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MOLECULAR CHARACTERIZATIONS AND DUALITIES OF VARIABLE EXPONENT HARDY SPACES ASSOCIATED WITH OPERATORS

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Abstract. Let L be a linear operator on $L^2(\mathbf{R}^n)$ generating an analytic semigroup $\{e^{-tL}\}_{t\geq 0}$ with kernels having pointwise upper bounds and $p(\cdot)\colon \mathbf{R}^n \to (0,1]$ be a variable exponent function satisfying the globally log-Hölder continuous condition. In this article, the authors introduce the variable exponent Hardy space associated with the operator L, denoted by $H_L^{p(\cdot)}(\mathbf{R}^n)$, and the BMO-type space $\mathrm{BMO}_{p(\cdot),L}(\mathbf{R}^n)$. By means of tent spaces with variable exponents, the authors then establish the molecular characterization of $H_L^{p(\cdot)}(\mathbf{R}^n)$ and a duality theorem between such a Hardy space and a BMO-type space. As applications, the authors study the boundedness of the fractional integral on these Hardy spaces and the coincidence between $H_L^{p(\cdot)}(\mathbf{R}^n)$ and the variable exponent Hardy spaces $H^{p(\cdot)}(\mathbf{R}^n)$.

1. Introduction

In recent years, function spaces with variable exponents attract much attentions (see, for example, [4, 14, 16, 18, 19, 20, 37, 44, 50, 51, 54, 55] and their references). The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbf{R}^n)$, with an exponent function $p(\cdot)\colon \mathbf{R}^n \to (0,\infty)$, which consists of all measurable functions f such that $\int_{\mathbf{R}^n} |f(x)|^{p(x)}\,dx < \infty$, is a generalization of the classical Lebesgue space. The study of variable exponent Lebesgue spaces can be traced back to Birnbaum–Orlicz [6] and Orlicz [40] (see also Luxemburg [34] and Nakano [38, 39]), but the modern development started with the articles [31] of Kováčik and Rákosník as well as [13] of Cruz-Uribe and [17] of Diening. The variable function spaces have been widely used in the study of harmonic analysis; see, for example, [14, 18]. Apart from theoretical considerations, such function spaces also have interesting applications in fluid dynamics [1, 42], image processing [9], partial differential equations and variational calculus [2, 26, 43].

Particularly, Nakai and Sawano [37] introduced Hardy spaces with variable exponents, $H^{p(\cdot)}(\mathbf{R}^n)$, and established their atomic characterizations which were further applied to consider dual spaces of such Hardy spaces. Later, in [44], Sawano extended the atomic characterization of the space $H^{p(\cdot)}(\mathbf{R}^n)$ in [37], which also improves the

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corresponding result in [37], and gave out more applications including the boundedness of the fractional integral operator and the commutators generated by singular integral operators and BMO functions, and an Olsen's inequality. After that, Zhuo et al. [55] established their equivalent characterizations via intrinsic square functions, including the intrinsic Lusin area function, the intrinsic g-function and the intrinsic g_λ^*-function. Independently, Cruz-Uribe and Wang [16] also investigated the variable exponent Hardy space with some slightly weaker conditions than those used in [37]. Recall that the theory of classical Hardy spaces $H^p(\mathbf{R}^n)$ with $p \in (0,1]$ and their duals are well studied and certainly play an important role in harmonic analysis as well as partial differential equations; see, for example, [11, 25, 36, 46].

On the other hand, in recent years, the study of function spaces, especially on Hardy spaces associated with different operators, has also inspired great interests (see, for example, [5, 22, 23, 24, 29, 30, 48, 33] and their references). Particularly, let L be a linear operator on $L^2(\mathbf{R}^n)$ and generate an analytic semigroup $\{e^{-tL}\}_{t\ge 0}$ with kernel having pointwise upper bounds, whose decay is measured by $\theta(L) \in (0,\infty]$. Then, by using the Lusin area function, Auscher, Duong and McIntosh [5] initially introduced the Hardy space $H_L^1(\mathbf{R}^n)$ associated with the operator L and established its molecular characterization. Based on this, Duong and Yan [23, 24] introduced the BMO-type space $\mathrm{BMO}_L(\mathbf{R}^n)$ associated with L and proved that the dual space of $H_L^1(\mathbf{R}^n)$ is just $\mathrm{BMO}_{L^*}(\mathbf{R}^n)$, where L^* denotes the *adjoint operator* of L in $L^2(\mathbf{R}^n)$. Later, Yan [48] further generalized these results to the Hardy spaces $H_L^p(\mathbf{R}^n)$ with $p \in (n/[n+\theta(L)], 1]$ and their dual spaces. Moreover, Jiang et al. [30] investigated the Orlicz-Hardy space and its dual space associated with such an operator L.

Let $p(\cdot)\colon \mathbf{R}^n \to (0,1]$ be a variable exponent function satisfying the globally log-Hölder continuous condition. Motivated by [37, 48], in this article, we introduce the variable exponent Hardy space associated with the operator L, denoted by $H_L^{p(\cdot)}(\mathbf{R}^n)$. More precisely, for all $f \in L^2(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$, let

$$S_L(f)(x) := \left\{ \int_{\Gamma(x)} \left| t^m L e^{-t^m L}(f)(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{\frac{1}{2}},$$

where m is a positive constant appearing in the pointwise upper bound of the heat kernel (see (2.2) below) and $\Gamma(x) := \{(y,t) \in \mathbf{R}^n \times (0,\infty)\colon |y-x| < t\}$. The *Hardy spaces* $H_L^{p(\cdot)}(\mathbf{R}^n)$ is defined to be the completion of the set $\{f \in L^2(\mathbf{R}^n)\colon S_L(f) \in L^{p(\cdot)}(\mathbf{R}^n)\}$ with respect to the quasi-norm

$$\|f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} := \|S_L(f)\|_{L^{p(\cdot)}(\mathbf{R}^n)} := \inf\left\{ \lambda \in (0,\infty)\colon \int_{\mathbf{R}^n} \left[\frac{S_L(f)(x)}{\lambda}\right]^{p(x)} dx \le 1 \right\}.$$

We then establish the molecular characterization of $H_L^{p(\cdot)}(\mathbf{R}^n)$ via variable exponent tent spaces. Using this molecular characterization, we further prove that the dual space of $H_L^{p(\cdot)}(\mathbf{R}^n)$ is the BMO-type space $\mathrm{BMO}_{p(\cdot),L^*}(\mathbf{R}^n)$, which is also introduced in this article. As more applications, we study the boundedness of the fractional integral $L^{-\gamma}$ ($\gamma \in (0, \frac{n}{m})$ with m as in Assumption (A) below) from $H_L^{p(\cdot)}(\mathbf{R}^n)$ to $H_L^{q(\cdot)}(\mathbf{R}^n)$ with $\frac{1}{q(\cdot)} := \frac{1}{p(\cdot)} - \frac{m\gamma}{n}$ and the coincidence between $H_L^{p(\cdot)}(\mathbf{R}^n)$ and variable exponent Hardy spaces $H^{p(\cdot)}(\mathbf{R}^n)$ introduced in [37].

A novel aspect of this article is to give a non-trivial combination of function spaces with variable exponents and the theory of operators including their functional calculi and semigroups, and these new function spaces prove necessary in the study of the boundedness of the associated operators (for example, fractional integrals $L^{-\gamma}$ with $\gamma \in (0, \frac{n}{m})$).

This article is organized as follows. In Section 2, we first recall some notation and definitions about variable exponent Lebesgue spaces, holomorphic functional calculi of operators and semigroups, also including some basic assumptions on the operator L considered in this article and the domain of the semigroup $\{e^{-tL}\}_{t\geq 0}$. Via the Lusin area function $S_L(f)$, we then introduce the variable exponent Hardy space associated with L, denoted by $H_L^{p(\cdot)}(\mathbf{R}^n)$.

In Section 3, we mainly establish a molecular characterization of the space $H_L^{p(\cdot)}(\mathbf{R}^n)$ (see Theorem 3.13 below). To this end, we first establish an atomic characterization of the variable exponent tent space $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$ (see Corollary 3.7 below). Then the molecular characterization of $H_L^{p(\cdot)}(\mathbf{R}^n)$ is obtained by using a project operator π_L corresponding to L, which is proved to be bounded from $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$ to $H_L^{p(\cdot)}(\mathbf{R}^n)$. We point out that [44, Lemma 4.1] of Sawano (a slight weaker variant of this lemma was early obtained by Nakai and Sawano [37, Lemma 4.11]), which is re-stated in Lemma 3.5 below, plays a key role in the proof of Theorem 3.13

Section 4 is devoted to proving a duality theorem. Indeed, in Theorem 4.3 below, we show that the dual space of $H_L^{p(\cdot)}(\mathbf{R}^n)$ is just the BMO-type space $\mathrm{BMO}_{p(\cdot),L^*}(\mathbf{R}^n)$, which is also introduced in this section. To show Theorem 4.3, we rely on several key estimates related to BMO-type spaces and $p(\cdot)$-Carleson measures (see Propositions 4.5, 4.6 and 4.7, and Lemma 4.9 below), and the duality of the variable exponent tent space (see Proposition 4.8 below). The main difficulty to establish these estimates is that the quasi-norm $\|\cdot\|_{L^{p(\cdot)}(\mathbf{R}^n)}$ has no the translation invariance, namely, for any cube $Q(x,r) \subset \mathbf{R}^n$, with $x \in \mathbf{R}^n$ and $r \in (0,\infty)$, and $z \in \mathbf{R}^n$, $\|\chi_{Q(x,r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}$ may not equal to $\|\chi_{Q(x+z,r)}\|_{L^{p(\cdot)}(\mathbf{R}^n)}$. To overcome this difficulty, we make full use of Lemma 3.14 below, which is just [55, Lemma 2.6] and presents a relation between two quasi-norms $\|\cdot\|_{L^{p(\cdot)}(\mathbf{R}^n)}$ corresponding to two cubes.

As applications of the molecular characterization of $H_L^{p(\cdot)}(\mathbf{R}^n)$ from Theorem 3.13, in Section 5, we investigate the boundedness of fractional integrals on $H_L^{p(\cdot)}(\mathbf{R}^n)$ (see Theorem 5.9 below) and show that the spaces $H_L^{p(\cdot)}(\mathbf{R}^n)$ and $H^{p(\cdot)}(\mathbf{R}^n)$ coincide with equivalent quasi-norms under some additional assumptions on L (see Theorem 5.3 below).

2. Preliminaries

In this section, we first recall some notation and notions on variable exponent Lebesgue spaces and some knowledge about holomorphic functional calculi as well as semigroups. Then we introduce the variable exponent Hardy spaces associated with operators, denoted by $H_L^{p(\cdot)}(\mathbf{R}^n)$, which generalize the Hardy spaces $H_L^p(\mathbf{R}^n)$ studied in [23, 48].

We begin with some notation which will be used in this article. Let $\mathbf{N} := \{1, 2, \dots\}$ and $\mathbf{Z}_+ := \mathbf{N} \cup \{0\}$. We denote by C a *positive constant* which is independent of the main parameters, but may vary from line to line. We use $C_{(\alpha,\dots)}$ to denote a positive constant depending on the indicated parameters $\alpha, \dots$. The *symbol* $A \lesssim B$ means $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. If E is a subset of $\mathbf{R}^n$, we denote by χ_E its *characteristic function* and by $E^{\complement}$ the set $\mathbf{R}^n \backslash E$. For $a \in \mathbf{R}$, $\lfloor a \rfloor$ denotes the largest integer m such that $m \leq a$. For all $x \in \mathbf{R}^n$ and $r \in (0,\infty)$, denote by $Q(x,r)$ the cube centered at x with side length r, whose sides are parallel to the axes of coordinates. For each cube $Q \subset \mathbf{R}^n$ and $a \in (0,\infty)$, we use x_Q to denote the center of Q and $\ell(Q)$ to denote the side length of Q, and denote by aQ the cube concentric with Q having the side length $a\ell(Q)$.

2.1. Variable exponent Lebesgue spaces. In what follows, a measurable function $p(\cdot)\colon \mathbf{R}^n \to (0,\infty)$ is called a *variable exponent*. For any variable exponent $p(\cdot)$, let

$$p_- := \operatorname*{ess\,inf}_{x\in\mathbf{R}^n} p(x) \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbf{R}^n} p(x). \tag{2.1}$$

Denote by $\mathcal{P}(\mathbf{R}^n)$ the *collection of variable exponents* $p(\cdot)\colon \mathbf{R}^n \to (0,\infty)$ *satisfying* $0 < p_- \le p_+ < \infty$.

For a measurable function f on $\mathbf{R}^n$ and a variable exponent $p(\cdot) \in \mathcal{P}(\mathbf{R}^n)$, the *modular* $\varrho_{p(\cdot)}(f)$ of f is defined by setting $\varrho_{p(\cdot)}(f) := \int_{\mathbf{R}^n} |f(x)|^{p(x)}\,dx$ and the *Luxemburg quasi-norm*

$$\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)} := \inf\left\{\lambda \in (0,\infty)\colon \varrho_{p(\cdot)}(f/\lambda) \le 1\right\}.$$

Then the *variable exponent Lebesgue space* $L^{p(\cdot)}(\mathbf{R}^n)$ is defined to be the set of all measurable functions f such that $\varrho_{p(\cdot)}(f) < \infty$ equipped with the quasi-norm $\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)}$. For more properties on the variable exponent Lebesgue spaces, we refer the reader to [14, 18].

Remark 2.1. Let $p(\cdot) \in \mathcal{P}(\mathbf{R}^n)$.

(i) If $p_- \in [1,\infty)$, then $L^{p(\cdot)}(\mathbf{R}^n)$ is a Banach space (see [18, Theorem 3.2.7]). In particular, for all $\lambda \in \mathbf{C}$ and $f \in L^{p(\cdot)}(\mathbf{R}^n)$, $\|\lambda f\|_{L^{p(\cdot)}(\mathbf{R}^n)} = |\lambda|\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)}$ and, for all $f, g \in L^{p(\cdot)}(\mathbf{R}^n)$,

$$\|f+g\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le \|f\|_{L^{p(\cdot)}(\mathbf{R}^n)} + \|g\|_{L^{p(\cdot)}(\mathbf{R}^n)}.$$

(ii) For any non-trivial function $f \in L^{p(\cdot)}(\mathbf{R}^n)$, it holds true that

$$\varrho_{p(\cdot)}(f/\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)}) = 1;$$

see, for example, [14, Proposition 2.21].

(iii) If $\int_{\mathbf{R}^n} [|f(x)|/\delta]^{p(x)}\,dx \le c$ for some $\delta \in (0,\infty)$ and some positive constant c independent of δ, then it is easy to see that $\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le C\delta$, where C is a positive constant independent of δ, but depending on p_- (or p_+) and c.

Recall that a measurable function $g \in \mathcal{P}(\mathbf{R}^n)$ is said to be *locally* log-*Hölder continuous*, denoted by $g \in C^{\log}_{\mathrm{loc}}(\mathbf{R}^n)$, if there exists a positive constant $C_{\log}(g)$ such that, for all $x,\ y \in \mathbf{R}^n$,

$$|g(x) - g(y)| \le \frac{C_{\log}(g)}{\log(e + 1/|x-y|)},$$

and g is said to satisfy the *globally* log-*Hölder continuous condition*, denoted by $g \in C^{\log}(\mathbf{R}^n)$, if $g \in C^{\log}_{\mathrm{loc}}(\mathbf{R}^n)$ and there exist a positive constant C_∞ and a constant $g_\infty \in \mathbf{R}$ such that, for all $x \in \mathbf{R}^n$,

$$|g(x) - g_\infty| \le \frac{C_\infty}{\log(e + |x|)}.$$

Remark 2.2. Let $n = 1$ and, for all $x \in \mathbf{R}$,

$$p(x) := \max\left\{1 - e^{3-|x|}, \min\left(6/5, \max\left\{1/2, 3/2 - x^2\right\}\right)\right\}.$$

Then $p(\cdot) \in C^{\log}(\mathbf{R})$; see [37, Example 1.3]. With a slight modification, another example was obtained in [49, Example 2.20] as follows. For all $x \in \mathbf{R}$, let

$$p(x) := \max\left\{1 - e^{3-|x|},\ \min\left(6/5, \max(1/2, k|x| + 1/2 - k)\right)\right\},$$

where $k := 7/[10(\sqrt{3/10} - 1)]$. Then $p(\cdot) \in C^{\log}(\mathbf{R})$.

For all $r \in (0, \infty)$, denote by $L^r_{\mathrm{loc}}(\mathbf{R}^n)$ the *set of all locally r-integrable functions* on $\mathbf{R}^n$ and, for any measurable set $E \subset \mathbf{R}^n$, by $L^r(E)$ the *set of all measurable functions f such that*

$$\|f\|_{L^r(E)} := \left\{\int_E |f(x)|^r\,dx\right\}^{1/r} < \infty.$$

Recall that the *Hardy–Littlewood maximal operator* $\mathcal{M}$ is defined by setting, for all $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$,

$$\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\,dy,$$

where the supremum is taken over all balls B of $\mathbf{R}^n$ containing x.

Remark 2.3. Let $p(\cdot) \in C^{\log}(\mathbf{R}^n)$ and $1 < p_- \le p_+ < \infty$. Then there exists a positive constant C such that, for all $f \in L^{p(\cdot)}(\mathbf{R}^n)$, $\|\mathcal{M}(f)\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le C\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)}$; see, for example, [18, Theorem 4.3.8].

2.2. Holomorphic functional calculi. Here, we first recall some notions of the bounded holomorphic functional calculus, which were introduced by McIntosh [35], and then make two assumptions on L required in this article. For two normed linear spaces $\mathcal{X}$ and $\mathcal{Y}$, let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ be the *collection of continuous linear operators from $\mathcal{X}$ to $\mathcal{Y}$* and, for any $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $\|T\|_{\mathcal{X}\to\mathcal{Y}}$ its operator norm.

Let $v \in (0, \pi)$. Define the closed sector S_v by $S_v := \{z \in \mathbf{C}\colon |\arg z| \le v\} \cup \{0\}$ and denote by S_v^0 the interior of S_v. Let $H(S_v^0)$ be the set of all holomorphic functions on S_v^0,

$$H^\infty(S_v^0) := \left\{b \in H(S_v^0)\colon \|b\|_\infty := \sup_{z \in S_v^0} |b(z)| < \infty\right\}$$

and

$$\Psi(S_v^0) := \{\psi \in H(S_v^0)\colon \exists\ s,\ C \in (0, \infty) \text{ such that}$$
$$|\psi(z)| \le C|z|^s(1 + |z|^{2s})^{-1},\ \forall\ z \in S_v^0\}.$$

Given $v \in (0, \pi)$, a closed operator $L \in \mathcal{L}(L^2(\mathbf{R}^n), L^2(\mathbf{R}^n))$ is said to be of *type v* if $\sigma(L) \subset S_v$, where $\sigma(L)$ denotes the spectra of L, and, for all $\gamma \in (v, \pi)$, there exists a positive constant C such that, for all $\lambda \notin S_\gamma$,

$$\|(L - \lambda I)^{-1}\|_{L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)} \le C|\lambda|^{-1}.$$

Let $\theta \in (v, \gamma)$ and Σ be the contour $\{\xi = re^{\pm i\theta}\colon r \in [0, \infty)\}$ parameterized clockwise around S_v. Then, for $\psi \in \Psi(S_v^0)$ and L being of type v, the operator $\psi(L)$ is defined by

$$\psi(L) := \frac{1}{2\pi i} \int_\Sigma (L - \lambda I)^{-1}\psi(\lambda)\,d\lambda,$$

where the integral is absolutely convergent in $\mathcal{L}(L^2(\mathbf{R}^n), L^2(\mathbf{R}^n))$ and, by the Cauchy theorem, the above definition is independent of the choices of v and γ satisfying $\theta \in (v, \gamma)$. If L is a one-to-one linear operator having dense range and $b \in H^\infty(S_\gamma^0)$, then define an operator $b(L)$ by $b(L) := [\psi(L)]^{-1}(b\psi)(L)$, where $\psi(z) := z(1+z)^{-2}$ for all $z \in S^0_\gamma$. It was proved in [35] that $b(L)$ is well defined on $L^2(\mathbf{R}^n)$. The operator L is said to have a *bounded H^∞ functional calculus* on $L^2(\mathbf{R}^n)$ if, for all $\gamma \in (v, \pi)$, there exists a positive constant $\widetilde{C}$ such that, for all $b \in H^\infty(S^0_\gamma)$, $b(L) \in \mathcal{L}(L^2(\mathbf{R}^n), L^2(\mathbf{R}^n))$ and

$$\|b(L)\|_{L^2(\mathbf{R}^n)\to L^2(\mathbf{R}^n)} \le \widetilde{C}\|b\|_\infty.$$

Let L be a linear operator of type v on $L^2(\mathbf{R}^n)$ with $v \in (0, \frac{\pi}{2})$. Then it generates a bounded holomorphic semigroup $\{e^{-zL}\}_{z\in D_v}$, where $D_v := \{z \in \mathbf{C} \colon 0 \le |\arg(z)| < \frac{\pi}{2} - v\}$ and, for all $z \in \mathbf{C}$, $\arg(z) \in (-\pi, \pi]$ is the argument of z; see, for example, [41, Theorem 1.45].

In this article, we make the following two assumptions on the operator L.

Assumption (A). Assume that, for each $t \in (0,\infty)$, the distribution kernel p_t of e^{-tL} belongs to $L^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ and satisfies that, for all $x,\ y \in \mathbf{R}^n$,

$$|p_t(x,y)| \le t^{-\frac{n}{m}} g\left(\frac{|x-y|}{t^{\frac{1}{m}}}\right), \tag{2.2}$$

where m is a positive constant and g is a positive, bounded and decreasing function satisfying that, for some $\varepsilon \in (0,\infty)$,

$$\lim_{r\to\infty} r^{n+\varepsilon} g(r) = 0. \tag{2.3}$$

Assumption (B). Assume that the operator L is one-to-one, has dense range in $L^2(\mathbf{R}^n)$ and a bounded H^∞ functional calculus on $L^2(\mathbf{R}^n)$.

Remark 2.4. (i) If $\{e^{-tL}\}_{t\ge 0}$ is a bounded analytic semigroup on $L^2(\mathbf{R}^n)$ whose kernels $\{p_t\}_{t\ge0}$ satisfy (2.2) and (2.3), then, for any $k \in \mathbf{N}$, there exists a positive constant $C_{(k)}$, depending on k, such that, for all $t \in (0,\infty)$ and almost every $x,\ y \in \mathbf{R}^n$,

$$\left|t^k \frac{\partial^k p_t(x,y)}{\partial t^k}\right| \le \frac{C_{(k)}}{t^{n/m}} g_k\left(\frac{|x-y|}{t^{1/m}}\right). \tag{2.4}$$

Here, it should be pointed out that, for all $k \in \mathbf{N}$, the function g_k may depend on k but always satisfies (2.3); see [41, Theorem 6.17] and [12].

(ii) Let $v \in (0,\pi)$. Then L has a bounded H^∞ functional calculus on $L^2(\mathbf{R}^n)$ if and only if, for any $\gamma \in (v,\pi)$ and nonzero function $\psi \in \Psi(S^0_\gamma)$, L satisfies the following square function estimate: there exists a positive constant C such that, for all $f \in L^2(\mathbf{R}^n)$,

$$C^{-1}\|f\|_{L^2(\mathbf{R}^n)} \le \left\{\int_0^\infty \|\psi_t(L)f\|^2_{L^2(\mathbf{R}^n)} \frac{dt}{t}\right\}^{1/2} \le C\|f\|_{L^2(\mathbf{R}^n)},$$

where $\psi_t(\xi) := \psi(t\xi)$ for all $t \in (0,\infty)$ and $\xi \in \mathbf{R}^n$; see [35].

