

COMPARISON BETWEEN TEICHMÜLLER METRIC AND LENGTH SPECTRUM METRIC UNDER PARTIAL TWISTS

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Abstract. Let d_L and d_T denote, respectively, the length spectrum metric and the Teichmüller metric on the Teichmüller space $T(S_0)$ of a Riemann surface S_0 . Wolpert showed that $d_L(\tau, \tau^*) \leq d_T(\tau, \tau^*)$ for any two points τ and τ^* in $T(S_0)$. If S_0 is a hyperbolic Riemann surface with non-elementary Fuchsian group, then there are two sequences $\{\tau_n\}$ and $\{\tau_n^*\}$ of points in $T(S_0)$ such that $d_L(\tau_n, \tau_n^*) \rightarrow 0$ as $n \rightarrow \infty$, but $d_T(\tau_n, \tau_n^*) \geq b$ for some positive constant b and any n . This property was proved in [11] for any hyperbolic compact Riemann surface and in [13] for any hyperbolic one with non-elementary Fuchsian group. It is further shown in [13] that the two sequences can be modified to keep $d_L(\tau_n, \tau_n^*) \rightarrow 0$ but have $d_T(\tau_n, \tau_n^*) \rightarrow \infty$ as $n \rightarrow \infty$. For all these results, each τ_n^* is constructed from a Riemann surface τ_n by taking a number of Dehn twists (full twists) along a closed curve on τ_n . When τ_n^* is constructed from τ_n through a Dehn twist, one can use the maximal dilatation of a quasiconformal self map of τ_n to control and compare $d_L(\tau_n, \tau_n^*)$ and $d_T(\tau_n, \tau_n^*)$. But when τ_n^* is constructed from τ_n in a similar pattern but with partial twist, the method of using a self map of τ_n to control and compare $d_L(\tau_n, \tau_n^*)$ and $d_T(\tau_n, \tau_n^*)$ fails. In this paper, we show how to control and compare $d_L(\tau, \tau^*)$ and $d_T(\tau, \tau^*)$ under such partial twists, which enables us to obtain continuous versions of the results of [11] and [13] by using partial twists to connect the points τ_n^* .

1. Introduction

Let S_0 be a Riemann surface. A marked Riemann surface is a pair (S, f) with $f: S_0 \rightarrow S$ being a quasiconformal mapping. Two pairs (S_1, f_1) and (S_2, f_2) are equivalent if there exists a conformal mapping $c: S_1 \rightarrow S_2$ such that $c \circ f_1$ is homotopic to f_2 . The Teichmüller space $T(S_0)$ is the set of equivalence classes $[S, f]$.

The *Teichmüller metric* on $T(S_0)$ is defined by

$$(1.1) \quad d_T([S_1, f_1], [S_2, f_2]) = \inf_f \log K(f),$$

where f ranges over all quasiconformal mappings between S_1 and S_2 homotopic to $f_2 \circ f_1^{-1}$ and $K(f)$ represents the maximal dilatation of f .

We consider another metric on $T(S_0)$, called the length spectrum metric. Let S be a Riemann surface of the same type as S_0 . A simple closed curve on S is said to be *essential* if it is neither homotopic to a point nor to a puncture and nor to a boundary

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component. Let Σ_S be the collection of simple closed curves on S containing one and exactly one representative from each homotopy class of essential curves. For each $\gamma \in \Sigma_S$, let $l_S(\gamma)$ denote the length of the shortest curve in the homotopy class of γ in the hyperbolic metric. The *length spectrum metric* is defined by

$$(1.2) \quad d_L([S_1, f_1], [S_2, f_2]) = \log \sup_{\gamma \in \Sigma_{S_1}} \left\{ \frac{l_{S_2}(f_2 \circ f_1^{-1}(\gamma))}{l_{S_1}(\gamma)}, \frac{l_{S_1}(\gamma)}{l_{S_2}(f_2 \circ f_1^{-1}(\gamma))} \right\}.$$

This metric was introduced and studied by Sorvali [17] in 1972. In 1975, Sorvali [18] proved that the Teichmüller metric d_T and the length spectrum metric d_L are metrically equivalent on the Teichmüller space of a torus and posed a question as to whether or not this is true in general. A related question is whether or not these two metrics induce the same topology on Teichmüller space.

A well known result by Wolpert (see [1]) states that if $f: S_1 \rightarrow S_2$ is a quasiconformal mapping, then

$$\frac{l_{S_2}(f(\gamma))}{l_{S_1}(\gamma)} \leq K(f)$$

for all $\gamma \in \Sigma_{S_1}$. This inequality implies that

$$(1.3) \quad d_L(\tau_1, \tau_2) \leq d_T(\tau_1, \tau_2) \quad \text{for any } \tau_1, \tau_2 \in T(S_0).$$

Thus Sorvali's question is to study whether or not there exists a positive constant C such that

$$(1.4) \quad d_T(\tau_1, \tau_2) \leq C d_L(\tau_1, \tau_2) \quad \text{for any } \tau_1, \tau_2 \in T(S_0),$$

or even weaker, whether or not the identity map

$$(1.5) \quad \text{id}: (T(S_0), d_L) \rightarrow (T(S_0), d_T)$$

is continuous.

In 1986, Li [10] showed that the identity map is continuous if S_0 is a compact Riemann surface. This result was later generalized by Liu in [12] to the case where S_0 is a Riemann surface of type (g, m, n) , where g , m and n are the genus, number of punctures and number of ideal boundaries, respectively, with $6g - 6 + m + 3n > 0$. Then it follows that these two metrics are topologically equivalent on the Teichmüller space $T(S_0)$. Furthermore, in 2003 Li [11] proved that these two metrics are not metrically equivalent on the Teichmüller space of a compact Riemann surface of genus $g \geq 2$. In order to prove this, he first constructed a sequence $\{\tau_n = [S_n, f_n]\}$ of points in $T(S_0)$ such that S_n contains a closed curve β_n with $\lim_{n \rightarrow \infty} l_{S_n}(\beta_n) = 0$. Then, by taking $[1/l_{S_n}(\beta_n)]$ number of Dehn twists along each β_n , he obtained another sequence $\{\tau_n^*\}$ of points in $T(S_0)$ such that $d_T(\tau_n, \tau_n^*) \geq d > 0$ for n sufficiently large. On the other hand, since $l_{S_n}(\beta_n) \rightarrow 0$, it follows from the Collar Lemma [7] that the effect of that number of Dehn twists on the hyperbolic length of any curve crossing β_n becomes smaller and smaller as n approaching ∞ . This means $d_L(\tau_n, \tau_n^*) \rightarrow 0$ as $n \rightarrow \infty$.

