} &= \{p \geq 1\colon w \in A_p\}, \\ I_{P_p} &= \{p \geq 1\colon (1,p)\text{-Poincaré inequality holds in } (\mathbf{R}, |\cdot|, \mu)\}. \end{aligned}$$

In [5], the following result has been proved.

Theorem 1.1. *Let μ be a measure on $\mathbf{R}$ and let $p \geq 1$. Then μ is p-admissible in $\mathbf{R}$ if and only if $d\mu = wdx$ and w is a Muckenhoupt A_p-weight.*

Therefore one has $I_{A_p} = I_{P_p}$. Thus we can estimate I_{P_p} through I_{A_p}. In [6], the estimate of I_{A_p} has been done for measures supported on finite intervals. Namely, Korenovskii proved the following. Let $w \in A_p$ supported on a finite interval, $p > 1$, $M_{p,w} > 1$ and $p_0 \in (1, p)$ be the root of equation

$$\frac{p-p_0}{p-1}(M_{p,w}p_0)^{1/(p-1)} = 1. \tag{1.5}$$

Then for all $q > p_0$, we have $w \in A_q$. Note that the range of q is sharp, i.e. the statement does not hold for $q \leq p_0$.

The main result of this note is as follows.

Theorem 1.2. *Assume that μ is a p-admissible measure in $\mathbf{R}$. Denote by p_0 the root of the following equation*

$$\frac{p-p_0}{p-1}(Mp_0)^{1/(p-1)} = 1,$$

where $M = (C_pC_\mu 4^\nu)^p\left(1 - \frac{2}{C_\mu 4^\nu}\right)$. Then $(1,q)$-Poincaré inequality holds for $q > p_0$.

The proof of Theorem 1.2 is based on Lemma 2.2, which gives a precise estimate of an inequality that plays an important role in [5].

2. Proof of Theorem 1.2

To begin with, we prove some properties of the root of (1.5).

Lemma 2.1. *For any $p, M_{p,w} > 1$, denote by p_0 the root of (1.5). Then p_0 is an increasing function of $M_{p,w}$.*

Proof. It suffices to show that the inverse function

$$x \mapsto M(x) = \frac{1}{x}\left(\frac{p-1}{p-x}\right)^{p-1}$$

is strictly increasing in $[1, p)$, which is easily verified by differentiation. □

To proceed, we need to estimate $M_{p,w}$ for $w \in A_p$ by ν, C_μ and C_p.

Lemma 2.2. *Let μ be a p-admissible measure on $\mathbf{R}$ for $p \geq 1$. Then for any finite interval $I = (a, b) \subset \mathbf{R}$ and nonnegative functions f on I we have*

$$\frac{1}{|I|} \int_I f(x)\, dx \leq C(p, I) \left(\frac{1}{\mu(I)} \int_I f^p \, d\mu \right)^{1/p},$$

where $C(p, I) = \frac{C_p}{2} \left(\frac{\mu(I)}{\mu(2I)} \right)^{1/p} \frac{\mu(2I)}{\mu(I_+)} \frac{\mu(I_+)+\mu(I_-)}{\mu(I_-)}$ and $I_+ = (b, \frac{3b-a}{2})$, $I_- = (\frac{3a-b}{2}, a)$ are the parts of $2I \backslash I$ lying to the right and to the left of I, respectively.

Proof. Let $f_k = \min\{f, k\}$ for $k \in \mathbf{N}$ and for simplicity we denote

$$u(x) = \int_{-\infty}^{x} f_k(t) \chi_I(t)\, dt.$$

Set $2I = ((3a-b)/2, (3b-a)/2)$. Since u is Lipschitz, we can apply the $(1, p)$-Poincaré inequality.

$$\begin{aligned} \frac{1}{\mu(2I)} \int_{2I} |u - u_{2I}|\, d\mu &\leq C_p |I| \left(\frac{1}{\mu(2I)} \int_{2I} |u'|^p \, d\mu \right)^{1/p} \\ &\leq C_p |I| \left(\frac{\mu(I)}{\mu(2I)} \right)^{1/p} \left(\frac{1}{\mu(I)} \int_I f^p \, d\mu \right)^{1/p}. \end{aligned}$$

Next we will estimate the left side of above inequality. Note first that

$$u(x) = \begin{cases} 0, & x < a, \\ \displaystyle\int_a^x f_k(t)\, dt, & a \leq x \leq b, \\ \displaystyle\int_a^b f_k(t)\, dt, & x > b. \end{cases}$$

Thus we have

$$\begin{aligned} \int_{2I} |u - u_{2I}|\, d\mu &= \int_{I_-} u_{2I}\, d\mu + \int_{I_+} (u(b) - u_{2I})\, d\mu + \int_I |u - u_{2I}|\, d\mu \\ &\geq (\mu(I_-) - \mu(I_+)) u_{2I} + \mu(I_+) u(b) + \mu(I) |u_I - u_{2I}|. \end{aligned}$$

By the definition, one has

$$u_{2I} = \frac{1}{\mu(2I)} \int_I u\, d\mu + \frac{\mu(I_+)}{\mu(2I)} u(b) = \frac{\mu(I)}{\mu(2I)} u_I + \frac{\mu(I_+)}{\mu(2I)} u(b). \tag{2.1}$$

