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ON THE EXISTENCE OF SOLUTIONS OF A FERMAT-TYPE DIFFERENCE EQUATION

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Abstract. The analogue of Fermat's last theorem for function fields has been investigated by many scholars recently, and Gundersen–Hayman [6] collected the best lower estimates that are known for $F_C(n)$, where $F_C(n)$ is the smallest positive integer k such that the equation

$$f_1^n + f_2^n + \ldots + f_k^n = 1$$

has a solution consisting of k nonconstant functions $f_1, f_2, \ldots, f_k$ in C, and C is the ring of meromorphic functions M, rational functions R, entire functions E or polynomials P, respectively. In this paper, we investigate a difference analogue of this problem for the rings of M, R, E, P with certain conditions, and obtain lower bounds for G_C, where $G_C(n)$ is the smallest positive integer k such that the equation

$$f_1(z)f_1(z+c)\cdots f_1(z+(n-1)c) + \ldots + f_k(z)f_k(z+c)\cdots f_k(z+(n-1)c) = 1$$

has a solution consisting of k nonconstant functions $f_1, f_2, \ldots, f_k$ in C.

1. Introduction

According to the famous Fermat's last theorem, which was proved by Wiles [21] and by Taylor–Wiles [19], there do not exist nonzero rational numbers x, y, and an integer n, where $n \geq 3$, such that

$$x^n + y^n = 1.$$

There are natural analogues of Fermat's last theorem in complex function theory. For example, let M, R, E and P denote the rings of meromorphic functions, rational functions, entire functions and polynomials, respectively. Thus if C is equal to M, R, E or P, and n is an integer satisfying $n \geq 2$, then $F_C(n)$ denotes the smallest positive integer k such that the equation

$$f_1^n + f_2^n + \cdots + f_k^n = 1 \tag{1.1}$$

has a solution consisting of k nonconstant functions $f_1, f_2, \ldots, f_k$ in C. Hence, the smallest k depends on n.

Many scholars have investigated this and related problems, for details please see [3, 4, 12, 14, 15, 18, 20, 23, 25] etc. Gundersen–Hayman [6] collected the best lower

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estimates that are known for every n as follows:

$$F_P(n) > 1/2 + \sqrt{n+1/4}; \quad F_R(n) > \sqrt{n+1};$$
$$F_M(n) \geq \sqrt{n+1}; \quad F_E(n) \geq 1/2 + \sqrt{n+1/4}.$$

A natural difference analogue of the Taylor series expansion is the factorial series [17, p. 272], which suggests to consider the difference monomial $x(x-1)\cdots(x-n+1)$ as a discrete analogue of x^n. Similar correspondence occurs frequently in the theory of difference equations as well, and can be seen for example by comparing the Riccati equation and its difference analogue.

The purpose of this paper is to formulate and study a difference analogue of Fermat's last theorem for function fields M, R, E, P. We will consider the difference equation

$$f_1\overline{f}_1\cdots\overline{f}_1^{[n-1]} + f_2\overline{f}_2\cdots\overline{f}_2^{[n-1]} + \cdots + f_k\overline{f}_k\cdots\overline{f}_k^{[n-1]} = 1, \tag{1.2}$$

where $\overline{f}^{[i]}$ stands for $f(z+ic)$, c is a nonzero constant and i is a positive integer, and denote by G_C the smallest positive integer k such that the equation (1.2) has a solution consisting of k nonconstant functions $f_1, \ldots, f_k$ in C.

We need the following definition and notations in order to state our results.

Definition 1.1. [24] Let f and g be meromorphic functions and a be a complex number. Let $z_n\,(n=1,2,\ldots)$ be zeros of $f-a$. If $z_n\,(n=1,2,\ldots)$ are also zeros of $g-a$ (ignoring multiplicity), we denote

$$f = a \Rightarrow g = a \quad \text{or} \quad g = a \Leftarrow f = a.$$

Let $\nu(n)$ be the multiplicity of the zero z_n. If $z_n\,(n=1,2,\ldots)$ are also $\nu(n)\,(n=1,2,\ldots)$ multiple zeros of $g-a$ at least, we write

$$f = a \rightarrow g = a \quad \text{or} \quad g = a \leftarrow f = a.$$

If $f = a \rightleftharpoons g = a$, it is said that f and g share a CM; If $f = a \Leftrightarrow g = a$, it is said that f and g share a IM; If $f = a \rightarrow \overline{f} = a$ except for at most finitely many a-points of f, it is said that a is an exceptional paired value of f with the separation c (as defined in [8]).

Let $\widetilde{M}$ be the collection of all nonconstant meromorphic functions of hyper-order < 1 such that any finite collection $\{f_1, \ldots, f_k\} \subset \widetilde{M}$ satisfies the following properties

(i) f_i and $1/f_j\ (i,j=1,\ldots,k, i\neq j)$ have no common zeros;
(ii) $f_i = \infty \rightleftharpoons \overline{f}_i = \infty$ for all $i=1,\ldots,k$;
(iii) 0 is an exceptional paired value of f_i for all $i=1,\ldots,k$.

In the case of meromorphic functions, compared to the lower bound of F_M, we obtain a corresponding result about $G_{\widetilde{M}}$.

Theorem 1.2. *Let $n\,(\geq 2)$ be an integer. Then*

$$G_{\widetilde{M}}(n) \geq \sqrt{n+1}.$$

Let $\widetilde{E}$ be the collection of all nonconstant entire functions of hyper-order < 1 such that any finite collection $\{f_1, \ldots, f_k\} \subset \widetilde{E}$ satisfies the property that $f_i = 0 \Rightarrow \overline{f}_i = 0$ for all $i=1,\ldots,k$.

Particularly, for the case of entire functions, analogously to the lower bound of F_E, we give a better lower estimate for $G_{\widetilde{E}}$.

