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WIMAN–VALIRON THEOREM FOR q-DIFFERENCES

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Abstract. Let q be any complex number other than 0 and 1. We first asymptotically express the logarithmic q-difference $\log f(qz) - \log f(z)$ in terms of the logarithmic derivative f'/f for any meromorphic function f of order strictly less than 1/2. Then we show the assumption that the order strictly less than 1/2 is sharp. Finally, we prove a q-difference analogue of the Wiman–Valiron theorem for entire functions of order strictly less than 1/2.

1. Introduction

Wiman–Valiron theory is a powerful tool in the study of entire function theory and complex differential equation theory (e.g. [6]). The most inspiring theorem in this theory is the classical Wiman–Valiron theorem which reveals the local behavior of entire functions and their derivatives when $|f(z)|$ is close to its maximum modulus at z (e.g. [3, 5, 9]).

By utilizing an analogue of Wiman–Valiron theory for difference equations, Bergweiler and Langley [1] investigated the zero distribution of $\Delta f(z) \overset{\text{def}}{=} f(z+1) - f(z)$, also in general $\Delta^k f(z)$, for entire functions of order less than one. They obtained that $\Delta^k f(z) \sim f^{(k)}(z)$ holds outside an exceptional set. Ishizaki and Yanagihara [4] showed an analogue of Wiman–Valiron theory by thoughtfully rewriting power series of entire functions of order less than 1/2 into binomial series. Chiang and Feng [2] established a relationship between $\log f(z+q) - \log f(z)$ and f'/f, further proved a difference analogue of Wiman–Valiron theory estimates for entire functions of order less than one, which can be used in the study of entire solutions of linear difference equations.

The first main result in this paper is to establish a relationship between $\log f(qz) - \log f(z)$ and f'/f for meromorphic functions of order less than 1/2 and show the bound 1/2 is the best upper bound for this relationship. The second main result, which is a by-product of the first main result, is a q-difference analogue of the Wiman–Valiron theorem for any entire function of order less than 1/2.

As usual, the order of a meromorphic function f in $\mathbf{C}$ is defined by

$$\rho_f := \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

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The standard notations in Nevanlinna theory and entire function theory are also employed. For example, $T(r, f)$ is the Nevanlinna characteristic function of f and $\nu(r, f)$ is the central index of f. Moreover, we say a set $F \subset R^+$ is of the finite logarithmic measure if and only if

$$\int_E \frac{1}{t}\,dt < \infty.$$

Also, for any complex-valued function $f(z)$, the notation $f(z) = O(g(|z|))$ throughout the paper is that there is an $r_0 > 0$ such that $|f(z)/g(|z|)| < C$ holds for a constant $C > 0$ and for all z with $|z| > r_0$. Furthermore, $C(> 0)$ is always a constant in each equation and its value may be different in its each appearance.

2. Main results

The following theorem reveals an interesting relationship between the logarithmic q-difference of f and the logarithmic derivative of f. It is a key ingredient in the proof of our q-difference analogue of the Wiman–Valiron theorem.

Theorem 2.1. *If f is a transcendental meromorphic function of order strictly less than $1/2$ and $q \in \mathbf{C}\backslash\{0, 1\}$, then we have*

$$\log \frac{f(qz)}{f(z)} = (q-1)z\frac{f'(z)}{f(z)} + O(1), \tag{2.1}$$

or, equivalently,

$$\frac{f(qz)}{f(z)} = e^{(q-1)z\frac{f'(z)}{f(z)}+O(1)},$$

for any r outside an exceptional set which is of finite logarithmic measure.

Note that $O(1)$ in above theorem depends on f, q.

Theorem 2.2. *Let q be any complex number. There is an entire function f of order $1/2$ such that, if $\sqrt{z} = -\sqrt{x}i$, for any $x \in \mathbf{R}^+$, we have, as $x \to \infty$,*

$$\log \frac{f(qz)}{f(z)} = (\sqrt{q}-1)\sqrt{x} + \log(1+o(1)) \quad \textit{and}$$
$$(q-1)z\frac{f'(z)}{f(z)} = \frac{q-1}{2}\sqrt{x}(1+o(1)).$$

Furthermore, for any $q \in \mathbf{C} \setminus \{1\}$, f does not satisfy (2.1) in Theorem 2.1.

Clearly, the theorem tells us that the condition $\rho < 1/2$ in Theorem 2.1 is the best possible.