2.3. An acting class of semigroups $\{e^{-tL}\}_{t\ge0}$. For all $\beta \in (0,\infty)$, let $\mathcal{M}_\beta(\mathbf{R}^n)$ be the set of all functions $f \in L^2_{\mathrm{loc}}(\mathbf{R}^n)$ satisfying

$$\|f\|_{\mathcal{M}_\beta(\mathbf{R}^n)} := \left\{\int_{\mathbf{R}^n} \frac{|f(x)|^2}{1+|x|^{n+\beta}}\,dx\right\}^{1/2} < \infty.$$

We point out that the space $\mathcal{M}_\beta(\mathbf{R}^n)$ was introduced by Duong and Yan in [24] and it is a Banach space under the norm $\|\cdot\|_{\mathcal{M}_\beta(\mathbf{R}^n)}$. For any given operator L satisfying Assumptions (A) and (B), let

$$\theta(L) := \sup\{\varepsilon \in (0,\infty) \colon (2.2) \text{ and } (2.3) \text{ hold true}\} \tag{2.5}$$

and

$$\mathcal{M}(\mathbf{R}^n) := \begin{cases} \mathcal{M}_{\theta(L)}(\mathbf{R}^n), & \text{if } \theta(L) < \infty, \\ \bigcup_{\beta\in(0,\infty)} \mathcal{M}_\beta(\mathbf{R}^n), & \text{if } \theta(L) = \infty. \end{cases}$$

Let $s \in \mathbf{Z}_+$. For any $f \in \mathcal{M}(\mathbf{R}^n)$ and $(x,t) \in \mathbf{R}^{n+1}_+ := \mathbf{R}^n \times (0,\infty)$, let

$$(2.6) \qquad P_{s,t}f(x) := f(x) - (I - e^{-tL})^{s+1} f(x) \quad \text{and} \quad Q_{s,t}f(x) := (tL)^{s+1}e^{-tL}f(x),$$

and, particularly, let

$$(2.7) \qquad P_t f(x) := P_{0,t}f(x) = e^{-tL}f(x) \quad \text{and} \quad Q_t f(x) := Q_{0,t}f(x) = tLe^{-tL}f(x).$$

Here, we point out that these operators in (2.6) were introduced by Blunck and Kunstmann [7] and Holfmann and Martell [27].

Remark 2.5. (i) For all $f \in \mathcal{M}(\mathbf{R}^n)$, the operators $P_{s,t}f$ and $Q_{s,t}f$ are well defined. Moreover, the kernels $p_{s,t}$ of $P_{s,t}$ and $q_{s,t}$ of $Q_{s,t}$ satisfy that there exists a positive constant C such that, for all $t \in (0,\infty)$ and $x,\ y \in \mathbf{R}^n$,

$$(2.8) \qquad |p_{s,t^m}(x,y)| + |q_{s,t^m}(x,y)| \le Ct^{-n} g\left(\frac{|x-y|}{t}\right),$$

where the function g satisfies the conditions as in Assumption (A); see, for example, [48].

(ii) A typical example of L satisfying $\theta(L) = \infty$ is that the kernels $\{p_t\}_{t\ge0}$ of $\{e^{-tL}\}_{t\ge0}$ have the pointwise Gaussian upper bound, namely, there exists a positive constant C such that, for all $t \in (0,\infty)$ and $x,\ y \in \mathbf{R}^n$, $|p_t(x,y)| \le \frac{C}{t^{n/2}} e^{-\frac{|x-y|^2}{t}}$. Obviously, if $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i}$ is the Laplacian operator and $L = -\Delta$, then the heat kernels have the pointwise Gaussian upper bound. There are several other operators whose heat kernels have the pointwise Gaussian upper bound; see, for example, [48, p. 4390, Remarks].

(iii) Let $s \in \mathbf{Z}_+$ and $p \in (1,\infty)$. Then, by (i) of this remark, we easily conclude that there exists a positive constant C such that, for all $t \in (0,\infty)$ and $f \in L^p(\mathbf{R}^n)$,

$$\|P_{s,t^m}(f)\|_{L^p(\mathbf{R}^n)} \le C\|f\|_{L^p(\mathbf{R}^n)}.$$

2.4. Definition of Hardy spaces $H_L^{p(\cdot)}(\mathbf{R}^n)$. For all functions $f \in L^2(\mathbf{R}^n)$, define the *Lusin area function* $S_L(f)$ by setting, for all $x \in \mathbf{R}^n$,

$$S_L(f)(x) := \left\{\int_{\Gamma(x)} |Q_{t^m}f(y)|^2 \,\frac{dy\,dt}{t^{n+1}}\right\}^{1/2},$$

here and hereafter, for all $x \in \mathbf{R}^n$, $\Gamma(x) := \{(y,t) \in \mathbf{R}^{n+1}_+ \colon |y-x| < t\}$ and Q_t is defined as in (2.7). In [5], Auscher et al. proved that, for any $p \in (1,\infty)$, there exists a positive constant $C_{(p)}$, depending on p, such that, for all $f \in L^p(\mathbf{R}^n)$,

$$(2.9) \qquad C_{(p)}^{-1}\|f\|_{L^p(\mathbf{R}^n)} \le \|S_L(f)\|_{L^p(\mathbf{R}^n)} \le C_{(p)}\|f\|_{L^p(\mathbf{R}^n)};$$

see also Duong and McIntosh [21] and Yan [47].

We now introduce the variable exponent Hardy spaces associated with operators.

Definition 2.6. Let L be an operator satisfying Assumptions (A) and (B), and $p(\cdot) \in C^{\log}(\mathbf{R}^n)$ satisfy $p_+ \in (0,1]$. A function $f \in L^2(\mathbf{R}^n)$ is said to be in $\widetilde{H}_L^{p(\cdot)}(\mathbf{R}^n)$ if $S_L(f) \in L^{p(\cdot)}(\mathbf{R}^n)$; moreover, define

$$\|f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} := \|S_L(f)\|_{L^{p(\cdot)}(\mathbf{R}^n)} := \inf\left\{\lambda \in (0,\infty)\colon \int_{\mathbf{R}^n} \left[\frac{S_L(f)(x)}{\lambda}\right]^{p(x)} dx \le 1\right\}.$$

Then the *variable Hardy space associated with operator* L, denoted by $H_L^{p(\cdot)}(\mathbf{R}^n)$, is defined to be the completion of $\widetilde{H}_L^{p(\cdot)}(\mathbf{R}^n)$ in the quasi-norm $\|\cdot\|_{H_L^{p(\cdot)}(\mathbf{R}^n)}$.

Remark 2.7. (i) By the theorem of completion of Yosida [53, p. 65], we find that $\widetilde{H}_L^{p(\cdot)}(\mathbf{R}^n)$ is dense in $H_L^{p(\cdot)}(\mathbf{R}^n)$, namely, for any $f \in H_L^{p(\cdot)}(\mathbf{R}^n)$, there exists a Cauchy sequence $\{f_k\}_{k\in\mathbf{N}}$ in $\widetilde{H}_L^{p(\cdot)}(\mathbf{R}^n)$ such that $\lim_{k\to\infty}\|f_k - f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} = 0$. Moreover, if $\{f_k\}_{k\in\mathbf{N}}$ is a Cauchy sequence in $\widetilde{H}_L^{p(\cdot)}(\mathbf{R}^n)$, then there exists an unique $f \in H_L^{p(\cdot)}(\mathbf{R}^n)$ such that $\lim_{k\to\infty}\|f_k - f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} = 0$. Moreover, $L^2(\mathbf{R}^n)\cap H_L^{p(\cdot)}(\mathbf{R}^n)$ is dense in $H_L^{p(\cdot)}(\mathbf{R}^n)$.

(ii) We point out that smooth functions with compact supports do not necessarily belong to $H_L^{p(\cdot)}(\mathbf{R}^n)$; see [48] and also Remark 4.4 below for more details.

(iii) Observe that, when $p(\cdot) \equiv p \in (0,\infty)$, $L^{p(\cdot)}(\mathbf{R}^n) = L^p(\mathbf{R}^n)$. If $p(\cdot) \equiv 1$, then $H_L^{p(\cdot)}(\mathbf{R}^n) = H_L^1(\mathbf{R}^n)$, which was introduced by Auscher et al. [5]; see also Duong and Yan [23]. If $p(\cdot) \equiv p \in (\frac{n}{n+\theta(L)}, 1)$, then the space $H_L^{p(\cdot)}(\mathbf{R}^n)$ is just the space $H_L^p(\mathbf{R}^n)$ introduced by Yan [48].

(iv) Different from the space $H_L^p(\mathbf{R}^n)$ which is just $L^p(\mathbf{R}^n)$ when $p \in (1,\infty)$ (see, for example, [48, p. 4400]), since it is still unclear whether (2.9) holds true or not with $L^p(\mathbf{R}^n)$ replaced by $L^{p(\cdot)}(\mathbf{R}^n)$ when $p_+ \in (1,\infty)$, it is also unclear whether $H_L^{p(\cdot)}(\mathbf{R}^n)$ and $L^{p(\cdot)}(\mathbf{R}^n)$ (or $H^{p(\cdot)}(\mathbf{R}^n)$) coincide or not. We will not push this issue in this article due to its length.

We end this section by comparing the variable exponent Hardy spaces associated with operators in this article with the Musielak–Orlicz–Hardy spaces associated with operators satisfying reinforced off-diagonal estimates in [8]. Indeed, in general, these two scales of Hardy-type spaces do not cover each other.

Remark 2.8. Let $\varphi\colon \mathbf{R}^n \times [0,\infty) \to [0,\infty)$ be a growth function in [32] and L an operator satisfying reinforced off-diagonal estimates in [8]. Then Bui et al. [8] introduced the Musielak–Orlicz–Hardy space associated with operator L via the Lusin area function, denoted by $H_{\varphi,L}(\mathbf{R}^n)$. Recall that the Musielak–Orlicz space $L^\varphi(\mathbf{R}^n)$ is defined to be the set of all measurable functions f on $\mathbf{R}^n$ such that

$$\|f\|_{L^\varphi(\mathbf{R}^n)} := \inf\left\{\lambda \in (0,\infty) : \int_{\mathbf{R}^n} \varphi(x, |f(x)|/\lambda)\,dx \le 1\right\} < \infty.$$

Observe that, if

$$\varphi(x,t) := t^{p(x)} \quad \text{for all } x \in \mathbf{R}^n \text{ and } t \in [0,\infty), \tag{2.10}$$

then $L^\varphi(\mathbf{R}^n) = L^{p(\cdot)}(\mathbf{R}^n)$. However, a general Musielak–Orlicz function φ satisfying all the assumptions in [32] (and hence [8]) may not have the form as in (2.10) (see [32]). On the other hand, it was proved in [49, Remark 2.23(iii)] that there exists a variable exponent function $p(\cdot) \in C^{\log}(\mathbf{R}^n)$, but $t^{p(\cdot)}$ is not a uniformly Muckenhoupt weight, which was required in [32] (and hence [8]). Thus, Musielak–Orlicz–Hardy spaces associated with operators in [8] and variable exponent Hardy spaces associated with operators in this article do not cover each other.

Moreover, in Theorem 5.3 below, we show that, under some additional assumptions on L, the spaces $H_L^{p(\cdot)}(\mathbf{R}^n)$ coincide with the variable exponent Hardy spaces $H^{p(\cdot)}(\mathbf{R}^n)$ which can not cover and also can not be covered by Musielak–Orlicz Hardy spaces in [32] based on the same reason as above.

3. Molecular characterizations of $H_L^{p(\cdot)}(\mathbf{R}^n)$

In this section, we aim to obtain the molecular characterizations of $H_L^{p(\cdot)}(\mathbf{R}^n)$. To this end, we first give out some properties of the tent spaces with variable exponents including their atomic characterizations, which are then applied to establish the molecular characterizations of $H_L^{p(\cdot)}(\mathbf{R}^n)$ by using a project operator π_L corresponding to L.

3.1. Atomic characterizations of tent spaces $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$. We begin with the definition of the tent space with variable exponent. Let $p(\cdot) \in \mathcal{P}(\mathbf{R}^n)$. For all measurable functions g on $\mathbf{R}_+^{n+1}$ and $x \in \mathbf{R}^n$, define

$$\mathcal{T}(g)(x) := \left\{\int_{\Gamma(x)} |g(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}$$

and

$$\mathcal{C}_{p(\cdot)}(g)(x) := \sup_{Q\ni x} \frac{|Q|^{1/2}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \left\{\int_{\widehat{Q}} |g(y,t)|^2 \frac{dy\,dt}{t}\right\}^{1/2},$$

where the supremum is taken over all cubes Q of $\mathbf{R}^n$ containing x and $\widehat{Q}$ denotes the tent over Q, namely, $\widehat{Q} := \{(y,t) \in \mathbf{R}_+^{n+1} \colon B(y,t) \subset Q\}$.

Definition 3.1. Let $p(\cdot) \in \mathcal{P}(\mathbf{R}^n)$.

(i) Let $q \in (0,\infty)$. Then the *tent space* $T_2^q(\mathbf{R}_+^{n+1})$ is defined to be the set of all measurable functions g on $\mathbf{R}_+^{n+1}$ such that $\|g\|_{T_2^q(\mathbf{R}_+^{n+1})} := \|\mathcal{T}(g)\|_{L^q(\mathbf{R}^n)} < \infty$.

(ii) The *tent space with variable exponent* $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$ is defined to be the set of all measurable functions g on $\mathbf{R}_+^{n+1}$ such that $\|g\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})} := \|\mathcal{T}(g)\|_{L^{p(\cdot)}(\mathbf{R}^n)} < \infty$.

(iii) The *space* $T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})$ is defined to be the set of all measurable functions g on $\mathbf{R}_+^{n+1}$ such that $\|g\|_{T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})} := \|\mathcal{C}_{p(\cdot)}(g)\|_{L^\infty(\mathbf{R}^n)} < \infty$.

Remark 3.2. (i) We point out that the spaces $T_2^q(\mathbf{R}_+^{n+1})$ and $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$ were introduced in [10] and [55], respectively. Moreover, if $p(\cdot) \equiv q \in (0,\infty)$, then $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1}) = T_2^q(\mathbf{R}_+^{n+1})$.

(ii) If $g \in T_2^2(\mathbf{R}_+^{n+1})$, then we easily see that $\|g\|_{T_2^2(\mathbf{R}_+^{n+1})} = \{\int_{\mathbf{R}_+^{n+1}} |g(x,t)|^2 \frac{dx\,dt}{t}\}^{\frac{1}{2}}$.

Let $q \in (1,\infty)$ and $p(\cdot) \in \mathcal{P}(\mathbf{R}^n)$. Recall that a measurable function a on $\mathbf{R}_+^{n+1}$ is called a $(p(\cdot),q)$-*atom* if a satisfies

(i) $\operatorname{supp} a \subset \widehat{Q}$ for some cube $Q \subset \mathbf{R}^n$;

(ii) $\|a\|_{T_2^q(\mathbf{R}_+^{n+1})} \le |Q|^{1/q}\|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)}^{-1}$.

Furthermore, if a is a $(p(\cdot),q)$-atom for all $q \in (1,\infty)$, then a is call a $(p(\cdot),\infty)$-*atom*. We point out that the $(p(\cdot),\infty)$-atom was introduced in [55].

For any $p(\cdot) \in \mathcal{P}(\mathbf{R}^n)$, $\{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$ and cubes $\{Q_j\}_{j\in\mathbf{N}}$ of $\mathbf{R}^n$, let

$$\mathcal{A}\left(\{\lambda_j\}_{j\in\mathbf{N}}, \{Q_j\}_{j\in\mathbf{N}}\right) := \left\|\left\{\sum_{j\in\mathbf{N}} \left[\frac{|\lambda_j|\chi_{Q_j}}{\|Q_j\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right]^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}, \tag{3.1}$$

here and hereafter, we let

$$\underline{p} := \min\{1, p_-\} \tag{3.2}$$

with p_- as in (2.1).

The following atomic decomposition of $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$ was proved in [55, Theorem 2.16].

Lemma 3.3. *Let $p(\cdot) \in C^{\log}(\mathbf{R}^n)$. Then, for all $f \in T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$, there exist $\{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$ and a sequence $\{a_j\}_{j\in\mathbf{N}}$ of $(p(\cdot),\infty)$-atoms such that, for almost every $(x,t) \in \mathbf{R}_+^{n+1}$,*

$$f(x,t) = \sum_{j\in\mathbf{N}} \lambda_j a_j(x,t); \tag{3.3}$$

moreover, the series in (3.3) *converges absolutely for almost all $(x,t) \in \mathbf{R}_+^{n+1}$ and there exists a positive constant C such that, for all $f \in T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$,*

$$\mathcal{B}(\{\lambda_j a_j\}_{j\in\mathbf{N}}) := \mathcal{A}(\{\lambda_j\}_{j\in\mathbf{N}}, \{Q_j\}_{j\in\mathbf{N}}) \le C\|f\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})}, \tag{3.4}$$

where, for each $j \in \mathbf{N}$, Q_j denotes the cube such that $\operatorname{supp} a_j \subset \widehat{Q}_j$.

By Lemma 3.3, we have the following conclusion.

Corollary 3.4. *Let $p(\cdot) \in C^{\log}(\mathbf{R}^n)$. Assume that $f \in T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$, then the decomposition* (3.3) *also holds true in $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$.*

To prove Corollary 3.4, we need the following useful lemma, which is just [44, Lemma 4.1].

Lemma 3.5. *Let $p(\cdot) \in C^{\log}(\mathbf{R}^n)$ and $q \in [1,\infty) \cap (p_+,\infty)$ with p_+ as in* (2.1). *Then there exists a positive constant C such that, for all sequences $\{Q_j\}_{j\in\mathbf{N}}$ of cubes of $\mathbf{R}^n$, numbers $\{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$ and functions $\{a_j\}_{j\in\mathbf{N}}$ satisfying that, for each $j \in \mathbf{N}$,* $\operatorname{supp} a_j \subset Q_j$ *and $\|a_j\|_{L^q(\mathbf{R}^n)} \le |Q_j|^{1/q}$,*

$$\left\|\left(\sum_{j=1}^{\infty} |\lambda_j a_j|^{\underline{p}}\right)^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le C \left\|\left(\sum_{j=1}^{\infty} |\lambda_j \chi_{Q_j}|^{\underline{p}}\right)^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)},$$

where $\underline{p}$ is as in (3.2).

Proof of Corollary 3.4. Let $f \in T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$. Then, by Lemma 3.3, we may assume that $f = \sum_{j\in\mathbf{N}} \lambda_j a_j$ almost everywhere on $\mathbf{R}_+^{n+1}$, where $\{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$ and $\{a_j\}_{j\in\mathbf{N}}$ is a sequence of $(p(\cdot),\infty)$-atoms such that, for each $j \in \mathbf{N}$, $\operatorname{supp} a_j \subset \widehat{Q}_j$ with some cube $Q_j \subset \mathbf{R}^n$, and

$$\mathcal{A}(\{\lambda_j\}_{j\in\mathbf{N}}, \{Q_j\}_{j\in\mathbf{N}}) \lesssim \|f\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})}. \tag{3.5}$$

Let $q \in [1,\infty) \cap (p_+,\infty)$. Then, by the definition of $(p(\cdot),\infty)$-atoms, we see that, for all $j \in \mathbf{N}$,

$$\|\mathcal{T}(a_j)\|_{L^q(\mathbf{R}^n)} = \|a_j\|_{T_2^q(\mathbf{R}_+^{n+1})} \le \frac{|Q_j|^{1/q}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}.$$

From this, Lemma 3.5 and the fact that, for all $\theta \in (0,1]$ and $\{\xi_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$,

$$\left(\sum_{j\in\mathbf{N}} |\xi_j|\right)^{\theta} \le \sum_{j\in\mathbf{N}} |\xi_j|^{\theta}, \tag{3.6}$$

we deduce that, for all $N\in\mathbf{N}$,

$$\begin{aligned}\left\|\mathcal{T}\left(f-\sum_{j=1}^N\lambda_ja_j\right)\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}&\le\left\|\sum_{j=N+1}^\infty|\lambda_j|\mathcal{T}(a_j)\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}\\&\lesssim\left\|\left\{\sum_{j=N+1}^\infty\left[\frac{|\lambda_j|\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right]^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}.\end{aligned}\tag{3.7}$$

This, combined with (3.5) and the dominated convergence theorem (see [14, Theorem 2.62]), implies that

$$\lim_{N\to\infty}\left\|\mathcal{T}\left(f-\sum_{j=1}^N\lambda_ja_j\right)\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}\lesssim\left\|\lim_{N\to\infty}\left\{\sum_{j=N+1}^\infty\left[\frac{|\lambda_j|\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right]^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}=0.$$

Therefore, (3.3) holds true in $T_2^{p(\cdot)}(\mathbf{R}^{n+1}_+)$, which completes the proof of Corollary 3.4. □

Remark 3.6. It was proved in [29, Proposition 3.1] that, if $f\in T_2^q(\mathbf{R}^{n+1}_+)$ with $q\in(0,\infty)$, then the decomposition (3.3) also holds true in $T_2^q(\mathbf{R}^{n+1}_+)$.

Using Corollary 3.4 and an argument similar to that used in the proof of (3.7), we obtain the following atomic characterization of $T_2^{p(\cdot)}(\mathbf{R}^{n+1}_+)$, the details being omitted.

Corollary 3.7. *Let $p(\cdot)\in C^{\log}(\mathbf{R}^n)$ satisfy $p_+\in(0,1]$. Then $f\in T_2^{p(\cdot)}(\mathbf{R}^{n+1}_+)$ if and only if there exist $\{\lambda_j\}_{j\in\mathbf{N}}\subset\mathbf{C}$ and a sequence $\{a_j\}_{j\in\mathbf{N}}$ of $(p(\cdot),\infty)$-atoms such that, for almost every $(x,t)\in\mathbf{R}^{n+1}_+$, $f(x,t)=\sum_{j\in\mathbf{N}}\lambda_ja_j(x,t)$ and*

$$\int_{\mathbf{R}^n}\left\{\sum_{j\in\mathbf{N}}\left[\frac{\lambda_j\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right]^{\underline{p}}\right\}^{\frac{p(x)}{\underline{p}}}dx<\infty,$$

where, for each j, Q_j denotes the cube appearing in the support of a_j; moreover, for all $f\in T_2^{p(\cdot)}(\mathbf{R}^{n+1}_+)$, $\|f\|_{T_2^{p(\cdot)}(\mathbf{R}^{n+1}_+)}\sim\mathcal{A}(\{\lambda_j\}_{j\in\mathbf{N}},\{Q_j\}_{j\in\mathbf{N}})$ with the implicit positive constants independent of f.

The following remark plays an important role in the proof of Theorem 4.3.

Remark 3.8. Let $p(\cdot)\in\mathcal{P}(\mathbf{R}^n)$ satisfy $p_+\in(0,1]$. Then, by [37, Remark 4.4], we know that, for any $\{\lambda_j\}_{j\in\mathbf{N}}\subset\mathbf{C}$ and cubes $\{Q_j\}_{j\in\mathbf{N}}$ of $\mathbf{R}^n$,

$$\sum_{j\in\mathbf{N}}|\lambda_j|\le\mathcal{A}(\{\lambda_j\}_{j\in\mathbf{N}},\{Q_j\}_{j\in\mathbf{N}}).$$

In what follows, let $T_{2,c}^{p(\cdot)}(\mathbf{R}^{n+1}_+)$ and $T_{2,c}^q(\mathbf{R}^{n+1}_+)$ with $q\in(0,\infty)$ be the *sets* of all functions, respectively, in $T_2^{p(\cdot)}(\mathbf{R}^{n+1}_+)$ and $T_2^q(\mathbf{R}^{n+1}_+)$ with compact supports.

