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A_1 WEIGHTS ON R, AN ALTERNATIVE APPROACH

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Abstract. We will prove that if ϕ belongs to the class $A_1(\mathbf{R})$ with constant $c \geq 1$ then the decreasing rearrangement of ϕ, belongs to the same class with constant not more than c. We also find for such ϕ the exact best possible range of those $p > 1$ for which $\phi \in L^p$, for any such ϕ. In this way we provide alternative proofs of the results that appear in a previous work of Bojarski, Sbordone and Wik.

1. Introduction

The theory of Muckenhoupt weights has been proved to be an important tool in analysis. One of the most important facts concerning these is their self improving property. A way to express this is through the so-called reverse Hölder or more generally reverse Jensen inequalities (see [3], [4] and [9]).

In this paper we are concerned with such weights and more precisely for those ϕ that belong to the class $A_1(J)$ where J is an interval on $\mathbf{R}$. This is defined as follows.

A function $\phi\colon J \to \mathbf{R}^+$ belongs to $A_1(J)$ if there exists a constant $c \geq 1$ such that the following condition is satisfied:

$$\frac{1}{|I|}\int_I \phi(x)\,dx \leq c \cdot \operatorname*{ess\,inf}_I(\phi). \tag{1.1}$$

for every I subinterval of J, where $|\cdot|$ is the Lesbesgue measure on $\mathbf{R}$. Moreover, if the constant c is the least for which (1.1) is satisfied for any $I \subseteq J$ we say that it is the A_1 constant of ϕ and is denoted by $[\phi]_1$. We say then that ϕ belongs to the A_1 class of J with constant c and we write $\phi \in A_1(J, c)$.

It is a known fact that if $\phi \in A_1(J, c)$ then there exists $p(c) > 1$ such that $\phi \in L^p$ for every $p \in [1, p(c))$. Moreover, ϕ satisfies a reverse Hölder inequality for every $p \in [1, p(c))$. That is for any such p there exists $C = C(p, c) > 1$ such that

$$\frac{1}{|I|}\int_I \phi^p(x)\,dx \leq C\left(\frac{1}{|I|}\int_I \phi(x)\,dx\right)^p, \tag{1.2}$$

for every I subinterval of J and every $\phi \in A_1(J, c)$.

The problem of the exact determination of the best possible exponent $p(c)$ has been treated in [2]. More precisely it is shown there the following:

Theorem A. *If $\phi \in A_1((0,1), c)$ and c is greater than 1, then $\phi \in L^p(0,1)$ for any p such that $1 \leq p < \frac{c}{c-1}$. Moreover, the following inequality is true*

$$\frac{1}{|I|}\int_I \phi^p(x)\,dx \leq \frac{1}{c^{p-1}(c+p-pc)}\left(\frac{1}{|I|}\int_I \phi(x)\,dx\right)^p \tag{1.3}$$

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for every I subinterval of $(0,1)$ and for any p in the range $[1, \frac{c}{c-1})$. Additionally, the constant that appears on the right of inequality (1.3) is best possible.

As a consequence of the above theorem we have that the best possible range for the L^p-integrability of any ϕ with $[\phi]_1 = c$ is $[1, \frac{c}{c-1})$.

The approach for proving the above theorem as is done in [2], is by using the decreasing rearrangement of ϕ which is defined by the following equation

$$\phi^*(t) = \sup_{\substack{e \subset (0,1) \\ |e| \geq t}} \Big[\inf_{x \in e} \phi(x) \Big], \tag{1.4}$$

for any $t \in (0,1]$. Then ϕ^* is a function equimeasurable to ϕ, non-increasing and left continuous. For a complete study of the notion of the non-increasing rearrangement of a function see [1] or [5].

The immediate step for proving Theorem A, as it appears in [2] is the following:

Theorem B. *If $\phi \in A_1((0,1), c)$ then $\phi^* \in A_1((0,1), c')$ for some c' such that $1 \leq c' \leq c$.*

This is treated in [2] initially for continuous functions ϕ and generalized to arbitrary ϕ by use of a covering lemma. Then applying several techniques the authors in [2] were able to prove Theorem A first for non-increasing functions and second for general ϕ by use of Theorem B.