Liu, Sun and Wei [13] generalized and improved Li's result to the Teichmüller space of any hyperbolic Riemann surface with non-elementary Fuchsian group by finding two sequences $\{\tau_n\}$ and $\{\tau_n^*\}$ such that $d_T(\tau_n, \tau_n^*) \rightarrow \infty$ but $d_L(\tau_n, \tau_n^*) \rightarrow 0$ as $n \rightarrow \infty$. Their construction follows Li's idea but uses more Dehn twists to obtain the sequence $\{\tau_n^*\}$. In their constructions, each τ_n^* can be represented by a quasiconformal mapping from S_n onto itself, which comes from cutting S_n along β_n and then gluing the two copies of β_n back after twisting one of the two copies three hundred and sixty degrees enough times. This feature makes it feasible to estimate the length

spectrum distance between τ_n and τ_n^* by using ratios of the lengths of closed curves and their images under a homeomorphism defined on the same Riemann surface. This convenience of representing τ_n^* is lost when a partial twist is performed to construct a new point in $T(S_0)$. In this case, the new point has to be represented by maps between two Riemann surfaces with different hyperbolic metrics. In this paper, we first show how to control the length spectrum distance between two points τ and τ^* in $T(S_0)$ when τ^* is constructed through a partial twist along a closed geodesic β on τ . The points τ_n are connected by a nice curve in $T(S_0)$. Now we use continuously changed partial twists along closed geodesics on the points on the nice curve to obtain another continuous curve in $T(S_0)$ to connect the points τ_n^* and in the meantime we have the properties of τ_n and τ_n^* in [11] and [13] preserved for the points on the two curves. More precisely, we prove the following theorem.

Theorem 1. *Let S_0 be a hyperbolic Riemann surface with non-elementary Fuchsian group. Then there exist three curves $\alpha(t)$, $\alpha^*(t)$ and $\hat{\alpha}(t)$, $0 \leq t < 1$, in $T(S_0)$ having the following properties.*

- (1) $\lim_{t \rightarrow 1} d_T(\alpha(t), \alpha^*(t)) = \infty$, but $\lim_{t \rightarrow 1} d_L(\alpha(t), \alpha^*(t)) = 0$.
- (2) There exist $M, m > 0$ such that $m \leq d_T(\alpha(t), \hat{\alpha}(t)) \leq M$ for all t , but $\lim_{t \rightarrow 1} d_L(\alpha(t), \hat{\alpha}(t)) = 0$.

Moreover, the curve α can be chosen to be a Teichmüller geodesic ray.

Our theorem can be viewed as continuous versions of the main results of [11] and [13].

Remark 1. In 2003, Shiga [16] studied the relation between the length spectrum metric and the Teichmüller metric on Riemann surfaces S_0 of infinite type. He showed that if S_0 has a sequence of closed geodesics with lengths approaching 0, then one of the sequences in the result of Liu, Sun and Wei [13] can be reduced to a point. That is, in the Teichmüller space $T(S_0)$ of such a Riemann surface S_0 there is a sequence of points $\{\tau_n\}_{n=0}^\infty$ such that $d_L(\tau_n, \tau_0) \rightarrow 0$ but $d_T(\tau_n, \tau_0) \rightarrow \infty$ as $n \rightarrow \infty$, where τ_0 is the base point of $T(S_0)$. This result implies that the topologies defined by the two metrics are not equivalent. It also raises a problem on how to connect the points $\{\tau_n\}_{n=1}^\infty$ by a continuous curve in $T(S_0)$ such that Shiga’s result continues to hold on the curve. Since Shiga’s construction of the points τ_n , $n \geq 1$, involves with different amounts of Dehn twists around infinitely many different closed geodesics, the technique developed in this paper doesn’t seem enough to solve the problem.

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2. Earthquakes

Earthquakes were introduced by Thurston in [21] to measure the difference between two conformal structures on a surface or two points in the Teichmüller space of a Riemann surface. For background on earthquake maps and their relations with quasymmetric circle homeomorphisms, we refer to [3, 4, 5, 15].

A lamination \mathcal{L} on \mathbf{D} is a collection of hyperbolic geodesics that foliates a closed subset of \mathbf{D} . The geodesics in \mathcal{L} are called leaves, whereas the connected components of $\mathbf{D} \setminus \mathcal{L}$ are called gaps. The strata of the lamination \mathcal{L} consists of gaps and leaves. A generalized left earthquake map $\tilde{E}: \mathbf{D} \rightarrow \mathbf{D}$ supported on \mathcal{L} is a possibly discontinuous injective map which is an isometry on each stratum of \mathcal{L} . Even more, for any two strata $A \neq B$, the comparison isometry

$$(2.1) \quad \text{Comp}(A, B) = (\tilde{E}|_A)^{-1} \circ (\tilde{E}|_B)$$

is a hyperbolic translation whose axis weakly separates A and B and which translates B to the left as viewed from A .

If in addition the map \tilde{E} is surjective, we call it a left earthquake map.

Thurston [21] showed that each left earthquake map $\tilde{E}: \mathbf{D} \rightarrow \mathbf{D}$ extends to a map defined on $\mathbf{D} \cup \mathbf{S}^1$. The extension is continuous at each point of \mathbf{S}^1 and $\tilde{E}|_{\mathbf{S}^1}$ is a homeomorphism. Conversely, every circle homeomorphism can be realized in this way.

Theorem 2. (Thurston) *Let h be an orientation-preserving homeomorphism of the unit circle \mathbf{S}^1 . Then there exists a lamination \mathcal{L} and a left earthquake map $\tilde{E}: \mathbf{D} \rightarrow \mathbf{D}$ along the leaves in \mathcal{L} such that $\tilde{E}|_{\mathbf{S}^1} = h$. The lamination is uniquely determined by h . Moreover, h determines the isometries of \tilde{E} on all gaps, and for any leaf L in \mathcal{L} , two possibly different isometries on L only differ by a hyperbolic isometry with axis L and translation length between 0 and the limit value of the translation lengths of the comparison maps for E on the two sides of L .*

Each generalized earthquake map $\tilde{E}: \mathbf{D} \rightarrow \mathbf{D}$ along the leaves in \mathcal{L} induces a transverse measure σ , called an earthquake measure on \mathcal{L} . The measure σ quantifies the amount of relative shearing along the lamination of the earthquake map.

The following result is due to Thurston [21]. A proof is given in [3].