Theorem 1.3. *Let $n\,(\geq 2)$ be an integer. Then*

$$G_{\widetilde{E}}(n) \geq 1/2 + \sqrt{n + 1/4}.$$

The condition that hyper-order is less than 1 cannot be deleted. For example, take $f(z) = \exp\{e^z\}$, $c = i\pi$ and $n = 2$. Since 0 and ∞ are Picard exceptional values of $f(z)$, they are automatically also exceptional paired values of $f(z)$. Moreover, the function also satisfies the conditions $f = 0 \Rightarrow \overline{f} = 0$ and $f = \infty \Rightarrow \overline{f} = \infty$. The hyper-order of $f(z)$ is 1, and

$$f(z) \cdot f(z + c) = \exp\{e^z\} \cdot \exp\{e^{z+i\pi}\} = \exp\{e^z\} \cdot \exp\{-e^z\} = 1.$$

But $k = 1$ is strictly less than $1/2 + \sqrt{2 + 1/4} = \frac{3+1}{2} = 2$ and $\sqrt{2+1}\,(> 1)$.

The following example shows the sharpness of the bound of G_C, where C is equal to $\widetilde{M}$ and $\widetilde{E}$.

Example 1.4. Let $c = 2\pi$, $f_1 = \sin z$ and $f_2 = \cos z$. Then $\overline{f}_1 = \sin(z + 2\pi) = \sin z$ $\overline{f}_2 = \cos(z + 2\pi) = \cos z$. Clearly $f_i\,(i = 1, 2)$ satisfy $f_i = 0 \Rightarrow \overline{f}_i = 0$ and

$$f_1\overline{f}_1 + f_2\overline{f}_2 = \sin^2 z + \cos^2 z = 1.$$

Also, 0 is an exceptional paired value of f_i for $i = 1, 2$. Thus we have $G_{\widetilde{M}}(2) \leq 2$ and $G_{\widetilde{E}}(2) \leq 2$. On the other hand, by Theorems 1.2 and 1.3, we have $G_C(n) > 1$ for $C = \widetilde{M}, \widetilde{E}$. Therefore, $G_{\widetilde{M}}(2) = G_{\widetilde{E}}(2) = 2$.

Let $\widetilde{R}$ be the collection of all nonconstant rational functions such that any finite collection $\{f_1, \ldots, f_k\} \subset \widetilde{R}$ satisfies the property that zeros and poles are of multiplicity positive integer multiple of n.

In the case of rational functions, compared to the lower bound for F_R, we get a corresponding estimate for $G_{\widetilde{R}}$.

Theorem 1.5. *Let $n\,(\geq 2)$ be an integer. Then*

$$G_{\widetilde{R}}(n) > \sqrt{n + 1}.$$

Let $\widetilde{P}$ be the collection of all nonconstant polynomial functions such that any finite collection $\{f_1, \ldots, f_k\} \subset \widetilde{P}$ satisfies the property that zeros are of multiplicity no less than n.

Also, as an analogue to the entire case, for the case of polynomials, we give a better lower estimate for $G_{\widetilde{P}}$.

Theorem 1.6. *Let $n\,(\geq 2)$ be an integer. Then*

$$G_{\widetilde{P}}(n) > 1/2 + \sqrt{n + 1/4}.$$

2. Lemmas

The following lemma on the growth of non-decreasing real-valued functions is a generalization of [9, Lemma 2.1]. It implies that shifting a characteristic or a counting function does not affect its growth significantly, provided that the hyper-order of the function is strictly less than one.

Lemma 2.1. [10] *Let* $T\colon [0,+\infty) \to [0,+\infty)$ *be a non-decreasing continuous function and let* $s \in (0,\infty)$. *If the hyper-order of* T *is strictly less than one, i.e.,*

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} = \varsigma < 1$$

and $\delta \in (0, 1-\varsigma)$, *then*

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{\delta}}\right),$$

where r *runs to infinity outside of a set of finite logarithmic measure.*

Recently, Halburd, Korhonen and Tohge generalized the difference analogue of Logarithmic Derivative Lemma of meromorphic functions of finite order (see [2, 7]) into meromorphic functions of hyper-order strictly less than one.

Lemma 2.2. [10] *Let* f *be a non-constant meromorphic function,* $\varepsilon > 0$ *and* $c \in \mathbf{C}$. *If the hyper-order of* $T(r,f)$, *i.e.,* $\sigma_2 = \sigma_2(f) < 1$, *then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\sigma_2-\varepsilon}}\right)$$

for all r *outside of a set of finite logarithmic measure.*

Throughout the remainder of the paper, we let $d(P)$ denote the degree of P, and let $\overline{d}(P)$ denote the number of distinct zeros of P, where $P\,(\not\equiv 0)$ is a polynomial. The following lemma is an application of Cartan's theorem.

Lemma 2.3. [6] *Let* $g_1, g_2, \cdots, g_p$ *be linearly independent entire functions, where* $p \geq 2$. *Suppose that for each complex number* z *we have*

$$\max\{|g_1(z)|, |g_2(z)|, \ldots, |g_p(z)|\} > 0, \tag{2.1}$$

and set $g_{p+1} = g_1 + g_2 + \ldots + g_p$. *We distinguish two cases.*

(a) *Suppose that all the quotients* g_j/g_m *are rational functions. Then there exist polynomials* $h_1, h_2, \ldots, h_{p+1}$, *and an entire function* ϕ, *such that*

$$g_j = h_j e^{\phi}, \quad j = 1, 2, \ldots, p+1.$$

Then $h_{p+1} = h_1 + h_2 + \ldots + h_p$ *and*

$$\max\{d(h_1), d(h_2), \ldots, d(h_p)\} \leq (p-1)\left\{\sum_{j=1}^{p+1} \overline{d}(h_j) - \frac{1}{2}p\right\}. \tag{2.2}$$

In particular, if all the functions $g_1, g_2, \ldots, g_p$ *are polynomials, then*

$$\max\{d(g_1), d(g_2), \ldots, d(g_p)\} \leq (p-1)\left\{\sum_{j=1}^{p+1} \overline{d}(g_j) - \frac{1}{2}p\right\}. \tag{2.3}$$

(b) *Suppose that at least one quotient* g_j/g_m *is a transcendental function. Set*

$$N(r) = \sup_{1\leq j\leq p+1} N(r, 0, g_j).$$

Then

$$\frac{N(r)}{\log r} \to \infty, \quad \text{as } r \to \infty, \tag{2.4}$$

and we have

$$(1 + o(1))N(r) \le (p-1)\sum_{j=1}^{p+1} \overline{N}(r,0,g_j) \quad \text{as } r \to \infty \ \text{n.e.} \tag{2.5}$$

where n.e. stands for nearly everywhere, which always means the inequality holds in the real axis outside of a finite logarithmic measure.