In this paper we provide alternative proofs of the Theorems A and B. We first prove Theorem B without any use of covering lemmas. Then we provide a proof of Theorem A for non-increasing functions ϕ. Our proof gives in an immediate way the inequality (1.3). At last we prove Theorem A in it's general form by using the above mentioned results.

Additionally, we need to say that related results concerning the dyadic analogue of the above problem can be seen in [8] and [10], while in [6] and [7] related problems for estimates for the range of p in higher dimensions have been treated. At last for further study of Muckenhoupt weights one can see [11].

2. Rearrangements of A_1 weights on $(0,1)$

We are now ready to state and prove the main theorem in this section.

Theorem 1. *Let $\phi\colon (0,1) \to \mathbf{R}^+$ which satisfies condition (1.1) for any subinterval I of $(0,1)$, and for a constant $c \geq 1$. Then ϕ^* satisfies this condition with the same constant.*

Proof. It is easy to see that in order to prove our result, we need to prove the following inequality:

$$\frac{1}{t} \int_0^t \phi^*(u)\, du \leq c\phi^*(t) \tag{2.1}$$

for any $t \in (0,1]$, due to the fact that ϕ^* is left continuous and non-increasing.

For any $\lambda > 0$ we consider the set $E_\lambda = \{x \in (0,1)\colon \phi(x) > \lambda\}$. Let now $\varepsilon > 0$. Then we can find for any such ε an open set $G_\varepsilon \subseteq (0,1)$ for which $G_\varepsilon \supseteq E_\lambda$ and $|G_\varepsilon \setminus E_\lambda| < \varepsilon$. Then G_ε can be decomposed as follows: $G_\varepsilon = \bigcup_{j=1}^{+\infty} I_{j,\varepsilon}$, where $(I_{j,\varepsilon})$ is a family of non-overlapping open subintervals of $(0,1)$. If any two of these have a common endpoint we replace them by their union. We apply the above procedure to the new family of intervals and at last we reach to a family $(I'_{j,\varepsilon})_j$ of non-overlapping open intervals such that, if $G'_\varepsilon = \bigcup_{j=1}^{+\infty} I'_{j,\varepsilon}$ we still have that $G_\varepsilon \supseteq E_\lambda$ and $|G_\varepsilon \backslash E_\lambda| < \varepsilon$.

Additionally, we have that for any j such that $I'_{j,\varepsilon} \neq (0,1)$ there exists an endpoint of it such that if we enlarge this interval in the direction of this point, thus producing the interval $I'_{j,\varepsilon,\delta}$, with δ small enough, we have that $\operatorname{ess\,inf}_{I'_{j,\varepsilon,\delta}}(\phi) \leq \lambda$. This follows by our construction and the definition of E_λ. Suppose now that $|E_\lambda| < 1$. Thus $I'_{j,\varepsilon,\delta} \neq (0,1)$ for any j, ε and δ. On each of these intervals we apply (1.1). So we conclude that

$$\frac{1}{|I'_{j,\varepsilon,\delta}|} \int_{I'_{j,\varepsilon,\delta}} \phi \leq c \cdot \operatorname*{ess\,inf}_{I'_{j,\varepsilon,\delta}}(\phi) \leq c\lambda,$$

for every $\varepsilon, \delta > 0$ and $j = 1, 2, \ldots$.

Letting $\delta \to 0^+$ we reach to the inequality $\frac{1}{|I'_{j,\varepsilon}|} \int_{I'_{j,\varepsilon}} \phi \leq c\lambda$ for any $j = 1, 2, \ldots$ and every $\varepsilon > 0$.