We now state our q-difference analogue for the Wiman–Valiron theorem for entire functions of order less than $1/2$.

Theorem 2.3. *Suppose that m is any positive integer and q a complex number with $q^m \in \mathbf{C}\backslash\{0, 1\}$. Let $f(z)$ be a transcendental entire function of order strictly less than $1/2$ and $F \subset \mathbf{R}^+$ a set of finite logarithmic measure. Then for any $0 < \delta < 1/4$ and any z with $|z| = r \notin F$ satisfying*

$$|f(z)| > M(r, f)\nu(r, f)^{\delta-1/4},$$

we have

$$\frac{f(q^m z)}{f(z)} = e^{(q^m-1)\nu(r,f)(1+o(1))}.$$

3. Proofs of our theorems

The following Poisson–Jensen formula plays an important role in proving our Theorem 2.1.

Lemma 3.1. [8, p. 163] *Let $f(z)$ be a meromorphic function in the complex plane, not identically zero. Let $\{a_\mu\}_{\mu\in\mathbf{N}}$ and $\{b_\nu\}_{\nu\in\mathbf{N}}$ be the sequences of zeros and poles, with due account of multiplicities, of $f(z)$, respectively. Then for $|z|<R<\infty$,*

$$\begin{aligned}\log f(z) = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\,d\theta - \sum_{|a_\mu|<R}\log\frac{R^2-\overline{a}_\mu z}{R(z-a_\mu)}\\ + \sum_{|b_\nu|<R}\log\frac{R^2-\overline{b}_\nu z}{R(z-b_\nu)} + iK,\end{aligned} \tag{3.1}$$

where

$$K = \arg f(0) - \sum_{|b_\nu|<R}\arg\left(-\frac{R}{b_\nu}\right) + \sum_{|a_\mu|<R}\arg\left(-\frac{R}{a_\mu}\right) + 2m\pi$$

and m is an integer depending on R, f and the choice of branch of the logarithm functions of both sides of (3.1).

We also need following lemmas.

Lemma 3.2. [2, Lemma 3.2] *Let us define*

$$\log w = \log|w| + i\arg w, \quad -\pi\le\arg w<\pi,$$

to be the principal branch of the logarithmic function in the complex plane. Then we have

$$\log(1+w)-w = O(|w|^2) + O\left(\left|\frac{w}{1+w}\right|^2\right)$$

for all w in $\mathbf{C}$.

The next result is a well-known lemma of Cartan [7, pp. 19–22].

Lemma 3.3. *Let $a_1,\dots,a_m$ be any finite collection of complex numbers, and let $d>0$ be any given positive number. Then there exists a finite collection of closed disks $D_1,\dots,D_q$ with corresponding radii $r_1,\dots,r_q$ that satisfy $r_1+\dots+r_q=2d$, such that if $z\notin D_j$ for all $j=1,\dots,q$, then there is a permutation of the points $a_1,\dots,a_m$, say, $b_1,\dots,b_m$, that satisfy*

$$|z-b_k| > k\frac{d}{m}$$

for $k=1,\dots,m$, where the permutation may depend on z.

In the sequel, we denote the number of zeros and poles of f in $|z|<r$ by $n(r)$ and set $m(r)=m(r,f)+m(r.1/f)$.

Lemma 3.4. *Let $q\in\mathbf{C}\setminus\{0\}$ and $\phi>1$ an increasing function on $\mathbf{R}$. If $f(z)$ is a transcendental meromorphic function, then we have*

$$\begin{aligned}\log\frac{f(qz)}{f(z)} = (q-1)z\frac{f'(z)}{f(z)} + O\left(\frac{|q-1|^2 m(r\phi(r))}{\phi^2(r)}\right)\\ + O\left(\frac{|q-1|^2 n(r\phi(r))}{\phi^2(r)}\right) + O\left(\sum_{|s_k|<r\phi(r)}\left|\frac{(q-1)z}{z-s_k}\right|\right) + O(1),\end{aligned}$$

for any z with $|z| = r$, and s_k is either a pole or a zero of f.