Theorem 3. (Thurston) *Let σ be a transverse measure defined on a lamination \mathcal{L} . Then there exists a generalized earthquake map $\tilde{E}: \mathbf{D} \rightarrow \mathbf{D}$ supported on \mathcal{L} such that σ is the induced earthquake measure by \tilde{E} . Moreover, up to post-composition by a hyperbolic isometry, σ determines the isometries of \tilde{E} on all gaps, and for any leaf L in \mathcal{L} , two possible isometries on L only differ by a hyperbolic isometry with axis L and translation length between 0 and the measure $\sigma(L)$ of L .*

In order to determine whether or not a generalized earthquake map is indeed an earthquake map we need to introduce the concept of the Thurston norm of an earthquake measure (σ, \mathcal{L}) .

Let \mathcal{L} be a lamination with a transverse measure σ . The *Thurston norm* of σ is defined to be

$$(2.2) \quad \|\sigma\|_{\text{Th}} = \sup \sigma(\alpha),$$

where the supremum is taken over all closed geodesic segments α of length 1 that are transverse to the lamination.

We say that σ is Thurston bounded if $\|\sigma\|_{\text{Th}}$ is finite. An earthquake map is called Thurston bounded if the induced earthquake measure is Thurston bounded.

Thurston [21] outlined the proofs of the following two theorems. Three different complete proofs are given in [3, 4, 15].

Theorem 4. (Thurston) *Let \mathcal{L} be a lamination with a transverse measure σ . If σ is Thurston bounded, then there exists a left earthquake map $\tilde{E}: \mathbf{D} \rightarrow \mathbf{D}$ supported*

on \mathcal{L} such that σ is the induced earthquake measure by \tilde{E} . Moreover, up to post-composition by a hyperbolic isometry, σ determines the isometries of \tilde{E} on all gaps, and for any leaf L in \mathcal{L} , two possible isometries on L only differ by a hyperbolic isometry with axis L and translation length between 0 and the measure $\sigma(L)$ of L .

Theorem 5. (Thurston) *Let h be an orientation-preserving circle homeomorphism, and let σ_h be the earthquake measure induced by a left earthquake map \tilde{E} with $\tilde{E}|_{\mathbb{S}^1} = h$. Then σ_h is Thurston bounded if and only if h is quasymmetric.*

For situations handled in this paper, we are only interested in discrete laminations, i.e., laminations consisting of countably many hyperbolic geodesics without any accumulation in \mathbf{D} . From now on, we assume that this is the case. We now construct generalized earthquakes for these types of laminations with transverse measures.

Let \mathcal{L} be a discrete lamination. Two gaps are called neighbors if there is a geodesic in \mathcal{L} , called the separating geodesic, that belongs to the boundary of both gaps. Given any two gaps A and B , there exists a unique minimal chain of gaps $A_0 = A, A_1, A_2, \dots, A_{n+1} = B$ such that A_i and A_{i+1} are neighbors. In this sense, the pattern of neighboring gaps determined by \mathcal{L} is a tree.

Let σ be a measure on a lamination \mathcal{L} . That is, σ assigns a positive number, called a weight, to each element of \mathcal{L} . Fix a gap A of the lamination \mathcal{L} and define $\tilde{E}|_A = \text{Id}$. Let B be any other gap and let $A_0 = A, A_1, \dots, A_{n+1} = B$ be the minimal chain of neighboring gaps between A and B . For each $i = 1, 2, \dots, n+1$, let T_i be the hyperbolic translation whose axis is the separating geodesic L_i between A_{i-1} and A_i and that translates A_i to the left by a distance λ_i , where λ_i is the weight of L_i . Define $\tilde{E}|_B = T_1 \circ T_2 \circ \dots \circ T_{n+1}$. Since any two gaps are connected by a unique minimal chain of neighboring gaps, this construction gives a map defined on the whole hyperbolic plane. Of course, this construction depends on the gap A , nonetheless, any two maps constructed in this way differ only by pre-composition by a conformal isometry of \mathbf{D} . It is easy to see that this map is injective and that the comparison isometry satisfies condition (2.1), however, \tilde{E} might not be surjective.

A Thurston bounded earthquake map is a quasi-isometry on \mathbf{D} with respect to the hyperbolic metric. The Thurston norm of the earthquake measure is a quantifier of the quasi-isometry. On the other hand, the cross-ratio distortion norm is a quantifier of the quasymmetry of the boundary homeomorphism determined by the earthquake map. Now we introduce the definition of the cross-ratio distortion norm and we give its quantitative relationship with the Thurston norm of the measure.

Let h be an orientation-preserving homeomorphism of the unit circle \mathbf{S}^1 . The *cross-ratio distortion norm* $\|h\|_{\text{cr}}$ of h is defined as

$$(2.3) \quad \|h\|_{\text{cr}} = \sup_Q \left| \log \frac{\text{cr}(h(Q))}{\text{cr}(Q)} \right|,$$

where the supremum is taken over all quadruples $Q = \{a, b, c, d\}$ of four points arranged in counter-clockwise order on the unit circle with $\text{cr}(Q) = 1$, and where

$$(2.4) \quad \text{cr}(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)}.$$

Theorem 6. (Gardiner–Hu–Lakic) *There exists a universal constant $C' > 0$ such that for any measured lamination (\mathcal{L}, σ) ,*

$$(2.5) \quad \frac{1}{C'} \|\tilde{E}|_{\mathbf{S}^1}\|_{\text{cr}} \leq \|\sigma\|_{\text{Th}} \leq C' \|\tilde{E}|_{\mathbf{S}^1}\|_{\text{cr}}.$$

The right inequality in the previous theorem was proved by Gardiner, Hu and Lakić in [3] and the left inequality was proved by Hu in [4].

For any orientation-preserving homeomorphism f of the unit circle \mathbf{S}^1 , let $\Phi(f)$ be the Douady–Earle extension of f (see [2]) and let $K_e(f)$ be the infimum of maximal dilatations of quasiconformal extensions of f . Hu and Muzician [6] proved the following two theorems.

Theorem 7. (Hu–Muzician) *There exists a universal constant $C''' > 0$ such that for any quasisymmetric circle homeomorphism f ,*

$$(2.6) \quad \log K(\Phi(f)) \leq C''' \|f\|_{\text{cr}}.$$

Theorem 8. (Hu–Muzician) *Let f be a quasisymmetric homeomorphism. Then*

$$(2.7) \quad \lim_{\|f\|_{\text{cr}} \rightarrow \infty} \frac{\|f\|_{\text{cr}}}{K_e(f)} \leq \pi.$$

The following property is also used in the next section.