Substituting (2.1) into the above inequality, we obtain

$$\begin{aligned} \int_{2I} |u - u_{2I}|\, d\mu \geq (\mu(I_-) - \mu(I_+)) \left(\frac{\mu(I)}{\mu(2I)} u_I + \frac{\mu(I_+)}{\mu(2I)} u(b) \right) + \mu(I_+) \mu(b) \\ + \mu(I) \left| \frac{\mu(I_+) + \mu(I_-)}{\mu 2I} u_I - \frac{\mu(I_+)}{\mu(2I)} u(b) \right|. \end{aligned}$$

If $u_I - u_{2I} \geq 0$, then $|u_I - u_{2I}| = u_I - u_{2I}$ and (2.1) gives

$$u_I \geq \frac{\mu(I_+)}{\mu(I_+) + \mu(I_-)} u(b).$$

Thus, after some elementary cancellations, one has

$$\int_{2I} |u - u_{2I}|\, d\mu \geq 2\mu(I_-)\frac{\mu(I)}{\mu(2I)}u_I + 2\mu(I_+)\frac{\mu(I_-)}{\mu(2I)}u(b)$$
$$\geq 2\mu(I_-)\frac{\mu(I_+)}{\mu(I_+) + \mu(I_-)}u(b).$$

Similarly, if $u_I - u_{2I} \leq 0$, then $|u_I - u_{2I}| = u_{2I} - u_I$ and

$$u_I \leq \frac{\mu(I_+)}{\mu(I_+) + \mu(I_-)}u(b)$$

and hence, after suitable simplifications,

$$\int_{2I} |u - u_{2I}|\, d\mu \geq 2\mu(I_+)\frac{\mu(I_-) + \mu(I)}{\mu(2I)}u(b) - 2\mu(I_+)\frac{\mu(I)}{\mu(2I)}u_I$$
$$\geq 2\mu(I_+)\frac{\mu(I_-)}{\mu(I_+) + \mu(I_-)}u(b).$$

Putting this into the Poincaré inequality results in

$$\frac{1}{|I|}\int_I f(x)\, dx \leq \frac{C_p}{2}\left(\frac{\mu(I)}{\mu(2I)}\right)^{1/p}\frac{\mu(2I)}{\mu(I_+)}\frac{\mu(I_+) + \mu(I_-)}{\mu(I_-)}\left(\frac{1}{\mu(I)}\int_I f^p d\mu\right)^{1/p}$$

and the monotone convergence theorem proves the lemma. □

According to the equivalence definition of the A_p weights (see p.200 of [8]), one gets

$$M_{p,w} \leq \frac{\mu(I)}{\mu(2I)}\left(\frac{C_p\mu(2I)}{2}\frac{\mu(I_+) + \mu(I_-)}{\mu(I_+)\mu(I_-)}\right)^p$$
$$\leq \left(1 - \frac{2\min\{\mu(I_+), \mu(I_-)\}}{\mu(2I)}\right)\left(\frac{C_p\mu(2I)}{\min\{\mu(I_+), \mu(I_-)\}}\right)^p$$
$$\leq \left(1 - \frac{2}{C_\mu 4^\nu}\right)(C_pC_\mu 4^\nu)^p = M.$$

Now we prove Theorem 1.2.

Proof. Theorem 1.2 follows directly from (1.5) and Lemmas 2.1 and 2.2. and that the new A_q constant is no larger than

$$M\left(\frac{q-1}{p-1}\frac{1}{Z(q)}\right)^{q-1}$$

where $Z(q) = 1 - \frac{p-q}{p-1}(Mq)^{1/(p-1)}$ (see p. 1200 of [6]). □

References

[1] Bojarski, B.: Remarks on the stability of reverse Hölder inequalities and quasi-conformal mappings. - Ann. Acad. Sci. Fenn. Math. 10, 1985, 291–296.

[2] Coifman, R., and C. Fefferman: Weighted norm inequalities for maximal functions and singular integrals. - Studia Math. 51, 1974, 241–250.

[3] Hajłasz, P., and P. Koskela: Sobolev met Poincaré. - Mem. Amer. Math. Soc. 688, Amer. Math. Soc., 2000.

[4] Keith, S., and X. Zhong: The Poincaré inequality is an open ended condition. - Ann. of Math. (2) 2, 2008, 575–599.

[5] Björn, J., S. Buckley, and S. Keith: Admissible measures in one dimension. - Proc. Amer. Math. Soc. 134, 2006, 703–705.

[6] Korenovskii, A. A.: The exact continuation of a reverse Hölder inequality and Muckenhoupt's conditions. - Math. Notes 52, 1992, 1192–1201.

[7] Malaksiano, N. A.: Exact inclusions of Gehring classes in Muckenhoupt classes. - Math. Notes 70, 2001, 673–681.

[8] Stein, E. M.: Harmonic analysis: Real variable methods, orthogonality and oscillatory integrals. - Princeton Univ. Press, 1993.

[9] Wik, I.: Reverse Hölder inequalities with constant close to 1. - Ric. Math. 39, 1990, 151–157.

Received 15 October 2015 • Accepted 18 January 2016