Proof. Let $\{a_\mu\}_{\mu\in\mathbf{N}}$ and $\{b_\nu\}_{\nu\in\mathbf{N}}$ be the sequences of zeros and poles, with due account of multiplicities, of $f(z)$, respectively. For $q \in \mathbf{C}\backslash\{0,1\}$, and $R > |qz|+|z/q|$, if we substitute qz for z into (3.1), then we conclude that

$$\begin{aligned}\log f(qz) = &\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\frac{Re^{i\theta}+qz}{Re^{i\theta}-qz}\,d\theta - \sum_{|a_\mu|<R}\log\frac{R^2-\overline{a}_\mu qz}{R(qz-a_\mu)}\\ &+\sum_{|b_\nu|<R}\log\frac{R^2-\overline{b}_\nu qz}{R(qz-b_\nu)}+O(1).\end{aligned}\tag{3.2}$$

where $O(1)$ also depends on q now. It is easy to check that

$$\begin{aligned}\log\frac{R^2-\overline{c}_k qz}{R(qz-c_k)} - \log\frac{R^2-\overline{c}_k z}{R(z-c_k)} &= \log\frac{R^2-\overline{c}_k qz}{R^2-\overline{c}_k z} - \log\frac{qz-c_k}{z-c_k}\\ &= \log\left(1-\frac{(q-1)\overline{c}_k z}{R^2-\overline{c}_k z}\right) - \log\left(1+\frac{(q-1)z}{z-c_k}\right),\end{aligned}\tag{3.3}$$

where $\{c_k\}_{k\in\mathbf{N}} = \{a_\mu\}_{\mu\in\mathbf{N}}\bigcup\{b_\nu\}_{\nu\in\mathbf{N}}$. Combining (3.3) with (3.1) and (3.2), we have

$$\begin{aligned}\log\frac{f(qz)}{f(z)} = &\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\frac{2(q-1)zRe^{i\theta}}{(Re^{i\theta}-qz)(Re^{i\theta}-z)}\,d\theta\\ &-\sum_{|a_\mu|<R}\left(\log\left(1-\frac{(q-1)\overline{a}_\mu z}{R^2-\overline{a}_\mu z}\right)-\log\left(1+\frac{(q-1)z}{z-a_\mu}\right)\right)\\ &+\sum_{|b_\nu|<R}\left(\log\left(1-\frac{(q-1)\overline{b}_\nu z}{R^2-\overline{b}_\nu z}\right)-\log\left(1+\frac{(q-1)z}{z-b_\nu}\right)\right)+O(1).\end{aligned}\tag{3.4}$$

In order to estimate f'/f, by differentiating the Poisson-Jensen formula (3.1) in Lemma 3.1, we have

$$\begin{aligned}\frac{f'(z)}{f(z)} = &\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\frac{2Re^{i\theta}}{(Re^{i\theta}-z)^2}\,d\theta\\ &+\sum_{|a_\mu|<R}\left(\frac{\overline{a}_\mu}{R^2-\overline{a}_\mu z}+\frac{1}{z-a_\mu}\right)-\sum_{|b_\nu|<R}\left(\frac{\overline{b}_\nu}{R^2-\overline{b}_\nu z}+\frac{1}{z-b_\nu}\right).\end{aligned}\tag{3.5}$$

Combining equalities (3.4) and (3.5), we deduce that

$$\begin{aligned}\log\frac{f(qz)}{f(z)}-(q-1)z\frac{f'(z)}{f(z)} = &\frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\frac{2Rz^2(q-1)^2e^{i\theta}}{(Re^{i\theta}-z)^2(Re^{i\theta}-qz)}\,d\theta\\ &-\sum_{|a_\mu|<R}\left\{\log\left(1-\frac{(q-1)\overline{a}_\mu z}{R^2-\overline{a}_\mu z}\right)+\frac{(q-1)z\overline{a}_\mu}{R^2-\overline{a}_\mu z}\right\}\\ &+\sum_{|a_\mu|<R}\left\{\log\left(1+\frac{(q-1)z}{z-a_\mu}\right)-\frac{(q-1)z}{z-a_\mu}\right\}\\ &+\sum_{|b_\nu|<R}\left\{\log\left(1-\frac{(q-1)\overline{b}_\nu z}{R^2-\overline{b}_\nu z}\right)+\frac{(q-1)z\overline{b}_\nu}{R^2-\overline{b}_\nu z}\right\}\end{aligned}$$

$$- \sum_{|b_\nu|<R} \left\{ \log \left(1 + \frac{(q-1)z}{z-b_\nu} \right) - \frac{(q-1)z}{z-b_\nu} \right\} + O(1)$$

$$:= S_1(z) + S_2(z) + S_3(z) + S_4(z) + S_5(z) + O(1). \tag{3.6}$$

We will estimate $|S_i(z)| (i = 1, \ldots, 5)$ separately. Let us estimate $|S_1(z)|$ at first. We note the fact that

$$\left| \frac{2Rz^2(q-1)^2 e^{i\theta}}{(Re^{i\theta} - z)^2(Re^{i\theta} - qz)} \right| \le \frac{2|q-1|^2 r^2 R}{(R-r)^2(R-|q|r)},$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \log |f(Re^{i\theta})| \right| d\theta = m(R, f) + m(R, 1/f).$$