Proposition 1. *Let $\{f_n\}$ be a sequence of orientation-preserving homeomorphisms of the unit circle \mathbf{S}^1 such that $K_e(f_n) \rightarrow 1$ as $n \rightarrow \infty$. Then*

$$\|f_n\|_{\text{cr}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof. Suppose that the proposition is not true. By passing to a subsequence, we may assume that there exists $\varepsilon > 0$ such that

$$(2.8) \quad \|f_n\|_{\text{cr}} > \varepsilon \quad \text{for all } n.$$

For each n , choose a quadrilateral Q_n such that

$$\text{cr}(Q_n) = 1 \quad \text{and} \quad |\log \text{cr}(f_n(Q_n))| > \varepsilon.$$

We may assume, by post-composing and pre-composing by Möbius transformations, that Q_n and $f_n(Q_n)$ are the quadrilaterals on \mathbf{H} with vertices at $\{-1, 0, 1, \infty\}$ and $\{x_n, 0, 1, \infty\}$ respectively. For each n , the conformal modulus $\text{Mod}(Q_n)$ of Q_n is equal to 1 and

$$\frac{\text{Mod}(Q_n)}{K_e(f_n)} \leq \text{Mod}(f_n(Q_n)) \leq K_e(f_n) \text{Mod}(Q_n).$$

Using the assumption that $K_e(f_n) \rightarrow 1$ as $n \rightarrow \infty$, we conclude that $\text{Mod}(f_n(Q_n)) \rightarrow 1$ and then $x_n \rightarrow -1$ as $n \rightarrow \infty$. Since $\text{cr}(f_n(Q_n)) = -x_n$, it follows that $\log \text{cr}(f_n(Q_n)) \rightarrow 0$ as $n \rightarrow \infty$, contradicting (2.8). \square

Remark 2. Using an elliptic integral depending on a parameter and the work on pp. 59–60 of [9], one can see how $\|f_n\|_{\text{cr}}$ depends on $K_e(f_n)$ asymptotically when $K_e(f_n) \rightarrow 1$ as $n \rightarrow \infty$, from which the previous proposition follows as well.

3. Partial twists on Riemann surfaces

Let β be a simple closed geodesic on a hyperbolic Riemann surface S that is not homotopic to a boundary component of S . As pointed out in the introduction, the point created by a Dehn twist along β in $T(S)$ is described by a quasiconformal map from S onto S , which comes from cutting S along β and then gluing the two copies of β back after twisting one copy of β three hundred and sixty degrees. The first goal of this section is to introduce how partial twists (not equal to multiples of 360°) along β create new points in $T(S)$. This is related to conjugations of the Fuchsian group representing S by earthquake maps corresponding to partial twists.

Let $\pi: \mathbf{D} \rightarrow S$ be the universal covering map, where \mathbf{D} denotes the unit disk. The preimage $\pi^{-1}(\beta)$ consists of a union of non intersecting geodesics. By assigning the weight λ to each one of these geodesics, we obtain a measured lamination $(\sigma_\lambda, \mathcal{L})$ on \mathbf{D} . We can see that the Thurston norm $\|\sigma_\lambda\|_{\text{Th}}$ is finite. Let $D > 0$ be the width of a collar neighborhood around β . Given any geodesic segment α of length 1 transversal to the lamination \mathcal{L} , $\sigma_\lambda(\alpha) = \lambda$ if α intersects only one leaf of \mathcal{L} ; if α intersects $n > 1$ leaves of \mathcal{L} , then $1 = l(\alpha) \geq nD/2$ and hence $\sigma_\lambda(\alpha) \leq \frac{\lambda}{D/2}$. In summary,

$$(3.1) \quad \lambda \leq \|\sigma_\lambda\|_{\text{Th}} \leq \max \left\{ \lambda, \frac{\lambda}{D/2} \right\}.$$

It follows from Theorems 4 and 5 that any left earthquake map $\tilde{E}: \mathbf{D} \rightarrow \mathbf{D}$ inducing $(\sigma_\lambda, \mathcal{L})$ is onto and $\tilde{E}|_{\mathbf{S}^1}: \mathbf{S}^1 \rightarrow \mathbf{S}^1$ is a quasimetric homeomorphism of the unit circle.

Let G be the Fuchsian group uniformizing S . For each element $g \in G$, $\tilde{E}g\tilde{E}^{-1}$ is conformal on every gap of the lamination \mathcal{L} . Moreover, it is continuous on \mathbf{D} . In order to prove this, consider a geodesic L' of the lamination $E(\mathcal{L})$. Since \tilde{E}^{-1} is a right earthquake, along L' it splits the hyperbolic plane into two half disks and moves the gap V' on one side of L' to the right with respect to the gap U' on the other side of L' . Because g is an isometry and it maps a leaf of \mathcal{L} to another leaf of \mathcal{L} , it follows that g maps $\tilde{E}^{-1}(V')$ and $\tilde{E}^{-1}(U')$ to two adjacent gaps V and U of \mathcal{L} separated by $\tilde{E}^{-1}(L')$, which is a leaf L of \mathcal{L} . In the meantime, $g \circ \tilde{E}^{-1}$ is also a right earthquake without changing the comparison along L' . Finally, since \tilde{E} is a left earthquake and $L \in \mathcal{L}$, it undoes the discontinuity of $g \circ \tilde{E}^{-1}$ along L' . Therefore, for every $g \in G$, the mapping $\tilde{E}g\tilde{E}^{-1}$ is conformal on \mathbf{D} and hence it defines an isomorphism between G and a new Fuchsian group $G' = \tilde{E}G\tilde{E}^{-1}$. Let $S' = \mathbf{D}/G'$. The earthquake \tilde{E} projects to a quasi-isometry

$$E_{\beta,\lambda}: S \rightarrow S',$$

which is called a *left earthquake map* of twist or shear λ along β . It is a discontinuous map as soon as $\lambda \neq 0$. In the following, we introduce a quasiconformal map to represent the same point of $E_{\beta,\lambda}: S \rightarrow S'$ in the Teichmüller space $T(S)$.

The Douady–Earle extension $\Phi(\tilde{E}|_{\mathbf{S}^1}): \overline{\mathbf{D}} \rightarrow \overline{\mathbf{D}}$ of $\tilde{E}|_{\mathbf{S}^1}$ is a quasiconformal mapping. Moreover, $\Phi(\tilde{E}|_{\mathbf{S}^1})$ and \tilde{E} induce the same group isomorphism between G and G' . This is due to the fact that the Douady–Earle extension of a homeomorphism of the unit circle is conformally natural; i.e.,

$$\Phi(T_1 \circ f \circ T_2) = T_1 \circ \Phi(f) \circ T_2$$

for any homeomorphism f of the unit circle and any two isometries T_1 and T_2 of \mathbf{D} (see [2]). Then $\Phi(\tilde{E}|_{\mathbf{S}^1})$ projects to a quasiconformal homeomorphism

$$\Phi_{\beta,\lambda}: S \rightarrow S',$$

which we call the Douady–Earle representation of the left earthquake map $E_{\beta,\lambda}: S \rightarrow S'$.