The following lemma tells us that if $f_1, \dots, f_k$ is a collection of functions for which the minimum $G_C(n) = k$ is attained, then the corresponding terms on the left hand side of (1.2) must be linearly independent.

Lemma 2.4. *Let $n\,(\ge 2)$ be an integer. Suppose the equation (1.2) has a solution consisting of k nonconstant functions $f_1, \dots, f_k$ in C. If $G_C(n) = k$, then the functions $f_i \cdots \overline{f}_i^{[n-1]}$ $(i = 1, \dots, k)$ are linearly independent.*

Proof. If $f_i \cdots \overline{f}_i^{[n-1]}$ $(i = 1, \dots, k)$ are linearly dependent, then there exists complex constants α_i $(i = 1, \dots, k)$, not all of which are 0, such that

$$\alpha_1 f_1 \cdots \overline{f}_1^{[n-1]} + \cdots + \alpha_k f_k \cdots \overline{f}_k^{[n-1]} = 0. \tag{2.6}$$

Without loss of generality, we suppose that α_{i_0} $(i_0 \in \{1, \dots, k\}) \ne 0$. Then from (2.6) we can get that

$$\begin{aligned} f_{i_0} \cdots \overline{f}_{i_0}^{[n-1]} = {} & \frac{\alpha_1}{\alpha_{i_0}} \cdot f_1 \cdots \overline{f}_1^{[n-1]} + \cdots + \frac{\alpha_{i_0-1}}{\alpha_{i_0}} f_{i_0-1} \cdots \overline{f}_{i_0-1}^{[n-1]} \\ & + \frac{\alpha_{i_0+1}}{\alpha_{i_0}} f_{i_0+1} \cdots \overline{f}_{i_0+1}^{[n-1]} + \cdots + \frac{\alpha_k}{\alpha_{i_0}} \cdot f_k \cdots \overline{f}_k^{[n-1]}. \end{aligned} \tag{2.7}$$

Substituting (2.7) into (1.2), we get that

$$\begin{aligned} & (1 + \frac{\alpha_1}{\alpha_{i_0}}) \cdot f_1 \cdots \overline{f}_1^{[n-1]} + \cdots + (1 + \frac{\alpha_{i_0-1}}{\alpha_{i_0}}) f_{i_0-1} \cdots \overline{f}_{i_0-1}^{[n-1]} \\ & + (1 + \frac{\alpha_{i_0+1}}{\alpha_{i_0}}) f_{i_0+1} \cdots \overline{f}_{i_0+1}^{[n-1]} + \cdots + (1 + \frac{\alpha_k}{\alpha_{i_0}}) f_k \cdots \overline{f}_k^{[n-1]} = 1. \end{aligned} \tag{2.8}$$

We set

$$g_j = \left(1 + \frac{\alpha_j}{\alpha_{i_0}}\right)^{1/n} f_j, \quad j = 1, \dots, i_0 - 1, i_0 + 1, \dots, k. \tag{2.9}$$

Substituting (2.9) into (2.8), we then have

$$g_1 \cdots \overline{g}_1^{[n-1]} + g_2 \cdots \overline{g}_2^{[n-1]} + \cdots + g_{i_0-1} \cdots \overline{g}_{i_0-1}^{[n-1]} + g_{i_0+1} \cdots \overline{g}_{i_0+1}^{[n-1]} + \cdots + g_k \cdots \overline{g}_k^{[n-1]} = 1,$$

where $g_j (j = 1, \dots, i_0 - 1, i_0 + 1, \dots, k)$ are non-constant functions in C. Thus we have $G_C(n) = k - 1$, a contradiction with the assumption $k = G_C(n)$. Thus the functions $f_i \cdots \overline{f}_i^{[n-1]}$ $(i = 1, \dots, k)$ are linearly independent. □

3. Proof of Theorem 1.3

Suppose that $f_1, f_2, \dots, f_k$ are nonconstant entire functions satisfying (1.2).

From the assumption $f_i = 0 \Rightarrow \overline{f}_i = 0$, we can get that $f_i(z)$ are transcendental. Since if there exists a z_0 such that $f_i(z_0) = 0$, then $f_i(z_0 + jc) = 0$, where $i = 1, 2, \dots, k$ and $j = 1, 2, \dots$, so $f_i(z)$ has infinitely many zeros. Thus $f_i(z)$ is transcendental. If

$f_i(z)$ has no zeros, then 0 is a Picard exceptional value of non-constant entire function $f_i(z)$, which implies that $f_i(z)$ is transcendental.

Firstly we prove that $k \geq 2$.

If $k = 1$, then we have $f_1 \cdots \overline{f}_1^{[n-1]} = 1$. Since f_1 is a transcendental entire function, then from Lemma 2.2 and $\sigma_2(f_1) < 1$, we have

$$\begin{aligned} nT(r, f_1) = T(r, f_1^n) &= T\left(r, \frac{1}{f_1^n}\right) + O(1) = T\left(r, \frac{\overline{f}_1 \cdots \overline{f}_1^{[n-1]}}{f_1^{n-1}}\right) + O(1) \\ &= m\left(r, \frac{\overline{f}_1 \cdots \overline{f}_1^{[n-1]}}{f_1^{n-1}}\right) + N\left(r, \frac{\overline{f}_1 \cdots \overline{f}_1^{[n-1]}}{f_1^{n-1}}\right) + O(1) \\ &\leq N\left(r, \frac{1}{f_1^{n-1}}\right) + S(r, f_1) \leq (n-1)T(r, f_1) + S(r, f_1), \end{aligned}$$

a contradiction. Thus we have $k \geq 2$.

From Lemma 2.4, we know that if $f_i \cdots \overline{f}_i^{[n-1]} (i = 1, \ldots, k)$ are linearly dependent, then we have $G_E(n) \neq k$. Since $G_E(n) \leq k$, so we have $G_E(n) < k$. This means that if $f_i \cdots \overline{f}_i^{[n-1]}$ $(i = 1, \ldots, k)$ are linearly dependent, then we can shorten the equation (1.2). Thus, in order to get the smallest k, from Lemma 2.4 we assume that the functions $f_i \cdots \overline{f}_i^{[n-1]}$ $(i = 1, \ldots, k)$ are linearly independent.