Proposition 3.9. *Let $p(\cdot)\in C^{\log}(\mathbf{R}^n)$. Then $T_{2,c}^{p(\cdot)}(\mathbf{R}^{n+1}_+)\subset T_{2,c}^2(\mathbf{R}^{n+1}_+)$ as sets.*

Proof. By [29, Lemma 3.3(i)], we know that, for any $q\in(0,\infty)$, $T_{2,c}^q(\mathbf{R}^{n+1}_+)\subset T_{2,c}^2(\mathbf{R}^{n+1}_+)$. Thus, to prove this proposition, it suffices to show that $T_{2,c}^{p(\cdot)}(\mathbf{R}^{n+1}_+)\subset T_{2,c}^{q_0}(\mathbf{R}^{n+1}_+)$ for some $q_0\in(0,\infty)$. To this end, suppose that $f\in T_{2,c}^{p(\cdot)}(\mathbf{R}^{n+1}_+)$ and supp $f\subset K$, where K is a compact set in $\mathbf{R}^{n+1}_+$. Let Q be a cube in $\mathbf{R}^n$ such that

$K \subset \widehat{Q}$. Then $\operatorname{supp} \mathcal{T}(f) \subset Q$. From this and the fact that $p_- \le p(x)$ for all $x \in \mathbf{R}^n$, we deduce that

$$\begin{aligned}\int_{\mathbf{R}^n} [\mathcal{T}(f)(x)]^{p_-}\,dx &\le \int_{\{x\in Q:\ \mathcal{T}(f)(x)<1\}} [\mathcal{T}(f)(x)]^{p_-}\,dx + \int_{\{x\in Q:\ \mathcal{T}(f)(x)\ge 1\}} \cdots \\ &\le |Q| + \int_{\mathbf{R}^n} [\mathcal{T}(f)(x)]^{p(x)}\,dx < \infty,\end{aligned}$$

which implies that $T_{2,c}^{p(\cdot)}(\mathbf{R}_+^{n+1}) \subset T_{2,c}^{p_-}(\mathbf{R}_+^{n+1})$ as sets and hence completes the proof of Proposition 3.9. □

3.2. Molecular characterizations of $H_L^{p(\cdot)}(\mathbf{R}^n)$. In this subsection, we establish the molecular characterizations of $H_L^{p(\cdot)}(\mathbf{R}^n)$. We begin with some notions. In what follows, for any $q \in (0,\infty)$, let $L^q(\mathbf{R}_+^{n+1})$ be the *set of all q-integrable functions on $\mathbf{R}_+^{n+1}$* and $L_{\mathrm{loc}}^q(\mathbf{R}_+^{n+1})$ the *set of all locally q-integrable functions on $\mathbf{R}_+^{n+1}$*. For any $p(\cdot) \in \mathcal{P}(\mathbf{R}^n)$, let

$$s_0 := \lfloor (n/m)(1/p_- - 1) \rfloor, \tag{3.8}$$

namely, s_0 denotes the largest integer smaller than or equal to $\frac{n}{m}(\frac{1}{p_-} - 1)$.

Let m be as in (2.2) and $s \in [s_0,\infty)$. Let $C_{(m,s)}$ be a positive constant, depending on m and s, such that

$$C_{(m,s)} \int_0^\infty t^{m(s+2)} e^{-2t^m} (1 - e^{-t^m})^{s_0+1} \,\frac{dt}{t} = 1. \tag{3.9}$$

Let $q \in (0,\infty)$. Recall that the *operator* π_L is defined by setting, for all $f \in T_{2,c}^q(\mathbf{R}_+^{n+1})$ and $x \in \mathbf{R}^n$,

$$\pi_L(f)(x) := C_{(m,s)} \int_0^\infty Q_{s,t^m}(I - P_{s_0,t^m})(f(\cdot,t))(x)\,\frac{dt}{t}.$$

Moreover, π_L is well defined and $\pi_L(f) \in L^2(\mathbf{R}^n)$ for all $f \in T_{2,c}^q(\mathbf{R}_+^{n+1})$ (see [48, p. 4395]).

Remark 3.10. Let $f \in L^2(\mathbf{R}^n)$. Then, by [48, (3.10)], we know that

$$f = C_{(m,s)} \int_0^\infty Q_{s,t^m}(I - P_{s_0,t^m}) Q_{t^m} f \,\frac{dt}{t},$$

where the integral converges in $L^2(\mathbf{R}^n)$; see also [3, 35].

Definition 3.11. Let $p(\cdot) \in C^{\log}(\mathbf{R}^n)$ and $s \in [s_0,\infty)$ with s_0 as in (3.8). A measurable function α on $\mathbf{R}^n$ is called a *$(p(\cdot),s,L)$-molecule* if there exists a $(p(\cdot),\infty)$-atom a supported on Q for some cube $Q \subset \mathbf{R}^n$ such that, for all $x \in \mathbf{R}^n$, $\alpha(x) := \pi_L(a)(x)$.

When it is necessary to specify the cube Q, then a is called a *$(p(\cdot),s,L)$-molecule associated with Q*.

Remark 3.12. Let $p(\cdot) \in C^{\log}(\mathbf{R}^n)$ with $p_- \in (\frac{n}{n+\theta(L)},\infty)$, where p_- and $\theta(L)$ are as in (2.1) and (2.5), respectively. Then, by Proposition 3.17(ii) below, we see that the $(p(\cdot),s,L)$-molecule is well defined. Indeed, if a is a $(p(\cdot),\infty)$-atom, by Corollary 3.7, we then know $a \in T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$, which, together with Proposition 3.17(ii) below, implies that $\pi_L(a) \in H_L^{p(\cdot)}(\mathbf{R}^n)$. Thus, the $(p(\cdot),s,L)$-molecule is well defined.

The molecular characterization of $H_L^{p(\cdot)}(\mathbf{R}^n)$ is stated as follows.

Theorem 3.13. *Let $p(\cdot) \in C^{\log}(\mathbf{R}^n)$ satisfy $p_+ \in (0,1]$ and $p_- \in (\frac{n}{n+\theta(L)}, 1]$, and $s \in [s_0, \infty)$ with p_+, p_-, $\theta(L)$ and s_0, respectively, as in (2.1), (2.5) and (3.8).*

(i) *If $f \in H_L^{p(\cdot)}(\mathbf{R}^n)$, then there exist $\{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$ and a sequence $\{\alpha_j\}_{j\in\mathbf{N}}$ of $(p(\cdot), s, L)$-molecules associated with cubes $\{Q_j\}_{j\in\mathbf{N}}$ such that $f = \sum_{j\in\mathbf{N}} \lambda_j \alpha_j$ in $H_L^{p(\cdot)}(\mathbf{R}^n)$ and*

$$\mathcal{B}(\{\lambda_j\alpha_j\}_{j\in\mathbf{N}}) := \mathcal{A}(\{\lambda_j\}_{j\in\mathbf{N}}, \{Q_j\}_{j\in\mathbf{N}}) \le C\|f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)}$$

with C being a positive constant independent of f.

(ii) *Suppose that $\{\lambda_k\}_{k\in\mathbf{N}} \subset \mathbf{C}$ and $\{\alpha_k\}_{k\in\mathbf{N}}$ is a family of $(p(\cdot), s, L)$-molecules satisfying $\mathcal{B}(\{\lambda_k\alpha_k\}_{k\in\mathbf{N}}) < \infty$. Then $\sum_{k\in\mathbf{N}} \lambda_k\alpha_k$ converges in $H_L^{p(\cdot)}(\mathbf{R}^n)$ and*

$$\left\|\sum_{k\in\mathbf{N}} \lambda_k\alpha_k\right\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} \le C\mathcal{B}(\{\lambda_k\alpha_k\}_{k\in\mathbf{N}})$$

with C being a positive constant independent of $\{\lambda_k\alpha_k\}_{k\in\mathbf{N}}$.

The proof of Theorem 4.3 strongly depends on several auxiliary estimates and will be presented later. The following Lemma 3.14 is just [55, Lemma 2.6] (For the case when $p_- > 1$, see also [28, Corollary 3.4]).

Lemma 3.14. *Let $p(\cdot) \in C^{\log}(\mathbf{R}^n)$. Then there exists a positive constant C such that, for all cubes Q_1 and Q_2 satisfying $Q_1 \subset Q_2$,*

$$C^{-1}\left(\frac{|Q_1|}{|Q_2|}\right)^{1/p_-} \le \frac{\|\chi_{Q_1}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_{Q_2}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \le C\left(\frac{|Q_1|}{|Q_2|}\right)^{1/p_+},$$

where p_- and p_+ are as in (2.1).

The following Fefferman–Stein vector-valued inequality of the Hardy–Littlewood maximal operator $\mathcal{M}$ on the space $L^{p(\cdot)}(\mathbf{R}^n)$ was obtained in [15, Corollary 2.1].

Lemma 3.15. *Let $r \in (1,\infty)$ and $p(\cdot) \in C^{\log}(\mathbf{R}^n)$. If $p_- \in (1,\infty)$ with p_- as in (2.1), then there exists a positive constant C such that, for all sequences $\{f_j\}_{j=1}^\infty$ of measurable functions,*

$$\left\|\left\{\sum_{j=1}^\infty [\mathcal{M}(f_j)]^r\right\}^{1/r}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le C\left\|\left(\sum_{j=1}^\infty |f_j|^r\right)^{1/r}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}.$$

Remark 3.16. Let $k \in \mathbf{N}$ and $p(\cdot) \in C^{\log}(\mathbf{R}^n)$. Then, by Lemma 3.15 and the fact that, for all cubes $Q \subset \mathbf{R}^n$, $r \in (0, p_-)$, $\chi_{2^kQ} \le 2^{kn/r}[\mathcal{M}(\chi_Q)]^{1/r}$, we conclude that there exists a positive constant C such that, for any $\{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$ and cubes $\{Q_j\}_{j\in\mathbf{N}}$ of $\mathbf{R}^n$,

$$\mathcal{A}(\{\lambda_j\}_{j\in\mathbf{N}}, \{2^kQ_j\}_{j\in\mathbf{N}}) \le C2^{kn(\frac{1}{r}-\frac{1}{p_+})}\mathcal{A}(\{\lambda_j\}_{j\in\mathbf{N}}, \{Q_j\}_{j\in\mathbf{N}}),$$

where p_- and p_+ are as in (2.1).

Proposition 3.17. *Let $p(\cdot) \in C^{\log}(\mathbf{R}^n)$ with $p_- \in (\frac{n}{n+\theta(L)}, \infty)$ and $q \in (1,\infty)$, where p_- and $\theta(L)$ are as in (2.1) and (2.5), respectively. Then*

(i) *the operator π_L is a bounded linear operator from $T_2^q(\mathbf{R}_+^{n+1})$ to $L^q(\mathbf{R}^n)$;*

(ii) *the operator π_L, well defined on the space $T_{2,c}^{p(\cdot)}(\mathbf{R}_+^{n+1})$, extends to a bounded linear operator from $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$ to $H_L^{p(\cdot)}(\mathbf{R}^n)$.*

Proof. To prove this proposition, it suffices to show (ii), since (i) is just [48, Lemma 3.4(a)]. Noticing that, due to Corollary 3.7, $T_{2,c}^{p(\cdot)}(\mathbf{R}_+^{n+1})$ is a dense subset of $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$, to prove (ii), we only need to show that π_L maps $T_{2,c}^{p(\cdot)}(\mathbf{R}_+^{n+1})$ continuously into $H_L^{p(\cdot)}(\mathbf{R}^n)$.

To this end, let $f \in T_{2,c}^{p(\cdot)}(\mathbf{R}_+^{n+1})$. Then, by Proposition 3.9, we know that $f \in T_{2,c}^{2}(\mathbf{R}_+^{n+1})$ and hence π_L is well defined on $T_{2,c}^{p(\cdot)}(\mathbf{R}_+^{n+1})$ by (i). This, combined with Lemma 3.3, Corollary 3.4 and Remark 3.6, implies that there exist sequences $\{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$ and $\{a_j\}_{j\in\mathbf{N}}$ of $(p(\cdot),\infty)$-atoms such that, for each $j\in\mathbf{N}$, supp $a_j \subset \widehat{Q}_j$ with some cube $Q_j \subset \mathbf{R}^n$, $f=\sum_{j\in\mathbf{N}}\lambda_j a_j$ in both $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$ and $T_2^{2}(\mathbf{R}_+^{n+1})$, and $\mathcal{A}(\{\lambda_j\}_{j\in\mathbf{N}},\{Q_j\}_{j\in\mathbf{N}}) \lesssim \|f\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})}$. Thus, it follows from (i) that, for all $N\in\mathbf{N}$,

$$\left\|\pi_L\left(f-\sum_{j=1}^N \lambda_j a_j\right)\right\|_{L^2(\mathbf{R}^n)} \lesssim \left\|\sum_{j=N+1}^\infty \lambda_j a_j\right\|_{T_2^2(\mathbf{R}_+^{n+1})} \to 0$$

as $N\to\infty$; furthermore,

$$\begin{aligned}\pi_L(f) &= \lim_{N\to\infty}\sum_{j=1}^N \pi_L(\lambda_j a_j)\\ &= \sum_{j\in\mathbf{N}}\lambda_j C_{(m,s)}\int_0^\infty Q_{s,t^m}(I-P_{s_0,t^m})(a_j(\cdot,t))\,\frac{dt}{t} =: \sum_{j\in\mathbf{N}}\lambda_j\alpha_j\end{aligned}\tag{3.10}$$

in $L^2(\mathbf{R}^n)$, where s_0 is as in (3.8) and $s\in[s_0,\infty)$.

Next, we prove $\|S_L(\pi_L(f))\|_{L^{p(\cdot)}(\mathbf{R}^n)} \lesssim \|f\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})}$. Observe that, for almost every $x\in\mathbf{R}^n$,

$$S_L(\pi_L(f))(x) = S_L\left(\sum_{j\in\mathbf{N}}\lambda_j\alpha_j\right)(x) \le \sum_{j\in\mathbf{N}} S_L(\lambda_j\alpha_j)(x)$$

due to (3.10), the Fatou lemma and the fact that S_L is bounded on $L^2(\mathbf{R}^n)$ (see (2.9)). Then, by Remark 2.1(i) and the Fatou lemma of $L^{p(\cdot)}(\mathbf{R}^n)$ (see [14, Theorem 2.61]), we see that

$$\|S_L(\pi_L(f))\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le \left\{\sum_{i=0}^\infty\left\|\left(\sum_{j\in\mathbf{N}}\left[S_L(\lambda_j\alpha_j)\chi_{U_i(Q_j)}\right]^{\underline{p}}\right)^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}^{\underline{p}}\right\}^{\frac{1}{\underline{p}}},\tag{3.11}$$

where $\underline{p}$ is as in (3.2) and, for any $j\in\mathbf{N}$, $U_0(Q_j):=4Q_j$ and $U_i(Q_j):=2^{i+2}Q_j\backslash(2^{i+1}Q_j)$ for all $i\in\mathbf{N}$.

By (2.9), (i) and Lemma 3.14, we find that, for all $q\in(1,\infty)$,

$$\|S_L(\alpha_j)\|_{L^q(4Q_j)} \lesssim \|\alpha_j\|_{L^q(\mathbf{R}^n)} \sim \|\pi_L(a_j)\|_{L^q(\mathbf{R}^n)} \lesssim \|a_j\|_{T_2^q(\mathbf{R}_+^{n+1})} \lesssim |Q_j|^{\frac{1}{q}}\|\chi_{4Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}^{-1}.$$

From this, Lemma 3.5 and Remark 3.16, we deduce that

$$\begin{aligned}&\left\|\left(\sum_{j\in\mathbf{N}}\left[|\lambda_j|S_L(\alpha_j)\chi_{4Q_j}\right]^{\underline{p}}\right)^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}\\ &\quad\lesssim \mathcal{A}\left(\{\lambda_j\}_{j\in\mathbf{N}},\{4Q_j\}_{j\in\mathbf{N}}\right) \lesssim \mathcal{A}\left(\{\lambda_j\}_{j\in\mathbf{N}},\{Q_j\}_{j\in\mathbf{N}}\right) \lesssim \|f\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})}.\end{aligned}\tag{3.12}$$

Since $p_- \in (\frac{n}{n+\theta(L)}, \infty)$, we choose $\varepsilon \in (0, \theta(L))$ such that $p_- \in (\frac{n}{n+\varepsilon}, \infty)$. For $i \in \mathbf{N}$, by [30, (4.12)], we know that, for all $x \in (4Q_j)^\complement$,

$$S_L(\alpha_j)(x) \lesssim (r_{Q_j})^{\varepsilon+\frac{n}{2}} |x - x_{Q_j}|^{-(n+\varepsilon)} \|a_j\|_{T_2^2(\mathbf{R}_+^{n+1})}.$$

Then, by this, the Hölder inequality and Lemma 3.14, we further find that, for any $q \in (p_+, \infty) \cap [2, \infty)$ with p_+ as in (2.1),

$$\begin{aligned}\|S_L(\alpha_j)\|_{L^q(U_i(Q_j))} &\lesssim 2^{i(\frac{n}{q}-n-\varepsilon)} |Q_j|^{\frac{1}{q}-\frac{1}{2}} \|a_j\|_{T_2^2(\mathbf{R}_+^{n+1})} \lesssim 2^{i(\frac{n}{q}-n-\varepsilon)} \|a_j\|_{T_2^q(\mathbf{R}_+^{n+1})} \\ &\lesssim 2^{-i(n+\varepsilon)} |2^i Q_j|^{\frac{1}{q}} \|\chi_{Q_j}\|^{-1}_{L^{p(\cdot)}(\mathbf{R}^n)}.\end{aligned} \tag{3.13}$$

Observe that, for all $r \in (\frac{n}{n+\varepsilon}, p_-)$, $\chi_{2^i Q_j} \le 2^{\frac{n}{r} i} [\mathcal{M}(\chi_{Q_j})]^{\frac{1}{r}}$. Thus, from this, (3.13) and Lemmas 3.5 and 3.15, we deduce that

$$\begin{aligned}&\left\| \left(\sum_{j\in\mathbf{N}} \left[|\lambda_j| S_L(\alpha_j) \chi_{U_i(Q_j)} \right]^{\underline{p}} \right)^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \\ &\quad \lesssim 2^{-i(n+\varepsilon)} \left\| \left\{ \sum_{j\in\mathbf{N}} \left[\frac{|\lambda_j| \chi_{2^i Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \right]^{\underline{p}} \right\}^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \\ &\quad \lesssim 2^{-i(n+\varepsilon-n/r)} \mathcal{A}(\{\lambda_j\}_{j\in\mathbf{N}}, \{Q_j\}_{j\in\mathbf{N}}) \lesssim 2^{-i(n+\varepsilon-n/r)} \|f\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})}.\end{aligned} \tag{3.14}$$

Combining (3.11), (3.12) and (3.14), together with $r > \frac{n}{n+\varepsilon}$, we conclude that

$$\|S_L(\pi_L(f))\|_{L^{p(\cdot)}(\mathbf{R}^n)} \lesssim \left\{ \sum_{i=0}^{\infty} 2^{-i(n+\varepsilon-\frac{n}{r})\underline{p}} \|f\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})} \right\}^{1/\underline{p}} \lesssim \|f\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})},$$

which implies that π_L is a bounded linear operator from $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$ to $H_L^{p(\cdot)}(\mathbf{R}^n)$ and hence completes the proof of Proposition 3.17. □

We now turn to the proof of Theorem 3.13.

Proof of Theorem 3.13. We first prove (i). Let $C_{(m,s)}$ be the constant as in (3.9) and $f \in H_L^{p(\cdot)}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$. Then, by Remark 3.10, we see that

$$f = C_{(m,s)} \int_0^\infty Q_{s,t^m}(I - P_{s_0,t^m}) Q_{t^m} f \, \frac{dt}{t} = \pi_L(Q_{t^m} f) \tag{3.15}$$

in $L^2(\mathbf{R}^n)$, where s_0 is as in (3.8) and $s \in [s_0, \infty)$. Since $f \in H_L^{p(\cdot)}(\mathbf{R}^n)$, it follows that $Q_{t^m} f \in T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$. Thus, by Lemma 3.3 and Corollary 3.4, we find that $Q_{t^m} f = \sum_{j\in\mathbf{N}} \lambda_j a_j$ in the sense of both pointwise and in $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$, where $\{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$ and $\{a_j\}_{j\in\mathbf{N}}$ are $(p(\cdot), \infty)$-atoms satisfying that, for each $j \in \mathbf{N}$, supp $a_j \subset Q_j$ with some cube $Q_j \subset \mathbf{R}^n$, and

$$\mathcal{A}(\{\lambda_j\}_{j\in\mathbf{N}}, \{Q_j\}_{j\in\mathbf{N}}) \lesssim \|Q_{t^m} f\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})} \sim \|f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)}.$$

For any $j \in \mathbf{N}$, let $\alpha_j := \pi_L(a_j)$. Then α_j is a $(p(\cdot), s, L)$-molecule and, by (3.15) and Proposition 3.17, we conclude that

$$f = \pi_L(Q_{t^m} f) = \sum_{j\in\mathbf{N}} \lambda_j \pi_L(a_j) =: \sum_{j\in\mathbf{N}} \lambda_j \alpha_j$$

in both $L^2(\mathbf{R}^n)$ and $H_L^{p(\cdot)}(\mathbf{R}^n)$.

Now, for any $f \in H_L^{p(\cdot)}(\mathbf{R}^n)$, since $H_L^{p(\cdot)}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ is dense in $H_L^{p(\cdot)}(\mathbf{R}^n)$, it follows that there exists a sequence $\{f_k\}_{k\in\mathbf{N}} \subset [H_L^{p(\cdot)}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)]$ such that, for all $k \in \mathbf{N}$,

$$\|f - f_k\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} \le 2^{-k}\|f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)}.$$

Let $f_0 := 0$. Then

$$f = \sum_{k\in\mathbf{N}} (f_k - f_{k-1}) \quad \text{in } H_L^{p(\cdot)}(\mathbf{R}^n). \tag{3.16}$$

From the above argument, we deduce that, for each $k \in \mathbf{N}$, there exist $\{\lambda_j^k\}_{k\in\mathbf{N}} \subset \mathbf{C}$ and a sequence $\{\alpha_j^k\}_{j\in\mathbf{N}}$ of $(p(\cdot), s, L)$-molecules such that

$$f_k - f_{k-1} = \sum_{j\in\mathbf{N}} \lambda_j^k \alpha_j^k \quad \text{in } H_L^{p(\cdot)}(\mathbf{R}^n) \tag{3.17}$$

and

$$\mathcal{A}(\{\lambda_j^k\}_{j\in\mathbf{N}}, \{Q_j^k\}_{j\in\mathbf{N}}) \lesssim \|f_k - f_{k-1}\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} \lesssim 2^{-k}\|f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)},$$

where, for any k, $j \in \mathbf{N}$, Q_j^k denotes the cube appearing in the definition of the $(p(\cdot), s, L)$-molecule α_j^k. By this, the Minkowski inequality, (ii) and (iii) of Remark 2.1 and (3.6), we see that

$$\begin{aligned}
&\int_{\mathbf{R}^n} \left\{ \sum_{k\in\mathbf{N}} \sum_{j\in\mathbf{N}} \left[\frac{|\lambda_j^k| \chi_{Q_j^k}}{\|f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} \|\chi_{Q_j^k}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \right]^{\underline{p}} \right\}^{\frac{p(x)}{\underline{p}}} dx \\
&\le \int_{\mathbf{R}^n} \left\{ \sum_{k\in\mathbf{N}} \left[\sum_{j\in\mathbf{N}} \left(\frac{|\lambda_j^k| \chi_{Q_j^k}}{\|f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} \|\chi_{Q_j^k}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \right)^{\underline{p}} \right]^{p(x)} \right\}^{\frac{1}{\underline{p}}} dx \\
&\lesssim \left(\sum_{k\in\mathbf{N}} \left\{ \int_{\mathbf{R}^n} 2^{-kp(x)} \left[\sum_{j\in\mathbf{N}} \left(\frac{|\lambda_j^k| \chi_{Q_j^k}}{2^{-k}\|f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} \|\chi_{Q_j^k}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \right)^{\underline{p}} \right]^{\frac{p(x)}{\underline{p}}} dx \right\}^{\underline{p}} \right)^{\frac{1}{\underline{p}}} \\
&\lesssim \left(\sum_{k\in\mathbf{N}} 2^{-k\underline{p}^2} \right)^{1/\underline{p}} < \infty,
\end{aligned}$$

which implies that $\mathcal{B}\left(\{\lambda_j^k \alpha_j^k\}_{j,k\in\mathbf{N}}\right) \lesssim \|f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)}$. Moreover, by (3.16) and (3.17), we conclude that

$$f = \sum_{k\in\mathbf{N}} \sum_{j\in\mathbf{N}} \lambda_j^k \alpha_j^k$$

in $H_L^{p(\cdot)}(\mathbf{R}^n)$ and hence the proof of (i) is completed.