Lemma 1. *Let g_1, g_2 be two hyperbolic transformations of the unit disk \mathbf{D} with axes L_1, L_2 and translation lengths $\tau(g_1), \tau(g_2)$ respectively. If L_1 intersects L_2 at one point in \mathbf{D} , then $g_3 = g_1 \circ g_2$ is hyperbolic. Moreover, the translation length $\tau(g_3)$ of g_3 satisfies*

$$\tau(g_2) - \tau(g_1) \leq \tau(g_3) \leq \tau(g_2) + \tau(g_1).$$

Proof. We label the repelling fixed point of g_1 by a and the attracting fixed point by c , the repelling fixed point of g_2 by b and the attracting fixed point by d . Then L_1 connects a to c and L_2 connects b to d . Let I be the circular arc between c and d not containing a and b , and let J be the circular arc between a and b not containing c and d . See Figure 1 for an illustration. By studying the images of the endpoints of I or J under g_2 and then g_1 , one can show that the intervals I and J satisfy $(g_1 \circ g_2)(I) \subset I$ and $J \subset (g_1 \circ g_2)(J)$. Therefore, $g_3 = g_1 \circ g_2$ has two fixed points, one in I and another one in J . Then g_3 must be a hyperbolic transformation with axis joining the intervals I and J .

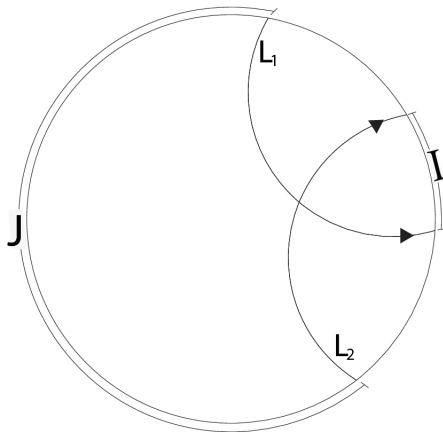


Figure 1. A reference figure for the proof of Lemma 1.

Let $p \in L_1 \cap L_2$. Then $g_2^{-1}(p) \in L_2$ and

$$\rho(g_2^{-1}(p), g_1(p)) \leq \rho(g_2^{-1}(p), p) + \rho(p, g_1(p)) = \tau(g_2) + \tau(g_1),$$

where $\rho(\cdot, \cdot)$ denotes the hyperbolic metric on \mathbf{D} . Since

$$\tau(g_3) = \inf_{z \in \mathbf{D}} \rho(z, g_3(z)) \leq \rho(g_2^{-1}(p), g_1(p)),$$

it follows that

$$\tau(g_3) \leq \tau(g_2) + \tau(g_1).$$

Applying a similar argument to $g_2 = g_3 \circ g_1^{-1}$, we obtain

$$\tau(g_2) \leq \tau(g_3) + \tau(g_1^{-1}) = \tau(g_3) + \tau(g_1).$$

Thus

$$\tau(g_2) - \tau(g_1) \leq \tau(g_3). \quad \square$$

Theorem 9. Let β be a simple closed geodesic on S , and let

$$\Phi_{\beta, \lambda}: S \rightarrow S'$$

be the Douady–Earle representation of a left earthquake map $E_{\beta, \lambda}: S \rightarrow S'$ of shear λ along β . Then for any $\gamma \in \Sigma_S$,

$$l_S(\gamma) - i(\gamma, \beta)\lambda \leq l_{S'}(\Phi_{\beta, \lambda}(\gamma)) \leq l_S(\gamma) + i(\gamma, \beta)\lambda,$$

where $i(\gamma, \beta)$ denotes the intersection number between γ and β .

Proof. Let $\pi: \mathbf{D} \rightarrow S$ be a universal covering map and let G be the Fuchsian group uniformizing S . Recall that $(\sigma_\lambda, \mathcal{L})$ is a measured lamination with \mathcal{L} consisting of all geodesics in $\pi^{-1}(\beta)$ and with the weight of each leaf equal to λ . Let $\tilde{E}: \mathbf{D} \rightarrow \mathbf{D}$ is a left earthquake on \mathbf{D} inducing $(\sigma_\lambda, \mathcal{L})$. Then S' is uniformized by $G' = \tilde{E}G\tilde{E}^{-1}$.

Let $g \in G$ such that its axis L projects to γ . Notice that $l_S(\gamma)$ and $l_{S'}(\Phi_{\beta,\lambda}(\gamma))$ are equal to the translation lengths of g and $g' = \tilde{E}g\tilde{E}^{-1}$, respectively. Let us assume that $\gamma \neq \beta$ and $i(\gamma, \beta) = 0$. Then L intersects none of the geodesics in the lamination $\mathcal{L} = \pi^{-1}(\beta)$. Without loss of generality, we may assume that $\tilde{E}|_L = \text{id}$. Thus, for any point $z \in L$, $g(z) \in L$ and hence $g'(z) = \tilde{E}|_A g \tilde{E}|_A^{-1}(z) = g(z)$. By the identity principle, $g' \equiv g$ and then they have the same translation length. It follows that

$$(3.2) \quad l_S(\gamma) = l_{S'}(\Phi_{\beta,\lambda}(\gamma)) \text{ if } \gamma \neq \beta \text{ and } i(\gamma, \beta) = 0.$$

If $\gamma = \beta$, then a similar argument shows that the equation (3.2) holds.

Assume now that $\gamma \neq \beta$ and $i(\gamma, \beta) = n > 0$. Let p be a point in \mathbf{D} which projects to a point in the intersection of γ and β . Since p and $g(p)$ project to the same point, the geodesic segment $[p, g(p)]$ crosses exactly $n + 1$ lines in the lamination \mathcal{L} , denoted by L_1, \dots, L_{n+1} , and n gaps, denoted by A_1, \dots, A_n . Without loss of generality, we may assume $\tilde{E}|_{A_0} = \text{id}$, where A_0 is the gap that is separated from A_1 by L_1 . Then

$$\tilde{E}|_{A_n} = T_1 \circ T_2 \circ \dots \circ T_n,$$

where T_k is the hyperbolic transformation with axis L_k and translation length λ . Notice that g maps an open neighborhood V contained in A_0 to a neighborhood contained in A_n . Thus,

$$g'|_V = (\tilde{E}g\tilde{E}^{-1})|_V = (T_1 \circ \dots \circ T_{n-1} \circ T_n \circ g)|_V.$$