Therefore, from the above equations, we have

$$|S_1(z)| \le \frac{2|q-1|^2 r^2 R}{(R-r)^2(R-|q|r)} (m(R,f) + m(R,1/f)).$$

Let $R = r\phi(r)$. We have, for $|z| = r$,

$$|S_1(z)| = O\left(\frac{|q-1|^2 m(R)}{\phi^2(r)} \right).$$

In the following, we proceed to estimate $|S_2(z)|$ and $|S_4(z)|$ by means of Lemma 3.2. It is easy to check that

$$\begin{aligned}
&\log \left(1 - \frac{(q-1)\overline{a}_\mu z}{R^2 - \overline{a}_\mu z} \right) + \frac{(q-1)z\overline{a}_\mu}{R^2 - \overline{a}_\mu z} \\
&= O\left(\left| \frac{(q-1)z\overline{a}_\mu}{R^2 - \overline{a}_\mu z} \right|^2 \right) + O\left(\left| \frac{\frac{(q-1)z\overline{a}_\mu}{R^2 - \overline{a}_\mu z}}{1 - \frac{(q-1)z\overline{a}_\mu}{R^2 - \overline{a}_\mu z}} \right|^2 \right) \\
&= O\left(\left(\frac{|q-1|r}{R-r} \right)^2 \right) + O\left(\left(\frac{|q-1|r}{R-|q|r} \right)^2 \right).
\end{aligned} \tag{3.7}$$

Similarly as in (3.7), we have

$$\log \left(1 - \frac{(q-1)\overline{b}_\nu z}{R^2 - \overline{b}_\nu z} \right) + \frac{(q-1)z\overline{b}_\nu}{R^2 - \overline{b}_\nu z} = O\left(\left(\frac{|q-1|r}{R-r} \right)^2 + \left(\frac{|q-1|r}{R-|q|r} \right)^2 \right).$$

Recall $n(r)$ is the number of zeros and poles of f in $|z| \le r$, counting multiplicities. We conclude that

$$\begin{aligned}
|S_i(z)| &= O\left(\sum_{|c_k|<R} \left(\frac{|q-1|r}{R-r} \right)^2 + \left(\frac{|q-1|r}{R-|q|r} \right)^2 \right) \\
&= O\left(\left(\frac{|q-1|r}{R-r} \right)^2 + \left(\frac{|q-1|r}{R-|q|r} \right)^2 \right) n(R) \quad (i = 2, 4).
\end{aligned}$$

Let $R = r\phi(r)$. We get for $j = 2, 4$ and $|z| = r$,

$$|S_j(z)| = O\left(\frac{|q-1|^2 n(R)}{\phi^2(r)} \right).$$

Now, we proceed to estimate $|S_3(z)|$ and $|S_5(z)|$. First we claim that, for any $w \in \mathbf{C}$,

$$w - \log(1+w) = O(|w|). \tag{3.8}$$

Indeed, $|w - \log(1+w)| \le |w| + \log(1+|w|) + C_1 \le C|w|$, where C_1, C are some constants. It follows from (3.6) and (3.8) that

$$|S_3(z)| = \left| \sum_{|a_\mu|<R} \left\{ \log\left(1 + \frac{(q-1)z}{z-a_\mu}\right) - \frac{(q-1)z}{z-a_\mu} \right\} \right| = O\left(\sum_{|a_\mu|<R} \left| \frac{(q-1)z}{z-a_\mu} \right| \right).$$

Similarly, we have

$$|S_5(z)| = O\left(\sum_{|a_\mu|<R} \left| \frac{(q-1)b_\mu}{z-b_\mu} \right| \right).$$

It follows the lemma is proved. □

Lemma 3.5. *Let $\{s_k\}$ be the sequence in $\mathbf{C}$. Then, when $|z| = r$,*

$$\sum_{|s_k|<R} \left| \frac{z}{z-s_k} \right| = O\left(\frac{(\log r)^2 n(r^2) \log n(r^2)}{r} \right),$$

for all sufficiently large r outside a set E of finite logarithmic measure.