Next, we show (ii). Without loss of generality, we may assume that, for each $k \in \mathbf{N}$, $\alpha_k := \pi_L(a_k)$, where a_k is a $(p(\cdot), \infty)$-atom supported on $\widehat{Q}_k$ for some cube $Q_k \subset \mathbf{R}^n$. Then, from Proposition 3.17(ii) and Corollary 3.7, we deduce that, for all

$N_1,\ N_2 \in \mathbf{N}$ with $N_1 < N_2$,

$$\left\| \sum_{k=N_1}^{N_2} \lambda_k \alpha_k \right\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} \sim \left\| \pi_L \left(\sum_{k=N_1}^{N_2} \lambda_k a_k \right) \right\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} \lesssim \left\| \sum_{k=N_1}^{N_2} \lambda_k a_k \right\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})}$$
$$\lesssim \left\| \left\{ \sum_{k=N_1}^{N_2} \left[\frac{|\lambda_k| \chi_{Q_k}}{\|\chi_{Q_k}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \right]^{\underline{p}} \right\}^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}(\mathbf{R}^n)},$$

which tends to zero as $N_1,\ N_2 \to \infty$ due to the dominated convergence theorem. Thus, $\sum_{k\in\mathbf{N}} \lambda_k \alpha_k$ converges in $H_L^{p(\cdot)}(\mathbf{R}^n)$ and, by the Fatou lemma of $L^{p(\cdot)}(\mathbf{R}^n)$, Proposition 3.17(ii) and Corollary 3.7, we further know that

$$\left\| \sum_{k=1}^{\infty} \lambda_k \alpha_k \right\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} \le \liminf_{N\to\infty} \left\| \sum_{k=1}^{N} \lambda_k \alpha_k \right\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} = \liminf_{N\to\infty} \left\| \pi_L \left(\sum_{k=1}^{N} \lambda_k a_k \right) \right\|_{H_L^{p(\cdot)}(\mathbf{R}^n)}$$
$$\lesssim \liminf_{N\to\infty} \left\| \sum_{k=1}^{N} \lambda_k a_k \right\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})} \lesssim \liminf_{N\to\infty} \left\| \left\{ \sum_{k=1}^{N} \left[\frac{|\lambda_k| \chi_{Q_k}}{\|\chi_{Q_k}\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \right]^{\underline{p}} \right\}^{\frac{1}{\underline{p}}} \right\|_{L^{p(\cdot)}(\mathbf{R}^n)}$$
$$\lesssim \mathcal{B}(\{\lambda_k \alpha_k\}_{k\in\mathbf{N}}),$$

which completes the proof of Theorem 3.13. □

In what follows, for all $s \in [s_0, \infty)$ with s_0 as in (3.8) and $p(\cdot) \in \mathcal{P}(\mathbf{R}^n)$, denote by $H_{L,\mathrm{fin}}^{p(\cdot)}(\mathbf{R}^n)$ the *set of finite linear combinations of $(p(\cdot), s, L)$-molecules*. For any $f \in H_{L,\mathrm{fin}}^{p(\cdot)}(\mathbf{R}^n)$, the quasi-norm is given by

$$\|f\|_{H_{L,\mathrm{fin}}^{p(\cdot)}(\mathbf{R}^n)} := \inf \left\{ \mathcal{B}(\{\lambda_j \alpha_j\}_{j=1}^N) \colon N \in \mathbf{N},\ f = \sum_{j=1}^{N} \lambda_j \alpha_j \right\},$$

where the infimum is taken over all finite molecular decompositions of f.

Corollary 3.18. *Let $p(\cdot) \in C^{\log}(\mathbf{R}^n)$ satisfy $p_+ \in (0,1]$ and $p_- \in (\frac{n}{n+\theta(L)}, 1]$, where p_-, p_+ and $\theta(L)$ are, respectively, as in (2.1) and (2.5). Then $H_{L,\mathrm{fin}}^{p(\cdot)}(\mathbf{R}^n)$ is dense in $H_L^{p(\cdot)}(\mathbf{R}^n)$.*

Proof. Let $f \in H_L^{p(\cdot)}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$. Then $Q_{t^m} f \in T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$ and hence, by Lemma 3.3, we have $Q_{t^m} f = \sum_k \lambda_k a_k$ in $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$, where $\{\lambda_k\}_{k\in\mathbf{N}} \subset \mathbf{C}$ and $\{a_k\}_{k\in\mathbf{N}}$ are $(p(\cdot), \infty)$-atoms. For every $k \in \mathbf{N}$, let $\alpha_k := \pi_L(a_k)$. Then $\{\alpha_k\}_{k\in\mathbf{N}}$ are $(p(\cdot), s, L)$-molecules. Thus, by Proposition 3.17(ii), we conclude that, for all $N \in \mathbf{N}$,

$$\left\| f - \sum_{k=1}^{N} \lambda_k \alpha_k \right\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} = \left\| \pi_L \left(Q_{t^m} f - \sum_{k=1}^{N} \lambda_k a_k \right) \right\|_{H_L^{p(\cdot)}(\mathbf{R}^n)}$$
$$\lesssim \left\| Q_{t^m} f - \sum_{k=1}^{N} \lambda_k a_k \right\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})} \to 0,$$

as $N \to \infty$, which implies that $H_{L,\mathrm{fin}}^{p(\cdot)}(\mathbf{R}^n)$ is dense in $H_L^{p(\cdot)}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ and hence in $H_L^{p(\cdot)}(\mathbf{R}^n)$. This finishes the proof of Corollary 3.18. □

4. BMO-type spaces and the duality of $H_L^{p(\cdot)}(\mathbf{R}^n)$

In this section, we mainly consider the duality of $H_L^{p(\cdot)}(\mathbf{R}^n)$. To this end, motivated by [30], we introduce the space $\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)$ associated with the operator L and the variable exponent $p(\cdot)$.

Definition 4.1. Let L satisfy Assumptions (A) and (B), $p(\cdot) \in C^{\log}(\mathbf{R}^n)$ with $p_+ \in (0,1]$ and $s \in [s_0,\infty)$, where p_+ and s_0 are, respectively, as in (2.1) and (3.8). Then the BMO-*type space* $\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)$ is defined to be the set of all functions $f \in \mathcal{M}(\mathbf{R}^n)$ such that $\|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)} < \infty$, where

$$\|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)} := \sup_{Q\subset\mathbf{R}^n} \frac{|Q|^{1/2}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \left\{\int_Q |f(x) - P_{s,(r_Q)^m} f(x)|^2\,dx\right\}^{\frac{1}{2}}$$

and the supremum is taken over all cubes Q of $\mathbf{R}^n$.

Remark 4.2. (i) The space $(\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n), \|\cdot\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)})$ is a vector space with the semi-norm vanishing on the space $\mathcal{K}_{(L,s)}(\mathbf{R}^n)$ which is defined by

$$\begin{aligned}\mathcal{K}_{(L,s)}(\mathbf{R}^n) :=\{f \in \mathcal{M}(\mathbf{R}^n)\colon\ & P_{s,t}f(x) = f(x) \text{ for almost every } x \in \mathbf{R}^n \\ & \text{and all } t \in (0,\infty)\}.\end{aligned}$$

In this article, the space $\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)$ is understood to be modulo $\mathcal{K}_{(L,s)}(\mathbf{R}^n)$; see [23, Section 6] for a discussion of $\mathcal{K}_{(L,0)}(\mathbf{R}^n)$ when L is a second order elliptic operator of divergence form or a Schrödinger operator.

(ii) If $p(\cdot) \equiv 1$ and $s = 0$, then $\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)$ is just $\mathrm{BMO}_L(\mathbf{R}^n)$ introduced by Duong and Yan [23]. If $p(\cdot) \in \mathcal{P}(\mathbf{R}^n)$ is defined by $\frac{1}{p(\cdot)} := \alpha + \frac{1}{2}$ for some constant $\alpha \in (0, \frac{\theta(L)}{n})$, then $\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)$ becomes the space $\mathfrak{L}_L(\alpha,2,s)$ studied in [48].

Now we state the main result of this section as follows.

Theorem 4.3. *Let $p(\cdot) \in C^{\log}(\mathbf{R}^n)$ satisfy $p_+ \in (0,1]$ and $p_- \in (\frac{n}{n+\theta(L)},1]$ with p_+, p_- and $\theta(L)$, respectively, as in (2.1) and (2.5). Let s_0 be as in (3.8) and L^* denote the adjoint operator of L. Then $(H_L^{p(\cdot)}(\mathbf{R}^n))^*$ coincides with $\mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbf{R}^n)$ in the following sense:*

(i) *If $g \in \mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbf{R}^n)$, then the linear mapping ℓ, which is initially defined on $H^{p(\cdot)}_{L,\mathrm{fin}}(\mathbf{R}^n)$ by*

$$\ell_g(f) := \int_{\mathbf{R}^n} f(x)g(x)\,dx, \tag{4.1}$$

extends to a bounded linear functional on $H_L^{p(\cdot)}(\mathbf{R}^n)$ and

$$\|\ell_g\|_{(H_L^{p(\cdot)}(\mathbf{R}^n))^*} \le C\|g\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbf{R}^n)},$$

where C is a positive constant independent of g.

(ii) *Conversely, let ℓ be a bounded linear functional on $H_L^{p(\cdot)}(\mathbf{R}^n)$. Then ℓ has the form as in (4.1) with a unique $g \in \mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbf{R}^n)$ for all $f \in H^{p(\cdot)}_{L,\mathrm{fin}}(\mathbf{R}^n)$ and*

$$\|g\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbf{R}^n)} \le \widetilde{C}\|\ell\|_{(H_L^{p(\cdot)}(\mathbf{R}^n))^*},$$

where $\widetilde{C}$ is a positive constant independent of ℓ.

Remark 4.4. Let $p(\cdot)$, s_0, L, L^* and $\theta(L)$ be as in Theorem 4.3.

(i) If $f \in H_L^{p(\cdot)}(\mathbf{R}^n)$, then, from Theorem 4.3, we deduce that f satisfies the cancelation condition $\int_{\mathbf{R}^n} f(x)g(x)\,dx = 0$ for all $g \in \mathcal{K}_{L^*,s_0}(\mathbf{R}^n)$, since, if $g \in \mathcal{K}_{L^*,s_0}(\mathbf{R}^n)$, then $\|g\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbf{R}^n)} = 0$. Observe that, if $g \in \mathcal{K}_{L^*,s_0}(\mathbf{R}^n)$, then g is not necessary to be zero almost everywhere and hence, if f is a smooth function with compact support, then $\int_{\mathbf{R}^n} f(x)g(x)\,dx$ may not equal zero. Therefore, smooth functions with compact supports are not necessary to be in $H_L^{p(\cdot)}(\mathbf{R}^n)$; see also [23, p. 962].

(ii) Observe that, by the proof of Corollary 3.18, we see that

$$H^{p(\cdot)}_{L,\mathrm{fin}}(\mathbf{R}^n) \subset [H_L^{p(\cdot)}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)]$$

and $H^{p(\cdot)}_{L,\mathrm{fin}}(\mathbf{R}^n)$ is dense in $H_L^{p(\cdot)}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$. From this, it follows that, if we require that (4.1) holds true for all $f \in H_L^{p(\cdot)}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ instead of all $f \in H^{p(\cdot)}_{L,\mathrm{fin}}(\mathbf{R}^n)$, then all conclusions of Theorem 4.3 also hold true.

To prove Theorem 4.3, we need some preparations.

Proposition 4.5. *Let $p(\cdot) \in C^{\log}(\mathbf{R}^n)$ satisfy $p_+ \in (0,1]$ and $p_- \in (\frac{n}{n+\theta(L)}, 1]$, and $s \in [s_0, \infty)$, where p_+, p_-, $\theta(L)$ and s_0 are, respectively, as in* (2.1), (2.5) *and* (3.8). *Then there exists a positive constant C such that, for all $t \in (0,\infty)$, $K \in (1,\infty)$, $f \in \mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$, when $p_+ = 1$,*

$$\begin{aligned} &|P_{s,t}f(x) - P_{s,Kt}f(x)| \\ &\le C(1+\log_2 K)\left\|\chi_{Q(x,(Kt)^{\frac{1}{m}})}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} |Q(x,(Kt)^{\frac{1}{m}})|^{-1}\|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)} \end{aligned} \tag{4.2}$$

and, when $p_+ \in (0,1)$,

$$\begin{aligned} &|P_{s,t}f(x) - P_{s,Kt}f(x)| \\ &\le C\left\|\chi_{Q(x,(Kt)^{\frac{1}{m}})}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} |Q(x,(Kt)^{\frac{1}{m}})|^{-1}\|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)}. \end{aligned} \tag{4.3}$$

Proof. Without loss of generality, we may assume that $\|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)} = 1$. We claim that, for all $t,\ v \in (0,\infty)$ with $\frac{t}{2} \le v \le 2t$, and $x \in \mathbf{R}^n$,

$$|P_{s,t}f(x) - P_{s,v}f(x)| \lesssim \left\|\chi_{Q(x,t^{\frac{1}{m}})}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} |Q(x,t^{\frac{1}{m}})|^{-1}. \tag{4.4}$$

If this claim holds true, then, by Lemma 3.14, we see that

$$\begin{aligned} |P_{s,t}f(x) - P_{s,Kt}f(x)| &\le \sum_{i=0}^{l-1} \left|P_{s,2^i t}f(x) - P_{s,2^{i+1}t}f(x)\right| + \left|P_{s,2^l t}f(x) - P_{s,Kt}f(x)\right| \\ &\lesssim \sum_{i=0}^{l-1} \frac{\|\chi_{Q(x,(2^i t)^{\frac{1}{m}})}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{|Q(x,(2^i t)^{\frac{1}{m}})|} + \frac{\|\chi_{Q(x,(Kt)^{\frac{1}{m}})}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{|Q(x,(Kt)^{\frac{1}{m}})|} \\ &\lesssim \left\{\sum_{i=0}^{l-1}\left[\frac{|Q(x,(2^i t)^{\frac{1}{m}})|}{|Q(x,(Kt)^{\frac{1}{m}})|}\right]^{\frac{1}{p_+}-1} + 1\right\} \frac{\|\chi_{Q(x,(Kt)^{\frac{1}{m}})}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{|Q(x,(Kt)^{\frac{1}{m}})|}, \end{aligned}$$

where $l := \lfloor \log_2 K \rfloor$. By this, we further conclude that, when $p_+ = 1$,

$$|P_{s,t}f(x) - P_{s,Kt}f(x)| \lesssim (1+\log_2 K)\left\|\chi_{Q(x,(Kt)^{\frac{1}{m}})}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} |Q(x,(Kt)^{\frac{1}{m}})|^{-1}$$

and, when $p_+ \in (0,1)$,

$$|P_{s,t}f(x) - P_{s,Kt}f(x)| \lesssim \left\|\chi_{Q(x,(Kt)^{\frac{1}{m}})}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} |Q(x,(Kt)^{\frac{1}{m}})|^{-1},$$

which implies that (4.2) and (4.3) hold true.

Therefore, to complete the proof of this proposition, it remains to prove the above claim. By the commutative properties of semigroups, we have

$$P_{s,t}f - P_{s,v}f = P_{s,t}(f - P_{s,v}f) - P_{s,v}(f - P_{s,t}f). \tag{4.5}$$

Since $\theta(L) \in (n[\frac{1}{p_-} - 1], \infty)$ and $p_+ \in (0,1]$, it follows that there exists $\varepsilon \in (0, \theta(L))$ such that

$$\varepsilon > n\left(\frac{1}{p_-} - 1\right) > n\left(\frac{1}{p_-} - \frac{1}{p_+}\right). \tag{4.6}$$

From (2.8), Assumption (A), the Hölder inequality, Lemma 3.14 and the fact that $\frac{t}{2} \le v \le 2t$, we deduce that, for all $x \in \mathbf{R}^n$,

$$\begin{aligned}
|P_{s,t}(f - P_{s,v}f)(x)| &\lesssim t^{-\frac{n}{m}} \int_{\mathbf{R}^n} g\left(\frac{|x-y|}{t^{1/m}}\right) |f(y) - P_{s,v}f(y)|\,dy\\
&\lesssim \left\{v^{-\frac{n}{m}} \int_{Q(x,v^{\frac{1}{m}})} |f(y) - P_{s,v}f(y)|^2\,dy\right\}^{1/2}\\
&\quad + v^{-\frac{n}{m}} \sum_{i=1}^{\infty} \int_{S_i} g\left(\frac{|x-y|}{t^{\frac{1}{m}}}\right) |f(y) - P_{s,v}f(y)|\,dy\\
&\lesssim \left\|\chi_{Q(x,t^{\frac{1}{m}})}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} |Q(x,t^{\frac{1}{m}})|^{-1} \|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)}\\
&\quad + \sum_{i=1}^{\infty} 2^{-\frac{n+\varepsilon}{m}i} v^{-\frac{n}{m}} \int_{Q(x,(2^i v)^{\frac{1}{m}})} |f(y) - P_{s,v}f(y)|\,dy,
\end{aligned} \tag{4.7}$$

where, for each $i \in \mathbf{N}$, $S_i := Q(x,(2^i v)^{\frac{1}{m}}) \backslash Q(x,(2^{i-1}v)^{\frac{1}{m}})$. Notice that, for any $i \in \mathbf{N}$, there exists a collection $\{Q_{i,j}\}_{j=1}^{N_i}$ of cubes with $N_i \sim 2^{ni/m}$ such that $\ell(Q_{i,j}) = v^{1/m}$ and $Q(x,(2^i v)^{1/m}) \subset \bigcup_{j=1}^{N_i} Q_{i,j}$. Thus, by the Hölder inequality and Lemma 3.14, we find that

$$\begin{aligned}
\int_{Q(x,(2^i v)^{\frac{1}{m}})} |f(y) - P_{s,v}f(y)|\,dy &\le \sum_{j=1}^{N_i} \int_{Q_{i,j}} |f(y) - P_{s,v}f(y)|\,dy\\
&\lesssim \sum_{j=1}^{N_i} \left\{\int_{Q_{i,j}} |f(y) - P_{s,v}f(y)|^2\,dy\right\}^{\frac{1}{2}} |Q_{i,j}|^{\frac{1}{2}}\\
&\lesssim \|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)} \sum_{j=1}^{N_i} \|\chi_{Q_{i,j}}\|_{L^{p(\cdot)}(\mathbf{R}^n)}\\
&\lesssim 2^{\frac{n}{m}i} 2^{-\frac{n}{m}(\frac{1}{p_+} - \frac{1}{p_-})i} \left\|\chi_{Q(x,v^{\frac{1}{m}})}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)},
\end{aligned}$$

which, together with Lemma 3.14 again and (4.6), implies that

$$\begin{aligned}
&\sum_{i=1}^{\infty} 2^{-i\frac{n+\varepsilon}{m}} v^{-\frac{n}{m}} \int_{Q(x,(2^i v)^{\frac{1}{m}})} |f(y) - P_{s,v}f(y)|\,dy \\
&\lesssim v^{-\frac{n}{m}} \sum_{i=1}^{\infty} 2^{-\frac{n}{m}[\frac{\varepsilon}{n}+\frac{1}{p_+}-\frac{1}{p_-}]i} \left\|\chi_{Q(x,t^{\frac{1}{m}})}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \sim \left\|\chi_{Q(x,t^{\frac{1}{m}})}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} |Q(x,t^{\frac{1}{m}})|^{-1}.
\end{aligned}$$

By this and (4.7), we further conclude that, for all $x \in \mathbf{R}^n$,

$$|P_{s,t}(f - P_{s,v}f)(x)| \lesssim \left\|\chi_{Q(x,t^{\frac{1}{m}})}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} |Q(x,t^{\frac{1}{m}})|^{-1}. \tag{4.8}$$

By an argument similar to that used in the proof of (4.8), we also see that, for all $x \in \mathbf{R}^n$,

$$|P_{s,v}(f - P_{s,t}f)(x)| \lesssim \left\|\chi_{Q(x,t^{\frac{1}{m}})}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} |Q(x,t^{\frac{1}{m}})|^{-1},$$

which, combined with (4.5) and (4.8), implies that (4.4) holds true. This finishes the proof of Proposition 4.5. □

Proposition 4.6. *Let $p(\cdot)$ and s be as in Proposition 4.5. Then, for any $\delta \in (n[\frac{1}{p_-}-1],\infty)$, there exists a positive constant C such that, for all $f \in \mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)$, $t \in (0,\infty)$ and $x \in \mathbf{R}^n$,*

$$\int_{\mathbf{R}^n} \frac{|f(y) - P_{s,t}f(y)|}{(t^{\frac{1}{m}} + |x-y|)^{n+\delta}}\,dy \le C t^{-\frac{n+\delta}{m}} \left\|\chi_{Q(x,t^{\frac{1}{m}})}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)}.$$

Proof. For all $t \in (0,\infty)$ and $x \in \mathbf{R}^n$, we write

$$\int_{\mathbf{R}^n} \frac{|f(y) - P_{s,t}f(y)|}{(t^{\frac{1}{m}} + |x-y|)^{n+\delta}}\,dy = \int_{Q(x,t^{\frac{1}{m}})} \frac{|f(y) - P_{s,t}f(y)|}{(t^{\frac{1}{m}} + |x-y|)^{n+\delta}}\,dy + \int_{[Q(x,t^{\frac{1}{m}})]^{\complement}} \cdots =: \mathrm{I}_1 + \mathrm{I}_2.$$

Obviously, by the Hölder inequality, we easily see that

$$\mathrm{I}_1 \lesssim t^{-\frac{n+\delta}{m}} \int_{Q(x,t^{\frac{1}{m}})} |f(y) - P_{s,t}f(y)|\,dy \lesssim t^{-\frac{n+\delta}{m}} \left\|\chi_{Q(x,t^{\frac{1}{m}})}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)}.$$

To estimate I_2, we first notice that

$$\begin{aligned}
\mathrm{I}_2 &\lesssim \sum_{k=1}^{\infty} (2^k t)^{-\frac{n+\delta}{m}} \int_{Q(x,(2^k t)^{\frac{1}{m}})} |f(y) - P_{s,t}f(y)|\,dy \\
&\lesssim \sum_{k=1}^{\infty} (2^k t)^{-\frac{n+\delta}{m}} \int_{Q(x,(2^k t)^{\frac{1}{m}})} |f(y) - P_{s,2^k t}f(y)|\,dy \\
&\quad + \sum_{k=1}^{\infty} (2^k t)^{-\frac{n+\delta}{m}} \sup_{y \in Q(x,(2^k t)^{\frac{1}{m}})} |P_{s,t}f(y) - P_{s,2^k t}f(y)| |Q(x,(2^k t)^{\frac{1}{m}})| \\
&=: \mathrm{I}_{2,1} + \mathrm{I}_{2,2}.
\end{aligned} \tag{4.9}$$

By the Hölder inequality, Lemma 3.14 and the fact that $\delta > n(\frac{1}{p_-} - 1)$, we find that

$$
\begin{aligned}
\mathrm{I}_{2,1} &\lesssim \sum_{k=1}^{\infty} (2^k t)^{-\frac{n+\delta}{m}} \left\| \chi_{Q(x,(2^k t)^{\frac{1}{m}})} \right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)} \\
&\lesssim t^{-\frac{n+\delta}{m}} \left\| \chi_{Q(x,t^{\frac{1}{m}})} \right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)} \sum_{k=1}^{\infty} 2^{-k(n+\delta-\frac{n}{p_-})} \qquad (4.10)\\
&\lesssim t^{-\frac{n+\delta}{m}} \left\| \chi_{Q(x,t^{\frac{1}{m}})} \right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)}.
\end{aligned}
$$

For $\mathrm{I}_{2,2}$, by Proposition 4.5 and Lemma 3.14, we know that

$$
\begin{aligned}
\mathrm{I}_{2,2} &\lesssim \sum_{k=1}^{\infty} k(2^k t)^{-\frac{n+\delta}{m}} \sup_{y\in Q(x,(2^k t)^{\frac{1}{m}})} \left\| \chi_{Q(y,(2^k t)^{\frac{1}{m}})} \right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)} \\
&\lesssim \sum_{k=1}^{\infty} k(2^k t)^{-\frac{n+\delta}{m}} \left\| \chi_{Q(x,(2^k t)^{\frac{1}{m}})} \right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)} \\
&\lesssim t^{-\frac{n+\delta}{m}} \left\| \chi_{Q(x,t^{\frac{1}{m}})} \right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)}.
\end{aligned}
$$

This, together with (4.9) and (4.10), implies that

$$
\mathrm{I}_2 \lesssim t^{-\frac{n+\delta}{m}} \left\| \chi_{Q(x,t^{\frac{1}{m}})} \right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|f\|_{\mathrm{BMO}^s_{p(\cdot),L}(\mathbf{R}^n)},
$$

which completes the proof of Proposition 4.6. □

Let $p(\cdot) \in \mathcal{P}(\mathbf{R}^n)$. Recall that a measure $d\mu$ on $\mathbf{R}^{n+1}_+$ is called a $p(\cdot)$-*Carleson measure* if

$$
\|d\mu\|_{p(\cdot)} := \sup_{Q\subset\mathbf{R}^n} \frac{|Q|^{1/2}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \left\{ \int_{\widehat{Q}} d|\mu| \right\}^{1/2} < \infty,
$$

where the supremum is taken over all cubes Q of $\mathbf{R}^n$ and $\widehat{Q}$ denotes the tent over Q; see [55].