By the identity principle,

$$g' = T_1 \circ \dots \circ T_{n-1} \circ T_n \circ g$$

on \mathbf{D} . It remains to show that the translation lengths $\tau(g')$ and $\tau(g)$ of g' and g satisfy

$$\tau(g) - n\lambda \leq \tau(g') \leq \tau(g) + n\lambda.$$

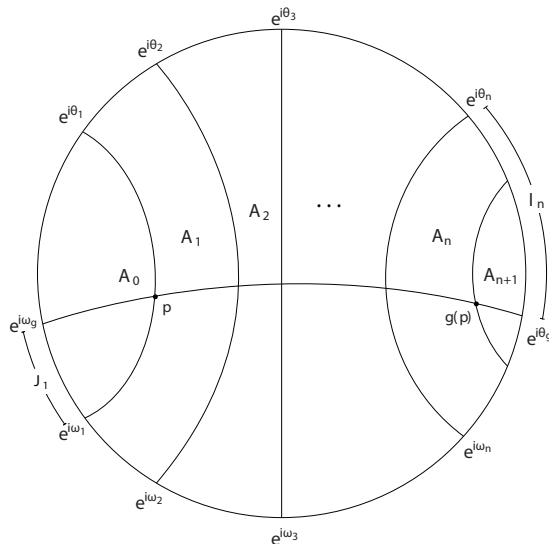


Figure 2. Illustration of a step in the proof of Theorem 9.

Clearly, the axis of g intersects L_n . By Lemma 1, we obtain

$$\tau(g) - \tau(T_n) \leq \tau(T_n \circ g) \leq \tau(g) + \tau(T_n).$$

Now we denote the attracting and repelling fixed points of g by $e^{i\theta_g}$ and $e^{i\omega_g}$ and the ones of T_k by $e^{i\theta_k}$ and $e^{i\omega_k}$, $k = 1, \dots, n + 1$. Let I_k be the circular arc on \mathbf{S}^1 between

$e^{i\theta_g}$ and $e^{i\theta_k}$ not containing $e^{i\omega_g}$ and $e^{i\omega_k}$, $k = 2, 3, \dots, n$, and let J_1 be the one on \mathbf{S}^1 between $e^{i\omega_g}$ and $e^{i\omega_1}$ not containing $e^{i\theta_g}$ and $e^{i\theta_1}$ (See Figure 2). Note that g maps L_1 to L_{n+1} . In particular, it maps $e^{i\omega_1}$ to $e^{i\omega_{n+1}}$. It follows that T_n maps $g(e^{i\omega_1}) = e^{i\omega_{n+1}}$ to a point closer to $e^{i\theta_n}$ than $g(e^{i\omega_1})$. On the other hand, g fixes $e^{i\omega_g}$ and then T_n maps it to a point on the circular arc between $e^{i\omega_g}$ and $e^{i\theta_n}$ not containing $e^{i\omega_1}$. We conclude that $(T_n \circ g)(J_1) \supset J_1$. A similar argument shows $(T_n \circ g)(I_n) \subset I_n$. These imply that $T_n \circ g$ has a fixed point in I_n and another one in J_1 . It follows that the axis of $T_n \circ g$ must cross the axis of T_{n-1} . Using Lemma 1 again, we obtain

$$\tau(T_n \circ g) - \tau(T_{n-1}) \leq \tau(T_{n-1} \circ T_n \circ g) \leq \tau(T_{n-1}) + \tau(T_n \circ g).$$

Similarly, we can see that $T_{n-1} \circ T_n \circ g(I_{n-1}) \subset I_{n-1}$ and $T_{n-1} \circ T_n \circ g(J_1) \supset J_1$. Hence $T_{n-1} \circ T_n \circ g$ has a fixed point in I_{n-1} and another one in J_1 . Then the axis of $T_{n-1} \circ T_n \circ g$ crosses the one of T_{n-2} . Lemma 1 can be applied again. Therefore, inductively, we are able to obtain

$$\tau(g) - n\lambda = \tau(g) - \sum_{k=1}^n \tau(T_k) \leq \tau(g') \leq \tau(g) + \sum_{k=1}^n \tau(T_k) = \tau(g) + n\lambda.$$

This means

$$(3.3) \quad l_S(\gamma) - i(\gamma, \beta)\lambda \leq l_{S'}(\Phi_{\beta, \lambda}(\gamma)) \leq l_S(\gamma) + i(\gamma, \beta)\lambda.$$

Combining the equations (3.2) and (3.3), we complete the proof. □

Remark 3. Our approach to prove the previous theorem is analytic. Thanks to one of the referees for providing the following geometric approach. Let us use the same notation introduced in the previous proof, but let p be a point on the hyperbolic plane projecting to a point on γ not equal to any intersection point between γ and β . Then the geodesic arc $[p, g(p)]$ crosses exactly n geodesic lines L_k , $k = 1, 2, \dots, n$, which are lifts of β . Let $a_k = [p, g(p)] \cap L_k$ for $k = 1, 2, \dots, n$ and set $a_0 = p$ and $a_{n+1} = g(p)$. The image $\tilde{E}([p, g(p)])$ is a union of geodesic arcs $\{\tilde{E}([a_k, a_{k+1}])\}_{k=0}^n$ in the hyperbolic plane. Set $b_k^- = \tilde{E}([a_{k-1}, a_k]) \cap \tilde{E}(L_k)$ and $b_k^+ = \tilde{E}([a_k, a_{k+1}]) \cap \tilde{E}(L_k)$ for $k = 1, 2, \dots, n$, where \tilde{E} also represents the extension to the hyperbolic plane of the part of \tilde{E} defined on the gap containing (a_{k-1}, a_k) for each $k = 1, 2, \dots, n + 1$. Then, by definition, the geodesic arc $[b_k^-, b_k^+]$ lies on the line $\tilde{E}(L_k)$ and has length λ (Lemma 3.6 of [8]). Therefore, the concatenation

$$\delta = \left(\bigcup_{k=1}^{n+1} \tilde{E}([a_{k-1}, a_k]) \right) \cup \left(\bigcup_{k=1}^n [b_k^-, b_k^+] \right)$$

has length $l_S(\beta) + n\lambda$. Because $\tilde{E} \circ g = g' \circ \tilde{E}$, the piecewise geodesic path

$$\bigcup_{k \in \mathbf{Z}} (g')^k(\tilde{E}(\delta))$$

is a cross cut, invariant under the action of g' and represents a closed curve on S' corresponding to β via markings. Thus, the length $l_{S'}(\beta)$ is at most $l_S(\beta) + n\lambda$.