Proof. Given any sufficiently large r, set $R = r^2$. Let $n(R)$ denote the number of the points s_k that lie in $|z| < R$. For any z with $|z| = r$ and $|qz| + |z/q| < R$, there is an j, a positive integer, such that

$$e^j \le r < e^{j+1}.$$

Consequently,

$$e^{2j} \le R < e^{2(j+1)} \quad \text{and} \quad j \le \log r < j+1.$$

By applying Lemma 3.3 with $d_j = \frac{e^{2j}}{j^2}$ and $m_j = n(R)$ to the points $s_1, \ldots, s_{m_j}$, we conclude that there exists a finite collection of closed disks $D_1, \ldots, D_{q_j}$, whose radii have a total sum equal to $2d_j$, such that if $z \notin \cup_{j=1}^{q_j} D_j$, then there is a permutation of the points $s_1, \ldots, s_{m_j}$, say, $\hat{s}_1, \ldots, \hat{s}_{m_j}$, that satisfy

$$|z - \hat{s}_k| > \frac{kd_j}{m_j},$$

for $k = 1, 2, \ldots, m_j$. Hence, if $z \notin \cup_{j=1}^{q_j} D_j$, then we have

$$\begin{aligned} \sum_{|s_k|<R} \frac{r}{|z-s_k|} &\le \sum_{k=1}^{m_j} \frac{rm_j}{d_j k} \le \frac{rm_j(1+\log m_j)}{d_j} \le \frac{rj^2 n(R)(1+\log n(R))}{e^{2j}} \\ &\le \frac{C(\log r)^2 n(r^2) \log n(r^2)}{r}. \end{aligned} \tag{3.9}$$

Next, we estimate the linear measure of the exceptional sets arising from the discs in the application of Cartan lemma in the above setting. For each n, let us define

$$E_n := \{|z| \colon z \in \cup_{k=1}^{q_n} D_k \text{ such that } e^{2n} \le |z| \le e^{2(n+1)}\}.$$

It is clear that $|E_n| \le 4d_n = 4e^{2n}/n^2$ and that

$$\int_{E_n} \frac{1}{t}\, dt \le \int_{E_n} \frac{1}{e^{2n}}\, dt \le \frac{4}{n^2}. \tag{3.10}$$

If we denote $E = \cup_n^\infty E_n$, then the logarithmic measure of E can be deduced from (3.10) as

$$\int_E \frac{1}{t}\,dt \le \sum_{n=1}^{\infty} \int_{E_n} \frac{1}{t}\,dt \le 4 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Further, for any sufficiently large $|z|$ with $|z| \notin E$, there is a j such that $|z| \in [e^{2j},\ e^{2(j+1)}) \setminus E_j$. Therefore, $z \notin \bigcup_{k=1}^{q_j} D_k$ and (3.9) holds. Thus the proof of the lemma is complete. □

Proof of Theorem 2.1. By using Lemma 3.4 with $\phi(r) = r$ and Lemma 3.5, we obtain

$$\log \frac{f(qz)}{f(z)} = (q-1)z\frac{f'(z)}{f(z)} + O\left(\frac{|q-1|^2 m(r^2)}{r^2}\right) + O\left(\frac{|q-1|^2 n(r^2)}{r^2}\right)$$
$$+ O\left(\frac{(\log r)^2 n(r^2)\log n(r^2)}{r}\right) + O(1), \tag{3.11}$$

for all sufficiently large r outside a set E of finite logarithmic measure.

Since f is of finite logarithmic order ρ, i.e.,

$$\rho = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log\log r} < 1/2,$$

there is an $\epsilon_0 > 0$ such that $\rho < 2\rho + 3\epsilon_0 < 1$. Thus, we have $T(r,f) < r^{\rho+\epsilon_0}$ for all large r. Consequently,

$$m(r^2) \le T(r^2, f) \le r^{2\rho+2\epsilon_0}, \quad n(r^2) \le r^{2\rho+2\epsilon_0},$$

for all sufficiently large r. Moreover, we have

$$\frac{m(r^2)}{r^2} \le \frac{1}{r} = o(1), \quad \frac{n(r^2)}{r^2} \le \frac{1}{r} = o(1)$$

and

$$\frac{(\log r)^2 n(r^2)\log n(r^2)}{r} \le \frac{(\log r)^3 r^{2\rho+3\epsilon_0}(2\rho+2\epsilon_0)}{r^{1+\epsilon_0}} \le \frac{(\log r)^3(2\rho+2\epsilon_0)}{r^{\epsilon_0}} = o(1),$$

as r goes to infinity. It follows from (3.11) that Theorem 2.1 is completely proved. □