Next we prove that at least one $f_i \cdots \overline{f}_i^{[n-1]}$ is transcendental. Since f_{i_0} is transcendental, also $f_{i_0}^n$ is transcendental. If $f_{i_0} \cdots \overline{f}_{i_0}^{[n-1]}$ is polynomial, we write $f_{i_0} \cdots \overline{f}_{i_0}^{[n-1]} = p(z)$. Thus from Lemma 2.2 and the fact that f_{i_0} is a transcendental entire function with $\sigma_2(f_{i_0}) < 1$, we get

$$\begin{aligned} nT(r, f_{i_0}) + S(r, f_{i_0}) = T(r, f_{i_0}^n) + S(r, f_{i_0}) &= T\left(r, \frac{p(z)}{f_{i_0}^n}\right) \\ &= T\left(r, \frac{\overline{f}_{i_0} \cdots \overline{f}_{i_0}^{[n-1]}}{f_{i_0}^{n-1}}\right) \leq (n-1)T(r, f_{i_0}) + S(r, f_{i_0}), \end{aligned}$$

a contradiction. So it follows that $f_{i_0} \cdots \overline{f}_{i_0}^{[n-1]}$ is transcendental. Dividing equation (1.2) by $f_{i_0} \cdots \overline{f}_{i_0}^{[n-1]}$, it therefore follows that at least one quotient

$$f_j \cdots \overline{f}_j^{[n-1]} / f_{i_0} \cdots \overline{f}_{i_0}^{[n-1]} \; (j \neq i_0)$$

must be transcendental. Next we apply Lemma 2.3 with $g_j = f_j \overline{f}_j \cdots \overline{f}_j^{[n-1]}$, $j = 1, \ldots, k$. Assumption (2.1) is satisfied for this set of functions, since otherwise we would get an immediate contradiction with (1.2). Then from Lemma 2.3, we find that

$$(1 + o(1))N(r) \leq (k-1) \sum_{j=1}^{k} \overline{N}\left(r, 0, f_j \cdots \overline{f}_j^{[n-1]}\right) \text{ as } r \to \infty \text{ n.e.,} \tag{3.1}$$

where $N(r) = \sup_{1 \leq j \leq k} N\left(r, 0, f_j \cdots \overline{f}_j^{[n-1]}\right)$. Since $\sigma_2(f_i) < 1$, we have

$$\limsup_{r \to \infty} \frac{\log \log \overline{N}(r, 0, f_i)}{\log r} \leq \sigma_2(f_i) < 1. \tag{3.2}$$

By the assumption $f_i = 0 \Rightarrow \overline{f}_i = 0$, from (3.1), (3.2) and Lemma 2.1, we get

$$\begin{aligned}(1+o(1))N(r) &\le (k-1)\sum_{j=1}^{k}\overline{N}(r,0,\overline{f}_j^{[n-1]})\\ &\le (k-1)\sum_{j=1}^{k}\overline{N}(r+(n-1)|c|,0,f_j)\\ &= (k-1)\sum_{j=1}^{k}(1+o(1))\overline{N}(r,0,f_j)\\ &\le (k-1)\frac{1}{n}\sum_{j=1}^{k}(1+o(1))N(r,0,f_j\cdots\overline{f}_j^{[n-1]})\\ &\le (k-1)(1+o(1))\frac{k}{n}N(r),\quad r\to\infty \text{ n.e.}\end{aligned} \tag{3.3}$$

From (2.4) in Lemma 2.3, we have $N(r) \to \infty$ as $r \to \infty$. Hence from (3.3), we see that $n \le k^2 - k$. This proves Theorem 1.3. □

4. Proof of Theorem 1.2

Suppose that each f_i is a nonconstant meromorphic function. Next we prove that f_i is transcendental.

Since zero is an exceptional paired value of f_i, then we get that either f_i has no zeros or it has infinitely many zeros. From $f_i(z)$ and $f_i(z+c)$ share ∞ CM, we get that f_i has no poles or it has infinitely many poles. Now we only need to consider the case when f_i has no zeros and poles (in other cases, f_i is transcendental obviously). But now, since 0 and ∞ are Picard exceptional values of a nonconstant meromorphic function f_i, it follows that f_i is transcendental.

Next we prove that at least one $f_i\cdots\overline{f}_i^{[n-1]}$ is transcendental. Since f_{i_0} is transcendental, then we have that $f_{i_0}^n$ is transcendental. Suppose that $f_{i_0}\cdots\overline{f}_{i_0}^{[n-1]}$ is polynomial, say, $p(z)$. Since f_{i_0} is a transcendental meromorphic function with hyper-order less than 1, then from Lemma 2.2 we get

$$\begin{aligned}nT(r,f_{i_0}) + S(r,f_{i_0}) &= T(r,f_{i_0}^n) + S(r,f_{i_0}) = T\left(r,\frac{p(z)}{f_{i_0}^n}\right)\\ &= N\left(r,\frac{\overline{f}_{i_0}\cdots\overline{f}_{i_0}^{[n-1]}}{f_{i_0}^{n-1}}\right) + S(r,f_{i_0}).\end{aligned} \tag{4.1}$$

By the assumption that 0 is an exceptional paired value of f_{i_0}, we have

$$N\left(r,\frac{\overline{f}_{i_0}\cdots\overline{f}_{i_0}^{[n-1]}}{f_{i_0}^{n-1}}\right) \le N\left(r,\overline{f}_{i_0}\right) + \cdots + N\left(r,\overline{f}_{i_0}^{[n-1]}\right) + S(r,f_{i_0}). \tag{4.2}$$

From $\sigma_2(f_{i_0}) < 1$, we have

$$\limsup_{r\to\infty}\frac{\log\log N(r,f_{i_0})}{\log r} \le \sigma_2(f_{i_0}) < 1.$$

Thus by a simple observation, Lemma 2.1, (4.1) and (4.2) we get

$$\begin{aligned} nT(r, f_{i_0}) &\le N(r+|c|, f_{i_0}) + \cdots + N(r+(n-1)|c|, f_{i_0}) + S(r, f_{i_0}) \\ &\le N(r, f_{i_0}) + \cdots + N(r, f_{i_0}) + S(r, f_{i_0}) \\ &\le (n-1)T(r, f_{i_0}) + S(r, f_{i_0}), \end{aligned}$$

which yields a contradiction. So we have that $f_{i_0} \cdots \overline{f}_{i_0}^{[n-1]}$ is transcendental.