Proposition 4.7. *Let $p(\cdot)$, s_0 and s be as in Proposition 4.5. If $f \in \mathrm{BMO}^{s_0}_{p(\cdot),L}(\mathbf{R}^n)$, then the measure*

$$
d\mu_f(x,t) := |Q_{s,t^m}(I - P_{s_0,t^m})f(x)|^2 \frac{dx\,dt}{t}, \quad \forall\ (x,t) \in \mathbf{R}^{n+1}_+
$$

is a $p(\cdot)$-Carleson measure on $\mathbf{R}^{n+1}_+$ and there exists a positive constant C, independent of f, such that $\|d\mu_f\|_{p(\cdot)} \le C\|f\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L}(\mathbf{R}^n)}$.

Proof. Since $s \ge s_0 = \lfloor \frac{n}{m}(\frac{1}{p_-} - 1) \rfloor$ and $\theta(L) \in (n[\frac{1}{p_-} - 1], \infty)$ with $\theta(L)$ as in (2.5), it follows that there exists $\varepsilon \in (n[\frac{1}{p_-} - 1], \theta(L))$ such that $m(s+1) > \varepsilon$. To prove this proposition, by definition, it suffices to show that, for any cube $R := R(x_R, r_R)$ with some $x_R \in \mathbf{R}^n$ and $r_R \in (0,\infty)$,

$$
\frac{|R|^{1/2}}{\|\chi_R\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \left\{ \int_{\widehat{R}} |Q_{s,t^m}(I - P_{s_0,t^m})f(x)|^2 \frac{dxdt}{t} \right\}^{1/2} \lesssim \|f\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L}(\mathbf{R}^n)}. \tag{4.11}
$$

Observe that

$$
I - P_{s_0,t^m} = (I - P_{s_0,t^m}) \left[I - P_{s_0,(r_R)^m} \right] + P_{s_0,(r_R)^m}(I - P_{s_0,t^m}).
$$

Then the estimate (4.11) is a direct consequence of

$$\text{(4.12)} \qquad \frac{|R|^{1/2}}{\|\chi_R\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \left\{ \int_{\widehat{R}} \left| Q_{s,t^m}(I - P_{s_0,t^m}) \left[I - P_{s_0,(r_R)^m} \right] f(x) \right|^2 \frac{dx\,dt}{t} \right\}^{1/2} \lesssim \|f\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L}(\mathbf{R}^n)}$$

and

$$\text{(4.13)} \qquad \frac{|R|^{1/2}}{\|\chi_R\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \left\{ \int_{\widehat{R}} |Q_{s,t^m} P_{s_0,(r_R)^m}(I - P_{s_0,t^m}) f(x)|^2 \frac{dx\,dt}{t} \right\}^{1/2} \lesssim \|f\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L}(\mathbf{R}^n)}.$$

Next we prove (4.12) and (4.13), respectively. To show (4.12), let

$$b_1 := \left[I - P_{s_0,(r_R)^m} \right] f \chi_{2R} \quad \text{and} \quad b_2 := \left[I - P_{s_0,(r_R)^m} \right] f \chi_{\mathbf{R}^n \setminus (2R)}.$$

By [30, (4.25)], we know that

$$\text{(4.14)} \qquad \mathrm{J} := \left\{ \int_{\widehat{R}} |Q_{s,t^m}(I - P_{s_0,t^m}) b_1(x)|^2 \frac{dx\,dt}{t} \right\}^{1/2} \lesssim \|b_1\|_{L^2(\mathbf{R}^n)},$$

which, combined with Proposition 4.5 and Lemma 3.14, implies that

$$\begin{aligned} \mathrm{J} &\lesssim \left\{ \int_{2R} \left| \left[I - P_{s_0,(2r_R)^m} \right] f(x) \right|^2 dx \right\}^{\frac{1}{2}} + |R|^{1/2} \sup_{x \in 2R} |P_{s_0,(r_R)^m} f(x) - P_{s_0,(2r_R)^m} f(x)| \\ &\lesssim |R|^{-1/2} \|\chi_R\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|f\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L}(\mathbf{R}^n)}. \end{aligned}$$

For b_2, we write

$$\text{(4.15)} \qquad Q_{s,t^m}(I - P_{s_0,t^m}) b_2 = Q_{s,t^m} b_2 - Q_{s,t^m} P_{s_0,t^m} b_2.$$

Let $(x,t) \in \widehat{R}$. Then, by (2.3), (2.8) and Proposition 4.6, we have

$$\begin{aligned} |Q_{s,t^m} b_2(x)| &\lesssim \int_{\mathbf{R}^n \setminus (2R)} \frac{t^\varepsilon}{|x-y|^{n+\varepsilon}} \left| \left[I - P_{s_0,(r_R)^m} \right] f(y) \right| dy \\ &\lesssim \int_{\mathbf{R}^n} \frac{t^\varepsilon}{(r_R + |x-y|)^{n+\varepsilon}} \left| \left[I - P_{s_0,(r_R)^m} \right] f(y) \right| dy \\ &\lesssim (t/r_R)^\varepsilon |R|^{-1} \|\chi_R\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|f\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L}(\mathbf{R}^n)}, \end{aligned}$$

which implies that

$$\text{(4.16)} \qquad \left\{ \int_{\widehat{R}} |Q_{s,t^m} b_2(x)|^2 \frac{dx\,dt}{t} \right\}^{1/2} \lesssim |R|^{-1/2} \|\chi_R\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|f\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L}(\mathbf{R}^n)}.$$

On the other hand, for all $k \in \{1, \ldots, s_0+1\}$, t, $v \in (0,\infty)$, $f \in \mathcal{M}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$, let

$$\text{(4.17)} \qquad \Psi^k_{t,v} f(x) := [kv^m + t^m]^{s+1} \left(\left. \frac{d^{s+1} P_\eta}{d\eta^{s+1}} \right|_{kv^m + t^m} f \right)(x).$$

Then, by Assumption (A) and (2.4), we conclude that $\psi^k_{t,v}$, the kernel of $\Psi^k_{t,v}$, satisfies that, for all t, $v \in (0,\infty)$ and x, $y \in \mathbf{R}^n$,

$$\text{(4.18)} \qquad |\psi^k_{t,v}(x,y)| \lesssim \frac{v^\varepsilon}{(v + t + |x-y|)^{n+\varepsilon}}.$$

From this, Proposition 4.6 and the fact that

$$P_{s_0,v^m} f = \sum_{k=1}^{s_0+1} (-1)^{k+1} C_{s_0+1}^k e^{-kv^m L} f, \tag{4.19}$$

where $C_{s_0+1}^k := \frac{(s_0+1)!}{k!(s_0+1-k)!}$, we deduce that, for all $(x,t) \in \widehat{R}$,

$$\begin{aligned}
|Q_{s,t^m} P_{s_0,t^m} b_2(x)| &= \left| \sum_{k=1}^{s_0+1} (-1)^k C_{s_0+1}^k \frac{t^{m(s+1)}}{[(k+1)t^m]^{s+1}} \Psi_{t,t}^k b_2(x) \right| \\
&\lesssim \int_{\mathbf{R}^n \setminus (2R)} \frac{t^\varepsilon}{(t+|x-y|)^{n+\varepsilon}} \left| \left[I - P_{s_0,(r_R)^m}\right] f(y) \right| dy \\
&\lesssim \left(\frac{t}{r_R}\right)^\varepsilon \int_{\mathbf{R}^n} \frac{(r_R)^\varepsilon}{(r_R+|x-y|)^{n+\varepsilon}} \left| \left[I - P_{s_0,(r_R)^m}\right] f(y) \right| dy \\
&\lesssim (t/r_R)^\varepsilon \|\chi_R\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|f\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L}(\mathbf{R}^n)},
\end{aligned} \tag{4.20}$$

which further implies that

$$\left\{ \int_{\widehat{R}} |Q_{s,t^m} P_{s_0,t^m} b_2(x)|^2 \frac{dx\,dt}{t} \right\}^{1/2} \lesssim |R|^{-1/2} \|\chi_R\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|f\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L}(\mathbf{R}^n)}. \tag{4.21}$$

Combining (4.14), (4.15), (4.16) and (4.21), we conclude that (4.12) holds true.

Similarly, by (4.17), (4.18), (4.19) and Proposition 4.6, we also see that, for all $(x,t) \in \widehat{R}$,

$$\begin{aligned}
&\left| Q_{s,t^m} P_{s_0,(r_R)^m} (I - P_{s_0,t^m})(f)(x) \right| \\
&= \left| \sum_{k=1}^{s_0+1} (-1)^k C_{s_0+1}^k \frac{t^{m(s+1)}}{[k(r_R)^m + t^m]^{s+1}} \Psi_{t,r_R}^k (I - P_{s_0,t^m})(f)(x) \right| \\
&\lesssim \frac{t^{m(s+1)}}{[k(r_R)^m + t^m]^{s+1}} \int_{\mathbf{R}^n} \frac{(r_R)^\varepsilon}{(r_R+t+|x-y|)^{n+\varepsilon}} |(I - P_{s_0,t^m})(f)(y)|\, dy \\
&\lesssim \left(\frac{t}{r_R}\right)^{m(s+1)-\varepsilon} \int_{\mathbf{R}^n} \frac{t^\varepsilon}{(t+|x-y|)^{n+\varepsilon}} |(I - P_{s_0,t^m})(f)(r)|\, dy \\
&\lesssim t^{-n} \left(\frac{t}{r_R}\right)^{m(s+1)-\varepsilon} \|\chi_{Q(x,t)}\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|f\|_{\mathrm{BMO}^{s}_{p(\cdot),L}(\mathbf{R}^n)},
\end{aligned}$$

which, together with Lemma 3.14, implies that, for all $(x,t) \in \widehat{R}$,

$$\left| Q_{s,t^m} P_{s_0,(r_R)^m} (I - P_{s_0,t^m})(f)(x) \right| \lesssim t^{-n} \left(\frac{t}{r_R}\right)^{m(s+1)-\varepsilon} \|\chi_R\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|f\|_{\mathrm{BMO}^{s}_{p(\cdot),L}(\mathbf{R}^n)}.$$

By this and the fact that $m(s+1) > \varepsilon$, we further conclude that (4.13) holds true. This finishes the proof of Proposition 4.7. □

Proposition 4.8. (i) *Let $q \in (1,\infty)$ and $q^* := \frac{q}{q-1}$. Then, for all $f \in T_2^q(\mathbf{R}_+^{n+1})$ and $g \in T_2^{q^*}(\mathbf{R}_+^{n+1})$,*

$$\int_{\mathbf{R}_+^{n+1}} |f(y,t) g(y,t)| \frac{dy\,dt}{t} \le \int_{\mathbf{R}^n} \mathcal{T}(f)(x) \mathcal{T}(g)(x)\, dx.$$

(ii) *Let $p(\cdot)\in C^{\log}(\mathbf{R}^n)$ satisfy $p_+\in(0,1]$. Then the dual space of $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$ is $T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})$ in the following sense: for any $h\in T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})$, the mapping*

$$\ell_h(f):=\int_{\mathbf{R}_+^{n+1}} h(y,t)f(y,t)\,\frac{dy\,dt}{t} \tag{4.22}$$

is a bounded linear functional on $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$; conversely, if ℓ is a bounded linear functional on $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$, then ℓ has the form as in (4.22) with a unique $h\in T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})$. Moreover, $\|h\|_{T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})}\sim\|\ell_h\|_{(T_2^{p(\cdot)}(\mathbf{R}_+^{n+1}))^*}$ with the implicit positive constants independent of h.

Proof. To prove this proposition, it suffices to show (ii), since (i) was already proved in [10, p. 316, Theorem 2]. We first show that $T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})\subset(T_2^{p(\cdot)}(\mathbf{R}_+^{n+1}))^*$. Let $h\in T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})$. Then, by the Hölder inequality and Remark 3.2(ii), we find that, for any $(p(\cdot),\infty)$-atom a supported on $\widehat{Q}$ with some cube $Q\subset\mathbf{R}^n$,

$$\begin{aligned}\int_{\mathbf{R}_+^{n+1}}|h(x,t)a(x,t)|\,\frac{dx\,dt}{t}&\le\frac{|Q|^{1/2}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\left\{\int_{\widehat{Q}}|h(x,t)|^2\,\frac{dx\,dt}{t}\right\}^{1/2}\\&\le\|\mathcal{C}_{p(\cdot)}(h)\|_{L^\infty(\mathbf{R}^n)}=\|h\|_{T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})}.\end{aligned} \tag{4.23}$$

For any $f\in T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$, by Lemma 3.3, we know that, for almost every $(x,t)\in\mathbf{R}_+^{n+1}$, $f(x,t)=\sum_{j\in\mathbf{N}}\lambda_ja_j(x,t)$, where $\{\lambda_j\}_{j\in\mathbf{N}}$ and $\{a_j\}_{j\in\mathbf{N}}$ are as in Lemma 3.3 satisfying (3.4). From this, (4.23) and Remark 3.8, we deduce that

$$\begin{aligned}|\ell_h(f)|&\le\sum_{j\in\mathbf{N}}|\lambda_j|\int_{\mathbf{R}_+^{n+1}}|h(x,t)||a_j(x,t)|\,\frac{dx\,dt}{t}\\&\le\sum_{j\in\mathbf{N}}|\lambda_j|\|h\|_{T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})}\lesssim\|h\|_{T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})}\|f\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})},\end{aligned}$$

which implies that ℓ_h is a bounded linear functional on $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$ and

$$\|\ell_h\|_{(T_2^{p(\cdot)}(\mathbf{R}_+^{n+1}))^*}\lesssim\|h\|_{T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})}.$$

Next, we prove that $(T_2^{p(\cdot)}(\mathbf{R}_+^{n+1}))^*\subset T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})$. Let $\ell\in(T_2^{p(\cdot)}(\mathbf{R}_+^{n+1}))^*$. For all $k\in\mathbf{N}$, let $\widetilde{O}_k:=\{(x,t)\in\mathbf{R}_+^{n+1}\colon |x|\le k,\,1/k\le t\le k\}$. Then $\{\widetilde{O}_k\}_{k\in\mathbf{N}}$ is a family of compact sets of $\mathbf{R}_+^{n+1}$ and $\mathbf{R}_+^{n+1}=\bigcup_{k\in\mathbf{N}}\widetilde{O}_k$. Observe that, for each $k\in\mathbf{N}$, if $f\in L^2(\mathbf{R}_+^{n+1})$ with $\operatorname{supp}f\subset\widetilde{O}_k$, then $\operatorname{supp}\mathcal{T}(f)\subset O_k^*:=\{x\in\mathbf{R}^n\colon|x|\le 2k\}$. It follows, from the Hölder inequality, that

$$\begin{aligned}\int_{O_k^*}\mathcal{T}(f)(x)\,dx&\le|O_k^*|^{1/2}\left\{\int_{O_k^*}\int_{\Gamma(x)}|f(y,t)|^2\frac{dy\,dt}{t^{n+1}}dx\right\}^{1/2}\\&\lesssim|O_k^*|^{1/2}\left\{\int_{\widetilde{O}_k}|f(y,t)|^2\frac{dy\,dt}{t}\right\}^{1/2}\sim|O_k^*|^{1/2}\|f\|_{L^2(\widetilde{O}_k)}.\end{aligned}$$

By this and the fact that $p_+\in(0,1]$, we further find that

$$\int_{\mathbf{R}^n}\left[\frac{\mathcal{T}(f)(x)}{|O_k^*|^{-\frac12}\|f\|_{L^2(\widetilde{O}_k)}}\right]^{p(x)}dx\le\int_{O_k^*}\left[1+\frac{|O_k^*|^{\frac12}\mathcal{T}(f)(x)}{\|f\|_{L^2(\widetilde{O}_k)}}\right]^{p(x)}dx\lesssim|O_k^*|,$$

which implies that $\|f\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})} \le C_{(k)}\|f\|_{L^2(\widetilde{O}_k)}$, where $C_{(k)}$ is a positive constant depending on k. Thus, ℓ also induces a bounded linear functional on $L^2(\widetilde{O}_k)$. By the Riesz theorem, there exists a unique $h_k \in L^2(\widetilde{O}_k)$ such that, for all $f \in L^2(\widetilde{O}_k)$,

$$\ell(f) = \int_{\mathbf{R}_+^{n+1}} f(x,t)h_k(x,t)\,\frac{dx\,dt}{t}.$$

Obviously, $h_{k+1}\chi_{\widetilde{O}_k} = h_k$ for all $k \in \mathbf{N}$. Let

$$h := h_1\chi_{\widetilde{O}_1} + \sum_{k=2}^{\infty} h_k\chi_{\widetilde{O}_k\setminus\widetilde{O}_{k-1}}.$$

Then $h \in L^2_{\mathrm{loc}}(\mathbf{R}_+^{n+1})$ and, for any $f \in L^2(\mathbf{R}_+^{n+1})$ having compact support,

$$\ell(f) = \int_{\mathbf{R}_+^{n+1}} f(y,t)h(y,t)\,\frac{dy\,dt}{t}.$$

Now, for any $f \in T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$, by Lemma 3.3, we have $f(x,t) = \sum_{j\in\mathbf{N}} \lambda_j a_j(x,t)$ for almost every $(x,t) \in \mathbf{R}_+^{n+1}$, where $\{\lambda_j\}_{j\in\mathbf{N}}$ and $\{a_j\}_{j\in\mathbf{N}}$ are as in Lemma 3.3 satisfying (3.4). For all $N \in \mathbf{N}$, let $f_N := \sum_{j=1}^{N} \lambda_j a_j$. Then $f_N \to f$ in $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$ as $N \to \infty$ due to Corollary 3.4. Moreover, it is easy to see that $f_N \in L^2(\mathbf{R}_+^{n+1})$ having compact support and hence

$$\ell(f_N) = \int_{\mathbf{R}_+^{n+1}} f_N(y,t)h(y,t)\,\frac{dy\,dt}{t}.$$

Observer that, for all $N \in \mathbf{N}$,

$$|f_N| \le \sum_{j=1}^{N} |\lambda_j||a_j| \le \sum_{j\in\mathbf{N}} |\lambda_j||a_j|$$

and, by (4.23) and Remark 3.8, we find that

$$\begin{aligned}\sum_{j\in\mathbf{N}} |\lambda_j| \int_{\mathbf{R}_+^{n+1}} |h(x,t)||a_j(x,t)|\,\frac{dx\,dt}{t} &\lesssim \|h\|_{T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})} \sum_{j\in\mathbf{N}} |\lambda_j| \\ &\lesssim \|h\|_{T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})} \mathcal{B}(\{\lambda_j a_j\}_{j\in\mathbf{N}}) \\ &\lesssim \|h\|_{T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})} \|f\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})}.\end{aligned}$$

Therefore, from the dominated convergence theorem, we deduce that

$$\ell(f) = \lim_{N\to\infty} \ell(f_N) = \int_{\mathbf{R}_+^{n+1}} f(y,t)h(y,t)\,\frac{dy\,dt}{t}.$$

To complete the proof of this proposition, it remains to show that $h \in T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})$. Indeed, for any cube $Q \subset \mathbf{R}^n$ and $j \in \mathbf{N}$, let $R_{Q,j} := \widehat{Q} \cap \{(x,t) \in \mathbf{R}_+^{n+1} \colon t \ge 1/j\}$ and

$$\eta_j := \frac{|Q|^{1/2}\overline{h}\chi_{R_{Q,j}}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)}\|h\chi_{R_{Q,j}}\|_{L^2(\mathbf{R}_+^{n+1})}}.$$

Then, by the Minkowski inequality, we find that

$$\|\eta_j\|_{T_2^2(\mathbf{R}_+^{n+1})} \sim \left\{\int_{\mathbf{R}^n}\int_{\Gamma(x)} |\eta_j(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\,dx\right\}^{\frac12}$$
$$\lesssim \left\{\int_{R_{Q,j}} |\eta_j(y,t)|^2 \frac{dy\,dt}{t}\right\}^{\frac12} \lesssim |Q|^{1/2}\|\chi_Q\|^{-1}_{L^{p(\cdot)}(\mathbf{R}^n)},$$

namely, η_j is a $(p(\cdot),2)$-atom up to a positive constant multiple. From this, the Fatou lemma and Corollary 3.7, we further deduce that

$$\begin{aligned}
&\frac{|Q|^{1/2}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\left\{\int_{\widehat{Q}} |h(y,t)|^2 \frac{dydt}{t}\right\}^{1/2}\\
&\le \liminf_{j\to\infty}\int_{\mathbf{R}_+^{n+1}} \frac{|Q|^{\frac12}\overline{h(y,t)}h(y,t)\chi_{R_{Q,j}}(y,t)}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)}\|h\chi_{R_{Q,j}}\|_{L^2(\mathbf{R}_+^{n+1})}}\frac{dy\,dt}{t} = \liminf_{j\to\infty} \ell(\eta_j)\\
&\lesssim \liminf_{j\to\infty}\|\ell\|_{(T_2^{p(\cdot)}(\mathbf{R}_+^{n+1}))^*}\|\eta_j\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})} \lesssim \|\ell\|_{(T_2^{p(\cdot)}(\mathbf{R}_+^{n+1}))^*},
\end{aligned}$$

which, together with the arbitrariness of cubes Q, implies that $h \in T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})$ and

$$\|h\|_{T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})} \lesssim \|\ell\|_{(T_2^{p(\cdot)}(\mathbf{R}_+^{n+1}))^*}.$$

This finishes the proof of Proposition 4.8. □

To prove Theorem 4.3, we also need the following estimate.