Since $G_\varepsilon = \bigcup_{j=1}^{+\infty} I'_{j,\varepsilon}$ is disjoint we must have that:

$$\frac{1}{|G_\varepsilon|} \int_{G_\varepsilon} \phi \leq \sup \left\{ \frac{1}{|I'_{j,\varepsilon}|} \int_{I'_{j,\varepsilon}} \phi \colon j = 1, 2, \ldots \right\} \leq c\lambda$$

for every $\varepsilon > 0$ and letting $\varepsilon \to 0^+$ we have as a result that

$$\frac{1}{|E_\lambda|} \int_{E_\lambda} \phi \leq c\lambda \leq c \cdot \operatorname*{ess\,inf}_{E_\lambda}(\phi).$$

By the definition of E_λ we have that

$$\frac{1}{|E_\lambda|} \int_{E_\lambda} \phi = \frac{1}{|E_\lambda|} \int_0^{|E_\lambda|} \phi^*(u)\, du$$

and of course

$$\operatorname*{ess\,inf}_{E_\lambda}(\phi) = \operatorname*{ess\,inf}_{(0,(E_\lambda)]}(\phi^*) = \phi^*(|E_\lambda|).$$

since ϕ^* is left continuous. As a consequence from the above we immediately see that

$$\frac{1}{|E_\lambda|} \int_0^{|E_\lambda|} \phi^*(u)\, du \leq c\phi^*(|E_\lambda|).$$

The same inequality holds even in the case where $|E_\lambda| = 1$, so $G_\varepsilon = (0,1)$. Then by relation (1.1) that holds for the interval $(0,1)$, we conclude (2.1). Thus we have proved that $\frac{1}{t} \int_0^t \phi^*(u)\, du \leq c\phi^*(t)$, for every t of the form $t = |E_\lambda|$ for some $\lambda > 0$.

Let now $t \in (0,1]$ and define

$$\phi^*(t) = \lambda_1, \quad t_1 = \min \Big\{ s \in (0,1] \colon \phi^*(s) = \lambda_1 \Big\} \leq t$$

Additionally $|E_{\lambda_1}| = t_1$. As a result

$$\begin{aligned} \frac{1}{t} \int_0^t \phi^*(u)\, du &\leq \frac{t_1}{t} \left(\frac{1}{|E_{\lambda_1}|} \int_0^{|E_{\lambda_1}|} \phi^*(u)\, du \right) + \frac{t - t_1}{t} \lambda_1 \\ &\leq \frac{t_1}{t} c\lambda_1 + \frac{t - t_1}{t} \lambda_1 \leq c\lambda_1 = \phi^*(t), \end{aligned}$$

where in the second inequality we have used the above results. Theorem 1 is now proved. □

3. L^p integrability for A_1 weights on $(0,1)$

We now proceed to the

Proof of Theorem A. We will need first a preliminary lemma which in fact can be proved under additional hypothesis by using an integration by parts argument. We prove it below in a more general form by using simple measure theoretic techniques.

Lemma 1. *Let $g\colon (0,1] \to \mathbf{R}^+$ be a non-increasing function. Then the following inequality is true for any $p>1$ and every $\delta \in (0,1)$*

$$\int_0^\delta \left(\frac{1}{t}\int_0^t g\right)^p dt = -\frac{1}{p-1}\left(\int_0^\delta g\right)^p \frac{1}{\delta^{p-1}} + \frac{p}{p-1}\int_0^\delta \left(\frac{1}{t}\int_0^t g\right)^{p-1} g(t)\,dt. \tag{3.1}$$

Proof. By using Fubini's theorem it is easy to see that

$$\int_0^\delta \left(\frac{1}{t}\int_0^t g\right)^p dt = \int_{\lambda=0}^{+\infty} p\lambda^{p-1}\left|\left\{t\in(0,\delta]\colon \frac{1}{t}\int_0^t g \ge \lambda\right\}\right| dt. \tag{3.2}$$

Let now $\frac{1}{\delta}\int_0^\delta g = f_\delta \ge f = \int_0^1 g$.

Then

$$\frac{1}{t}\int_0^t g > f_\delta, \quad \forall\, t\in(0,\delta) \quad \text{while} \quad \frac{1}{t}\int_0^t g \le f_\delta, \quad \forall\, t\in[\delta,1].$$

Let now λ be such that: $0<\lambda<f_\delta$. Then for every $t\in(0,\delta]$ we have that $\frac{1}{t}\int_0^t g \ge \frac{1}{\delta}\int_0^\delta g = f_\delta > \lambda$. Thus

$$\left|\left\{t\in(0,\delta]\colon \frac{1}{t}\int_0^t g \ge \lambda\right\}\right| = |(0,\delta]| = \delta.$$

Now for every $\lambda \ge f_\delta$ there exists unique $a(\lambda)\in(0,\delta]$ such that $\frac{1}{a(\lambda)}\int_0^{a(\lambda)} g = \lambda$. It's existence is quaranteeded by the fact that $\lambda > f_\delta$, that g is non-increasing and that $g(0^+) = +\infty$ which may without loss of generality be assumed (otherwise we work with the λ's on the interval $(0,\|g\|_\infty])$.