As in the proof of Theorem 1.3, by dividing with $f_{i_0} \cdots \overline{f}_{i_0}^{[n-1]}$ on both sides of (1.2), we obtain that at least one quotient $\frac{f_j \cdots \overline{f}_j^{[n-1]}}{f_{i_0} \cdots \overline{f}_{i_0}^{[n-1]}}$ $(j \ne i_0)$ must be transcendental.

For every meromorphic function $f_i (i = 1, 2, \ldots, k)$ with $\sigma_2(f_i) < 1$, there exists linearly independent entire functions g_i and h_i with no common zeros such that

$$f_i(z) = \frac{g_i(z)}{h_i(z)}. \tag{4.3}$$

Since entire functions g_i and h_i have no common zeros, we have that g_i and f_i have the same zeros and that h_i and $1/f_i$ have the same zeros. Thus we have

$$\lambda_2(g_i) = \limsup_{r\to\infty} \frac{\log^+ \log^+ N(r, 0, g_i)}{\log r} = \lambda_2(f_i) \le \sigma_2(f_i) < 1 \tag{4.4}$$

and

$$\lambda_2(h_i) = \limsup_{r\to\infty} \frac{\log^+ \log^+ N(r, 0, h_i)}{\log r} = \lambda_2\left(\frac{1}{f_i}\right) \le \sigma_2(f_i) < 1. \tag{4.5}$$

Substituting (4.3) into (1.2), and multiplying both sides by $h_1 \cdots \overline{h}_1^{[n-1]} h_2 \cdots \overline{h}_2^{[n-1]} \cdots h_k \cdots \overline{h}_k^{[n-1]}$, we have

$$l_1 + l_2 + \cdots + l_k = l_{k+1}, \tag{4.6}$$

where

$$\begin{aligned} l_1 &= g_1 \cdots \overline{g}_1^{[n-1]} h_2 \cdots \overline{h}_2^{[n-1]} \cdots h_k \cdots \overline{h}_k^{[n-1]}, \\ l_2 &= g_2 \cdots \overline{g}_2^{[n-1]} h_1 \cdots \overline{h}_1^{[n-1]} h_3 \cdots \overline{h}_3^{[n-1]} \cdots h_k \cdots \overline{h}_k^{[n-1]}, \\ &\cdots \\ l_k &= g_k \cdots \overline{g}_k^{[n-1]} h_1 \cdots \overline{h}_1^{[n-1]} h_2 \cdots \overline{h}_2^{[n-1]} \cdots h_{k-1} \cdots \overline{h}_{k-1}^{[n-1]}, \\ l_{k+1} &= h_1 \cdots \overline{h}_1^{[n-1]} \cdots h_k \cdots \overline{h}_k^{[n-1]}. \end{aligned} \tag{4.7}$$

We now define a function $d(z)$ as follows.

Case 1. Suppose that l_i $(i = 1, \ldots, k)$ have infinitely many non-zero common zeros. Let $\{a_n\}$ be the non-zero common zeros of l_i $(i = 1, \ldots, k)$ such that $|a_n| \to \infty$ (otherwise, if $|a_n| < M$ as $n \to \infty$, it would follow that $l_i \equiv 0$). If $\{p_n\}$ is any sequence of non-negative integers such that for all $r > 0$,

$$\sum_{n=1}^{\infty} (r/|a_n|)^{1+p_n} < \infty,$$

then from the Weierstrass theorem, we have that the function

$$d(z)=\prod_{n=1}^{\infty}E_{p_n}(z/a_n)$$

is entire with zeros only at points a_n, where

$$E_{p_n}=\begin{cases}(1-z), & \text{if } p_n=0,\\ (1-z)\exp\left(\frac{z^1}{1}+\frac{z^2}{2}+\cdots+\frac{z^{p_n}}{p_n}\right), & \text{otherwise.}\end{cases}$$

If the number z_0 occurs in the sequence $\{a_n\}$ exactly m times, then function $d(z)$ has a zero at $z=z_0$ of multiplicity m.

Case 2. Suppose that l_i $(i=1,\dots,k)$ have finitely many non-zero common zeros. We set $d(z)=\prod_{n=1}^{n_0}(z-a_n)$.

Suppose also that l_i $(i=1,\dots,k)$ have a common zero at $z=0$ of order $m\ge 0$ (a zero of order $m=0$ at $z=0$ means $f(0)\ne 0$).

With $d(z)$ defined according to Cases 1 and 2, let

$$\widetilde{l}_i(z)=\frac{l_i(z)}{z^m d(z)},\quad i=1,\dots,k+1. \tag{4.8}$$

Then for each complex number $z\in\mathbf{C}$, we have $\max\{|\widetilde{l}_1(z)|,|\widetilde{l}_2(z)|,\dots,|\widetilde{l}_k(z)|\}>0$ and

$$\widetilde{l}_1+\widetilde{l}_2+\cdots+\widetilde{l}_k=\widetilde{l}_{k+1}, \tag{4.9}$$

where $\widetilde{l}_i(z)$ $(i=1,\dots,k+1)$ are entire functions.

From (4.7), (4.8) and (4.3), we obtain that

$$\frac{\widetilde{l}_j}{\widetilde{l}_{i_0}}=\frac{g_j\cdots\overline{g}_j^{[n-1]}}{h_j\cdots\overline{h}_j^{[n-1]}}\cdot\frac{h_{i_0}\cdots\overline{h}_{i_0}^{[n-1]}}{g_{i_0}\cdots\overline{g}_{i_0}^{[n-1]}}=\frac{f_j\cdots\overline{f}_j^{[n-1]}}{f_{i_0}\cdots\overline{f}_{i_0}^{[n-1]}}, \tag{4.10}$$

$j=1,\dots,i_0-1,i_0+1,\dots,n$. In the beginning of the proof, we obtained that at least one quotient $\frac{f_j\cdots\overline{f}_j^{[n-1]}}{f_{i_0}\cdots\overline{f}_{i_0}^{[n-1]}}$ $(j\ne i_0)$ must be transcendental, thus it follows from (4.10) that at least one $\frac{\widetilde{l}_j}{\widetilde{l}_{i_0}}$ $(j\ne i_0)$ must be transcendental.