Lemma 4.9. *Let $p(\cdot)$ and s_0 be as in Proposition 4.5. Then there exists a positive constant C such that, for all $f \in L^2(\mathbf{R}^n)$ satisfying* supp $f \subset Q := Q(x_Q, r_Q)$ *with some $x_Q \in \mathbf{R}^n$ and $r_Q \in (0,\infty)$,*

$$\left\|(I - P_{s_0,r_Q^m})f\right\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} \le C|Q|^{-1/2}\|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)}\|f\|_{L^2(\mathbf{R}^n)}.$$

Proof. Obviously, we have

$$\begin{aligned}
&\left\|S_L([I - P_{s_0,r_Q^m}]f)\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}\\
&\lesssim \left\|S_L([I - P_{s_0,r_Q^m}]f)\chi_{4Q}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} + \left\|S_L([I - P_{s_0,r_Q^m}]f)\chi_{\mathbf{R}^n\setminus(4Q)}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}\\
&=: \mathrm{J}_1 + \mathrm{J}_2.
\end{aligned} \tag{4.24}$$

By the boundedness of S_L in $L^2(\mathbf{R}^n)$ (see (2.9)) and Remark 2.5(iii), we see that

$$\|S_L([I - P_{s_0,r_Q^m}]f)\|_{L^2(\mathbf{R}^n)} \lesssim \|f\|_{L^2(\mathbf{R}^n)},$$

which, together with Lemmas 3.5 and 3.14, implies that

$$\mathrm{J}_1 \lesssim |Q|^{-1/2}\|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)}\|f\|_{L^2(\mathbf{R}^n)}. \tag{4.25}$$

To deal with the term J_2, since $p_- \in (\frac{n}{n+\theta(L)}, 1]$ with p_- and $\theta(L)$ as in (2.1) and (2.5), respectively, we choose $\varepsilon \in (0, \theta(L))$ such that $p_- \in (\frac{n}{n+\varepsilon}, 1]$. Notice that, for all $x \notin 4Q$,

$$S_L(I - P_{s_0,r_Q^m})(f)(x) \lesssim \frac{(r_Q)^{\frac n2+\varepsilon}}{|x - x_Q|^{n+\epsilon}}\|f\|_{L^2(\mathbf{R}^n)};$$

see [30, (4.20)]. Then, by this, Lemma 3.14 and the fact that $\varepsilon \in (n(\frac{1}{p_-}-1), \theta(L))$, we further know that, for any $r \in (0, p_-)$,

$$
\begin{aligned}
\mathrm{J}_2 &\sim \left\| \sum_{k=0}^{\infty} S_L(I - P_{s_0, r_Q^m})(f)\chi_{(4^{k+1}Q)\setminus(4^kQ)} \right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \\
&\lesssim \left\{ \sum_{k=0}^{\infty} \left\| \frac{(r_Q)^{\frac{n}{2}+\varepsilon}}{|\cdot - x_Q|^{n+\epsilon}} \chi_{(4^{k+1}Q)\setminus(4^kQ)} \right\|_{L^{p(\cdot)}(\mathbf{R}^n)}^r \right\}^{\frac{1}{r}} \|f\|_{L^2(\mathbf{R}^n)} \\
&\lesssim \left\{ \sum_{k=0}^{\infty} 4^{-k(n+\varepsilon)r} \|\chi_{4^kQ}\|_{L^{p(\cdot)}(\mathbf{R}^n)} \right\}^{\frac{1}{r}} |Q|^{-\frac{1}{2}} \|f\|_{L^2(\mathbf{R}^n)} \qquad (4.26) \\
&\lesssim \left\{ \sum_{k=0}^{\infty} 4^{-k(n+\varepsilon-n/p_-)r} \right\}^{\frac{1}{r}} \|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)} |Q|^{-\frac{1}{2}} \|f\|_{L^2(\mathbf{R}^n)} \\
&\sim |Q|^{-\frac{1}{2}} \|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|f\|_{L^2(\mathbf{R}^n)}.
\end{aligned}
$$

Combining the estimates (4.24), (4.25) and (4.26), we conclude the desired result and then complete the proof of Lemma 4.9. □

For all $s \in [s_0, \infty)$ with s_0 as in (3.8), $t \in (0, \infty)$, $f \in \mathcal{M}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$, let

$$P_{s,t}^* f(x) := f(x) - (I - e^{-tL^*})^{s+1} f(x) \quad \text{and} \quad Q_{s,t}^* f(x) := (tL^*)^{s+1} e^{-tL^*} f(x),$$

where L^* denotes the adjoint operator of L in $L^2(\mathbf{R}^n)$. Suppose that α is a $(p(\cdot), s, L)$-molecule. Then, by Theorem 3.13(ii), we see that $\alpha \in H_L^{p(\cdot)}(\mathbf{R}^n)$ and hence $G := Q_{t^m}\alpha \in T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$. Let $f \in \mathcal{M}(\mathbf{R}^n)$ be such that

$$\mu_f(x,t) := |Q_{s,t^m}^*(I - P_{s_0,t^m}^*) f(x)|^2 \frac{dx\,dt}{t}, \quad \forall\ (x,t) \in \mathbf{R}_+^{n+1}$$

is a $p(\cdot)$-Carleson measure on $\mathbf{R}_+^{n+1}$ and, for all $(x,t) \in \mathbf{R}_+^{n+1}$, let

$$F(x,t) := Q_{s,t^m}^*(I - P_{s_0,t^m}^*) f(x).$$

Then $\|F\|_{T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})} \lesssim \|\mu_f\|_{p(\cdot)} < \infty$. From this and Proposition 4.8(ii), we deduce that the integral

$$J(F,G) := \int_{\mathbf{R}_+^{n+1}} F(x,t) G(x,t) \frac{dx\,dt}{t}$$

converges absolutely and hence

$$\int_{\mathbf{R}_+^{n+1}} \left| Q_{t^m}\alpha(x) Q_{s,t^m}^*(I - P_{s_0,t^m}^*) f(x) \right| \frac{dx\,dt}{t} < \infty.$$

Indeed, by an argument similar to that used in the proof of [23, Proposition 5.1], we have the following technical lemma, the details being omitted.

Lemma 4.10. *Let $p(\cdot) \in C^{\log}(\mathbf{R}^n)$, s_0 and s be as in Proposition 4.5. Suppose that α is a $(p(\cdot), s, L)$-molecule and $f \in \mathcal{M}(\mathbf{R}^n)$ satisfies that*

$$\mu_f(x,t) := |Q_{s,t^m}^*(I - P_{s_0,t^m}^*) f(x)|^2 \frac{dx\,dt}{t}$$

for all $(x,t) \in \mathbf{R}_+^{n+1}$ is a $p(\cdot)$-Carleson measure on $\mathbf{R}_+^{n+1}$. Then

$$\int_{\mathbf{R}^n} f(x)\alpha(x)\,dx = C_{(m,s)} \int_{\mathbf{R}_+^{n+1}} Q_{t^m}\alpha(x) Q_{s,t^m}^*(I - P_{s_0,t^m}^*) f(x) \frac{dx\,dt}{t},$$

where $C_{(m,s)}$ *is as in* (3.9).

We are now ready to prove Theorem 4.3.

Proof of Theorem 4.3. We first prove (i). Let $g \in \mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbf{R}^n)$ and $f \in H^{p(\cdot)}_{L,\mathrm{fin}}(\mathbf{R}^n)$. Then f has an expression $f = \sum_{j=1}^N \lambda_j a_j$, where $N \in \mathbf{N}$, $\{\lambda_j\}_{j=1}^N \subset \mathbf{C}$ and $\{\alpha_j\}_{j=1}^N$ are $(p(\cdot), s, L)$-molecules associated with cubes $\{R_j\}_{j=1}^N$ of $\mathbf{R}^n$ satisfying

$$\left\|\left\{\sum_{j=1}^N \left[\frac{|\lambda_j|}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\chi_{R_j}\right]^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \lesssim \|f\|_{H^{p(\cdot)}_{L,\mathrm{fin}}(\mathbf{R}^n)}.$$

For each $j \in \mathbf{N}$, since $\alpha_j \in H^{p(\cdot)}_L(\mathbf{R}^n)$, it follows that $Q_{t^m}\alpha_j \in T^{p(\cdot)}_2(\mathbf{R}^{n+1}_+)$. By this and Corollary 3.7, we know that, for any $j \in \mathbf{N}$, there exist $\{\lambda_j^k\}_{k\in\mathbf{N}} \subset \mathbf{C}$ and a sequence $\{a_j^k\}_{k\in\mathbf{N}}$ of $(p(\cdot),\infty)$-atoms such that $Q_{t^m}\alpha_j = \sum_{k\in\mathbf{N}} \lambda_j^k a_j^k$ almost everywhere on $\mathbf{R}^{n+1}_+$, supp $a_j^k \subset \widehat{R}_j^k$ with some cube $R_j^k \subset \mathbf{R}^n$ for all $k \in \mathbf{N}$, and

$$\mathcal{B}(\{\lambda_j^k a_j^k\}_{k\in\mathbf{N}}) \lesssim \|Q_{t^m}\alpha_j\|_{T^{p(\cdot)}_2(\mathbf{R}^{n+1}_+)} \sim \|\alpha_j\|_{H^{p(\cdot)}_L(\mathbf{R}^n)}.$$

Thus, from Lemma 4.10, the Hölder inequality, Proposition 4.7, Remarks 3.2(ii) and 3.8, we deduce that

$$\begin{aligned}
&\left|\int_{\mathbf{R}^n} \alpha_j(x)g(x)\,dx\right| \sim \left|\int_{\mathbf{R}^{n+1}_+} Q_{t^m}\alpha_j(y)Q^*_{s,t^m}(I-P^*_{s_0,t^m})g(y)\right| \frac{dy\,dt}{t}\\
&\lesssim \sum_{k\in\mathbf{N}} \int_{\widehat{R}_j^k} |\lambda_j^k a_j^k(y,t)Q^*_{s,t^m}(I-P^*_{s_0,t^m})g(y)|\,\frac{dy\,dt}{t}\\
&\lesssim \sum_{k\in\mathbf{N}} |\lambda_j^k| \left\{\int_{\widehat{R}_j^k} |a_j^k(y,t)|^2\,\frac{dy\,dt}{t}\right\}^{\frac{1}{2}} |R_j^k|^{\frac{1}{2}} \|\chi_{R_j^k}\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|g\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbf{R}^n)}\\
&\lesssim \sum_{k\in\mathbf{N}} |\lambda_j^k| \|g\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbf{R}^n)} \lesssim \mathcal{B}(\{\lambda_j^k a_j^k\}_{k\in\mathbf{N}})\|g\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbf{R}^n)}\\
&\lesssim \|g\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbf{R}^n)} \|\alpha_j\|_{H^{p(\cdot)}_L(\mathbf{R}^n)} \lesssim \|g\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbf{R}^n)}.
\end{aligned}$$

By this and Remark 3.8, we further obtain

$$\begin{aligned}
\left|\int_{\mathbf{R}^n} f(x)g(x)\,dx\right| &\le \sum_{j=1}^N |\lambda_j| \left|\int_{\mathbf{R}^n} \alpha_j(x)g(x)\,dx\right| \lesssim \sum_{j=1}^N |\lambda_j| \|g\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbf{R}^n)}\\
&\lesssim \left\|\left\{\sum_{j=1}^N \left[\frac{|\lambda_j|}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\chi_{R_j}\right]^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \|g\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbf{R}^n)}\\
&\lesssim \|f\|_{H^{p(\cdot)}_L(\mathbf{R}^n)} \|g\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbf{R}^n)}.
\end{aligned}$$

Therefore, by Corollary 3.18 and a density argument, we conclude that ℓ_g is a bounded linear functional on $H^{p(\cdot)}_L(\mathbf{R}^n)$ and $\|\ell_g\|_{(H^{p(\cdot)}_L(\mathbf{R}^n))^*} \lesssim \|g\|_{\mathrm{BMO}^{s_0}_{p(\cdot),L^*}(\mathbf{R}^n)}$.

Next we show (ii). For any $\eta \in T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$, by Proposition 3.17(ii), we know that $\pi_L(\eta) \in H_L^{p(\cdot)}(\mathbf{R}^n)$ and hence, for any $\ell \in (H_L^{p(\cdot)}(\mathbf{R}^n))^*$, we have

$$\begin{aligned}|(\ell \circ \pi_L)(\eta)| = |\ell(\pi_l(\eta))| &\lesssim \|\ell\|_{(H_L^{p(\cdot)}(\mathbf{R}^n))^*} \|\pi_L(\eta)\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} \\ &\lesssim \|\ell\|_{(H_L^{p(\cdot)}(\mathbf{R}^n))^*} \|\pi_L\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1}) \to H_L^{p(\cdot)}(\mathbf{R}^n)} \|\eta\|_{T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})}.\end{aligned}$$

In other words, $\ell \circ \pi_L$ is a bounded linear functional on $T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$. Thus, by Proposition 4.8(ii), we find that there exists a function $h \in T_{2,\infty}^{p(\cdot)}(\mathbf{R}^n)$ such that, for all $\eta \in T_2^{p(\cdot)}(\mathbf{R}_+^{n+1})$,

$$(\ell \circ \pi_L)(\eta) = \int_{\mathbf{R}_+^{n+1}} \eta(x,t) h(x,t) \, \frac{dxdt}{t}. \tag{4.27}$$

On the other hand, by Remark 3.10, we see that, for all $f \in H_L^{p(\cdot)}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$, $f = \pi_L(Q_{t^m} f)$ in $L^2(\mathbf{R}^n)$. From this and (4.27), we deduce that

$$\begin{aligned}\ell(f) = (\ell \circ \pi_L)(Q_{t^m} f) &= \int_{\mathbf{R}_+^{n+1}} h(x,t) Q_{t^m} f(x) \, \frac{dxdt}{t} \\ &= \int_{\mathbf{R}^n} \left\{ \int_0^\infty (Q_{t^m}^* h)(x,t) \, \frac{dt}{t} \right\} f(x) \, dx =: \int_{\mathbf{R}^n} g(x) f(x) \, dx.\end{aligned}$$

To complete the proof of Theorem 4.3, it remains to prove that $g \in \mathrm{BMO}_{p(\cdot),L^*}^{s_0}(\mathbf{R}^n)$. For any $Q := Q(x_Q, r_Q) \subset \mathbf{R}^n$ with some $x_Q \in \mathbf{R}^n$ and $r_Q \in (0,\infty)$, by Lemma 4.9, we conclude that

$$\begin{aligned}\left\{ \int_Q |g(x) - P_{s_0,r_Q^m}^* g(x)|^2 \, dx \right\}^{\frac12} &= \sup_{\|u\|_{L^2(Q)} \le 1} \left| \int_{\mathbf{R}^n} \left[g(x) - P_{s_0,r_Q^m}^* g(x) \right] u(x) \, dx \right| \\ &= \sup_{\|u\|_{L^2(Q)} \le 1} \left| \int_{\mathbf{R}^n} g(x) \left[(I - P_{s_0,r_Q^m}) u(x) \right] \, dx \right| \\ &= \sup_{\|u\|_{L^2(Q)} \le 1} \left| \ell([I - P_{s_0,r_Q^m}] u) \right| \\ &\lesssim \|\ell\|_{(H_L^{p(\cdot)}(\mathbf{R}^n))^*} \sup_{\|u\|_{L^2(Q)} \le 1} \left\| (I - P_{s_0,r_Q^m}) u \right\|_{H_L^{p(\cdot)}(\mathbf{R}^n)} \\ &\lesssim \|\ell\|_{(H_L^{p(\cdot)}(\mathbf{R}^n))^*} |Q|^{-1/2} \|\chi_Q\|_{L^{p(\cdot)}(\mathbf{R}^n)}.\end{aligned}$$

From this, we deduce that $g \in \mathrm{BMO}_{p(\cdot),L^*}^{s_0}(\mathbf{R}^n)$ and $\|g\|_{\mathrm{BMO}_{p(\cdot),L^*}^{s_0}(\mathbf{R}^n)} \lesssim \|\ell\|_{(H_L^{p(\cdot)}(\mathbf{R}^n))^*}$, which complete the proof of Theorem 4.3. □

From Proposition 4.7, Theorem 4.3 and an argument similar to that used in the proof of [30, Theorem 4.5], we easily deduce the following characterization of the BMO-type spaces, the details being omitted.

Corollary 4.11. *Let $p(\cdot) \in C^{\log}(\mathbf{R}^n)$, s_0 and s be as in Proposition 4.5. Then $g \in \mathrm{BMO}_{p(\cdot),L^*}^{s_0}(\mathbf{R}^n)$ if and only if $g \in \mathcal{M}(\mathbf{R}^n)$ and $|Q_{s,t^m}^*(I - P_{s_0,t^m}^*) g(x)|^2 \frac{dx\,dt}{t}$ for all $(x,t) \in \mathbf{R}_+^{n+1}$ is a $p(\cdot)$-Carleson measure. Moreover,*

$$\|g\|_{\mathrm{BMO}_{p(\cdot),L^*}^{s_0}(\mathbf{R}^n)} \sim \left\| Q_{s,t^m}^*(I - P_{s_0,t^m}^*) g \right\|_{T_{2,\infty}^{p(\cdot)}(\mathbf{R}_+^{n+1})}$$

with the implicit positive constants independent of g.

5. Applications

In this section, we give out two applications of the molecular characterizations of the spaces $H_L^{p(\cdot)}(\mathbf{R}^n)$ established in Theorem 3.13. One is to investigate the coincidence between the spaces $H_L^{p(\cdot)}(\mathbf{R}^n)$ and $H^{p(\cdot)}(\mathbf{R}^n)$, where $H^{p(\cdot)}(\mathbf{R}^n)$ denotes the Hardy space with variable exponent introduced by Nakai and Sawano in [37]. Another is to study the boundedness of the fractional integral $L^{-\gamma}$ on $H_L^{p(\cdot)}(\mathbf{R}^n)$.

5.1. The coincidence between $H_L^{p(\cdot)}(\mathbf{R}^n)$ and $H^{p(\cdot)}(\mathbf{R}^n)$. We begin with recalling the definition of the Hardy space with variable exponent introduced in [37]. Let $\mathcal{S}(\mathbf{R}^n)$ be the *space of all Schwartz functions* and $\mathcal{S}'(\mathbf{R}^n)$ its *topological dual space*. For any $N \in \mathbf{N}$, let

$$\mathcal{F}_N(\mathbf{R}^n) := \left\{ \psi \in \mathcal{S}(\mathbf{R}^n) \colon \sum_{\beta \in \mathbf{Z}_+^n,\, |\beta| \le N} \sup_{x \in \mathbf{R}^n} (1+|x|)^N |D^\beta \psi(x)| \le 1 \right\},$$

where, for all $\beta := (\beta_1, \ldots, \beta_n) \in \mathbf{Z}_+^n$, $|\beta| := \beta_1 + \cdots + \beta_n$ and $D^\beta := (\frac{\partial}{\partial x_1})^{\beta_1} \cdots (\frac{\partial}{\partial x_n})^{\beta_n}$. Then, for all $f \in \mathcal{S}'(\mathbf{R}^n)$, the *grand maximal function* f_N^* is defined by setting, for all $x \in \mathbf{R}^n$,

$$f_N^*(x) := \sup\left\{ |f * \psi_t(x)| \colon t \in (0,\infty) \text{ and } \psi \in \mathcal{F}_N(\mathbf{R}^n) \right\},$$

where, for all $t \in (0,\infty)$ and $\xi \in \mathbf{R}^n$, $\psi_t(\xi) := t^{-n}\psi(\xi/t)$.

Definition 5.1. Let $p(\cdot) \in C^{\log}(\mathbf{R}^n)$ and $N \in (\frac{n}{p_-} + n + 1, \infty)$. Then the *Hardy space with variable exponent* $p(\cdot)$, denoted by $H^{p(\cdot)}(\mathbf{R}^n)$, is defined to be the set of all $f \in \mathcal{S}'(\mathbf{R}^n)$ such that $f_N^* \in L^{p(\cdot)}(\mathbf{R}^n)$, equipped with the quasi-norm $\|f\|_{H^{p(\cdot)}(\mathbf{R}^n)} := \|f_N^*\|_{L^{p(\cdot)}(\mathbf{R}^n)}$.

Remark 5.2. In [37, Theorem 3.3], it was proved that the space $H^{p(\cdot)}(\mathbf{R}^n)$ is independent of N as long as N is sufficiently large. Although the range of N is not presented explicitly in [37, Theorem 3.3], it was pointed out in [55, Remark 1.3(ii)] that $N \in (\frac{n}{p_-} + n + 1, \infty)$ does the work.

In what follows, suppose that L is a linear operator of type ν on $L^2(\mathbf{R}^n)$ with $\nu \in (0, \frac{\pi}{2})$. Then it generates an analytic semigroup $\{e^{-zL}\}_z$, where $z \in \mathbf{C}$ satisfies $0 \le |\arg(z)| < \frac{\pi}{2} - \nu$. Following [48], we assume that the kernels of $\{e^{-tL}\}_{t>0}$, $\{p_t\}_{t>0}$, satisfy the following conditions: there exist positive constants C, m and $\tau \in (n(\frac{1}{p_-} - 1), 1]$ such that, for all $t \in (0,\infty)$ and x, y, $h \in \mathbf{R}^n$,

$$|p_t(x,y)| \le C \frac{t^{1/m}}{(t^{1/m} + |x-y|)^{n+1}}, \tag{5.1}$$

$$|p_t(x+h,y) - p_t(x,y)| + |p_t(x,y+h) - p_t(x,y)| \le C \frac{t^{1/m}}{(t^{1/m} + |x-y|)^{n+1+\gamma}} |h|^\tau \tag{5.2}$$

when $2|h| \le t^{1/m} + |x-y|$, and

$$\int_{\mathbf{R}^n} p_t(z,y)\,dz = 1 = \int_{\mathbf{R}^n} p_t(x,z)\,dz. \tag{5.3}$$

Theorem 5.3. *Let L be a linear operator of type v on $L^2(\mathbf{R}^n)$ with $v \in (0, \frac{\pi}{2})$ and its heat kernel satisfy* (5.1), (5.2) *and* (5.3). *Assume that $p(\cdot) \in C^{\log}(\mathbf{R}^n)$ satisfies*

$p_+ \in (0,1]$, $p_- \in (\frac{n}{n+1}, 1]$ and $\frac{2}{p_-} - \frac{1}{p_+} < \frac{n+1}{n}$, where p_- and p_+ are as in (2.1). *Then $H^{p(\cdot)}(\mathbf{R}^n)$ and $H_L^{p(\cdot)}(\mathbf{R}^n)$ coincide with equivalent quasi-norms.*

Remark 5.4. Obviously, if $L = -\Delta$, then its heat kernel satisfies (5.1), (5.2) and (5.3). It was also pointed out by Yan [48, p. 4405, Remark] that the assumptions (5.1), (5.2) and (5.3) are satisfied by the divergence form operator $L := -\mathrm{div}\,(A\nabla)$ when A has real entries, or when the dimension $n = 1$ or 2 in the case of complex entries; see also [23, 24, 30] for some other examples.

To prove Theorem 5.3, we need the atomic characterization of $H^{p(\cdot)}(\mathbf{R}^n)$. Let $p(\cdot) \in \mathcal{P}(\mathbf{R}^n)$, $q \in [1,\infty] \cap (p_+, \infty]$ and $d := \max\{0, \lfloor n(1/p_- - 1)\rfloor\}$. Recall that a function a on $\mathbf{R}^n$ is called a *$(p(\cdot), q, d)$-atom* if a satisfies

(i) $\mathrm{supp}\, a \subset R$ for some cube $R \subset \mathbf{R}^n$;

(ii) $\|a\|_{L^q(\mathbf{R}^n)} \le \frac{|R|^{1/q}}{\|\chi_R\|_{L^{p(\cdot)}(\mathbf{R}^n)}}$;

(iii) $\int_{\mathbf{R}^n} a(x) x^\beta \, dx = 0$ for all $\beta \in \mathbf{Z}_+^n$ with $|\beta| \le d$.

Definition 5.5. Let $p(\cdot) \in C^{\log}(\mathbf{R}^n)$, $q \in [1,\infty] \cap (p_+, \infty]$ and

$$d := \max\{0, \lfloor n(1/p_- - 1)\rfloor\}$$

with p_- and p_+ as in (2.1). Then the *atomic Hardy space* $H_{\mathrm{at}}^{p(\cdot),q}(\mathbf{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbf{R}^n)$ such that f can be written as $f = \sum_{j\in\mathbf{N}} \lambda_j a_j$ in $\mathcal{S}'(\mathbf{R}^n)$, where $\{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$ and $\{a_j\}_{j\in\mathbf{N}}$ are $(p(\cdot), q, d)$-atoms satisfying that, for each $j \in \mathbf{N}$, $\mathrm{supp}\, a_j \subset R_j$ for some cube $R_j \subset \mathbf{R}^n$ and $\mathcal{A}(\{\lambda_j\}_{j\in\mathbf{N}}, \{R_j\}_{j\in\mathbf{N}}) < \infty$, where $\mathcal{A}(\{\lambda_j\}_{j\in\mathbf{N}}, \{R_j\}_{j\in\mathbf{N}})$ is as in (3.1) with $\{Q_j\}_{j\in\mathbf{N}}$ replaced by $\{R_j\}_{j\in\mathbf{N}}$.

Moreover, for any $f \in H_{\mathrm{at}}^{p(\cdot),q}(\mathbf{R}^n)$, its quasi-norm is defined by

$$\|f\|_{H_{\mathrm{at}}^{p(\cdot),q}(\mathbf{R}^n)} := \inf\{\mathcal{A}(\{\lambda_j\}_{j\in\mathbf{N}}, \{R_j\}_{j\in\mathbf{N}})\},$$

where the infimum is taken over all admissible decompositions of f as above.

The following lemma was originally established by Nakai and Sawano in [37, Theorem 4.6] and further improved by Sawano in [44, Theorem 1.1].