4. Proof of main results

Let S be a Riemann surface with corresponding Fuchsian group G and let γ be an essential simple closed curve on S . A ring domain R on S is a set conformally equivalent to an annulus $\{z: a < |z| < 1\}$ in the complex plane, where a is a nonnegative real number uniquely determined by R . The conformal modulus of R is defined

by $\text{Mod}(R) = (2\pi)^{-1}|\log a|$. A ring domain R is said to be of homotopy type γ if there exists a simple closed curve $\gamma_0 \subset R$ freely homotopic to γ and separating the boundary components of R . If γ is not homotopic to a puncture then any element $g \in G$ covering γ is hyperbolic and $R'_{S,\gamma} = \mathbf{H}/\langle g \rangle$ is a ring domain that covers S . It is straightforward to check that $\text{Mod}(R'_{S,\gamma}) = \pi/l_S(\gamma)$. Moreover, any ring domain R in S , of homotopy type γ , can be conformally embedded into $R'_{S,\gamma}$ in such a way that both ring domains have the same homotopy type (see [20] or [14]). Then

$$(4.1) \quad \text{Mod}(R) \leq \text{Mod}(R'_{S,\gamma}) = \frac{\pi}{l_S(\gamma)}$$

for any ring domain R in S of homotopy type γ .

Let φ be a holomorphic quadratic differential on S . An arc γ is a trajectory of φ if it is a maximal horizontal arc, i.e., if $\arg \varphi(z)dz^2 = 0$ along γ . Every closed trajectory γ_0 of φ is embedded in a unique maximal ring domain R_0 swept out by closed trajectories. Two ring domains R_0 and R_1 associated with γ_0 and γ_1 are disjoint or identical.

The quadratic differential φ is said to be of closed trajectories if its non closed trajectories cover a set of measure zero. The associated ring domains of such φ divide S into ring domains swept out by closed trajectories. We say that φ is simple if it has only one associated ring domain.

A set of homotopically non trivial simple closed curves on S is called an admissible system of simple closed curves provided that they are mutually disjoint and belong to different homotopy classes. Strebel [20] proved the following theorem.

Theorem 10. (Strebel) *Let S be a Riemann surface and $\{\gamma_1, \dots, \gamma_n\}$ an admissible system of simple closed curves on S that are not homotopic to any puncture on S . Then for arbitrary positive numbers $b_k, k = 1, 2, \dots, n$, there exists a holomorphic quadratic differential φ on S with closed trajectories such that its associated ring domains R_k are of homotopy types γ_k and heights b_k respectively for $k = 1, 2, \dots, n$. Furthermore, the quadratic differential φ is uniquely determined and has a finite norm*

$$\|\varphi\| = \sum_{k=1}^n b_k^2/M_k,$$

where M_k is the modulus of $R_k, k = 1, \dots, n$.

For each simple closed curve $\beta \in \Sigma_S$, let φ_β be the unique simple quadratic differential with

$$\|\varphi_\beta\| = \iint_S |\varphi_\beta| dx dy = 1$$

such that its closed trajectories are freely homotopic to β . We will denote the associated ring domain of φ_β by $R_{S,\beta}$. Notice that (4.1) implies

$$(4.2) \quad \text{Mod}(R_{S,\beta}) \leq \text{Mod}(R'_{S,\beta}) = \frac{\pi}{l_S(\beta)}.$$

Let S_0 be a hyperbolic Riemann surface that is not an annulus. Let β be a simple closed geodesic on S_0 and φ_β be the corresponding simple quadratic differential. For each $t \in (0, 1)$, $\mu_t = -t \frac{\varphi_\beta}{|\varphi_\beta|}$ defines a Beltrami differential on S_0 . We use μ_t to give a new conformal structure on S_0 .

For any local parameter $z: U \rightarrow V$ on S_0 , let $z': V \rightarrow z'(V)$ be a quasiconformal mapping with complex dilatation $\mu_t|_V$. Since μ_t is a Beltrami differential, the new parameters z' give a new complex structure on S_0 . We will denote the new Riemann

surface by S_t . The identity map $\text{id}_t : S_0 \rightarrow S_t$ is a Teichmüller mapping with initial quadratic differential φ_β and some terminal quadratic differential ψ_t . The mapping id_t can be expressed as

$$\text{id}_t : \zeta \mapsto \zeta' = \frac{\zeta - t\bar{\zeta}}{1 - t}$$

where ζ and ζ' are natural parameters of φ_β and ψ_t respectively. It follows that id_t stretches the vertical trajectories of φ_β by a factor $K_t = \frac{1+t}{1-t}$ and leaves the lengths of the horizontal trajectories invariant. Then the quadratic differential ψ_t is also simple and its associated ring domain $R_{S_t, \beta}$ coincides with $R_{S_0, \beta}$. Nonetheless, the conformal moduli are different; in fact it is easily seen that

$$\text{Mod}(R_{S_t, \beta}) = K_t \text{Mod}(R_{S_0, \beta}).$$

Let β_t be the geodesic on S_t homotopic to β , it follows from the above equality and (4.2) that

$$(4.3) \quad l_{S_t}(\beta_t) \leq \frac{\pi}{\text{Mod}(R_{S_t, \beta})} = \frac{\pi}{K_t \text{Mod}(R_{S_0, \beta})}$$

It follows that $l_{S_t}(\beta_t) \rightarrow 0$ as $t \rightarrow 1$.

Consider the geodesic ray (in the Teichmüller metric) given by

$$\alpha(t) = [S_t, \text{id}_t], \quad t \in [0, 1),$$

and another continuous path defined by

$$(4.4) \quad \alpha^*(t) = [S'_t, \Phi_{\beta_t, \lambda_t} \circ \text{id}_t], \quad t \in [0, 1),$$

where λ_t is a positive parameter continuously depending on t and $\Phi_{\beta_t, \lambda_t}$ is the Douady–Earle representation of the left earthquake E_{β_t, λ_t} of twist λ_t along β_t . We show the following theorem.