Next, we prove $k\ge 2$. If not, $k=1$, then we have $\widetilde{l}_1=\widetilde{l}_2$, thus $l_1=l_2$, i.e. $f_1\cdots\overline{f}_1^{[n-1]}=1$. Since $\sigma_2(f_1)<1$, we have

$$\limsup_{r\to\infty}\frac{\log\log N(r,f_1)}{\log r}\le\sigma_2(f_1)<1.$$

Combining this with the fact that 0 is an exceptional paired value of transcendental meromorphic function f_1, from Lemma 2.2 and Lemma 2.1, we have

$$\begin{aligned}nT(r,f_1)&=T\left(r,\frac{1}{f_1^n}\right)+O(1)=T\left(r,\frac{\overline{f}_1\cdots\overline{f}_1^{[n-1]}}{f_1^{n-1}}\right)+O(1)\\&\le N\left(r,\frac{\overline{f}_1}{f_1}\right)+\cdots+N\left(r,\frac{\overline{f}_1^{[n-1]}}{f_1}\right)+S(r,f_1)\end{aligned}$$

$$\leq N\left(r,\overline{f}_1\right)+\cdots+N\left(r,\overline{f}_1^{[n-1]}\right)+S(r,f_1)$$
$$\leq (n-1)N(r,f_1)+S(r,f_1)\leq (n-1)T(r,f_1)+S(r,f_1),$$

a contradiction. Thus we have $k\geq 2$.

In order to get the smallest k, we assume that the functions $\widetilde{l}_1, \ldots, \widetilde{l}_k$ are linearly independent. Otherwise, if the functions $\widetilde{l}_1, \ldots, \widetilde{l}_k$ are linearly dependent, then there exists $\alpha_1,\ldots,\alpha_k$, not all of which are zeros, such that

$$\alpha_1\widetilde{l}_1+\cdots+\alpha_k\widetilde{l}_k=0. \tag{4.11}$$

Without loss of generality, we suppose that $\alpha_{i_0}\neq 0$. Then from (4.11), we have

$$\widetilde{l}_{i_0}=\frac{\alpha_1}{\alpha_{i_0}}\widetilde{l}_1+\cdots+\frac{\alpha_{i_0-1}}{\alpha_{i_0}}\widetilde{l}_{i_0-1}+\frac{\alpha_{i_0+1}}{\alpha_{i_0}}\widetilde{l}_{i_0+1}+\cdots+\frac{\alpha_k}{\alpha_{i_0}}\widetilde{l}_k. \tag{4.12}$$

Dividing (4.12) by $\widetilde{l}_{k+1}$ on both sides, from (4.7) and (4.8) we have that

$$\begin{aligned}f_{i_0}\cdots\overline{f}_{i_0}^{[n-1]}&=\frac{\alpha_1}{\alpha_{i_0}}f_1\cdots\overline{f}_1^{[n-1]}+\cdots+\frac{\alpha_{i_0-1}}{\alpha_{i_0}}f_{i_0-1}\cdots\overline{f}_{i_0-1}^{[n-1]}\\&\quad+\frac{\alpha_{i_0+1}}{\alpha_{i_0}}f_{i_0+1}\cdots\overline{f}_{i_0+1}^{[n-1]}+\cdots+\frac{\alpha_k}{\alpha_{i_0}}f_k\cdots\overline{f}_k^{[n-1]}.\end{aligned}$$

This gives that $f_1\cdots\overline{f}_1^{[n-1]},\ldots,f_k\cdots\overline{f}_k^{[n-1]}$ are linearly dependent, which contradicts with Lemma 2.4. So in order to get the smallest k, we assume that $\widetilde{l}_1, \ldots, \widetilde{l}_k$ are linearly independent.

Thus we can apply Lemma 2.3 to (4.9), and this yields

$$(1+o(1))N(r)\leq (k-1)\sum_{j=1}^{k+1}\overline{N}(r,0,\widetilde{l}_j)\quad\text{as } r\to\infty \text{ n.e.}, \tag{4.13}$$

where $N(r)=\sup_{1\leq j\leq k+1}N(r,0,\widetilde{l}_j)$. Since $f_i=g_i/h_i$, where g_i and h_i are entire functions with no common zeros, from f_i and $\overline{f}_i$ share ∞ CM, we get that h_i and $\overline{h}_i$ share 0 CM. Also, from 0 is an exceptional paired value of f_i, we get that 0 is an exceptional paired value of g_i.

Next we analyze the multiplicity of zeros of $h_1\cdots\overline{h}_1^{[n-1]}\cdots h_{j-1}\cdots\overline{h}_{j-1}^{[n-1]}h_{j+1}\cdots\overline{h}_{j+1}^{[n-1]}\cdots h_k\cdots\overline{h}_k^{[n-1]}/(z^m d(z)))$. From (4.7) and (4.6), we know that all the zeros of $z^m d(z)$ arise from the zeros of at least one h_i ($i=1,\ldots,k$). Since f_i and $1/f_j$ do not have common zeros, we have that g_i and h_j do not have common zeros. Since h_i and $\overline{h}_i$ share 0 CM, so we have that $g_i\cdots\overline{g}_i^{[n-1]}$ and $h_j\cdots\overline{h}_j^{[n-1]}$ do not have common zeros. Thus $g_i\cdots\overline{g}_i^{[n-1]}$ and $h_1\cdots\overline{h}_1^{[n-1]}\cdots h_k\cdots\overline{h}_k^{[n-1]}$ do not have common zeros. So we have that the common zeros of $l_1,\ldots,l_k$ (i.e., the zeros of $z^m d(z)$) arise from the common zeros of

$$h_1\cdots\overline{h}_1^{[n-1]}\cdots h_{i-1}\cdots\overline{h}_{i-1}^{[n-1]}h_{i+1}\cdots\overline{h}_{i+1}^{[n-1]}\cdots h_k\cdots\overline{h}_k^{[n-1]}\quad (i=1,\ldots,k).$$