Lemma 5.6. *Let $p(\cdot) \in C^{\log}(\mathbf{R}^n)$ and $q \in [1,\infty] \cap (p_+, \infty]$ with p_+ as in* (2.1). *Then the spaces $H^{p(\cdot)}(\mathbf{R}^n)$ and $H_{\mathrm{at}}^{p(\cdot),q}(\mathbf{R}^n)$ coincide with equivalent quasi-norms.*

Proof of Theorem 5.3. To prove this theorem, by Lemma 5.6, it suffices to show that $H_{\mathrm{at}}^{p(\cdot),2}(\mathbf{R}^n)$ and $H_L^{p(\cdot)}(\mathbf{R}^n)$ coincide with equivalent quasi-norms. Since $p_- \in (\frac{n}{n+1}, 1]$, it follows that $d = \max\{0, \lfloor n(1/p_- - 1)\rfloor\} = 0$ in this case.

We first show that $H_{\mathrm{at}}^{p(\cdot),2}(\mathbf{R}^n) \subset H_L^{p(\cdot)}(\mathbf{R}^n)$. To this end, let q_t be the kernel of the operator Q_t. Then, by [23, Lemma 6.10] (see also [48, p. 4404]), we find that, for any $\gamma \in (n[\frac{1}{p_-} - 1], \tau)$ and $\delta \in (0,1)$, there exists a positive constant C such that, for all $t \in (0,\infty)$ and x, y, $h \in \mathbf{R}^n$,

$$|q_t(x,y)| \le C \frac{t^{\delta/m}}{(t^{1/m} + |x-y|)^{n+\delta}}, \tag{5.4}$$

$$|q_t(x+h,y) - q_t(x,y)| + |q_t(x,y+h) - q_t(x,y)| \le C \frac{t^{\delta/m}}{(t^{1/m} + |x-y|)^{n+\delta+\gamma}} |h|^\gamma \tag{5.5}$$

when $2|h| \le t^{1/m} + |x-y|$, and

$$\int_{\mathbf{R}^n} q_t(z,y)\, dz = 1 = \int_{\mathbf{R}^n} q_t(x,z)\, dz. \tag{5.6}$$

Let $f \in H_{\rm at}^{p(\cdot),2}(\mathbf{R}^n)$. Then, by Definition 5.5, we see that f has an atomic decomposition $f=\sum_{j\in\mathbf{N}}\lambda_j a_j$, where $\{\lambda_j\}_{j\in\mathbf{N}}\subset\mathbf{C}$ and $\{a_j\}_{j\in\mathbf{N}}$ are $(p(\cdot),2,0)$-atoms such that, for each $j\in\mathbf{N}$, supp $a_j\subset R_j$ with some cube $R_j\subset\mathbf{R}^n$, and

$$\mathcal{B}(\{\lambda_j a_j\}_{j\in\mathbf{N}})\lesssim\|f\|_{H_{\rm at}^{p(\cdot),2}(\mathbf{R}^n)}. \tag{5.7}$$

Thus, we have

$$\begin{aligned}\|S_L(f)\|_{L^{p(\cdot)}(\mathbf{R}^n)}&\le\left\|\sum_{j\in\mathbf{N}}|\lambda_j|S_L(a_j)\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}\\&\le\left\|\sum_{j\in\mathbf{N}}|\lambda_j|S_L(a_j)\chi_{4R_j}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}+\left\|\sum_{j\in\mathbf{N}}|\lambda_j|S_L(a_j)\chi_{(4R_j)^\complement}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}\\&=:\mathrm{I}+\mathrm{II}.\end{aligned}$$

For I, since, due to (2.9), $\|S_L(a_j)\|_{L^2(\mathbf{R}^n)}\lesssim\|a_j\|_{L^2(\mathbf{R}^n)}\lesssim\frac{|R_j|^{1/2}}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}$, it follows, from Lemma 3.5, that $\mathrm{I}\lesssim\mathcal{B}(\{\lambda_j a_j\}_{j\in\mathbf{N}})\lesssim\|f\|_{H_{\rm at}^{p(\cdot),2}(\mathbf{R}^n)}$.

Next, we estimate the term II. For all $x\in(4R_j)^\complement$, we have

$$\begin{aligned}S_L(a_j)(x)&\le\left\{\int_0^{r_{R_j}}\int_{B(x,t)}|Q_{t^m}(a_j)(y)|^2\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}+\left\{\int_{r_{R_j}}^\infty\int_{B(x,t)}\cdots\right\}^{1/2}\\&=:\mathrm{II}_1(x)+\mathrm{II}_2(x).\end{aligned}$$

Observe that, when $x\in(4R_j)^\complement$, $|x-y|<t$ and $z\in R_j:=Q(x_{R_j},r_{R_j})$ for some $x_{R_j}\in\mathbf{R}^n$ and $r_{R_j}\in(0,\infty)$, we see that

$$t+|y-z|\ge|x-z|\ge\frac{1}{2}|x-x_{R_j}|.$$

By this, (5.4) and the Hölder inequality, we find that, for all $x\in(4R_j)^\complement$,

$$\begin{aligned}\mathrm{II}_1(x)&\lesssim\left\{\int_0^{r_{R_j}}\int_{|x-y|<t}\left[\int_{R_j}\frac{t^\delta}{(t+|y-z|)^{n+\delta}}|a_j(z)|\,dz\right]^2\frac{dy\,dt}{t^{n+1}}\right\}^{\frac12}\\&\lesssim\frac{(r_{R_j})^\delta}{|x-x_{R_j}|^{n+\delta}}\|a_j\|_{L^2(R_j)}|R_j|^{\frac12}\lesssim\frac{(r_{R_j})^\delta}{|x-x_{R_j}|^{n+\delta}}\frac{|R_j|}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}.\end{aligned}\tag{5.8}$$

Choose $\delta\in(n[\frac{1}{p_-}-1],1)$ and $r\in(0,p_-)$ such that $n+\delta>\frac{n}{r}$. Then, from (5.7), (5.8), Lemma 3.15 and the fact that, for all $k\in\mathbf{N}$, $\chi_{4^kR_j}\le 2^{k\frac{n}{r}}[\mathcal{M}(\chi_{R_j})]^{1/r}$, we deduce that

$$\begin{aligned}&\left\|\sum_{j\in\mathbf{N}}|\lambda_j|\mathrm{II}_1(\cdot)\chi_{(4R_j)^\complement}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}\\&\quad\lesssim\left\|\sum_{k\in\mathbf{N}}\sum_{j\in\mathbf{N}}\frac{|\lambda_j||R_j|}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\frac{(r_{R_j})^\delta}{|\cdot-x_{R_j}|^{n+\delta}}\chi_{(4^kR_j)\setminus(4^{k-1}R_j)}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left\{\sum_{k\in\mathbf{N}} 4^{-k(n+\delta-\frac{n}{r})} \left\|\sum_{j\in\mathbf{N}}\left(\frac{|\lambda_j|}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}[\mathcal{M}(\chi_{R_j})]^{\frac{1}{r}}\right)^{\underline{p}}\right\|_{L^{\frac{p(\cdot)}{\underline{p}}}(\mathbf{R}^n)}\right\}^{\frac{1}{\underline{p}}}\\
&\lesssim \left\|\left\{\sum_{j\in\mathbf{N}}\left[\mathcal{M}\left(\frac{|\lambda_j|^r}{\|\chi_{R_j}\|^r_{L^{p(\cdot)}(\mathbf{R}^n)}}\chi_{R_j}\right)\right]^{\underline{p}/r}\right\}^{r/\underline{p}}\right\|^r_{L^{\frac{p(\cdot)}{r}}(\mathbf{R}^n)}\\
&\lesssim \left\|\left\{\sum_{j\in\mathbf{N}}\left[\frac{|\lambda_j|}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\chi_{R_j}\right]^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \lesssim \|f\|_{H^{p(\cdot),2}_{\mathrm{at}}(\mathbf{R}^n)}.
\end{aligned}
\tag{5.9}
$$

On the other hand, by (5.5), (5.6) and the vanishing moment condition of a_j, we obtain

$$
\begin{aligned}
\mathrm{II}_2(x) &\le \left\{\int_{r_{R_j}}^\infty \int_{|y-x|<t}\left[\int_{R_j}|q_{t^m}(y,z)-q_{t^m}(y,x_{R_j})||a_j(z)|\,dz\right]^2\frac{dy\,dt}{t^{n+1}}\right\}^{\frac12}\\
&\lesssim \left\{\int_{r_{R_j}}^\infty \int_{|y-x|<t}\left[\int_{R_j}\frac{|z-x_{R_j}|^\gamma t^\delta}{(t+|y-z|)^{n+\delta+\gamma}}|a_j(z)|\,dz\right]^2\frac{dy\,dt}{t^{n+1}}\right\}^{\frac12}\\
&\lesssim \frac{(r_{R_j})^\gamma}{|\cdot-x_{R_j}|^{n+\gamma-\gamma_1}}\left\{\int_{r_{R_j}}^\infty\left[\int_{R_j}|a_j(z)|\,dz\right]^2 t^{-2\gamma_1}\frac{dt}{t}\right\}^{1/2},
\end{aligned}
$$

where $\gamma_1\in(0,\gamma)$ such that $\gamma-\gamma_1\in(n[\frac{1}{p_-}-1],1)$, which, together with the Hölder inequality, implies that, for all $x\in(4R_j)^\complement$,

$$
\mathrm{II}_2(x)\lesssim\frac{(r_{R_j})^{\gamma-\gamma_1}}{|x-x_{R_j}|^{n+\gamma-\gamma_1}}\frac{|R_j|}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}.
$$

By this and an argument similar to that used in the proof of (5.9), we conclude that

$$
\left\|\sum_{j\in\mathbf{N}}|\lambda_j|\mathrm{II}_2(\cdot)\chi_{(4R_j)^\complement}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}\lesssim\|f\|_{H^{p(\cdot),2}_{\mathrm{at}}(\mathbf{R}^n)}.
$$

This, combined with (5.9), shows that $\mathrm{II}\lesssim\|f\|_{H^{p(\cdot),2}_{\mathrm{at}}(\mathbf{R}^n)}$. Therefore, $f\in H^{p(\cdot)}_L(\mathbf{R}^n)$ and

$$
\|f\|_{H^{p(\cdot)}_L(\mathbf{R}^n)}=\|S_L(f)\|_{L^{p(\cdot)}(\mathbf{R}^n)}\lesssim\|f\|_{H^{p(\cdot),2}_{\mathrm{at}}(\mathbf{R}^n)},
$$

which further implies that $H^{p(\cdot),2}_{\mathrm{at}}(\mathbf{R}^n)\subset H^{p(\cdot)}_L(\mathbf{R}^n)$.

Conversely, we prove that $H^{p(\cdot)}_L(\mathbf{R}^n)\subset H^{p(\cdot),2}_{\mathrm{at}}(\mathbf{R}^n)$. Let α be a $(p(\cdot),s,L)$-molecule and $\alpha=\pi_L(a)$, where a is a $(p(\cdot),\infty)$-atom supported on $\widehat{R}$ for some cube $R\subset\mathbf{R}^n$. Let $R:=Q(x_R,r_R)$ with $x_R\in\mathbf{R}^n$ and $r_R\in(0,\infty)$, $D_0(R):=2R$ and, when $k\in\mathbf{N}$, $D_k(R):=(2^{k+1}R)\setminus(2^kR)$. Moreover, for any $k\in\mathbf{Z}_+$, we let $l_k:=\int_{D_k(R)}\alpha(x)\,dx$ and

$$
h_k:=\alpha\chi_{D_k(R)}-\frac{\chi_{D_k(R)}}{|D_k(R)|}\int_{D_k(R)}\alpha(x)\,dx.
$$

Then, for all $x \in \mathbf{R}^n$, we have

$$\begin{aligned}\alpha(x) &= \sum_{k\in\mathbf{Z}_+} h_k(x) + \sum_{k\in\mathbf{Z}_+} \frac{l_k}{|D_k(R)|}\chi_{D_k(R)}(x)\\ &= \sum_{k\in\mathbf{Z}_+} h_k(x) + \sum_{k\in\mathbf{Z}_+} N_{k+1}\left[\widetilde{\chi}_{k+1}(x) - \widetilde{\chi}_k(x)\right] =: \mathrm{J}_1 + \mathrm{J}_2,\end{aligned}$$

where, for any $k \in \mathbf{Z}_+$, $N_k := \sum_{j=k}^{\infty} l_j$ and $\widetilde{\chi}_k := \frac{\chi_{D_k(R)}}{|D_k(R)|}$.

We first deal with J_1. Obviously, for all $k \in \mathbf{Z}_+$, supp $h_k \subset 2^{k+1}R$ and $\int_{\mathbf{R}^n} h_k(x)\,dx = 0$. Moreover, by the Hölder inequality and Proposition 3.17(i), we see that

$$\|h_0\|_{L^2(\mathbf{R}^n)} \lesssim \|\alpha\|_{L^2(\mathbf{R}^n)} \lesssim \|a\|_{T_2^2(\mathbf{R}^{n+1}_+)} \lesssim |R|^{-1/2}\|\chi_R\|_{L^{p(\cdot)}(\mathbf{R}^n)}.$$

Since supp $a \subset \widehat{R}$, it follows that, for all $x \in \mathbf{R}^n$,

$$\begin{aligned}|\alpha(x)| &\lesssim \int_0^{r_R} |Q_{s,t^m}(I - P_{s_0,t^m})(a(\cdot,t))(x)|\,\frac{dt}{t}\\ &\lesssim \int_0^{r_R} |Q_{s,t^m}(a(\cdot,t))(x)|\,\frac{dt}{t} + \int_0^{r_R} |(Q_{s,t^m}P_{s_0,t^m})(a(\cdot,t))(x)|\,\frac{dt}{t},\end{aligned}$$

where s_0 is as in (3.8) and $s \in [s_0,\infty)$. By (5.4), (2.4) and the Hölder inequality, we find that, for all $k \in \mathbf{N}$ and $x \in D_k(R)$,

$$\begin{aligned}\int_0^{r_R} |Q_{s,t^m}(a(\cdot,t))(x)|\,\frac{dt}{t} &\lesssim \int_0^{r_R}\int_R \frac{t^\delta}{(t+|x-y|)^{n+\delta}}|a(y,t)|\,\frac{dy\,dt}{t}\\ &\lesssim \frac{(r_R)^{\frac{n}{2}+\delta}}{|x-x_R|^{n+\delta}}\|a\|_{T_2^2(\mathbf{R}^{n+1}_+)} \lesssim \frac{2^{-k(n+\delta)}}{\|\chi_R\|_{L^{p(\cdot)}(\mathbf{R}^n)}}.\end{aligned} \tag{5.10}$$

By an argument similar to that used in the proof of (4.20), we also have

$$\int_0^{r_R} |(Q_{s,t^m}P_{s_0,t^m})(a(\cdot,t))(x)|\,\frac{dt}{t} \lesssim \frac{2^{-k(n+\delta)}}{\|\chi_R\|_{L^{p(\cdot)}(\mathbf{R}^n)}},$$

which, combined with (5.10), implies that, for all $k \in \mathbf{N}$ and $x \in D_k(R)$,

$$|\alpha(x)| \lesssim \frac{2^{-k(n+\delta)}}{\|\chi_R\|_{L^{p(\cdot)}(\mathbf{R}^n)}}. \tag{5.11}$$

From this, together with Lemma 3.14, it follows that

$$\begin{aligned}\|h_k\|_{L^2(\mathbf{R}^n)} &\lesssim \|\alpha\|_{L^2(D_k(R))} \lesssim 2^{-k(\frac{n}{2}+\delta)}\|\chi_R\|^{-1}_{L^{p(\cdot)}(\mathbf{R}^n)}|R|^{\frac12}\\ &\lesssim 2^{-k(n+\delta-\frac{n}{p_-})}\frac{|2^{k+1}R|^{\frac12}}{\|2^{k+1}R\|_{L^{p(\cdot)}(\mathbf{R}^n)}}.\end{aligned} \tag{5.12}$$

Thus, for each $k \in \mathbf{N}$, $2^{k(n+\delta-\frac{n}{p_-})}h_k$ is a $(p(\cdot),2,0)$-atom up to a positive constant multiple. By (5.12) and the fact that $\delta > n(\frac{1}{p_-}-1)$, we find that

$$\left\|\sum_{k=0}^{\infty} h_k\right\|_{L^2(\mathbf{R}^n)} \lesssim \sum_{k=0}^{\infty}\|h_k\|_{L^2(\mathbf{R}^n)} \lesssim \|\chi_R\|^{-1}_{L^{p(\cdot)}(\mathbf{R}^n)}|R|^{\frac12},$$

which implies that $\sum_{k=0}^{\infty} h_k$ converges in $\mathcal{S}'(\mathbf{R}^n)$. Moreover, from Remark 2.1(i) and the Fatou lemma of $L^{p(\cdot)}(\mathbf{R}^n)$ (see [14, Theorem 2.61]), we deduce that

$$(5.13)\quad \begin{aligned} &\left\|\left\{\sum_{k=0}^{\infty}\left[\frac{2^{-k(n+\delta-n/p_-)}\chi_{2^{k+1}R}}{\|\chi_{2^{k+1}R}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right]^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \\ &\lesssim \left\{\sum_{k=0}^{\infty}\left\|\left[\frac{2^{-k(n+\delta-n/p_-)}\chi_{2^{k+1}R}}{\|\chi_{2^{k+1}R}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right]^{\underline{p}}\right\|_{L^{\frac{p(\cdot)}{\underline{p}}}}\right\}^{\frac{1}{\underline{p}}} \lesssim 1. \end{aligned}$$

Therefore, $\mathrm{J}_1 = \sum_{k=0}^{\infty} h_k \in H_{\mathrm{at}}^{p(\cdot),2}(\mathbf{R}^n)$.

Next, we consider the term J_2. Obviously, for any $k \in \mathbf{Z}_+$,

$$\operatorname{supp} N_{k+1}(\widetilde{\chi}_{k+1} - \widetilde{\chi}_k) \subset 2^{k+1}R$$

and

$$\int_{\mathbf{R}^n} N_{k+1}\left[\widetilde{\chi}_{k+1}(x) - \widetilde{\chi}_k(x)\right] dx = N_{k+1}\int_{\mathbf{R}^n}\left[\widetilde{\chi}_{k+1}(x) - \widetilde{\chi}_k(x)\right] dx = 0.$$

On the other hand, by (5.11) and Lemma 3.14, we know that, for each $k \in \mathbf{Z}_+$,

$$(5.14)\quad \begin{aligned} \|N_{k+1}(\widetilde{\chi}_{k+1} - \widetilde{\chi}_k)\|_{L^2(\mathbf{R}^n)} &\lesssim \frac{1}{|2^k R|}\int_{(2^{k+1}R)^{\complement}} |\alpha(x)|\, dx \\ &\lesssim 2^{-k(\frac{n}{2}+\delta)}\frac{|R|^{\frac{1}{2}}}{\|\chi_R\|_{L^{p(\cdot)}(\mathbf{R}^n)}} \\ &\lesssim 2^{-k(n+\delta-\frac{n}{p_-})}\frac{|2^{k+1}R|^{\frac{1}{2}}}{\|2^{k+1}R\|_{L^{p(\cdot)}(\mathbf{R}^n)}}. \end{aligned}$$

Therefore, for each $k \in \mathbf{Z}_+$, $2^{k(n+\delta-n/p_-)}N_{k+1}(\widetilde{\chi}_{k+1} - \widetilde{\chi}_k)$ is a $(p(\cdot), 2, 0)$-atom up to a positive constant multiple. Moreover, from (5.14), we deduce that

$$\left\|\sum_{k\in\mathbf{Z}_+} N_{k+1}(\widetilde{\chi}_{k+1} - \widetilde{\chi}_k)\right\|_{L^2(\mathbf{R}^n)} \lesssim \|\chi_R\|_{L^{p(\cdot)}(\mathbf{R}^n)}^{-1}|R|^{\frac{1}{2}}$$

and hence $\mathrm{J}_2 = \sum_{k\in\mathbf{Z}_+} N_{k+1}(\widetilde{\chi}_{k+1} - \widetilde{\chi}_k)$ converges in $\mathcal{S}'(\mathbf{R}^n)$. By this and (5.13), we conclude that $\mathrm{J}_2 \in H_{\mathrm{at}}^{p(\cdot),2}(\mathbf{R}^n)$. Therefore, for the molecule α, we have

$$(5.15)\quad \alpha = \sum_{k\in\mathbf{Z}_+} \frac{1}{2^{k(n+\delta-n/p_-)}}\widetilde{h}_k + \sum_{k\in\mathbf{Z}_+} \frac{1}{2^{k(n+\delta-n/p_-)}}\widetilde{N}_{k+1}\left(\widetilde{\chi}_{k+1} - \widetilde{\chi}_k\right)$$

in $L^2(\mathbf{R}^n)$ and hence in $\mathcal{S}'(\mathbf{R}^n)$, where, for every $k \in \mathbf{Z}_+$, $\widetilde{h}_k$ and $\widetilde{N}_{k+1}(\widetilde{\chi}_{k+1} - \widetilde{\chi}_k)$ are $(p(\cdot), 2, 0)$-atoms up to a positive constant multiple and, moreover, $\alpha \in H_{\mathrm{at}}^{p(\cdot),2}(\mathbf{R}^n)$.

Now, for all $f \in H_L^{p(\cdot)}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$, by Theorem 3.13(i), we find that f has an atomic decomposition $f = \sum_{j\in\mathbf{N}} \lambda_j\alpha_j$, where the summation converges in $L^2(\mathbf{R}^n)$ and also in $H_L^{p(\cdot)}(\mathbf{R}^n)$, $\{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$ and $\{\alpha_j\}_{j\in\mathbf{N}}$ are $(p(\cdot), s, L)$-molecules, as in Definition 3.11, such that

$$\mathcal{B}(\{\lambda_j\alpha_j\}_{j\in\mathbf{N}}) \lesssim \|f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)}.$$

Moreover, we may assume that, for each $j \in \mathbf{N}$, α_j is a molecule associated with some cube $R_j := Q(x_j, r_j)$ for some $x_j \in \mathbf{R}^n$ and $r_j \in (0, \infty)$. Then, from what we have proved as in (5.15), we deduce that

$$f=\sum_{j\in\mathbf{N}}\sum_{k\in\mathbf{Z}_+}\frac{\lambda_j}{2^{k(n+\delta-\frac{n}{p_-})}}\widetilde{h}_{j,k}+\sum_{j\in\mathbf{N}}\sum_{k\in\mathbf{Z}_+}\frac{\lambda_j}{2^{k(n+\delta-\frac{n}{p_-})}}\widetilde{N}_{j,k+1}\left(\widetilde{\chi}_{j,k+1}-\widetilde{\chi}_{j,k}\right)$$

in $L^2(\mathbf{R}^n)$ and hence also in $\mathcal{S}'(\mathbf{R}^n)$, where, for each $j\in\mathbf{N}$ and $k\in\mathbf{Z}_+$, $\widetilde{h}_{j,k}$ and $\widetilde{N}_{j,k+1}(\widetilde{\chi}_{j,k+1}-\widetilde{\chi}_{j,k})$ are $(p(\cdot),2,0)$-atoms, supported on $2^{k+1}R_j$, up to a positive constant multiple. On the other hand, by Lemma 3.14, we see that, for any $j\in\mathbf{N}$ and $k\in\mathbf{Z}_+$,

$$\frac{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}{\|\chi_{2^kR_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\lesssim\left(\frac{|R_j|}{|2^kR_j|}\right)^{\frac{1}{p_+}}\sim 2^{-k\frac{n}{p_+}}.$$

Then, by choosing $\delta\in(0,1)$ such that $\delta\in(n[\frac{2}{p_-}-1-\frac{1}{p_+}],1)$, Lemma 3.15 and the fact that, for any $j\in\mathbf{N}$, $k\in\mathbf{Z}_+$, $r\in(0,\underline{p})$ and $x\in\mathbf{R}^n$,

$$\chi_{2^kR_j}(x)\le 2^{kn/r}\left[\mathcal{M}(\chi_{R_j})(x)\right]^{\frac{1}{r}},$$

we deduce that

$$\begin{aligned}
&\left\|\left\{\sum_{j\in\mathbf{N}}\sum_{k\in\mathbf{Z}_+}\left[\frac{|\lambda_j|2^{-k(n+\delta-\frac{n}{p_-})}\chi_{2^{k+1}R_j}}{\|\chi_{2^{k+1}R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right]^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}\\
&\lesssim\left\|\left\{\sum_{j\in\mathbf{N}}\sum_{k\in\mathbf{Z}_+}\left[\frac{|\lambda_j|2^{-k(n+\delta+\frac{n}{p_+}-\frac{n}{p_-}-\frac{n}{r})}[\mathcal{M}(\chi_{R_j})]^{\frac{1}{r}}}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right]^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}\\
&\lesssim\left\|\left\{\sum_{j\in\mathbf{N}}\left[\frac{|\lambda_j|[\mathcal{M}(\chi_{R_j})]^{\frac{1}{r}}}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right]^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}\lesssim\mathcal{B}(\{\lambda_j\alpha_j\}_{j\in\mathbf{N}})\lesssim\|f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)}.
\end{aligned}$$

Therefore, $f\in H_{\mathrm{at}}^{p(\cdot),2}(\mathbf{R}^n)$ and hence $H_L^{p(\cdot)}(\mathbf{R}^n)\subset H_{\mathrm{at}}^{p(\cdot),2}(\mathbf{R}^n)$. This finishes the proof of Theorem 5.3. $\square$

Remark 5.7. When $p(\cdot)$ is a constant exponent, Theorem 5.3 goes back to [48, Theorem 6.1] (see also [30, Theorem 6.1]). We point out that the proof of Theorem 5.3 borrows some ideas from the proof of [30, Theorem 6.1].