Then

$$\left\{t\in(0,\delta]\colon \frac{1}{t}\int_0^t g \ge \lambda\right\} = (0,a(\lambda)].$$

Thus from (3.2) we conclude that

$$\begin{aligned}\int_0^\delta \left(\frac{1}{t}\int_0^t g\right)^p dt &= \int_{\lambda=0}^{f_\delta} p\lambda^{p-1}\cdot\delta\cdot d\lambda + \int_{\lambda=f_\delta}^{+\infty} p\lambda^{p-1} a(\lambda)\,d\lambda \\ &= \delta(f_\delta)^p + \int_{\lambda=f_\delta}^{+\infty} p\lambda^{p-1}\frac{1}{\lambda}\left(\int_0^{a(\lambda)} g(u)\,du\right)d\lambda,\end{aligned} \tag{3.3}$$

by the definition of $a(\lambda)$.

As a consequence (3.3) becomes

$$\begin{aligned}\int_0^\delta\Big(\frac{1}{t}\int_0^t g\Big)^p dt &= \frac{1}{\delta^{p-1}}\Big(\int_0^\delta g\Big)^p + \int_{\lambda=f_\delta}^{+\infty} p\lambda^{p-2}\Big(\int_0^{a(\lambda)} g(u)\,du\Big)d\lambda\\
&= \frac{1}{\delta^{p-1}}\Big(\int_0^\delta g\Big)^p + \int_{\lambda=f_\delta}^{+\infty} p\lambda^{p-2}\Big(\int_{\substack{\{u\in(0,\delta]:\\ \frac{1}{u}\int_0^u g\geq\lambda\}}} g(u)\,du\Big)d\lambda\\
&= \frac{1}{\delta^{p-1}}\Big(\int_0^\delta g\Big)^p + \frac{p}{p-1}\int_0^\delta g(t)\Big[\lambda^{p-1}\Big]_{\lambda=f_\delta}^{\frac{1}{t}\int_0^t g}dt\\
&= \frac{1}{\delta^{p-1}}\Big(\int_0^\delta g\Big)^p + \frac{p}{p-1}\Big[\int_0^\delta\Big(\frac{1}{t}\int_0^t g\Big)^{p-1}g(t) - \Big(\int_0^\delta g(t)\,dt\Big)f_\delta^{p-1}\Big]\\
&= -\frac{1}{p-1}\frac{1}{\delta^{p-1}}\Big(\int_0^\delta g\Big)^p + \frac{p}{p-1}\int_0^\delta\Big(\frac{1}{t}\int_0^t g\Big)^{p-1}g(t)\,dt,\end{aligned}$$

where in the third equality we have used Fubini's theorem and the fact that $\frac{1}{\delta}\int_0^\delta g = f_\delta$. Lemma 1 is now proved. □

In the proof of Theorem A we will need the following:

Lemma 2. *Let $g\colon (0,1] \to \mathbf{R}^+$ be non-increasing such that*

$$\frac{1}{t}\int_0^t g(u)\,du \leq cg(t), \quad \text{for every } t\in(0,1].$$

Then for every $\delta\in(0,1]$ we have the following inequality

$$\frac{1}{\delta}\int_0^\delta g^p \leq \frac{1}{c^{p-1}(c+p-pc)}\Big(\frac{1}{\delta}\int_0^\delta g\Big)^p,$$

for every p such that $1\leq p<\dfrac{p}{p-1}$. Moreover the above inequality is sharp.