From h_i and $\overline{h}_i$ share 0 CM, thus we have

$$N\left(r,0,\frac{h_1\cdots\overline{h}_1^{[n-1]}\cdots h_{i-1}\cdots\overline{h}_{i-1}^{[n-1]}h_{i+1}\cdots\overline{h}_{i+1}^{[n-1]}\cdots h_k\cdots\overline{h}_k^{[n-1]}}{z^m d(z)}\right)$$
$$\geq n\overline{N}\left(r,0,\frac{h_1\cdots\overline{h}_1^{[n-1]}\cdots h_{i-1}\cdots\overline{h}_{i-1}^{[n-1]}h_{i+1}\cdots\overline{h}_{i+1}^{[n-1]}\cdots h_k\cdots\overline{h}_k^{[n-1]}}{z^m d(z)}\right),$$

i.e.,

$$N\left(r,0,\widetilde{l}_i/g_i\cdots\overline{g}_i^{[n-1]}\right)\geq n\overline{N}\left(r,0,\widetilde{l}_i/g_i\cdots\overline{g}_i^{[n-1]}\right). \tag{4.14}$$

From above analysis, we also have

$$\begin{aligned}N(r,0,\widetilde{l}_{k+1})&=N(r,0,h_1\cdots\overline{h}_1^{[n-1]}\cdots h_k\cdots\overline{h}_k^{[n-1]}/(z^m d(z)))\\&\geq n\overline{N}\left(r,0,h_1\cdots\overline{h}_1^{[n-1]}\cdots h_k\cdots\overline{h}_k^{[n-1]}/(z^m d(z))\right)\\&=n\overline{N}(r,0,\widetilde{l}_{k+1}).\end{aligned} \tag{4.15}$$

From (4.14), (4.15), (4.13) and 0 is the exceptional paired value of g_i, we obtain that

$$(1+o(1))N(r)\leq(k-1)\left(\sum_{j=1}^{k}\overline{N}\left(r,0,\frac{\widetilde{l}_j}{g_j\cdots\overline{g}_j^{[n-1]}}\cdot g_j\cdots\overline{g}_j^{[n-1]}\right)+\overline{N}(r,0,\widetilde{l}_{k+1})\right)$$
$$\leq(k-1)\left(\sum_{j=1}^{k}\left(\overline{N}\left(r,0,g_j\cdots\overline{g}_j^{[n-1]}\right)+\overline{N}\left(r,0,\frac{\widetilde{l}_j}{g_j\cdots\overline{g}_j^{[n-1]}}\right)\right)+\overline{N}(r,0,\widetilde{l}_{k+1})\right)$$
$$\leq(k-1)\left(\sum_{j=1}^{k}\left(\overline{N}(r,0,\overline{g}_j^{[n-1]})+\frac{1}{n}N\left(r,0,\frac{\widetilde{l}_j}{g_j\cdots\overline{g}_j^{[n-1]}}\right)\right)+\frac{1}{n}N(r,0,\widetilde{l}_{k+1})\right)$$
$$\leq(k-1)\left(\sum_{j=1}^{k}\left(\overline{N}(r+(n-1)|c|,0,g_j)+\frac{1}{n}N\left(r,0,\frac{\widetilde{l}_j}{g_j\cdots\overline{g}_j^{[n-1]}}\right)\right)+\frac{1}{n}N(r,0,\widetilde{l}_{k+1})\right).$$

From (4.4), we have

$$\limsup_{r\to\infty}\frac{\log\log\overline{N}(r,0,g_i)}{\log r}\leq\lambda_2(g_i)<1.$$

Thus by Lemma 2.1, and from 0 is an exceptional paired value of g_i, we have

$$\begin{aligned}&(1+o(1))N(r)\\&\leq(k-1)\left(\sum_{j=1}^{k}\left(\overline{N}(r,0,g_j)+\frac{1}{n}N\left(r,0,\frac{\widetilde{l}_j}{g_j\cdots\overline{g}_j^{[n-1]}}\right)\right)+\frac{1}{n}N(r,0,\widetilde{l}_{k+1})\right)\\&\leq(k-1)\frac{1}{n}\left(\sum_{j=1}^{k}\left(N(r,0,g_j\cdots\overline{g}_j^{[n-1]})+N\left(r,0,\frac{\widetilde{l}_j}{g_j\cdots\overline{g}_j^{[n-1]}}\right)\right)\right.\\&\qquad\left.+N(r,0,\widetilde{l}_{k+1})\right).\end{aligned} \tag{4.16}$$

Since $\widetilde{l}_j$, g_j and $\frac{\widetilde{l}_j}{g_j\cdots\overline{g}_j^{[n-1]}}$ are all entire functions, we have

$$N(r,0,g_j\cdots\overline{g}_j^{[n-1]})+N\left(r,0,\frac{\widetilde{l}_j}{g_j\cdots\overline{g}_j^{[n-1]}}\right)=N\left(r,0,\widetilde{l}_j\right). \tag{4.17}$$

Therefore, combining (4.16) with (4.17), we have

$$(1+o(1))N(r)\le(k-1)\frac{1}{n}\left(\sum_{j=1}^{k+1}N(r,0,\widetilde{l}_j)\right)\le(k-1)\frac{1}{n}(k+1)N(r), \tag{4.18}$$

as $r\to\infty$ n.e. From Lemma 2.3, we get that $N(r)\to\infty$ as $r\to\infty$ n.e. Combining this with (4.18), we obtain that $n\le k^2-1$. □

5. Proof of Theorem 1.6

Suppose that $f_1,\dots,f_k$ are nonconstant polynomial functions whose zeros are of multiplicity no less than n.

First, we prove that $k>1$. If $k=1$, then from (1.2) we have that

$$\deg(f_1\dots\overline{f}_1^{[n-1]})=n\deg(f_1)=\deg 1=0,$$

a contradiction. Thus we have $k\ge 2$.

In order to get the smallest k, we can assume using Lemma 2.4 that the polynomial functions $f_i\dots\overline{f}_i^{[n-1]}(i=1,\dots,k)$ are linearly independent.