5.2. Fractional integrals $L^{-\gamma}$ on spaces $H_L^{p(\cdot)}(\mathbf{R}^n)$. Let L satisfy Assumptions (A) and (B) as in Section 2. In this subsection, we establish the boundedness of the fractional integral on variable exponent Hardy spaces associated with the operator L. Recall that, for any $\gamma\in(0,\frac{n}{m})$ with m as in Assumption (A), the generalized fractional integral $L^{-\gamma}$ associated with L is defined by setting, for all $f\in L^2(\mathbf{R}^n)$ and $x\in\mathbf{R}^n$,

$$L^{-\gamma}(f)(x):=\frac{1}{\Gamma(\gamma)}\int_0^\infty t^{\gamma-1}e^{-tL}(f)(x)\,dt,$$

where $\Gamma(\gamma)$ is an appropriate positive constant; see [48, p. 4400]. Notice that, if $L:=-\Delta$ with Δ being the Laplacian, then $L^{-\gamma}$ becomes the classical fractional integral; see, for example, [45, Chapter 5]. We also point out that the Hardy–Littlewood–Sobolev inequality related to the semigroup itself of an operator was studied by Yoshikawa [52].

Remark 5.8. Let L satisfy Assumptions (A) and (B). For $\gamma \in (0, \frac{n}{m})$ with m as in Assumption (A), define the operator $\widetilde{L}^{-\gamma}$ by setting, for all $f \in L^2(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$,

$$\widetilde{L}^{-\gamma}(f)(x) := \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} |e^{-tL}(f)(x)|\, dt.$$

It was proved in [30, Lemma 5.1(ii)] that, if $\gamma \in (0, \frac{n}{m})$ and p_1, $p_2 \in (1, \infty)$ satisfy $\frac{1}{p_2} = \frac{1}{p_1} - \frac{m\gamma}{n}$, then $\widetilde{L}^{-\gamma}$ is bounded from $L^{p_1}(\mathbf{R}^n)$ into $L^{p_2}(\mathbf{R}^n)$.

The main result of this subsection is stated as follows.

Theorem 5.9. *Let L satisfy Assumptions (A) and (B), $\gamma \in (0, \frac{n}{m})$ with m as in Assumption (A), $p(\cdot) \in C^{\log}(\mathbf{R}^n)$ satisfy $\frac{n}{n+\theta(L)} < p_- \le p_+ \le 1$ with p_-, p_+ and $\theta(L)$, respectively, as in* (2.1) *and* (2.5)*. Assume that $q(\cdot)$ is defined by setting, for all $x \in \mathbf{R}^n$,*

$$\frac{1}{q(x)} := \frac{1}{p(x)} - \frac{m\gamma}{n}.$$

Then the fractional integral $L^{-\gamma}$ maps $H_L^{p(\cdot)}(\mathbf{R}^n)$ continuously into $H_L^{q(\cdot)}(\mathbf{R}^n)$.

To prove Theorem 5.9, we need the following technical lemma, which is just [44, Lemma 5.2] and plays a key role in the proof of Theorem 5.9.

Lemma 5.10. *Let $\delta \in (0, n)$ and $p(\cdot) \in C^{\log}(\mathbf{R}^n)$ satisfy $p_+ \in (0, \frac{n}{\delta})$. Assume that $q(\cdot) \in \mathcal{P}(\mathbf{R}^n)$ is defined by setting, for all $x \in \mathbf{R}^n$, $\frac{1}{q(x)} := \frac{1}{p(x)} - \frac{\delta}{n}$. Then there exists a positive constant C such that, for all sequences $\{R_j\}_{j\in\mathbf{N}}$ of cubes of $\mathbf{R}^n$ and $\{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$,*

$$\left\|\sum_{j\in\mathbf{N}} |\lambda_j| |R_j|^{\frac{\delta}{n}} \chi_{R_j}\right\|_{L^{q(\cdot)}(\mathbf{R}^n)} \le C \left\|\sum_{j\in\mathbf{N}} |\lambda_j| \chi_{R_j}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}.$$

Proof of Theorem 5.9. To prove this theorem, we only need to show that, for all $f \in H_L^{p(\cdot)}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$,

$$\|S_L(L^{-\gamma}(f))\|_{L^{q(\cdot)}(\mathbf{R}^n)} \lesssim \|f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)}, \tag{5.16}$$

since $H_L^{p(\cdot)}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ is dense in $H_L^{p(\cdot)}(\mathbf{R}^n)$.

Let $f \in H_L^{p(\cdot)}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$. Then, by Theorem 3.13(ii), we see that there exist $\{\lambda_j\}_{j\in\mathbf{N}} \subset \mathbf{C}$ and a sequence $\{\alpha_j\}_{j\in\mathbf{N}}$ of $(p(\cdot), s_0, L)$-molecules associated with cubes $\{R_j\}_{j\in\mathbf{N}}$ such that $f = \sum_{j\in\mathbf{N}} \lambda_j \alpha_j$ in $H_L^{p(\cdot)}(\mathbf{R}^n)$ and also in $L^2(\mathbf{R}^n)$, and

$$\mathcal{A}(\{\lambda_j\}_{j\in\mathbf{N}}, \{R_j\}_{j\in\mathbf{N}}) \lesssim \|f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)}.$$

Observe that $L^{-\gamma}$ is bounded from $L^2(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ for some $q \in (1, \infty)$ such that $\frac{1}{q} = \frac{1}{2} - \frac{m\gamma}{n}$ (see [48, Theorem 5.1] and also Remark 5.8). It follows that, for almost every $x \in \mathbf{R}^n$,

$$|L^{-\gamma}(f)(x)| \lesssim \sum_{j\in\mathbf{N}} |\lambda_j| \int_0^\infty t^{\gamma-1} |e^{-tL}(\alpha_j)(x)|\, dt =: \sum_{j\in\mathbf{N}} |\lambda_j| \widetilde{L}^{-\gamma}(\alpha_j)(x)$$

and hence

$$
\begin{aligned}
\left\|S_L(L^{-\gamma}(f))\right\|_{L^{q(\cdot)}(\mathbf{R}^n)} &\lesssim \left\|\sum_{j\in\mathbf{N}}|\lambda_j|S_L(\widetilde{L}^{-\gamma}(\alpha_j))\chi_{4R_j}\right\|_{L^{q(\cdot)}(\mathbf{R}^n)} \\
&\quad+\left\|\sum_{j\in\mathbf{N}}|\lambda_j|S_L(\widetilde{L}^{-\gamma}(\alpha_j))\chi_{(4R_j)^\complement}\right\|_{L^{q(\cdot)}(\mathbf{R}^n)} =: \mathrm{I}_1+\mathrm{I}_2.
\end{aligned}\tag{5.17}
$$

To deal with I_1, let $r \in (1,2)$. Then, by the Hölder inequality, (2.9), Proposition 3.17 and Remark 5.8, we find that

$$
\begin{aligned}
\left\|S_L(\widetilde{L}^{-\gamma}(\alpha_j))\chi_{4R_j}\right\|_{L^r(\mathbf{R}^n)} &\lesssim |R_j|^{\frac{1}{r}-\frac{1}{q}}\left\|S_L(\widetilde{L}^{-\gamma}(\alpha_j))\chi_{4R_j}\right\|_{L^q(\mathbf{R}^n)} \\
&\lesssim |R_j|^{\frac{1}{r}-\frac{1}{q}}\left\|\widetilde{L}^{-\gamma}(\alpha_j)\chi_{4R_j}\right\|_{L^q(\mathbf{R}^n)} \\
&\lesssim |R_j|^{\frac{1}{r}-\frac{1}{q}}\|\alpha_j\|_{L^2(\mathbf{R}^n)} \lesssim [\ell(R_j)]^{m\gamma}\frac{|R_j|^{\frac{1}{r}}}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}.
\end{aligned}
$$

This, combined with Lemmas 3.5 and 5.10, implies that

$$
\begin{aligned}
\mathrm{I}_1 &\lesssim \left\|\sum_{j\in\mathbf{N}}\left[\frac{|\lambda_j|}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right]^{\underline{q}}[\ell(R_j)]^{m\gamma\underline{q}}\chi_{4R_j}\right\|_{L^{\frac{q(\cdot)}{\underline{q}}}}^{\frac{1}{\underline{q}}} \\
&\lesssim \left\|\sum_{j\in\mathbf{N}}\left[\frac{|\lambda_j|\chi_{4R_j}}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right]^{\underline{q}}\right\|_{L^{\frac{p(\cdot)}{\underline{q}}}}^{\frac{1}{\underline{q}}} \lesssim \left\|\left\{\sum_{j\in\mathbf{N}}\left[\frac{|\lambda_j|\chi_{4R_j}}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right]^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)} \\
&\sim \mathcal{A}(\{\lambda_j\}_{j\in\mathbf{N}},\{R_j\}_{j\in\mathbf{N}}) \lesssim \|f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)},
\end{aligned}\tag{5.18}
$$

where $\underline{q} := \min\{1, q_-\}$ with q_- as in (2.1) via $p(\cdot)$ replaced by $q(\cdot)$.

Next, we estimate I_2. Since $\frac{n}{n+\theta(L)} < p_-$, it follows that there exists $\varepsilon \in (0,\theta(L))$ such that $\frac{n}{n+\varepsilon} < p_-$. Moreover, we may choose $r_0 \in (0, p_-)$ such that

$$
\varepsilon \in (n[1/r_0-1], \theta(L)).
$$

Assume that, for each $j \in \mathbf{N}$, $R_j := Q(x_j, r_j)$ for some $x_j \in \mathbf{R}^n$ and $r_j \in (0,\infty)$. Then, by an argument similar to that used in the proof of [48, (5.3)], we conclude that, for all $j \in \mathbf{N}$ and $x \in (4R_j)^\complement$,

$$
S_L(\widetilde{L}^{-\gamma}(\alpha_j))(x) \lesssim \frac{|r_j|^{\varepsilon+m\gamma+n}}{|x-x_j|^{n+\varepsilon}}\frac{1}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}.\tag{5.19}
$$

For any $k,\, j \in \mathbf{N}$, let $D_k(R_j) := (2^{k+2}R_j)\backslash(2^{k+1}R_j)$. Then, by (5.19) and Lemma 5.10, we see that

$$
\begin{aligned}
\mathrm{I}_2 &\lesssim \left\|\sum_{k,\,j\in\mathbf{N}}\frac{|\lambda_j|}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\frac{r_j^{\varepsilon+m\gamma+n}}{|\cdot-x_j|^{n+\varepsilon}}\chi_{D_k(R_j)}\right\|_{L^{q(\cdot)}(\mathbf{R}^n)} \\
&\lesssim \left\|\sum_{k,\,j\in\mathbf{N}}\frac{|\lambda_j|}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\frac{(2^k r_j)^{m\gamma}}{2^{k(n+\varepsilon+m\gamma)}}\chi_{2^{k+2}R_j}\right\|_{L^{q(\cdot)}(\mathbf{R}^n)} \\
&\lesssim \left\|\sum_{k,\,j\in\mathbf{N}}\frac{|\lambda_j|}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\frac{\chi_{2^{k+2}R_j}}{2^{k(n+\varepsilon+m\gamma)}}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}.
\end{aligned}
$$

Thus, from Lemma 3.15, Remark 2.1(i) and the fact that, for any $k\in\mathbf{N}$,

$$\chi_{2^{k+2}R_j}\lesssim 2^{kn/r_0}[\mathcal{M}(\chi_{R_j})]^{1/r_0},$$

we deduce that

$$\begin{aligned}\mathrm{I}_2&\lesssim\left[\sum_{k\in\mathbf{N}}\frac{2^{kn\underline{p}/r_0}}{2^{k\underline{p}(n+\varepsilon+m\gamma)}}\left\|\sum_{j\in\mathbf{N}}\left\{\frac{|\lambda_j|}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}[\mathcal{M}(\chi_{R_j})]^{\frac{1}{r_0}}\right\}^{\underline{p}}\right\|_{L^{\frac{p(\cdot)}{\underline{p}}}(\mathbf{R}^n)}\right]^{\frac{1}{\underline{p}}}\\&\lesssim\left\|\left\{\sum_{j\in\mathbf{N}}\left[\frac{|\lambda_j|\chi_{R_j}}{\|\chi_{R_j}\|_{L^{p(\cdot)}(\mathbf{R}^n)}}\right]^{\underline{p}}\right\}^{\frac{1}{\underline{p}}}\right\|_{L^{p(\cdot)}(\mathbf{R}^n)}\lesssim\|f\|_{H_L^{p(\cdot)}(\mathbf{R}^n)}.\end{aligned}$$

This, together with (5.17) and (5.18), implies that (5.16) holds true, which shows that $L^{-\gamma}$ is bounded from $H_L^{p(\cdot)}(\mathbf{R}^n)$ to $H_L^{q(\cdot)}(\mathbf{R}^n)$ and hence completes the proof of Theorem 5.9. □

Remark 5.11. In the case of constant exponents, Theorem 5.9 was obtained by Yan [48, Theorem 5.1].

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References

[1] Acerbi, E., and G. Mingione: Regularity results for stationary electro-rheological fluids. - Arch. Ration. Mech. Anal. 164, 2002, 213–259.

[2] Acerbi, E., and G. Mingione: Gradient estimates for the $p(x)$-Laplacean system. - J. Reine Angew. Math. 584, 2005, 117–148.

[3] Albrecht, D., X. T. Duong, and A. McIntosh: Operator theory and harmonic analysis. - In: Instructional Workshop on Analysis and Geometry, Part III (Canberra, 1995), Proc. Centre Math. Appl. Austral. Nat. Univ. 34, 1996, 77–136.

[4] Almeida, A., and P. Hästö: Besov spaces with variable smoothness and integrability. - J. Funct. Anal. 258, 2010, 1628–1655.

[5] Auscher, P., X. T. Duong, and A. McIntosh: Boundedness of Banach space valued singular integral operators and Hardy spaces. - Unpublished manuscript, 2005.

[6] Birnbaum, Z., and W. Orlicz: Über die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen. - Studia Math. 3, 1931, 1–67.

[7] Blunck, S., and P. C. Kunstmann: Weak type (p,p) estimates for Riesz transforms. - Math. Z. 247, 2004, 137–148.

[8] Bui, T. A., J. Cao, L. D. Ky, D. Yang, and S. Yang: Musielak–Orlicz–Hardy spaces associated with operators satisfying reinforced off-diagonal estimates. - Anal. Geom. Metr. Spaces 1, 2013, 69–129.

[9] Chen, Y., S. Levine, and M. Rao: Variable exponent, linear growth functionals in image restoration. - SIAM J. Appl. Math. 66, 2006, 1383–1406.

[10] Coifman, R. R., Y. Meyer, and E. M. Stein: Some new function spaces and their applications to harmonic analysis. - J. Funct. Anal. 62, 1985, 304–335.

[11] Coifman, R. R., and G. Weiss: Extensions of Hardy spaces and their use in analysis. - Bull. Amer. Math. Soc. 83, 1977, 569–645.

[12] Coulhon, T., and X. T. Duong: Maximal regularity and kernel bounds: observations on a theorem by Hieber and Prüss. - Adv. Differential Equations 5, 2000, 343–368.

[13] Cruz-Uribe, D.: The Hardy–Littlewood maximal operator on variable-L^p spaces. - In: Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), Colecc. Abierta 64, Univ. Sevilla Secr. Publ., Seville, 2003, 147–156.

[14] Cruz-Uribe, D. V., and A. Fiorenza: Variable Lebesgue spaces. Foundations and harmonic analysis, applied and numerical harmonic analysis. - Birkhäuser/Springer, Heidelberg, 2013.

[15] Cruz-Uribe, D., A. Fiorenza, J. M. Martell, and C. Pérez: The boundedness of classical operators on variable L^p spaces. - Ann. Acad. Sci. Fenn. Math. 31, 2006, 239–264.

[16] Cruz-Uribe, D., and L.-A. D. Wang: Variable Hardy spaces. - Indiana Univ. Math. J. 63, 2014, 447–493.

[17] Diening, L.: Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}(\mathbf{R}^n)$. - Math. Inequal. Appl. 7, 2004, 245–253.

[18] Diening, L., P. Harjulehto, P. Hästö, and M. Růžička: Lebesgue and Sobolev spaces with variable exponents. - Lecture Notes in Math. 2017, Springer, Heidelberg, 2011.

[19] Diening, L., P. Hästö, and S. Roudenko: Function spaces of variable smoothness and integrability. - J. Funct. Anal. 256, 2009, 1731–1768.

[20] Drihem, D.: Some properties of variable Besov-type spaces. - Funct. Approx. Comment. Math. 52, 2015, 193–221.

[21] Duong, X. T., and A. MacIntosh: Singular integral operators with non-smooth kernels on irregular domains. - Rev. Mat. Iberoam. 15, 1999, 233–265.

[22] Duong, X. T., J. Xiao, and L. Yan: Old and new Morrey spaces with heat kernel bounds. - J. Fourier Anal. Appl. 13, 2007, 87–111.

[23] Duong, X. T., and L. Yan: Duality of Hardy and BMO spaces associated with operators with heat kernel bounds. - J. Amer. Math. Soc. 18, 2005, 943–973.

[24] Duong, X. T., and L. Yan: New function spaces of BMO type, the John–Nirenberg inequality, interpolation, and applications. - Comm. Pure Appl. Math. 58, 2005, 1375–1420.

[25] Fefferman, C., and E. M. Stein: H^p spaces of several variables. - Acta Math. 129, 1972, 137–193.

[26] Harjulehto, P., P. Hästö, and V. Latvala: Minimizers of the variable exponent, non-uniformly convex Dirichlet energy. - J. Math. Pures Appl. (9) 89, 2008, 174–197.

[27] Hofmann, S., and J. M. Martell: L^p bounds for Riesz transforms and square roots associated to second order elliptic operators. - Publ. Mat. 47, 2003, 497–515.

[28] Izuki, M.: Vector-valued inequalities on Herz spaces and characterizations of Herz–Sobolev spaces with variable exponent. - Glas. Mat. Ser. III 45:65, 2010, 475–503.

[29] Jiang, R., and D. Yang: New Orlicz–Hardy spaces associated with divergence form elliptic operators. - J. Funct. Anal. 258, 2010, 1167–1224.

[30] Jiang, R., D. Yang, and Y. Zhou: Orlicz–Hardy spaces associated with operators. - Sci. China Ser. A 52, 2009, 1042–1080.

[31] Kováčik, O., and J. Rákosník: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. - Czechoslovak Math. J. 41:116, 1991, 592–618.

[32] Ky, L. D.: New Hardy spaces of Musielak–Orlicz type and boundedness of sublinear operators. - Integral Equations Operator Theory 78, 2014, 115–150.

[33] Liu, S., and L. Song: Boundedness of functional calculi of Schrödinger operators on generalized Lebesgue spaces $L^p(\mathbf{R}^n)$. - Manuscripta Math. 138, 2012, 119–139.

[34] Luxemburg, W.: Banach Function spaces. - Thesis, Technische Hogeschool te Delft, 1955.

[35] McIntosh, A.: Operators which have an H^∞ functional calculus. - In: Miniconference on operator theory and partial differential equations (North Ryde, 1986), Proc. Centre Math. Anal. Austral. Nat. Univ. 14, 1986, 210–231.

[36] Müller, S.: Hardy space methods for nonlinear partial differential equations. - Tatra Mt. Math. Publ. 4, 1994, 159–168.

[37] Nakai, E., and Y. Sawano: Hardy spaces with variable exponents and generalized Campanato spaces. - J. Funct. Anal. 262, 2012, 3665–3748.

[38] Nakano, H.: Modulared semi-ordered linear spaces. - Maruzen Co., Ltd., Tokyo, 1950.

[39] Nakano, H.: Topology of linear topological spaces. - Maruzen Co., Ltd., Tokyo, 1951.

[40] Orlicz, W.: Über eine gewisse Klasse von Räumen vom Typus B. - Bull. Int. Acad. Pol. Ser. A 8, 1932, 207–220.

[41] Ouhabaz, E. M.: Analysis of heat equations on domains. - London Math. Soc. Monogr. Ser. 31. Princeton Univ. Press, Princeton, NJ, 2005.

[42] Růžička, M.: Electrorheological fluids: Modeling and mathematical theory. - Lecture Notes in Math. 1748, Springer-Verlag, Berlin, 2000.

[43] Sanchón, M., and J. M. Urbano: Entropy solutions for the $p(x)$-Laplace equation. - Trans. Amer. Math. Soc. 361, 2009, 6387–6405.

[44] Sawano, Y.: Atomic decompositions of Hardy spaces with variable exponents and its application to bounded linear operators. - Integral Equations Operator Theory 77, 2013, 123–148.

[45] Stein, E. M.: Singular integrals and differentiability properties of functions. - Princeton Math. Ser. 30, Princeton Univ. Press, Princeton, N.J., 1970.

[46] Stein, E. M.: Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals. - Princeton Math. Ser. 43, Monographs in Harmonic Analysis III. Princeton Univ. Press, Princeton, NJ, 1993.

[47] Yan, L.: Littlewood–Paley functions associated to second order elliptic operators. - Math. Z. 246, 2004, 655–666.

[48] Yan, L.: Classes of Hardy spaces associated with operators, duality theorem and applications. - Trans. Amer. Math. Soc. 360, 2008, 4383–4408.

[49] Yang, D., W. Yuan, and C. Zhuo: Musielak–Orlicz Besov-type and Triebel–Lizorkin-type spaces. - Rev. Mat. Complut. 27, 2014, 93–157.

[50] Yang, D., C. Zhuo, and W. Yuan: Triebel–Lizorkin type spaces with variable exponents. - Banach J. Math. Anal. 9:4, 2015, 146–202.

[51] Yang, D., C. Zhuo, and W. Yuan: Besov-type spaces with variable smoothness and integrability. - J. Funct. Anal. 269, 2015, 1840–1898.

[52] Yoshikawa, A.: An operator theoretical remark on the Hardy–Littlewood–Sobolev inequality. - J. Fac. Sci. Univ. Tokyo Sect. I A Math. 17, 1970, 559–566.

[53] Yosida, K.: Functional analysis. Sixth edition. - Grundlehren Math. Wiss. 123, Springer-Verlag, Berlin-New York, 1978.

[54] Zhuo, C., Y. Sawano, and D. Yang: Hardy spaces with variable exponents on RD-spaces and applications. - Dissertationes Math. (Rozprawy Mat.) (to appear).

[55] Zhuo, C., D. Yang, and Y. Liang: Intrinsic square function characterizations of Hardy spaces with variable exponents. - Bull. Malays. Math. Sci. Soc. (2) (to appear).

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