Proof of Theorem 2.2. Let

$$f(z) = \cos\sqrt{z} = \frac{e^{\sqrt{z}i} + e^{-\sqrt{z}i}}{2}.$$

Then f is an entire function of order $1/2$. Let q be any complex number. For any $x \in R^+$, there is z such that $\sqrt{z} = -\sqrt{x}i$ and, as $x \to +\infty$,

$$\log\frac{f(qz)}{f(z)} = \log\frac{e^{\sqrt{qx}} + e^{-\sqrt{qx}}}{e^{\sqrt{x}} + e^{-\sqrt{x}}} = (\sqrt{q}-1)\sqrt{x} + \log(1+o(1));$$

and

$$(q-1)z\frac{f'(z)}{f(z)} = \frac{(q-1)\sqrt{x}}{2}\frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{e^{\sqrt{x}} + e^{-\sqrt{x}}} = \frac{q-1}{2}\sqrt{x}(1+o(1)).$$

Let $q \in \mathbf{C}$. Suppose f satisfies (2.1) in Theorem 2.1. Then we have

$$(\sqrt{q}-1)\sqrt{x} + \log(1+o(1)) = \frac{q-1}{2}\sqrt{x}(1+o(1)) + O(1).$$

Dividing both sides by $\sqrt{x}$ and letting x go to infinity, we obtain

$$\sqrt{q} - 1 = \frac{q-1}{2} = \frac{(\sqrt{q}+1)(\sqrt{q}-1)}{2}.$$

The above equality holds if and only if $\sqrt{q} = 1$ if and only if $q = 1$. Thus the theorem is proved. □

To prove our q-difference analogy of the Wiman–Valiron theorem, we need the classical Wiman–Valiron theorem as follows.

Lemma 3.6. [9, Theorem 30] and [3, Theorem 12] *Let $f(z)$ be a transcendental entire function and $F \subset \mathbf{R}^+$ a set of finite logarithmic measure. Then for any $0 < \delta < 1/4$ and any z with $|z| = r \notin F$ satisfying*

$$|f(z)| > M(r, f)\nu(r, f)^{\delta - 1/4},$$

we have

$$\frac{f^{(m)}(z)}{f(z)} = \left(\frac{\nu(r, f)}{z}\right)^m (1 + o(1)).$$

Remark. It is clear from the statement of the theorem that the set F only depends on f.

Proof of Theorem 2.3. By using Theorem 2.1 and Lemma 3.6, we obtain that

$$\frac{f(q^m z)}{f(z)} = e^{(q^m - 1)\frac{zf'(z)}{f(z)} + O(1)} = e^{(q^m - 1)\nu(r,f)(1+o(1)) + O(1)},$$

which completes the proof. □

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References

[1] Bergweiler, W., and J. K. Langley: Zeros of differences of meromorphic functions. - Math. Proc. Cambridge Philos. Soc. 142:1, 2007, 133–147.

[2] Chiang, Y.-M., and S.-J. Feng: On the growth of logarithmic differences, difference quotients and logarithmic derivatives of meromorphic functions. - Trans. Amer. Math. Soc. 361:7, 2009, 3767–3791.

[3] Hayman, W. K.: The local growth of power series: a survey of the Wiman–Valiron method. - Canad. Math. Bull. 17:3, 1974, 317–358.

[4] Ishizaki, K., and N. Yanagihara: Wiman–Valiron method for difference equations. - Nagoya Math. J. 175, 2004, 75–102.

[5] Jank, G., and L. Volkmann: Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen. - Birkhäuser Verlag, Basel, 1985 (in German).

[6] Laine, I.: Nevanlinna theory and complex differential equations. - de Gruyter Stud. Math. 15, Walter de Gruyter & Co., Berlin, 1993.

[7] Levin, B. Ja.: Distribution of zeros of entire functions. Revised edition. - Transl. Math. Monogr. 5, translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman, Amer. Math. Soc., Providence, R.I., 1980.

[8] Nevanlinna, R.: Analytic functions. - Grundlehren Math. Wiss. 162, translated from the second German edition by Phillip Emig, Springer-Verlag, New York-Berlin 1970.

[9] Valiron, G.: Lectures on the general theory of integral functions. - Chelsea Publishing Company, 1949

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