Proof. Fix $\delta\in(0,1]$ and $p\in[1,\frac{c}{c-1})$. Then by Lemma 1

$$\text{(3.4)}\quad \begin{aligned}&\int_0^\delta\Big(\frac{1}{t}\int_0^t g\Big)^p dt = -\frac{1}{p-1}\Big(\int_0^\delta g\Big)^p\frac{1}{\delta^{p-1}} + \frac{p}{p-1}\int_0^\delta\Big(\frac{1}{t}\int_0^t g\Big)^{p-1}g(t)\,dt\\
&\Rightarrow \frac{1}{\delta}\int_0^\delta\Big[\Big(\frac{1}{t}\int_0^t g\Big)^{p-1}g(t) - \frac{p-1}{p}\Big(\frac{1}{t}\int_0^t g\Big)^p\Big]dt \leq \frac{1}{p}\Big(\frac{1}{\delta}\int_0^\delta g\Big)^p.\end{aligned}$$

We now define the following function h_y, of the variable x for any fixed constant $y>0$

$$h_h(x) = x^{p-1}y - \frac{p-1}{p}x^p, \quad \text{for } x\in[y,cy].$$

Then

$$h_y'(x) = (p-1)x^{p-2}y - (p-1)x^{p-1} = (p-1)x^{p-2}(y-x)\leq 0, \quad \forall\, x\geq y.$$

Thus, h_y is decreasing on the interval $[y,cy]$. We conclude that for any x such that $y\leq x\leq cy$ we have $h_y(x)\geq h_y(cy)$.

Applying the above conclusion in the case where $x = \frac{1}{t}\int_0^t g$, $y = g(t)$ (noting that $y \le x \le cy$, for any fixed t) we reach to the inequality:

$$\tag{3.5} \begin{aligned} \left(\frac{1}{t}\int_0^t g\right)^{p-1} g(t) - \frac{p-1}{p}\left(\frac{1}{t}\int_0^t g\right)^p &\ge c^{p-1}g^p(t) - \frac{p-1}{p}c^p g^p(t) \\ &= c^{p-1}\left[1 - \frac{p-1}{p}c\right] g^p(t), \quad \forall\, t \in (0,1]. \end{aligned}$$

Applying (3.5) in (3.4) we have as a result that

$$\tag{3.6} \begin{aligned} & c^{p-1}\left[1 - \frac{p-1}{p}c\right]\frac{1}{\delta}\int_0^\delta g^p(t)\,dt \le \frac{1}{p}\left(\frac{1}{\delta}\int_0^\delta g\right)^p \\ & \Rightarrow \frac{1}{\delta}\int_0^\delta g^p \le \frac{1}{c^{p-1}[p+c-pc]}\left(\frac{1}{\delta}\int_0^\delta g\right)^p, \end{aligned}$$

which is the inequality that is stated above.

Additionally (3.6) is sharp as can be seen by using the function $g(t) = \frac{1}{c}t^{\frac{1}{c}-1}$, $t \in (0,1]$, for $c > 1$, and $g = const$ for $c = 1$. Lemma 2 is now proved. □

We are now ready for the

Proof of Theorem A. Let $I \subseteq (0,1)$ be an interval. We set $\phi_I \colon I \to \mathbf{R}^+$ by $\phi_I(x) = \phi(x)$, $x \in I$. Then ϕ_I satisfies on I the condition (1.1) with constant c. That is $\phi \in A_1(I)$ with A_1-constant less or equal than c. Then by the results of Section 2 and a dilation argument we conclude that

$$\phi_I^* = g_I \colon (0,|I|] \to \mathbf{R}^+$$

satisfies

$$\frac{1}{t}\int_0^t g_I \le c g_I(t), \quad \text{for any } t \in (0,|I|].$$

Then by Lemma 2 and considering the results of this Section we have the inequality:

$$\tag{3.7} \frac{1}{t}\int_0^t g_I^p(u)\,du \le \frac{1}{c^{p-1}(c+p-pc)}\left(\frac{1}{t}\int_0^t g_I(u)\,du\right)^p$$

for any $t \in (0,|I|]$.

By the fact now that $g_I = (\phi/I)^*$ and (3.7) we see immediately: (for $t = |I|$) that

$$\frac{1}{|I|}\int_I \phi^p \le \frac{1}{c^{p-1}(c+p-pc)}\left(\frac{1}{|I|}\int_I \phi\right)^p.$$

At last, we mention that the result is best possible since Lemma 2 is proved to be sharp.

Theorem A is now proved. □

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