Theorem 11. *Let $\alpha(t)$ and $\alpha^*(t)$ be as above. If $\lambda_t = o(|\log l_{S_t}(\beta_t)|)$ as $t \rightarrow 1$, then*

$$\lim_{t \rightarrow 1} d_L(\alpha(t), \alpha^*(t)) = 0.$$

Proof. Suppose γ is a closed geodesic in S_t . Then Theorem 9 implies

$$(4.5) \quad l_{S'_t}(\Phi_{\beta_t, \lambda_t}(\gamma)) = l_{S_t}(\gamma) \quad \text{if } i(\gamma, \beta_t) = 0.$$

On the other hand, if $i(\gamma, \beta_t) > 0$, then Theorem 9 implies

$$1 - \frac{i(\gamma, \beta_t) o(|\log l_{S_t}(\beta_t)|)}{l_{S_t}(\gamma)} \leq \frac{l_{S'_t}(\Phi_{\beta_t, \lambda_t}(\gamma))}{l_{S_t}(\gamma)} \leq 1 + \frac{i(\gamma, \beta_t) o(|\log l_{S_t}(\beta_t)|)}{l_{S_t}(\gamma)}.$$

By inequality (4.3), $l_{S_t}(\beta_t)$ is small when t is close to 1. In this situation, we know, by the Collar Lemma, that β_t has a collar neighborhood of approximate width $D_t \approx \log(16/l_{S_t}(\beta_t))$. Then

$$l_{S_t}(\gamma) \geq i(\gamma, \beta) \log(16/l_{S_t}(\beta_t)).$$

Thus, if t is close to 1 and $i(\gamma, \beta) > 0$, then

$$(4.6) \quad 1 - \frac{o(|\log l_{S_t}(\beta_t)|)}{\log \frac{16}{l_{S_t}(\beta_t)}} \leq \frac{l_{S'_t}(\Phi_{\beta_t, \lambda_t}(\gamma))}{l_{S_t}(\gamma)} \leq 1 + \frac{o(|\log l_{S_t}(\beta_t)|)}{\log \frac{16}{l_{S_t}(\beta_t)}}.$$

By using (4.5) if $i(\gamma, \beta_t) = 0$ or (4.6) if $i(\gamma, \beta_t) > 0$ and letting $t \rightarrow 1$, we obtain

$$\lim_{t \rightarrow 1} d_L(\alpha(t), \alpha^*(t)) = 0. \quad \square$$

Remark 4. As a Teichmüller geodesic, the curve $\alpha(t)$ leaves any compact subset of $T(S_0)$ under the metric d_T as $t \rightarrow 1$. It also leaves any compact subset under the metric d_L since $l_{S_t}(\beta_t) \rightarrow 0$ as $t \rightarrow 1$; that is,

$$d_L(\alpha(t), \alpha(0)) \rightarrow \infty \text{ as } t \rightarrow 1.$$

Using the previous theorem, we obtain

$$d_L(\alpha^*(t), \alpha(0)) \rightarrow \infty \text{ as } t \rightarrow 1.$$

Then the inequality (1.3) implies

$$d_T(\alpha^*(t), \alpha(0)) \rightarrow \infty \text{ as } t \rightarrow 1.$$

Therefore, the curve $\alpha^*(t)$ leaves any compact subset of $T(S_0)$ under both d_T and d_L metrics as well.

Theorem 12. *Let $\alpha(t)$ and $\alpha^*(t)$ be as above.*

- (1) *If $\lambda_t \rightarrow \infty$ as $t \rightarrow 1$, then $\lim_{t \rightarrow 1} d_T(\alpha(t), \alpha^*(t)) = \infty$.*
- (2) *If there exist two positive constants $C_1 < C_2$ such that $C_1 \leq \lambda_t \leq C_2$ for all t , then there exist two positive constants $m < M$ such that $m \leq d_T(\alpha(t), \alpha^*(t)) \leq M$ for all t .*

Proof. Let $\pi_t: \mathbf{D} \rightarrow S_t$ be the universal covering map and let G_t be the group uniformizing S_t . Denote by $\tilde{E}_t: \mathbf{D} \rightarrow \mathbf{D}$ the left earthquake corresponding to E_{β_t, λ_t} . By inequality (3.1) and Theorem 6, we obtain

$$\|\tilde{E}_t\|_{\text{cr}} \geq \frac{1}{C'} \|\sigma_{\lambda_t}\|_{\text{Th}} \geq \frac{1}{C'} \lambda_t.$$

If $\lambda_t \rightarrow \infty$ as $t \rightarrow 1$, then the cross-ratio distortion norm approaches ∞ as $t \rightarrow 1$. It follows from Theorem 8 that the maximal dilatation of any extension of \tilde{E}_t is approaching ∞ as $t \rightarrow 1$. Since any lift F_t of the extremal quasiconformal mapping in the homotopy class of $\Phi_{\beta_t, \lambda_t}$ satisfies $F_t|_{\mathbf{R}} = (T \circ \tilde{E}_t)|_{\mathbf{R}}$ for some Möbius transformation T , it follows that $K(F_t) \rightarrow \infty$ as $t \rightarrow 1$. That is,

$$\lim_{t \rightarrow 1} d_T(\alpha(t), \alpha^*(t)) = \infty$$

and the first part of the theorem follows.

Now suppose that $\lambda_t \leq C$ for all t . By inequality (3.1) and Theorem 6, we now obtain

$$C' \max \{C_2, 2C_2/D_t\} \geq C' \|\sigma_{\lambda_t}\|_{\text{Th}} \geq \|\tilde{E}_t\|_{\text{cr}} \geq \frac{1}{C'} \|\sigma_{\lambda_t}\|_{\text{Th}} \geq \frac{C_1}{C'} > 0,$$

where as before, $D_t \approx \log(16/l_{S_t}(\beta_t))$ is the width of a collar neighborhood around the geodesic β_t and $\tilde{E}_t: \mathbf{D} \rightarrow \mathbf{D}$ is a left earthquake map corresponding to E_{β_t, λ_t} . Since the cross-ratio distortion norm of \tilde{E}_t is bounded from above, it follows from Theorem 7 that there exists a constant M such that $K_e(\tilde{E}_t) \leq e^M$ for all t . In particular

$$d_T(\alpha(t), \alpha^*(t)) \leq M \text{ for all } t.$$

On the other hand, since the cross-ratio distortion norm of \tilde{E}_t is bounded from below, we know, by Proposition 1, that there exists a constant $m > 0$ such that $e^m \leq K_e(\tilde{E}_t)$ for all t . Thus,

$$m \leq d_T(\alpha(t), \alpha^*(t)) \text{ for all } t. \quad \square$$

From Theorem 4 in [19], we know that the curve $\alpha(t), t \in [0, 1)$, is a Teichmüller geodesic in $T(S_0)$ emanating from the base point. Now let λ_t be a nonnegative parameter continuously depending on $t \in [0, 1)$ and satisfying that $\lambda_t = o(|\log(l_{S_t}(\beta_t))|)$ and $\lambda_t \rightarrow \infty$ as $t \rightarrow 1$. Then we obtain the first part of Theorem 1 by applying Theorem 11 and the first part of Theorem 12 to the curve $\alpha^*(t)$ given in (4.4). On the other hand, by letting λ_t be bounded from above and from below by positive constants, we obtain the second part of Theorem 1 by using Theorem 11 and the second part of Theorem 12.

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