Obviously, from (1.2) we can get that $f_i\dots\overline{f}_i^{[n-1]}(i=1,\dots,k)$ do not have common zeros. Set $d=\max\{d(f_1),\dots,d(f_k)\}$. Then for $i=1,\dots,n-1$, we have

$$\max\left\{d(\overline{f}_1^{[i]}),\dots,d(\overline{f}_k^{[i]})\right\}=d,$$

and $d>0$. Choosing $g_j=f_j\cdots\overline{f}_j^{[n-1]}$ $(j=1,\dots,k)$ and $g_{k+1}=1$, we have that the assumption (2.1) is satisfied. Thus, it follows by (2.3) in Lemma 2.3 that

$$\begin{aligned}n\cdot d&=n\cdot\max\{d(f_1),\dots,d(f_k)\}=\max\left\{d(f_1\dots\overline{f}_1^{[n-1]}),\dots,d(f_k\dots f_k^{[n-1]})\right\}\\&=\max\{d(g_1),\dots,d(g_k)\}\le(k-1)\left\{\sum_{j=1}^{k}\overline{d}(g_j)-\frac{1}{2}k\right\}\\&=(k-1)\left\{\sum_{j=1}^{k}\overline{d}(f_j\dots\overline{f}_j^{[n-1]})-\frac{1}{2}k\right\}.\end{aligned} \tag{5.1}$$

Since the zeros of $f_1,\dots,f_k$ are of multiplicity no less than n, so by (5.1) we have

$$\begin{aligned}n\cdot d&\le(k-1)\left\{\sum_{j=1}^{k}(\overline{d}(f_j)+\ldots+\overline{d}(\overline{f}_j^{[n-1]}))-\frac{1}{2}k\right\}\\&\le(k-1)\left\{\sum_{j=1}^{k}\frac{1}{n}(d(f_j)+\ldots+d(\overline{f}_j^{[n-1]}))-\frac{1}{2}k\right\}\\&\le(k-1)\left\{k\cdot d-\frac{1}{2}k\right\}<(k-1)k\cdot d,\end{aligned}$$

which yields $n < k^2 - k$. □

6. Proof of Theorem 1.5

Suppose that $f_1, \ldots, f_k$ are nonconstant rational functions whose zeros and poles are of multiplicity positive integer multiple of n, where $n\,(\geq 2)$ is a positive integer.

Suppose that at least one f_j is not polynomial. We set

$$f_j = \frac{P_j}{Q_j}, \tag{6.1}$$

where P_j and Q_j are polynomials without common zeros, and the zeros of P_j and Q_j are of multiplicity positive integer multiple of n. Substituting (6.1) into (1.2), then we obtain that

$$R_1 + R_2 + \cdots + R_k = R_{k+1}, \tag{6.2}$$

where

$$R_i = P_i \cdots \overline{P}_i^{[n-1]} Q_1 \cdots \overline{Q}_1^{[n-1]} \cdots Q_{i-1} \cdots \overline{Q}_{i-1}^{[n-1]} Q_{i+1} \cdots \overline{Q}_{i+1}^{[n-1]} \cdots Q_k \cdots \overline{Q}_k^{[n-1]}, \tag{6.3}$$

$i = 1, \ldots, k$, and

$$R_{k+1} = Q_1 \cdots \overline{Q}_1^{[n-1]} \cdots Q_k \cdots \overline{Q}_k^{[n-1]}. \tag{6.4}$$

If $R_1, \ldots, R_k$ have common zeros $\{z_i\}$ with multiplicities m_i, then the number of common zeros must be finite. We let $d(z) = \prod (z - z_i)^{m_i}$, then $d(z)$ is a polynomial. By assumption, it follows that $n | m_i$.

We set

$$\widetilde{R}_i = \frac{R_i}{d}, \quad i = 1, \ldots, k+1. \tag{6.5}$$

Then we have that $\widetilde{R}_1, \ldots, \widetilde{R}_{k+1}$ are polynomials. Combining (6.2) with (6.5) we have

$$\widetilde{R}_1 + \widetilde{R}_2 + \cdots + \widetilde{R}_k = \widetilde{R}_{k+1}, \tag{6.6}$$

where $\widetilde{R}_1, \ldots, \widetilde{R}_k$ have no common zeros. In addition, by $n | m_i$ and the assumption that the zeros of P_j and Q_j are of multiplicity positive integer multiple of n, it follows from (6.3), (6.4) and (6.5) that

$$N\left(r, \frac{1}{\widetilde{R}_i}\right) \geq n\overline{N}\left(r, \frac{1}{\widetilde{R}_i}\right), \quad \text{for } i = 1, \ldots, k+1. \tag{6.7}$$

Next, we prove that $k > 1$. If $k = 1$, then we have $R_1 = R_2$. From (6.3) and (6.4), it follows that

$$P_1 \cdots \overline{P}_1^{[n-1]} \equiv Q_1 \cdots \overline{Q}_1^{[n-1]}. \tag{6.8}$$

Since P_1 and Q_1 have no common zeros, and zeros of $\overline{P}_1^{[j]}$ and $\overline{Q}_1^{[j]}$ are zeros of P_1 and Q_1 after shifting with jc in the same direction respectively, it follows that (6.8) yields a contradiction.

In order to get the smallest k, we assume that the functions $\widetilde{R}_1, \ldots, \widetilde{R}_k$ are linearly independent. Otherwise, if the functions $\widetilde{R}_1, \ldots, \widetilde{R}_k$ are linearly dependent, then there exists $\alpha_1, \ldots, \alpha_k$, not all of which are zeros, such that

$$\alpha_1 \widetilde{R}_1 + \ldots + \alpha_k \widetilde{R}_k = 0. \tag{6.9}$$

Following a similar argument as in the proof of Theorem 1.2, we get that $f_1 \cdots \overline{f}_1^{[n-1]}$, $\ldots, f_k \cdots \overline{f}_k^{[n-1]}$ are linearly dependent, which contradicts with the Lemma 2.4. So in order to get the smallest k, we assume that $\widetilde{R}_1, \ldots, \widetilde{R}_k$ are linearly independent.

We set $d' = \max\{d(\widetilde{R}_1), \ldots, d(\widetilde{R}_k)\}$. Then from (6.6), we know that

$$d(\widetilde{R}_{k+1}) = d(\widetilde{R}_1 + \widetilde{R}_2 + \cdots + \widetilde{R}_k) \le \max\{d(\widetilde{R}_1), \ldots, d(\widetilde{R}_k)\} = d'.$$

Thus combining this with (6.7), we can apply Lemma 2.3 (a) to (6.6), and obtain

$$d' \le (k-1)\left\{\sum_{j=1}^{k+1} \overline{d}(\widetilde{R}_j) - \frac{1}{2}k\right\} \le (k-1)\left\{(k+1)\frac{d'}{n} - \frac{1}{2}k\right\} < (k^2-1)\frac{d'}{n},$$

which yields $n < k^2 - 1$. □

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